

UNIVERSAL LOG STRUCTURES ON SEMI-STABLE VARIETIES

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Abstract. Given a morphism of schemes which is flat, proper, and “fiber-by-fiber semi-stable”, we study the problem of extending the morphism to a morphism of fine log schemes, which is log smooth, integral, and vertical. The problem is rephrased in terms of a functor on the category of fine log schemes over the base, and the main result of the paper is that this functor is representable by a fine log scheme whose underlying scheme maps naturally to the base by a monomorphism of finite type. In the course of the proof, we also generalize results of Kato on the existence of log structures of embedding and semi-stable type.

1. Introduction. In light of the powerful tools of logarithmic geometry developed in recent years, an important question to ask, when given a singular morphism $f : \underline{X} \rightarrow \underline{S}$ of schemes, is whether there exist fine log structures on \underline{X} and \underline{S} making f a log smooth morphism. In addition, if such log structures exist one would like to know to what extent they are unique. This paper is an attempt to understand the situation for morphisms which are fiber-by-fiber semi-stable (see below for the precise definition). Our inspiration comes from the well-understood case of semi-stable curves, and the main result of this paper can be viewed as a generalization of the statement that the Deligne-Mumford compactification of the moduli space of curves of genus $g > 1$ is a moduli stack for certain log curves ([10]). This work was heavily influenced by the papers of Kato ([9], [10]).

Let $f : \underline{X} \rightarrow \underline{S}$ be a flat and proper morphism of schemes of finite type over an excellent Dedekind ring, and suppose that for every geometric point $\bar{s} \rightarrow \underline{S}$ the fiber $\underline{X}_{\bar{s}}$ is a semi-stable variety in the sense of the following definition:

DEFINITION 1.1. A scheme \underline{Y}/k over a separably closed field k is a *semi-stable variety* if for each closed point $y \in \underline{Y}$ there exists an étale neighborhood (\underline{U}, y') of y , integers $r \geq l$, and an étale morphism

$$\underline{U} \rightarrow k[X_1, \dots, X_r]/(X_1 \cdots X_l),$$

sending y' to the point $X_1 = \cdots = X_r = 0$ (we make the convention that if $l = 0$, then $(X_1 \cdots X_l) = (0)$).

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Let \mathbf{FLog} denote the category of fine log schemes over \underline{S} , and define the *functor of integral and vertical log structures*

$$\mathbf{IVLS} : \mathbf{FLog} \rightarrow \mathbf{Set}$$

as follows: for any fine log scheme T over \underline{S} , $\mathbf{IVLS}(T)$ is the set of isomorphism classes of pairs $(\mathcal{M}_{X_T}, f_T^b)$, where \mathcal{M}_{X_T} is a fine log structure on $\underline{X}_T := \underline{X} \times_{\underline{S}} \underline{T}$ and $f_T^b : \mathrm{pr}_2^* \mathcal{M}_T \rightarrow \mathcal{M}_{X_T}$ is a morphism of log structures making

$$(\mathrm{pr}_2, f_T^b) : (\underline{X}_T, \mathcal{M}_{X_T}) \rightarrow T$$

a log smooth, vertical, and integral morphism (recall that a morphism is vertical if $\mathcal{M}_{X_T/T} := \mathrm{Coker}(\mathrm{pr}_2^* \mathcal{M}_T \rightarrow \mathcal{M}_{X_T})$ is a sheaf of groups). The main result of this paper is the following theorem:

THEOREM 1.2. *The functor \mathbf{IVLS} is representable by a log scheme, and the map on underlying schemes $\mathbf{IVLS} \rightarrow \underline{S}$ is a monomorphism of finite type.*

The paper is organized as follows.

Section 2 is devoted to showing that if $f : X \rightarrow S$ is a proper, log smooth, integral, and vertical morphism all of whose geometric fibers are semi-stable varieties, then there exists a canonical cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X^\sharp \\ f \downarrow & & \downarrow f^\sharp \\ S & \xrightarrow{\phi} & S^\sharp, \end{array}$$

where $f^\sharp : X^\sharp \rightarrow S^\sharp$ has a special form and the underlying maps of schemes of ϕ and ψ are isomorphisms. This result will be used to show that the functor \mathbf{IVLS} is equivalent to the functor defined by a pair (F, \mathcal{M}_F) , where F is a functor on the category of \underline{S} -schemes and \mathcal{M}_F is a “log structure” on F .

More precisely, let $\mathcal{Log}_{\underline{S}}$ denote the fibered category over the category of \underline{S} -schemes whose fiber over an \underline{S} -scheme \underline{T} is the groupoid of fine log structures \mathcal{M}_T on \underline{T} . It is shown in ([17], 1.1) that $\mathcal{Log}_{\underline{S}}$ is an algebraic stack. By definition, a log structure on a fibered category \mathcal{S} is a morphism of fibered categories $\mathcal{S} \rightarrow \mathcal{Log}_{\underline{S}}$. Viewing functors on the category of \underline{S} -schemes as fibered categories, we get a notion of a log structure on a functor.

Now, suppose (F, \mathcal{M}_F) is a functor with a log structure. Then for any element $t \in F(\underline{T})$ over some \underline{S} -scheme \underline{T} , we get a log structure $t^* \mathcal{M}_F$ on \underline{T} by viewing t as a morphism of functors $\underline{T} \rightarrow F$ and defining $t^* \mathcal{M}_F$ to be the log structure obtained from the composite $\mathcal{M}_F \circ t : \underline{T} \rightarrow \mathcal{Log}_{\underline{S}}$. This enables us to define a functor F^{log} on \mathbf{FLog} as follows: for any $T = (\underline{T}, \mathcal{M}_T)$, $F^{\mathrm{log}}(T)$ is the set of pairs (t, t^b) , where $t \in F(\underline{T})$ and $t^b : t^* \mathcal{M}_F \rightarrow \mathcal{M}_T$ is a morphism of log structures. To say that \mathbf{IVLS} is equivalent to the functor defined by a pair (F, \mathcal{M}_F) means that there exists an isomorphism of functors $\mathbf{IVLS} \simeq F^{\mathrm{log}}$. The importance of this is that the proof of Theorem 1.2 is reduced to showing that the (ordinary) functor F is representable by a scheme, and that $F \rightarrow \underline{S}$ is a monomorphism of finite type.

In Section 3 we generalize results of Kato ([9]) on the existence of log structures of semi-stable and embedding type (see the section for definitions) to an arbitrary base scheme. Our generalization of Kato’s work includes cohomological obstructions for the existence of log structures of semi-stable and embedding type and so might also be of some interest for schemes over fields. This section can be read independently from the rest of the paper.

In Section 4 we use the results of Sections 2 and 3 to prove a theorem about effectivity of certain formal log schemes. This theorem will be used in Section 5.

Finally, in Section 5 we bring it all together to prove Theorem 1.2. The proof is based on Artin’s method ([5], 5.4) and the results of ([17], [18]).

1.1. CONVENTIONS. Throughout this paper we denote schemes by underlined letters (e.g., \underline{X}) and log schemes by unadorned letters (e.g., X). For a log scheme X , we denote the underlying scheme by \underline{X} . The reader is assumed to be familiar with logarithmic geometry at the level of ([11], [16]) and with the stack-theoretic approach introduced in ([17]). Our conventions about algebraic stacks are those of ([12]), except we only assume that our stacks are locally quasi-separated and not necessarily quasi-separated ($\mathcal{L}og_S$ is not quasi-separated).

2. Special elements of IVLS. Let $f : X \rightarrow S$ be a smooth, proper, integral, and vertical morphism of noetherian log schemes, and suppose that for every geometric point $\bar{x} \rightarrow \underline{S}$, the underlying scheme of $X_{\bar{x}}$ is a semi-stable variety (in the sense of Definition 1.1). The purpose of this section is to show that under these assumptions the log structure \mathcal{M}_X on \underline{X} is induced by base change from a smooth morphism

$$f^\sharp : X^\sharp \rightarrow S^\sharp$$

of a special form whose underlying morphism of schemes is that of the original f .

DEFINITION 2.1. A log smooth morphism $f : X \rightarrow S$ is *essentially semi-stable* if for each geometric point $\bar{x} \rightarrow \underline{X}$ the monoids $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}}$ and $\overline{\mathcal{M}}_{X,\bar{x}}$ are free monoids, and if for suitable isomorphisms $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \simeq \mathbf{N}^r$ and $\overline{\mathcal{M}}_{X,\bar{x}} \simeq \mathbf{N}^{r+s}$ the map

$$(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \rightarrow \overline{\mathcal{M}}_{X,\bar{x}}$$

is of the form

$$(2.1) \quad e_i \mapsto \begin{cases} e_i & \text{if } i \neq r, \\ e_r + e_{r+1} + \cdots + e_{r+s} & \text{if } i = r, \end{cases}$$

where e_i denotes the i -th standard generator of \mathbf{N}^r .

LEMMA 2.2. *If $f : X \rightarrow S$ is essentially semi-stable, then étale locally on X and S there exist charts $\mathbf{N}^r \rightarrow \mathcal{M}_S, \mathbf{N}^{r+s} \rightarrow \mathcal{M}_X$ such that the map $\mathbf{N}^r \rightarrow \mathbf{N}^{r+s}$ given by formula 2.1 is a chart for f , and such that the map*

$$\mathcal{O}_S \otimes_{\mathbf{Z}[\mathbf{N}^r]} \mathbf{Z}[\mathbf{N}^{r+s}] \rightarrow \mathcal{O}_X$$

is smooth.

PROOF. Observe that if $s \in \underline{S}$ is a point, then the stalk $\overline{\mathcal{M}}_{S,\overline{s}}$ is a free monoid and hence in some étale neighborhood of s there exists a chart $N^r \rightarrow \mathcal{M}_S$ such that the induced map $N^r \rightarrow \overline{\mathcal{M}}_{S,\overline{s}}$ is bijective. If $x \in \underline{X}$ is a point lying over s , then by ([15], 2.25) there exists in some étale neighborhood of x a chart

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M}_X \\ \uparrow & & \uparrow \\ N^r & \longrightarrow & \mathcal{M}_S \end{array}$$

such that the induced map

$$\mathcal{O}_S \otimes_{\mathbf{Z}[N^r]} \mathbf{Z}[P] \rightarrow \mathcal{O}_X$$

is smooth and such that the map $P \rightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ is bijective. From the bijectivity of $P \rightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ we conclude that P is a free monoid, and that the map $N^r \rightarrow P$ has the desired form (after perhaps applying an automorphism of N^r). \square

REMARK 2.3. The above lemma shows that the notion of an essentially semi-stable morphism is a natural generalization of the notion of a *normal crossing log variety* introduced in ([14], §2).

LEMMA 2.4. *An essentially semi-stable morphism $f : X \rightarrow S$ is integral and vertical.*

PROOF. To see the integrality we have to check that the map $N^r \rightarrow N^{r+s}$ described in formula 2.1 is integral. For this observe that if $\Delta : N \rightarrow N^{s+1}$ is the diagonal map, then there exists a natural cocartesian diagram

$$\begin{array}{ccc} N & \xrightarrow{e_1 \mapsto e_r} & N^r \\ \Delta \downarrow & & \downarrow \\ N^{s+1} & \xrightarrow{e_i \mapsto e_{i+r-s}} & N^{r+s}, \end{array}$$

and since the map Δ is integral, so is the map $N^r \rightarrow N^{r+s}$ ([11], 4.1 (i)). To see that it is vertical, let Q be the cokernel of $N^r \rightarrow N^{r+s}$. By definition of cokernel, to give a map $Q \rightarrow M$ to an integral monoid M is equivalent to giving a map $N^{r+s} \rightarrow M$ such that the composite $N^r \rightarrow N^{r+s} \rightarrow M$ is zero. A map $N^{r+s} \rightarrow M$ is equivalent to giving elements

$$m_1, \dots, m_{r+s} \in M,$$

and the condition that the map $N^r \rightarrow M$ be zero means that

$$m_1 = m_2 = \dots = m_{r-1} = m_r + \dots + m_{r+s} = 0.$$

Thus giving an arrow $Q \rightarrow M$ is equivalent to giving elements m_r, \dots, m_{r+s} in M^* such that $m_r + \dots + m_{r+s} = 0$. This in turn is equivalent to giving elements $m_r, \dots, m_{r+s-1} \in M^*$. But the functor on the category of integral monoids

$$M \mapsto \{r + s - 1\text{-tuples of elements } m \in M^*\}$$

is represented by \mathbf{Z}^{r+s-1} , and hence $Q \simeq \mathbf{Z}^{r+s-1}$. \square

Recall that if P is a sharp monoid (this means that $P^* = \{0\}$), then an element $p \in P - \{0\}$ is *irreducible* if for any equality $p_1 + p_2 = p$ in P we have $p_1 = 0$ or $p_2 = 0$. The basic results about irreducible elements is the following proposition whose proof we omit (see [16], I.2.1.2):

PROPOSITION 2.5. *If P is a fine sharp monoid, then the set $\text{Irr}(P)$ of irreducible elements in P is a finite set which generates P .*

Suppose $\underline{S} = \text{Spec}(k)$, where k is a separably closed field, and that $f : X \rightarrow S$ is essentially semi-stable. Let $x \in \underline{X}$ be a singular point. Then by Lemma 2.2 there exists a chart

$$\begin{array}{ccccc} N^{r+s} & \longrightarrow & \mathcal{M}_X & \longrightarrow & \mathcal{O}_X \\ \uparrow & & \uparrow & & \uparrow \\ N^r & \longrightarrow & \mathcal{M}_S & \xrightarrow{\alpha} & k \end{array}$$

in an étale neighborhood of x such that

$$k \otimes_{\mathbb{Z}[N^r]} \mathbb{Z}[N^{r+s}] \simeq k[x_r, \dots, x_{r+s}] / (x_r \cdots x_{r+s} - \alpha(e_r)) \rightarrow \mathcal{O}_X$$

is smooth. Since x is a singular point, it follows that $\alpha(e_r) = 0$ and hence the map

$$\overline{\mathcal{M}}_S \rightarrow \overline{\mathcal{M}}_{X,\bar{x}}$$

is of the form $N^{r'} \rightarrow N^{r'+s}$ as in Definition 2.1 for some $r' \leq r$. It follows that if $\text{Irr}(\overline{\mathcal{M}}_S)$ denotes the set of irreducible elements in $\overline{\mathcal{M}}_S$, then there is a unique element in $\text{Irr}(\overline{\mathcal{M}}_S)$ whose image in $\overline{\mathcal{M}}_{X,\bar{x}}$ is not irreducible. This defines a canonical map

$$s_X : \{\text{singular points of } X\} \rightarrow \text{Irr}(\overline{\mathcal{M}}_S).$$

DEFINITION 2.6. An essentially semi-stable morphism of log schemes $f : X \rightarrow S$ is *special at a geometric point \bar{s}* if the map

$$s_{X_{\bar{s}}} : \{\text{singular points of } X_{\bar{s}}\} \rightarrow \text{Irr}(\overline{\mathcal{M}}_{S,\bar{s}})$$

induces a bijection between the set of connected components of the singular locus of $X_{\bar{s}}$ and $\text{Irr}(\overline{\mathcal{M}}_{S,\bar{s}})$. If f is special at every geometric point $\bar{s} \rightarrow \underline{S}$, then we call f a *special morphism*.

THEOREM 2.7. *Let $f : X \rightarrow S$ be a smooth, proper, integral, and vertical morphism of noetherian log schemes, and suppose that for every geometric point $\bar{s} \rightarrow \underline{S}$ the underlying scheme of $X_{\bar{s}}$ is a semi-stable variety. Then there exist a pair of log structures $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp)$ on \underline{S} and \underline{X} , respectively, and a morphism*

$$f^\sharp : X^\sharp = (\underline{X}, \mathcal{M}_X^\sharp) \rightarrow S^\sharp = (\underline{S}, \mathcal{M}_S^\sharp),$$

which is special, together with morphisms of log structures

$$\phi : \mathcal{M}_S^\sharp \rightarrow \mathcal{M}_S, \quad \psi : \mathcal{M}_X^\sharp \rightarrow \mathcal{M}_X,$$

which make the diagram

$$\begin{array}{ccc} X & \longrightarrow & X^\sharp \\ f \downarrow & & \downarrow f^\sharp \\ S & \longrightarrow & S^\sharp \end{array}$$

cartesian. Moreover, the datum $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ is unique up to a unique isomorphism.

The proof occupies the remainder of this section.

Let us begin by considering the case when \underline{S} is the spectrum of a strictly henselian local ring \mathcal{O} , and \mathcal{M}_S admits a chart $Q \rightarrow \mathcal{M}_S$ such that $Q \rightarrow \overline{\mathcal{M}}_{S,s_0}$ is bijective (here s_0 denotes the closed point). Denote by T the set of connected components of the singular locus of the closed fiber.

Recall ([16], I.1.1) that if P is a monoid, then a congruence relation on P is a subset $E \subset P \oplus P$ which is both a submonoid and a set-theoretic equivalence relation. A subset $S \subset E$ generates the congruence relation E if E is the smallest congruence relation on P containing S . For any equivalence relation E on P , the surjection $P \rightarrow P/E$ induces a structure of a monoid on P/E if and only if E is a congruence relation. Therefore, there is a natural bijection between isomorphism classes of surjective maps of monoids $P \rightarrow P'$ and the set of congruence relations on P .

LEMMA 2.8. *Let P be a fine sharp monoid, $Q \hookrightarrow P$ an integral morphism (such a morphism is automatically injective by [11], paragraph preceding 4.7), and let $\pi : P \rightarrow P/Q$ be the projection map.*

1. *For each $i \in P/Q$, there exists a unique element $p_i \in P$ such that*

$$\pi^{-1}(i) = \{p_i + q \mid q \in Q\}.$$

2. *If $p \in P$ is an irreducible element not in Q and if $i = \pi(p)$, then $p = p_i$ (notation is in the part 1).*

3. *If $\{p_{i_1}, \dots, p_{i_n}\}$ denotes the set of irreducible elements of P which are not in Q (we index these irreducible elements by their images in P/Q), then the congruence relation on $Q \oplus N^n$ defined by the surjection*

$$Q \oplus N^n \rightarrow P, \quad (q, (\underline{n})) \mapsto q + \sum_{j=1}^n n_j p_{i_j}$$

is generated by equalities

$$\sum_{j=1}^n m_j p_{i_j} = q + \sum_{j=1}^n n_j p_{i_j}$$

in P , where for each j either m_j or n_j is equal to 0.

PROOF. 1. Since P is fine and sharp, there exists a partial ordering on P^+ defined by $p_1 \leq p_2$ if there exists $p_3 \in P$ such that $p_1 + p_3 = p_2$. By ([16], I.2.1.5.3) the set $\pi^{-1}(i)$ contains a unique finite set of minimal elements with respect to this partial ordering.

Suppose $p_1, p_2 \in \pi^{-1}(i)$ and suppose p_1 is a minimal element. By construction of the quotient, there exists $q_1, q_2 \in Q$ such that $p_1 + q_1 = p_2 + q_2$. Since the map is integral, there exists $q_3, q_4 \in Q$ and $p \in P$ such that

$$p_1 = q_3 + p, \quad p_2 = q_4 + p, \quad q_1 + q_3 = q_2 + q_4.$$

By definition of quotient, $p \in \pi^{-1}(i)$ and $p \leq p_1$. Since p_1 was minimal, it follows that $p_1 = p$ and that $p_2 = p_1 + q_4$.

2. By definition of p_i , there exists $q \in Q$ such that $p = p_i + q$. Since p is irreducible and has non-zero image in P/Q this means that $q = 0$.

3. Since $\text{Irr}(P)$ generates P , the congruence relation is generated by equalities

$$q + \sum_j n_{ij} p_{ij} = q' + \sum_j m_{ij} p_{ij}$$

in P . By the integrality of $Q \rightarrow P$, for any such equality there exist $p \in P$ and $q_3, q_4 \in Q$ such that

$$\sum_j n_{ij} p_{ij} = p + q_3, \quad \sum_j m_{ij} p_{ij} = p + q_4, \quad q + q_3 = q + q_4.$$

In addition, writing $p = \sum b_{ij} p_{ij} + q_5$, we see that the congruence relation on $Q \oplus N^n$ defined by the surjection

$$Q \oplus N^n \rightarrow P$$

is generated by equalities in P of the form

$$\sum_j n_{ij} p_{ij} = \sum_j b_{ij} p_{ij} + q.$$

Moreover, since P is cancellative, we can assume that for each j either $n_{ij} = 0$ or $b_{ij} = 0$. □

COROLLARY 2.9. *Let t be a singular point of X , and let $n \in \mathcal{M}_{X/S, \bar{t}}$. Then there exists a unique element $p_n \in \overline{\mathcal{M}}_{X, \bar{t}}$ such that*

$$\pi^{-1}(n) = \{p_n + q \mid q \in \overline{\mathcal{M}}_{S, s_0}\}.$$

PROOF. The map $\overline{\mathcal{M}}_{S, s_0} \rightarrow \overline{\mathcal{M}}_{X, \bar{t}}$ is integral by assumption. □

As above let s_0 denote the closed point of S , and let X_{s_0} denote the fiber. By ([15], 2.25), for every singular closed point t of X_{s_0} there exists a chart

$$(2.2) \quad \begin{array}{ccc} P & \xrightarrow{\beta} & \mathcal{M}_X \\ \uparrow & & \uparrow \\ Q & \xrightarrow{\alpha} & \mathcal{M}_S \end{array}$$

in some étale neighborhood of t such that

- (1) $P \rightarrow \overline{\mathcal{M}}_{X, \bar{t}}$ is bijective,

(2) the map

$$\mathcal{O} \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P] \rightarrow \mathcal{O}_X$$

is smooth.

Let $\{p_{i_1}, \dots, p_{i_n}\}$ be the set of irreducible elements in P which are not in Q (as in Lemma 2.8 we index these irreducible elements by their images in P/Q). If $i \in P/Q$ is any element and p_i is as in 2.8, then p_i can be written as

$$p_i = \sum m_j p_{i_j}.$$

Indeed, writing p_i as a sum of irreducible elements, we have

$$p_i = \sum m_j p_{i_j} + q, \quad q \in Q,$$

and by definition of p_i we must have $q = 0$. Hence the map

$$\gamma : \mathcal{O}[Y_{i_1}, \dots, Y_{i_n}] \rightarrow \mathcal{O} \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P], \quad Y_{i_j} \mapsto 1 \otimes e(p_{i_j})$$

is surjective. Let J be the kernel of γ .

PROPOSITION 2.10. (1) *There exists an element $q_0 \in Q$ such that*

$$\sum_{j=1}^n p_{i_j} = q_0.$$

(2) *J is generated by the element*

$$\prod_{j=1}^n Y_{i_j} - \alpha(q_0).$$

(3) *Let $Q \oplus_N N^n$ be the pushout of the diagram*

$$\begin{array}{ccc} N & \xrightarrow{\Delta} & N^n \\ 1 \mapsto q_0 \downarrow & & \\ Q & & \end{array}$$

Then the natural map

$$Q \oplus_N N^n \rightarrow P, \quad (q, (\underline{n}_j)) \mapsto q + \sum n_{i_j} p_{i_j}$$

is an isomorphism.

PROOF. We first reduce to the case when $\mathcal{O} = k(s_0)$. The reduction step is trivial for (1) and (3). Let us show that (2) holds if the proposition is true on the closed fiber. Since

$$\sum_{j=1}^n p_{i_j} = q_0, \quad q_0 \in Q,$$

there is an inclusion of ideals

$$\left(\prod Y_{i_j} - \alpha(q_0) \right) \subset J,$$

which defines a closed immersion

$$\mathrm{Spec}(\mathcal{O}[Y_{i_1}, \dots, Y_{i_n}]/J) \hookrightarrow \mathrm{Spec}(\mathcal{O}[Y_{i_1}, \dots, Y_{i_n}]/\left(\prod Y_{i_j} - \alpha(q_0)\right)).$$

Since both of these schemes are flat over \mathcal{O} and the reduction is an isomorphism, the closed immersion is an isomorphism also by ([1], VII.1.9). Therefore (2) holds and hence it suffices to consider the case when $\mathcal{O} = k$ for some separably closed field k .

LEMMA 2.11. *The ideal $J = \mathrm{Ker}(\gamma)$ is generated by elements of the form*

$$\prod_j Y_{i_j}^{n_{i_j}} - \alpha(q) \prod_j Y_{i_j}^{m_{i_j}},$$

where in P

$$\sum_j n_{i_j} p_{i_j} = \sum_j m_{i_j} p_{i_j} + q, \quad q \in Q,$$

and for each j either n_{i_j} or m_{i_j} is equal to 0.

PROOF. This is a restatement of 2.8.3. Indeed, it suffices to show that the kernel of the map

$$\mathbf{Z}[Q \oplus N^n] \rightarrow \mathbf{Z}[P]$$

induced by the surjection

$$Q \oplus N^n \rightarrow P, \quad (q, (\underline{n}_j)) \mapsto q + \sum_j n_j p_{i_j}$$

is the ideal defined by any set of generators for the congruence relation on $Q \oplus N^n$. This follows from the universal property of monoid algebras; for any algebra R , there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Alg}}(\mathbf{Z}[P], R) \rightarrow \mathrm{Hom}_{\mathrm{Mon}}(P, R),$$

where R is viewed as a multiplicative monoid. □

LEMMA 2.12. (1) *The completion $\widehat{\mathcal{O}}_{X,t}$ is isomorphic to*

$$k[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n}]]/(Y_{i_1} \cdots Y_{i_n}),$$

where $\{T_1, \dots, T_s\}$ are independent variables.

(2) *Let $L \subset P$ be the submonoid generated by the set $\{p_i \mid i \in P/Q\}$. Then L is isomorphic to N^n .*

PROOF. Let Y' be the log scheme with underlying scheme $\mathrm{Spec}(k \otimes_{\mathbf{Z}[Q]} \mathbf{Z}[P])$ and log structure induced by $P \rightarrow \mathbf{Z}[P]$. Then there exists a diagram of log schemes

$$X \xrightarrow{h} Y' \longrightarrow S,$$

where h is log smooth and strict. Since h is strict, the map

$$\Omega_{X/S}^1 \rightarrow \Omega_{X/Y'}^1$$

is surjective, so in some neighborhood of t there exist elements f_1, \dots, f_s in the maximal ideal of t such that df_1, \dots, df_s map to a basis for $\Omega_{X/Y'}^1 \otimes k(t)$. This defines a map

$$h' : X \rightarrow Y' \times_{\underline{S}} \underline{A}_{\underline{S}}^s$$

of log schemes which is strict, and for which the map

$$h'^* \Omega_{Y' \times_{\underline{S}} \underline{A}_{\underline{S}}^s/Y'}^1 \otimes k(t) \rightarrow \Omega_{X/Y'}^1 \otimes k(t)$$

is an isomorphism. It follows that in some neighborhood of t the map h' is étale ([15], 2.20). Thus we can extend the map γ to an étale map

$$X \rightarrow \text{Spec}(k[T_1, \dots, T_s] \otimes_k (k[Y_{i_1}, \dots, Y_{i_n}]/J))$$

sending t to the point $T_1 = \dots = T_s = Y_{i_1} = \dots = Y_{i_n} = 0$.

First, we claim that the resulting map

$$k[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n}]] \rightarrow \widehat{\mathcal{O}}_{X,t}$$

induces an isomorphism

$$k[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n}]]/(T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n})^2 \simeq \widehat{\mathcal{O}}_{X,t}/\mathfrak{m}^2.$$

To see this it suffices to show that a set of defining equations for J are contained in the ideal $(Y_{i_1}, \dots, Y_{i_n})^2$. But J is generated by equations arising from non-trivial equalities

$$\sum n_{i_j} p_{i_j} = q + \sum m_j p_{i_j}$$

in P . If $\sum n_{i_j} = 1$, then the equality is

$$p_{i_{j_0}} = q + \sum m_j p_{i_j}$$

for some j_0 . By the irreducibility of p_{i_j} , this implies that the equation comes from an equality

$$p_{i_j} = q \quad \text{or} \quad p_{i_j} = p_{i_{j'}}.$$

Since $p_{i_j} \notin Q$, we must have $p_{i_j} = p_{i_{j'}}$ and hence the equation is trivial.

Fix an isomorphism

$$\widehat{\mathcal{O}}_{X,t} \simeq k[[X_1, \dots, X_l]]/(X_1 \cdots X_h)$$

for some l and h (this is possible by assumption), and let \mathfrak{m} denote the maximal ideal of $\widehat{\mathcal{O}}_{X,t}$. It follows from the above that

$$l = \dim(\mathfrak{m}/\mathfrak{m}^2) = n + s.$$

For each $k \geq 1$, the dimension of $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is equal to the number of monomials of degree k in X_1, \dots, X_l which do not contain the string $X_1 \cdots X_h$. In particular, the dimension of $\mathfrak{m}^h/\mathfrak{m}^{h+1}$ is equal to one less than the number of monomials of degree h in $l = n + s$ variables. This implies that we have at least one equation involving monomials in the Y_{i_j} of degree h . Fix one such equation $f(\underline{Y}) = 0$. By Lemma 2.11, we can choose our equation so that it is either of the form

$$f(\underline{Y}) = Y_{i_1} \cdots Y_{i_n} - Y_{i'_1} \cdots Y_{i'_m}$$

or of the form

$$f(\underline{Y}) = Y_{i_1} \cdots Y_{i_h}.$$

Let R be the ring

$$k[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_h}]]/(f),$$

and let $\psi : R \rightarrow \widehat{\mathcal{O}}_{X,t}$ be the natural map.

SUB-LEMMA 2.13. *Let $\psi : A \rightarrow B$ be a surjective local homomorphism between complete noetherian local rings. If the induced map*

$$\mathrm{gr}^*(\psi) : \mathrm{gr}_{\mathfrak{m}_A}^*(A) \rightarrow \mathrm{gr}_{\mathfrak{m}_B}^*(B)$$

is an isomorphism, then ψ is an isomorphism.

PROOF. Because A and B are complete with respect to \mathfrak{m}_A , it suffices to show that $A/\mathfrak{m}_A^n \rightarrow B/\mathfrak{m}_B^n$ is an isomorphism for all n . We may therefore assume that A and B are artinian.

Since the map ψ is an isomorphism modulo \mathfrak{m}_A , it is enough to show that B is flat over A (by Nakayama's lemma). The result therefore follows from the local criterion for flatness ([1], page 91). \square

To prove that ψ is an isomorphism, it therefore suffices to show that for every integer k the dimensions of $\mathfrak{m}_R^k/\mathfrak{m}_R^{k+1}$ and $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ are equal. But we can compute these dimensions as follows: If we define $N(k, l)$ to be the number of monomials of degree k in l variables, then the dimensions of $\mathfrak{m}_R^k/\mathfrak{m}_R^{k+1}$ and $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ are both equal to

$$N(k, l) - N(k - h, l) = N(k, n + s) - N(k - h, n + s).$$

Hence ψ is an isomorphism.

Next, we claim that f cannot be of the form

$$f(\underline{Y}) = Y_{i_1} \cdots Y_{i_h} - Y_{i'_1} \cdots Y_{i'_m}.$$

Since P/Q is a group, for every $p_{i_j} \in P$ there exists an element p_{-i_j} such that

$$\pi(p_{i_j}) + \pi(p_{-i_j}) = 0$$

in P/Q . Writing $p_{-i_j} = \sum n_a p_{i_a}$, we conclude that there exist natural numbers n_a such that

$$p_{i_j} + \sum_{a=1}^n n_a p_{i_a} \in \pi^{-1}(0).$$

By Lemma 2.8, $\pi^{-1}(0) = Q$ and hence there exists an element $q \in Q$ such that

$$p_{i_j} + \sum_{a=1}^n n_a p_{i_a} = q.$$

Since p_{i_j} is not a unit, q is non-zero, and hence there exist natural numbers n_a such that $Y_{i_j} \prod_{a=1}^n Y_{i_a}^{n_a} = 0$.

Since $\widehat{\mathcal{O}}_{X,t}$ is reduced, if $\prod_j Y_{i_j}^{m_j} = 0$ with the $m_j \geq 1$, then $\prod_j Y_{i_j} = 0$. It follows that $\prod_j Y_{i_j} = 0$. This is impossible if

$$f(\underline{Y}) = Y_{i_1} \cdots Y_{i_h} - Y_{i'_1} \cdots Y_{i'_m}.$$

Therefore we have an isomorphism

$$k[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n}]]/(Y_{i_1} \cdots Y_{i_h}) \simeq \widehat{\mathcal{O}}_{X,t}$$

for some h .

We claim that $h = n$. To see this note that

$$\sum_{j=1}^h p_{i_j} = p + q$$

for some $p \in P$ and a nonzero $q \in Q$. If p_{i_a} is another irreducible element with $a > h$, then since P/Q is a group, there exists an element p_{-i_a} as in 2.8 such that $p_{i_a} + p_{-i_a}$ maps to zero in P/Q . Write

$$p_{-i_a} = \sum_{j=1}^n b_{i_j} p_{i_j}.$$

Then at least one of the b_{i_j} with $1 \leq j \leq h$ must be zero, since otherwise we can write

$$p_{-i_a} = \sum_{j=1}^h p_{i_j} + \sum_{j=1}^h (b_{i_j} - 1) p_{i_j} + \sum_{j>h} b_{i_j} p_{i_j} = p + q + \sum_{j>h} b_{i_j} p_{i_j} + \sum_{j=1}^h (b_{i_j} - 1) p_{i_j}$$

contradicting the minimality of p_{-i_a} . It follows that $\prod_j Y_{i_j}^{b_{i_j}} \notin J$. On the other hand, $p_{i_a} + p_{-i_a} \in \pi^{-1}(0) = Q$ and since p_{i_a} is not a unit, $p_{i_a} + p_{-i_a}$ is a non-zero element in Q . Therefore $Y_{i_a} \prod_j Y_{i_j}^{b_{i_j}} \in J$, and hence Y_{i_a} is a zero divisor in the ring

$$k[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n}]]/(Y_{i_1} \cdots Y_{i_h}).$$

This is a contradiction so we must have $h = n$. This completes the proof of the part 1 of the lemma.

To see the second part, suppose we have an equality

$$\sum m_j p_{i_j} = \sum m'_j p_{i_j}$$

in P . Since P is cancellative, we can assume that at least one $m_j = 0$ and hence we get

$$\prod Y_{i_j}^{m_j} = \prod Y_{i_j}^{m'_j} \neq 0$$

in the ring

$$k(s_0)[[T_1, \dots, T_s, Y_{i_1}, \dots, Y_{i_n}]]/\left(\prod Y_{i_j}\right).$$

It follows that $m_j = m'_j$ for all j . This completes the proof of Lemma 2.12. \square

From Lemmas 2.8.3 and 2.12 it follows that the congruence relation on $Q \oplus N^n$ defined by the map to P is generated by equalities

$$\sum_j n_{i_j} p_{i_j} = q,$$

where all the n_{i_j} are greater than 0. In addition, from 2.12 we see that we must have an equality

$$\sum_{j=1}^n p_{i_j} = q_0$$

for some $q_0 \in Q$. This proves (1) of the proposition. Now suppose that we had an equality in P

$$\sum_j n_{i_j} p_{i_j} = q'$$

for some $q' \in Q$ and all $n_{i_j} \geq 1$. Then, after reordering the p_{i_j} when necessary, we can assume that n_{i_1} is the smallest of the n_{i_j} . This gives

$$q' = \sum_j n_{i_j} p_{i_j} = n_{i_1} q_0 + \sum_j (n_{i_j} - n_{i_1}) p_{i_j}.$$

The integrality of $Q \rightarrow P$ now implies that $\sum_j (n_{i_j} - n_{i_1}) p_{i_j}$ is in Q , which from above is only possible if $n_{i_j} = n_{i_1}$ for all j . This implies (2).

To see (3), note that the map

$$Q \oplus_N N^n \rightarrow P$$

is surjective by the definition of the p_{i_j} . To see that it is injective, suppose that $(q, (\underline{n}_j))$ and $(q', (\underline{m}_j))$ map to the same element. Without loss of generality, we may assume that at least one $n_j = 0$ and at least one $m_{j'} = 0$ (since there exists $q_0 \in Q$ such that $(q_0, 0) = (0, (1, \dots, 1))$ in $Q \oplus_N N^n$). Then we get an equality

$$q + \sum n_j p_{i_j} = q' + \sum m_j p_{i_j}$$

in P , and by the integrality we get that there exists $p \in P$ and $q_3, q_4 \in Q$ such that

$$\sum n_j p_{i_j} = p + q_3, \quad \sum m_j p_{i_j} = p + q_4, \quad q_3 + q = q_4 + q'.$$

Since $\sum p_{i_j} \in Q$, we may assume that $p = \sum_j b_{i_j} p_{i_j}$ with at least one $b_{i_j} = 0$. If q_3 is non-zero, we obtain that $\prod Y_{i_j}^{n_j} = 0$ in $k[Y_{i_1}, \dots, Y_{i_n}]/J$, which from above is impossible, since at least one n_j is zero. Thus $q_3 = 0$ and hence

$$\sum m_j p_{i_j} = \sum n_j p_{i_j} + q_4.$$

By the same reasoning using the fact that at least one $m_{j'} = 0$, we conclude that $q_4 = 0$ and hence $q = q'$ and $(\underline{n}_j) = (\underline{m}_j)$. This completes the proof of Proposition 2.10. \square

Let $L \subset P$ be the submonoid of 2.12.2, and define $K \subset Q$ to be the submonoid generated by the element $\sum_j p_{ij}$. Equivalently,

$$K = L \cap Q \subset P.$$

If $t' \in \underline{X}$ is another singular point lying over some $\zeta \in \text{Spec}(\mathcal{O})$, then the above discussion shows that the submonoid $L_{t'} \subset \overline{\mathcal{M}}_{X,t'}$ generated by the irreducible elements of $\overline{\mathcal{M}}_{X,t'}$ not in the image of $\overline{\mathcal{M}}_{S,\zeta}$ is isomorphic to N^r for some r , and that the intersection $K_{t'} = L_{t'} \cap \overline{\mathcal{M}}_{S,\zeta}$ is a free monoid of rank 1. To check this one may change base to an algebraic closure \bar{k} of $k(\zeta)$ and hence can assume that there exists a chart $Q \rightarrow \mathcal{M}_S$ such that $Q \rightarrow \overline{\mathcal{M}}_S$ is bijective. If t' is a closed point, the result now follows from the above discussion. If t' is not a closed point, then choose a specialization t'' of t' and note that the stalk at t' is obtained by the cospecialization map from t'' ([16], II.2.3).

LEMMA 2.14. *Suppose a chart as in 2.2 has been chosen over some étale neighborhood of t , which we also denote by \underline{X} . Let L_t and K_t be the submonoids of P and Q , respectively, given by the above construction. Then for any other singular point t' of \underline{X} lying over some $\zeta \in \text{Spec}(\mathcal{O})$, the submonoids $L_{t'} \subset \overline{\mathcal{M}}_{X,t'}$ and $K_{t'} \subset \overline{\mathcal{M}}_{S,\zeta}$ are equal to the images of L_t and K_t .*

PROOF. By Proposition 2.10, there is an isomorphism

$$Q \oplus_{K_t} L_t \simeq P,$$

and hence an isomorphism

$$\overline{\mathcal{M}}_{X,t'} \simeq \overline{\mathcal{M}}_{S,\zeta} \oplus_{K_t} (L_t / (p_{ij} = 0 \text{ if } p_{ij} \text{ maps to a unit in } \mathcal{O}_{X,t'})).$$

This implies the lemma. □

Let D be a connected component of the singular locus of the closed fiber. For any point $t \in D$ we get from the above construction a free monoid $K_t \subset \overline{\mathcal{M}}_{S,s_0}$ of rank 1. Proposition 2.14 implies that this monoid is independent of the choice of t . To see this suppose that $t' \in D$ is a second singular point. Choose étale covers $\{U_i\}$ of D and singular points t_i in the image of U_i such that over each U_i there exists a chart as in 2.14 using t_i . We can also assume that $t_1 = t$ and $t_r = t'$ for some r . Since D is connected, there exists a sequence U_1, \dots, U_r such that for each i there exists a singular point x_i of $U_i \times_X U_{i+1}$ mapping to D . It follows from 2.14 that the submonoid $K_{t_i} \subset \overline{\mathcal{M}}_S$ defined by t_i is equal to the submonoid defined by x_i and x_{i-1} . We conclude that $K_t = K_{t'}$.

LEMMA 2.15. *There exists a unique fine sub-log structure $\tilde{K}_D \subset \mathcal{M}_S$ whose image in $\overline{\mathcal{M}}_{S,s_0}$ is equal to K_t for some $t \in D$ (and hence for all). Also, there exists a unique fine sub-log structure $\tilde{L}_D \subset \mathcal{M}_X$ whose image in $\overline{\mathcal{M}}_{X,t'}$ (t' any point of \underline{X}) is equal to the image of \tilde{K}_D if t' is not singular or if t' does not specialize to a point of D and is equal to $L_{t'}$ (notation as in 2.14) otherwise.*

PROOF. \tilde{K}_D and \tilde{L}_D are uniquely determined if they exist, since we have specified their values on stalks, and \tilde{K}_D exists, since \underline{S} is the spectrum of a strictly henselian local ring.

To show that \tilde{L}_D exists, we first claim that the set of singular points of \underline{X} which specialize to D is a connected component of the singular locus of \underline{X} . To see this, let $\underline{X}^{\text{sing}}$ be the closed subset of singular points of \underline{X} with the reduced subscheme structure. Then $\underline{X}^{\text{sing}}$ is a proper scheme over $\text{Spec}(\mathcal{O})$, and hence by ([2], 3.1) there is an equivalence of categories between the category of finite étale schemes over $\underline{X}^{\text{sing}}$ and the category of finite étale schemes over $\underline{X}_0^{\text{sing}}$ (the reduction of $\underline{X}^{\text{sing}}$). In a neighborhood of any point of the closed fiber, \underline{X} is isomorphic to

$$\text{Spec}(\mathcal{O}[X_1, \dots, X_d]/(X_1 \cdots X_r - t))$$

for some $d, r \in \mathbf{Z}$ and $t \in \mathcal{O}$, and hence $\underline{X}_0^{\text{sing}}$ is equal to the singular locus of the closed fiber \underline{X}_0 with the reduced structure. Now by the above equivalence, the connected component $D \subset \underline{X}_0^{\text{sing}}$ lifts to a finite étale scheme $\tilde{D} \rightarrow \underline{X}^{\text{sing}}$. Since the reduction of this map is of degree 1, this is in fact a closed immersion, and since it is also étale, the scheme \tilde{D} is a connected component of $\underline{X}^{\text{sing}}$ (\tilde{D} is connected by [2], 3.3). Moreover, the points of \tilde{D} are precisely the singular points which specialize to points of D .

It follows that if we choose a chart as in 2.2 in some étale neighborhood of a point t of D , then the log structure defined by L_t (see 2.14) has the properties required of \tilde{L}_D . Moreover, these locally constructed log structures will glue, since they are sub-log structures of \mathcal{M}_X . \square

Define \mathcal{M}_S^\sharp to be the amalgamation over all D

$$\bigoplus_{\mathcal{O}_S^*} \tilde{K}_D,$$

and similarly define \mathcal{M}_X^\sharp to be

$$\bigoplus_{\mathcal{O}_X^*} \tilde{L}_D.$$

Then the natural diagram

$$\begin{array}{ccc} \mathcal{M}_X^\sharp & \xrightarrow{\psi} & \mathcal{M}_X \\ \uparrow & & \uparrow \\ f^* \mathcal{M}_S^\sharp & \xrightarrow{\phi} & f^* \mathcal{M}_S \end{array}$$

is co-cartesian. By construction, f^\sharp is smooth, essentially semi-stable, and the pullback to the closed fiber is special.

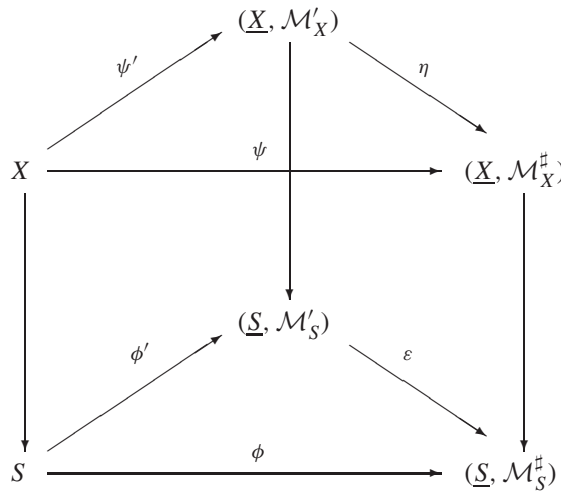
LEMMA 2.16. *Suppose $(\mathcal{M}'_S, \mathcal{M}'_X, \phi', \psi')$ is a second collection of data for which*

$$f' : (\underline{X}, \mathcal{M}'_X) \rightarrow (\underline{S}, \mathcal{M}'_S)$$

is essentially semi-stable and special at s_0 , and for which the diagram

$$\begin{array}{ccc} \mathcal{M}'_X & \xrightarrow{\psi'} & \mathcal{M}_X \\ \uparrow & & \uparrow \\ f'^* \mathcal{M}'_S & \xrightarrow{\phi'} & f^* \mathcal{M}_S \end{array}$$

is co-cartesian. Then there exists a unique pair of isomorphisms $\varepsilon : \mathcal{M}'_S \simeq \mathcal{M}^\sharp_S$ and $\eta : \mathcal{M}'_X \simeq \mathcal{M}^\sharp_X$ such that the diagram



commutes.

PROOF. If we apply the above construction to $(\underline{X}, \mathcal{M}'_X)/(\underline{S}, \mathcal{M}'_S)$, then it follows from chasing through the construction that we recover the data $(\mathcal{M}'_S, \mathcal{M}'_X, \phi', \psi')$. Hence \mathcal{M}'_S and \mathcal{M}'_X have canonical decompositions

$$\mathcal{M}'_S = \oplus_{\mathcal{O}_S^*} \tilde{K}'_D, \quad \mathcal{M}'_X = \oplus_{\mathcal{O}_S^*} \tilde{L}'_D.$$

In addition, the images of \tilde{L}'_D and \tilde{K}'_D in \mathcal{M}_X and \mathcal{M}_S are the sub-log structures obtained by applying the construction to X/S . From this the existence of (ε, η) follows. The uniqueness follows from the fact that any pair of isomorphisms (ε, η) as in the lemma must preserve the decompositions together with the fact that the \tilde{K}_D and \tilde{L}_D are sub-log structures of \mathcal{M}_S and \mathcal{M}_X , respectively. \square

To complete the proof of Proposition 2.17 in the case when S is the spectrum of a strictly henselian local ring \mathcal{O} , and \mathcal{M}_S admits a chart $Q \rightarrow \mathcal{M}_S$ such that $Q \rightarrow \overline{\mathcal{M}}_{S,s_0}$ is bijective, it suffices to show that the morphism

$$(\underline{X}^\sharp, \mathcal{M}^\sharp_X) \rightarrow (\underline{S}^\sharp, \mathcal{M}^\sharp_S)$$

constructed above is special at all points of \underline{S} and not just at s_0 . For later use, we prove a slightly stronger statement:

PROPOSITION 2.17. *Let $f : X \rightarrow S$ be as in 2.7 (we no longer assume that S is the spectrum of a strictly henselian local ring), and suppose in addition that f is essentially semi-stable. Then the set*

$$V := \{s \in \underline{S} \mid f \text{ is special at } \bar{s}\}$$

is open.

PROOF. By ([1], V.4.4) it suffices to show that V is a constructible set and that if $s \in V$, then every generization of s is also in V .

LEMMA 2.18. V is constructible.

PROOF. The condition is local on S , so we may assume that $S = \text{Spec}(A)$ for some ring A and that \mathcal{M}_S is induced from some map

$$N^r \rightarrow A.$$

Let $t_1, \dots, t_r \in A$ be the images of the standard generators. For each integer n and for each choice of n distinct elements $i_1, \dots, i_n \in \{1, \dots, r\}$, let $A_{n,i}$ be the ring

$$A_{t_{i_1} \dots t_{i_n}} / (t_j)_{j \notin \{i_1, \dots, i_n\}}.$$

Let $S_{n,i} = \text{Spec}(A_{n,i})$. Then each $S_{n,i}$ is a constructible subscheme of S , and the union of the $S_{n,i}$ is all of S .

To prove the lemma it therefore suffices to consider the case of the $S_{n,i}$. This reduces us to the case when the log structure on S is given by a map

$$N^r \rightarrow A$$

sending all elements to zero. In addition, by base changing to the irreducible components with the reduced structure, we may assume that A is an integral domain. In this case, \underline{X} is étale locally isomorphic to

$$A[X_1, \dots, X_l] / X_1 \cdots X_s$$

for some l and s . Let $\underline{X}^{\text{sing}}$ be the singular locus of \underline{X} with the reduced induced structure. From the local description of \underline{X} we see that the geometric fibers of $\underline{X}^{\text{sing}}$ are all reduced and that $\underline{X}^{\text{sing}}$ is flat over A . It follows that the function

$$s \mapsto \text{number of connected components of } \underline{X}_s^{\text{sing}}$$

can be identified with the function

$$s \mapsto h^0(\underline{X}^{\text{sing}} \times_S \text{Spec}(k(s)), \mathcal{O}_{\underline{X}^{\text{sing}} \times_S \text{Spec}(k(s))}).$$

By the semi-continuity theorem ([8], III.12.8), we conclude that the function

$$s \mapsto \text{number of connected components of } \underline{X}_s^{\text{sing}}$$

is upper semi-continuous. It follows that the set of points s where this number is equal to r is constructible. On the other hand, the set of points s where the map in Definition 2.6 is surjective is open by the description of the charts in Lemma 2.2. Therefore the set of points where the map in 2.6 is bijective is constructible. \square

LEMMA 2.19. V is stable under generization.

PROOF. We can assume that $S = \text{Spec}(A)$, where A is a local ring and that we want to prove that if the closed fiber is special then the fiber over some prime $\mathfrak{p} \subset A$ is also. Replacing A by A/\mathfrak{p} we may assume that A is a local integral domain and that we wish to prove that the generic fiber is special if the closed fiber is special. In addition, by replacing A by an étale cover, we may assume that we have a global chart

$$N^T \rightarrow \mathcal{M}_S,$$

where T is the set of connected components of the singular locus of $\underline{X}_{\bar{s}_0}$. Let $n_t \in N^T$ be an irreducible element such that $\alpha(n_t) = 0$, and let \underline{Z} denote the set of singular points of \underline{X} which map to n_t under the map in definition 2.6. \underline{Z} is a closed set, and we view \underline{Z} as a subscheme with the reduced structure. What has to be shown is that the generic fiber of \underline{Z} is non-empty and geometrically connected.

Let t_0 be a point of the closed fiber whose corresponding element of N^T is n_t . In an étale neighborhood of t_0 there exists an étale morphism

$$\underline{U} \rightarrow \text{Spec}(A[X_1, \dots, X_n]/X_1 \cdots X_s)$$

for some n and s . From this it follows that the generic fiber of \underline{Z} is not empty. From this local description it also follows that for each point $s \in \underline{S}$ the fiber \underline{Z}_s is reduced and that \underline{Z} is flat over A . Therefore, the generic fiber of \underline{Z} is geometrically connected if and only if

$$h^0(\underline{Z} \otimes_A \text{Frac}(A), \mathcal{O}_{\underline{Z} \otimes_A \text{Frac}(A)}) = 1,$$

where $\text{Frac}(A)$ denotes the field of fractions of A . This number is at least 1 since $\underline{Z} \otimes_A \text{Frac}(A)$ is not empty. On the other hand, the semi-continuity theorem ([8], III.12.8) combined with the fact that the closed fiber of \underline{Z} is geometrically connected implies that the generic fiber of \underline{Z} is geometrically connected. This implies the lemma. □

Proposition 2.17 now follows from the two lemmas. □

We can now complete the proof of Theorem 2.7. By the uniqueness we may assume that S is affine. Note first that if s is a point and $S_s = \text{Spec}(\mathcal{O}_{S,\bar{s}})$, then by ([12], 4.18) there is an equivalence of categories

$$(2.3) \quad \varinjlim_{\underline{U} \rightarrow \underline{S}} (\text{fine log structures on } \underline{X} \times_{\underline{S}} \underline{U}) \rightarrow (\text{fine log structures on } \underline{X} \times_{\underline{S}} \underline{S}_s),$$

where the limit is taken over étale neighborhoods $\underline{U} \rightarrow \underline{S}$ of \bar{s} .

It follows that if we have two collections of data

$$(\mathcal{M}_{S,1}^\sharp, \mathcal{M}_{X,1}^\sharp, \phi_1, \psi_1), \quad (\mathcal{M}_{S,2}^\sharp, \mathcal{M}_{X,2}^\sharp, \phi_2, \psi_2),$$

then there exists exactly one isomorphism between them. Indeed, if $s \in S$ is a point, then after replacing S by an fppf-cover we can assume that we have a chart $Q \rightarrow \mathcal{M}_S$ such that the map $Q \rightarrow \overline{\mathcal{M}}_{S,\bar{s}}$ is bijective. Over the strict henselization of the local ring of s there is a unique isomorphism, which can by the equivalence 2.3 be extended to some étale neighborhood of s . It follows that fppf-locally there exists a unique isomorphism, and since a faithfully flat

morphism of finite presentation is a morphism of effective descent for fine log structures ([17], A.2), there exists a global isomorphism.

To prove existence, it suffices by ([17], A.2) to prove existence in some fppf-neighborhood of a point $s \in S$. Thus we may assume that there exists a chart $Q \rightarrow \mathcal{M}_S$ such that $Q \rightarrow \overline{\mathcal{M}}_{S,\bar{s}}$ is bijective. We have shown that there exist data $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ over the strict henselization of the local ring at s . From the equivalence 2.3 it follows that we can extend the data $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ over the henselization to some étale neighborhood of s . After perhaps shrinking S , we can assume that the resulting morphism is still log smooth. We claim that after perhaps shrinking S some more, the morphism f^\sharp will be essentially semi-stable. For this we may assume that there exists a chart $N^T \rightarrow \mathcal{M}_S^\sharp$, where T denotes the set of connected component of the singular locus of $\underline{X}_{\bar{s}}$. Consider the set \underline{V} of points $t \in \underline{X}$ for which the map

$$(f^{-1}\overline{\mathcal{M}}_S^\sharp)_{\bar{t}} \rightarrow \overline{\mathcal{M}}_{X,\bar{t}}^\sharp$$

is as in the definition of essentially semi-stable. By Lemma 2.2, the set \underline{V} is open and by construction \underline{V} contains every point of the fiber over s . Since f is proper, the image of \underline{V}^c in \underline{S} is a closed set not containing s . Therefore by replacing S by the complement of $f(\underline{V}^c)$, we may assume that f^\sharp is essentially semi-stable everywhere.

It was also shown above that the set of points where the morphism f^\sharp is special is open on the base, and hence after shrinking S some more, we have constructed datum $(\mathcal{M}_S^\sharp, \mathcal{M}_X^\sharp, \phi, \psi)$ satisfying the conditions of 2.7 in an fppf-neighborhood of s . This concludes the proof of Theorem 2.7.

3. Semi-stable log structures and log structures of embedding type. Fix a scheme \underline{S} and a global section $t \in \Gamma(\underline{S}, \mathcal{O}_{\underline{S}})$. Let \mathcal{M}_S denote the log structure on \underline{S} associated to the map $N \rightarrow \mathcal{O}_{\underline{S}}$ sending 1 to t , and let S be the log scheme $(\underline{S}, \mathcal{M}_S)$.

DEFINITION 3.1. A log smooth morphism $f : X \rightarrow S$ is *semi-stable* if for every geometric point $\bar{x} \rightarrow \underline{X}$ the stalk $\overline{\mathcal{M}}_{X,\bar{x}}$ is a free monoid and the map

$$N \rightarrow \overline{\mathcal{M}}_{S,f(\bar{x})} \rightarrow \overline{\mathcal{M}}_{X,\bar{x}}$$

is the diagonal map. If the morphism $f : \underline{X} \rightarrow \underline{S}$ is fixed, then we will also refer to a pair (\mathcal{M}_X, f^b) , consisting of a fine log structure \mathcal{M}_X on X together with a morphism of log structures $f^b : f^*\mathcal{M}_S \rightarrow \mathcal{M}_X$ making $(\underline{X}, \mathcal{M}_X) \rightarrow S$ semi-stable, as a *semi-stable log structure on \underline{X} relative to S* (or just a *semi-stable log structure* if the reference to S is clear).

REMARK 3.2. For any integer $r > 0$, the diagonal map $N \rightarrow N^r$ is invariant under all automorphisms of N^r , and hence for any free monoid M there is a canonical diagonal map $N \rightarrow M$. This is the map referred to in the definition.

REMARK 3.3. The above definition of a semi-stable log structure is closely related to the notion of a *log structure of semi-stable type* introduced in ([9], 11.6). The two notions are, however, different. If $\underline{S} = \text{Spec}(k)$ is the spectrum of an algebraically closed field, then a log structure of semi-stable type on \underline{X} in the sense of (loc. cit.) is a log structure \mathcal{M}_X on

\underline{X} for which there exists a morphism $f^b : f^* \mathcal{M}_S \rightarrow \mathcal{M}_X$ making (\mathcal{M}_X, f^b) a semi-stable log structure in the sense of Definition 3.1. However, a log structure \mathcal{M}_X of semi-stable type does not come equipped with a specified such morphism. This distinction becomes especially important when we speak of isomorphisms of semi-stable log structures. For us, an isomorphism $(\mathcal{M}_X, f^b) \rightarrow (\mathcal{M}'_X, f'^b)$ between two semi-stable log structures on \underline{X} relative to S is an isomorphism of log structures $\iota : \mathcal{M}_X \rightarrow \mathcal{M}'_X$ such that $f'^b = \iota \circ f^b$.

We will also study a generalization of Kato’s notion of a log structure of embedding type:

DEFINITION 3.4. Let $f : \underline{X} \rightarrow \underline{S}$ be a morphism of schemes. A log structure \mathcal{M}_X on \underline{X} is of *embedding type relative to S* (or just of *embedding type* if the reference to S is clear) if the morphism $(\underline{X}, \mathcal{M}_X) \rightarrow (\underline{S}, \mathcal{O}_S^*)$ factors étale locally through a semi-stable morphism $(\underline{X}, \mathcal{M}_X) \rightarrow S$. If $f : X \rightarrow (\underline{S}, \mathcal{O}_S^*)$ is a morphism of log schemes, then we also say that f is of *embedding type relative to S* if \mathcal{M}_X is a log structure of embedding type relative to S on \underline{X} .

REMARK 3.5. By an isomorphism $\mathcal{M}_X \rightarrow \mathcal{M}'_X$ of log structures of embedding type relative to S , we simply mean an isomorphism of log structures on \underline{X} .

It follows from ([15], 2.25) that if $f : X \rightarrow S$ is semi-stable (resp. $f : X \rightarrow (\underline{S}, \mathcal{O}_S^*)$ is of embedding type relative to S), then in an étale neighborhood of any closed point the log structure \mathcal{M}_X admits a chart of the following form:

DEFINITION 3.6. Suppose \mathcal{M}_X is a semi-stable log structure relative to S (resp. a log structure of embedding type relative to S) on \underline{X} . A *standard chart* for \mathcal{M}_X at a closed point $x \in \underline{X}$ is a strict, étale morphism of log schemes over S (resp. over $(\underline{S}, \mathcal{O}_S^*)$)

$$(\underline{X}, \mathcal{M}_X) \longrightarrow (\text{Spec}(\mathcal{O}_S[X_1, \dots, X_l]/(X_1 \cdots X_r - t)), \text{standard log str.})$$

for some l and r , which sends x to the point $X_1 = \cdots = X_l = 0$ if $r > 1$, and sends x to the point $X_1 = t, X_2 = \cdots = X_l = 0$ if $r = 1$.

REMARK 3.7. The standard log structure referred to in the definition is that induced by the map

$$N^r \longrightarrow \mathcal{O}_S[X_1, \dots, X_l]/(X_1 \cdots X_r - t), \quad (n_j)_{j=1}^r \mapsto \prod_{j=1}^r X_j^{n_j}.$$

The diagonal map $N \rightarrow N^r$ naturally gives

$$(\text{Spec}(\mathcal{O}_S[X_1, \dots, X_l]/(X_1 \cdots X_r - t)), \text{standard log str.}),$$

the structure of a log scheme over S .

For the remainder of this section we will fix a morphism of schemes $f : \underline{X} \rightarrow \underline{S}$ such that étale locally on \underline{X} there exists a log structure \mathcal{M}_X of embedding type relative to S .

PROPOSITION 3.8. *Any two log structures on \underline{X} of embedding type relative to S are locally isomorphic. Moreover, if $\phi : \mathcal{M}_X \rightarrow \mathcal{M}_X$ is an automorphism of a log structure of embedding type relative to S , then the induced map $\bar{\phi} : \overline{\mathcal{M}}_X \rightarrow \overline{\mathcal{M}}_X$ is the identity.*

PROOF. Let \mathcal{M}_1 and \mathcal{M}_2 be two log structures of embedding type relative to S , and let $x \in \underline{X}$ be a closed point. We construct an isomorphism $\mathcal{M}_1 \simeq \mathcal{M}_2$ in some étale neighborhood of x .

To do this, we may assume that we have standard charts

$$\pi_1 : \underline{X} \rightarrow \text{Spec}(\mathcal{O}_S[X_1, \dots, X_d]/(X_1 \cdots X_r - t)),$$

$$\pi_2 : \underline{X} \rightarrow \text{Spec}(\mathcal{O}_S[Y_1, \dots, Y_{d'}]/(Y_1 \cdots Y_{r'} - t))$$

sending x to the points $X_1 = \cdots = X_d = 0$ and $Y_1 = \cdots = Y_{d'} = 0$. To prove the first part of the proposition, it suffices to show that $d = d'$, $r = r'$, and that there exists a unique automorphism σ of the set $\{1, \dots, r\}$ such that in some étale neighborhood of x there exist solutions to the equations

$$(3.1) \quad \pi_1^*(X_i) = U_i \pi_2^* Y_{\sigma(i)}, \quad U_i V_i = 1, \quad i = 1, \dots, r.$$

By the Artin approximation theorem ([2], 1.10), it suffices to show that there exist solutions in the completion $\widehat{\mathcal{O}}_{X,x}$. Let \mathfrak{m} denote the maximal ideal of $\widehat{\mathcal{O}}_{X,x}$, $\mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}$ the ideal in $\widehat{\mathcal{O}}_{X,x}$ generated by the maximal ideal of $\widehat{\mathcal{O}}_{S,f(x)}$, and let

$$\phi : \widehat{\mathcal{O}}_{S,f(x)}[[X_1, \dots, X_d]]/(X_1 \cdots X_r - t) \simeq \widehat{\mathcal{O}}_{S,f(x)}[[Y_1, \dots, Y_{d'}]]/(Y_1 \cdots Y_{r'} - t)$$

be the isomorphism induced by π_1 and π_2 . By looking at the dimension of

$$\widehat{\mathcal{O}}_{X,x}/(\mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}, \mathfrak{m}^2),$$

we see that $d = d'$.

Consider first the case when $t = 0$. In this case, the ring

$$\widehat{\mathcal{O}}_{S,f(x)}[[Y_1, \dots, Y_{d'}]]/(Y_1 \cdots Y_{r'})$$

is naturally graded by degree in the Y_i . For $i = 1, \dots, r$ write

$$\phi(X_i) = L_i(\underline{Y}) + H_i(\underline{Y}),$$

where $L_i(\underline{Y})$ consists of linear terms and $H_i(\underline{Y})$ consists of terms of degree 2 or greater.

LEMMA 3.9. 1. *For any ring Λ and any $i \in \{1, \dots, r\}$, the kernel of multiplication by $Y_1 \cdots \widehat{Y}_i \cdots Y_r$ on*

$$\Lambda[Y_1, \dots, Y_d]/(Y_1 \cdots Y_r) \quad (\text{resp. } \Lambda[[Y_1, \dots, Y_d]]/(Y_1 \cdots Y_r))$$

is equal to the ideal generated by Y_i .

2. *There exists a unique permutation σ of $\{1, \dots, r\}$ such that for each $i \in \{1, \dots, r\}$ there exists a unit $u_i \in \widehat{\mathcal{O}}_{S,f(x)}^*$ such that $L_i(\underline{Y}) = u_i Y_{\sigma(i)}$.*

PROOF. 1. As a module over Λ , the ring $\Lambda[Y_1, \dots, Y_d]/(Y_1 \cdots Y_r)$ is isomorphic to the direct sum

$$\bigoplus_{l \in I} \Lambda \underline{Y}^l,$$

where $I \subset N^d$ is the subset of d -tuples (l_1, \dots, l_d) with at least one $l_i = 0$ for $1 \leq i \leq r$. Part 1 follows from the fact that multiplication by $Y_1 \cdots \widehat{Y}_i \cdots Y_r$ preserves this decomposition. Moreover, the case of $\Lambda[[Y_1, \dots, Y_d]]/(Y_1 \cdots Y_r)$ also follows, since this ring is flat over $\Lambda[Y_1, \dots, Y_d]/(Y_1 \cdots Y_r)$.

2. Let \mathfrak{n} be the maximal ideal of $\widehat{\mathcal{O}}_{S, f(x)}$. We first reduce to the case when $\mathfrak{n} = 0$. Evidently, it suffices to prove the uniqueness of σ in this special case. Now suppose that we have found units $u_i \in (\mathcal{O}_{S, f(x)}/\mathfrak{n}^n)^*$ such that

$$L_i(\underline{Y}) \equiv u_i Y_{\sigma(i)} \pmod{\mathfrak{n}^n}.$$

We construct units $u'_i \in (\mathcal{O}_{S, f(x)}/\mathfrak{n}^{n+1})^*$ such that

$$L_i(\underline{Y}) \equiv u'_i Y_{\sigma(i)} \pmod{\mathfrak{n}^{n+1}}$$

and such that there exists an integer n_0 independent of n , for which

$$u_i \equiv u'_i \pmod{\mathfrak{n}^{n-n_0}}.$$

To do this, choose a lifting \tilde{u}_i of u_i to $\widehat{\mathcal{O}}_{S, f(x)}^*$, and let $D_i(\underline{Y}) = L_i(\underline{Y}) - \tilde{u}_i Y_{\sigma(i)}$. Then we get

$$0 \equiv \sum_{i=1}^r \tilde{u}_1 \cdots \widehat{\tilde{u}_i} \cdots \tilde{u}_r D_i(\underline{Y}) Y_1 \cdots \widehat{Y}_{\sigma(i)} \cdots Y_r \pmod{(\mathfrak{n}^{n+1}, (Y_1 \cdots Y_r)^{r+1})}.$$

Since the $D_i(\underline{Y})$ are linear, this implies that

$$D_i(\underline{Y}) \equiv Y_{\sigma(i)} F_i \pmod{\mathfrak{n}^{n+1}}$$

for some $F_i \in \mathcal{O}_{S, f(x)}$. By the Artin-Rees lemma, there exists an integer n_0 independent of n , such that

$$(Y_{\sigma(i)}) \cap \mathfrak{n}^n \cdot \widehat{\mathcal{O}}_{X, x} \subset \mathfrak{n}^{n-n_0} \cdot (Y_{\sigma(i)}),$$

and hence we may choose F_i to be in \mathfrak{n}^{n-n_0} . We define u'_i to be $\tilde{u}_i + F_i$. This proves that it suffices to consider the case when $\mathfrak{n} = 0$.

Assuming now that $\mathfrak{n} = 0$, observe that the X_i map to non-zero elements in $\mathfrak{m}/\mathfrak{m}^2$, and hence the terms $L_i(\underline{Y})$ are non-zero. Since $t = 0$, we find that

$$0 = \prod_{i=1}^r \phi(X_i) = \prod_{i=1}^r L_i(\underline{Y}) + \text{terms of degree } \geq r + 1.$$

From this we conclude that $r \geq r'$ and hence by symmetry $r = r'$. Moreover, by looking at degrees in the Y_i , one finds that $L_i(\underline{Y}) = u_i Y_{\sigma(i)}$ for a unique permutation σ of $\{1, \dots, r\}$ and some unit $u_i \in \widehat{\mathcal{O}}_{S, f(x)}^*$. \square

REMARK 3.10. Part 1 of the lemma implies that the kernel of multiplication by $Y_1 \cdots \widehat{Y}_i \cdots Y_r$ on $\mathcal{O}_S[Y_1, \dots, Y_d]/(Y_1 \cdots Y_r - t)$ is equal to the ideal $K_t \cdot (Y_i)$, where K_t is the kernel of multiplication by t on \mathcal{O}_S . Indeed, the case $t = 0$ implies that any element killed by $Y_1 \cdots \widehat{Y}_i \cdots Y_r$ can be written as

$$f_1 Y_i + f_2 t = (f_1 + f_2 Y_1 \cdots \widehat{Y}_i \cdots Y_r) Y_i,$$

where $t(f_1 + f_2 Y_1 \cdots \widehat{Y}_i \cdots Y_r) = 0$. Since $\mathcal{O}_S[Y_1, \dots, Y_d]/(Y_1 \cdots Y_r - t)$ is flat over \mathcal{O}_S , this implies that $(f_1 + f_2 Y_1 \cdots \widehat{Y}_i \cdots Y_r)$ is in $K_t \cdot \mathcal{O}_X$.

Next, we show that every term in $H_i(\underline{Y})$ can be written as a multiple of $Y_{\sigma(i)}$. We proceed by induction on the degree of monomials. Suppose true for monomials of degree smaller than n in $H_i(\underline{Y})$. Then we can find a unit u_i which is a polynomial of degree less than or equal to $n - 2$ such that

$$\phi(X_i) \equiv u_i Y_{\sigma(i)} \pmod{(Y_1, \dots, Y_{d'})^n}.$$

By looking at monomial terms of degree $r - 1 + n$ in $\prod_{i=1}^r \phi(X_i)$ we find that $Y_1 \cdots \widehat{Y}_{\sigma(i)} \cdots Y_r$ kills the degree n part of $H_i(\underline{Y})$. From this it follows that the degree n part of $H_i(\underline{Y})$ can be written as a multiple of $Y_{\sigma(i)}$. This completes the induction step and the proof of the case when $t = 0$.

Now, suppose $t \neq 0$. By the Artin-Rees lemma there exists an integer m_0 such that for all n

$$(Y_{\sigma(i)}) \cap \mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^n \subset \mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n-m_0} \cdot (Y_{\sigma(i)}).$$

Given an integer n and units u_i such that

$$\phi(X_i) \equiv u_i Y_{\sigma(i)} \pmod{\mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^n},$$

we construct units u'_i such that

$$\phi(X_i) \equiv u'_i Y_{\sigma(i)} \pmod{\mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n+1}}$$

and such that

$$u'_i \equiv u_i \pmod{\mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n-m_0}}.$$

Let $H_i(\underline{Y}) = \phi(X_i) - u_i Y_{\sigma(i)}$. Then

$$t \equiv \prod_{i=1}^r u_i Y_{\sigma(i)} + \sum_{i=1}^r u_1 \cdots \widehat{u}_i \cdots u_r Y_1 \cdots \widehat{Y}_{\sigma(i)} \cdots Y_r H_i(\underline{Y}) \pmod{\mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n+1}},$$

and hence

$$Y_1 \cdots \widehat{Y}_{\sigma(i)} \cdots Y_r H_i(\underline{Y}) \equiv 0 \pmod{(t, \mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n+1})}.$$

Since $t \in (Y_{\sigma(i)})$, it follows that

$$H_i(\underline{Y}) \equiv 0 \pmod{((Y_{\sigma(i)}) \cap \mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^n, \mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n+1})}.$$

Hence by definition of m_0 , there exists $F_i \in \mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n-m_0}$ such that

$$H_i(\underline{Y}) \equiv Y_{\sigma(i)} F_i \pmod{\mathfrak{m}_{\widehat{\mathcal{O}}_{S,f(x)}}^{n+1}}.$$

This completes the proof of the existence of solutions to the equations 3.1 in the completion $\widehat{\mathcal{O}}_{X,x}$ and hence the proof of existence of an isomorphism between \mathcal{M}_1 and \mathcal{M}_2 in an étale neighborhood of x . The second statement in the proposition follows from the uniqueness of σ . □

The proposition and its proof imply that there are various canonically defined sheaves on \underline{X} :

CONSTRUCTION 3.11. The uniqueness of $\bar{\sigma}$ in the proposition implies that for any log structure \mathcal{M}_X of embedding type relative to S , the sheaf $\overline{\mathcal{M}}_X$ is invariant under all automorphisms. Hence there is a canonically defined sheaf $\overline{\mathcal{M}}$ on all of \underline{X} , since there exists a log structure of embedding type on \underline{X} étale locally.

CONSTRUCTION 3.12. If

$$(3.2) \quad \pi : \underline{X} \rightarrow \text{Spec}(\mathcal{O}_S[X_1, \dots, X_d]/(X_1 \cdots X_r - t))$$

is a standard chart for a log structure of embedding type relative to S on \underline{X} , then the ideal

$$J := (\dots, X_1 \cdots \widehat{X}_i \cdots X_r, \dots)_{i=1}^r$$

is independent of the choice of chart, and hence there exists a globally defined ideal sheaf $J \subset \mathcal{O}_X$. We let $\underline{D} \subset \underline{X}$ be the resulting closed subscheme.

CONSTRUCTION 3.13. If ϕ is a chart as in 3.6, let $G \subset \mathcal{O}_X^*$ be the subsheaf of units u which locally can be written as

$$u = 1 + \sum_{i=1}^r a_i X_1 \cdots \widehat{X}_i \cdots X_r,$$

where $ta_i = 0$. Then G is independent of the choices and has a natural structure of a group; if we define $\underline{Z} \subset \underline{X}$ to be the closed subscheme defined by J times the kernel of multiplication by t on \mathcal{O}_X , then G is the kernel of the map $\mathcal{O}_X^* \rightarrow \mathcal{O}_Z^*$.

The following is the main result in this section about log structures of embedding type relative to S :

THEOREM 3.14. 1. *There is a canonical class $o \in H^2(\underline{X}_{et}, G)$ whose vanishing is necessary and sufficient for there to exist a log structure of embedding type relative to S .*

2. *If $o = 0$, then the set of isomorphism classes of log structures of embedding type is naturally a torsor under $H^1(\underline{X}_{et}, G)$.*

3. *If \mathcal{M}_X is a log structure of embedding type, then there is a natural isomorphism of sheaves*

$$\underline{\text{Aut}}(\mathcal{M}_X) \simeq G,$$

where $\underline{\text{Aut}}(\mathcal{M}_X)$ denotes the sheaf on $\underline{X}_{\text{ét}}$ of automorphisms of \mathcal{M}_X .

PROOF. It suffices to prove the third statement, since we have already seen that any two log structure of embedding type are locally isomorphic. For then the stack over the étale site of \underline{X} whose objects are log structures of embedding type relative to S is a gerbe ([13], IV.2) bound by G . Statements 1 and 2 therefore follows from 3 and ([7], IV.3.4).

LEMMA 3.15. Fix a standard chart for \mathcal{M}_X as in 3.6, and let $K_t = \text{Ker}(\times t : \mathcal{O}_S \rightarrow \mathcal{O}_S)$.

1. $\text{Ker}(\times X_i : \mathcal{O}_X \rightarrow \mathcal{O}_X) = K_t \cdot (X_1 \cdots \widehat{X}_i \cdots X_r)$.
2. The natural map

$$\bigoplus_{i=1}^r K_t \cdot (X_1 \cdots \widehat{X}_i \cdots X_r) \rightarrow \sum_{i=1}^r K_t \cdot (X_1 \cdots \widehat{X}_i \cdots X_r)$$

is an isomorphism.

PROOF. All sheaves involved are quasi-coherent sheaves so that it suffices to consider the ring $\mathcal{O}_S[X_1, \dots, X_d]/(X_1 \cdots X_r - t)$. As a module over \mathcal{O}_S , this ring is isomorphic to

$$(3.3) \quad \bigoplus_{\underline{l} \in I} \mathcal{O}_S \underline{X}^{\underline{l}},$$

where $I \subset N^d$ is the subset of d -tuples (l_1, \dots, l_d) with $l_i = 0$ for some $i \in [1, r]$. For each $i \in [1, r]$, let $I_i \subset I$ denote the subset of d -tuples \underline{l} for which $l_i = 0$ and $l_{i'} \neq 0$ for all $i' \in [1, r] - \{i\}$. Then

$$K_t \cdot (X_1 \cdots \widehat{X}_i \cdots X_r) \simeq \bigoplus_{\underline{l} \in I_i} K_t \underline{X}^{\underline{l}}.$$

From this the statement 2 follows.

To see the statement 1, observe that multiplication by X_i preserves the decomposition in 3.3 if $t = 0$, and hence the result is clear in this special case. For the general case, note that the case $t = 0$ implies that if $f \in \text{Ker}(\times X_i)$, then there exists f_1 and f_2 such that

$$f = f_1 X_1 \cdots \widehat{X}_i \cdots X_r + f_2 t = (f_1 + f_2 X_i) X_1 \cdots \widehat{X}_i \cdots X_r.$$

Since \underline{X} is flat over \underline{S} and f is killed by X_i , the element $f_1 + f_2 X_i$ is in $K_t \cdot \mathcal{O}_X$. □

Let $\Delta : N \rightarrow \overline{\mathcal{M}}_X$ be the diagonal map and let $\phi : \mathcal{M}_X \rightarrow \mathcal{M}_X$ be an automorphism. Then for any lifting $\widetilde{\Delta}(1)$ of $\Delta(1)$ to \mathcal{M}_X

$$\phi(\widetilde{\Delta}(1)) = \lambda(u) + \widetilde{\Delta}(1),$$

where u is a unit independent of the choice of $\widetilde{\Delta}(1)$ and $\lambda : \mathcal{O}_X^* \rightarrow \mathcal{M}_X$ denotes the natural inclusion. This defines an element $u \in \mathcal{O}_X^*$. Now, if we choose a chart as in 3.6, then locally the automorphism ϕ is induced from solutions $\{u_i\}$ to the equations

$$U_i X_i = X_i, \quad U_i V_i = 1, \quad i = 1, \dots, r.$$

Hence $u_i - 1 \in \text{Ker}(\times X_i : \mathcal{O}_X \rightarrow \mathcal{O}_X)$. From the lemma we conclude that

$$u_i = 1 + a_i X_1 \cdots \widehat{X}_i \cdots X_r,$$

where $ta_i = 0$, and hence

$$u = \prod_{i=1}^r (1 + a_i X_1 \cdots \widehat{X}_i \cdots X_r) = 1 + \sum_{i=1}^r a_i X_1 \cdots \widehat{X}_i \cdots X_r \in G.$$

Conversely, given an element

$$1 + \sum_{i=1}^r a_i X_1 \cdots \widehat{X}_i \cdots X_r$$

with $ta_i = 0$, we can define an automorphism of \mathcal{M}_X by sending an element $m \in \mathcal{M}_X$ with $\alpha(m) = X_i$ to $\lambda(1 + a_i X_1 \cdots \widehat{X}_i \cdots X_r) + m$. This proves the theorem. \square

Let H be the kernel of $\mathcal{O}_X^* \rightarrow \mathcal{O}_D^*$. There is a natural inclusion $G \hookrightarrow H$, which is an isomorphism if $t = 0$. The following proposition explains the relationship between Theorem 3.14 and the results of Kato ([9], theorem 11.7):

PROPOSITION 3.16. 1. *The sheaf $\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}, \mathcal{O}_X)$ is an invertible \mathcal{O}_D -module.*

2. *Let $o \in H^2(\underline{X}_{et}, G)$ be the class constructed in Theorem 3.14, and let $o' \in H^1(X_{et}, \mathcal{O}_D^*)$ be the class of the invertible \mathcal{O}_D -module*

$$\mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}, \mathcal{O}_X), \mathcal{O}_D).$$

Then the image of o in $H^2(\underline{X}_{et}, H)$ is equal to $\delta(o')$, where

$$\delta : H^1(\underline{X}_{et}, \mathcal{O}_D^*) \rightarrow H^2(\underline{X}_{et}, H)$$

is the map obtained from the exact sequence

$$0 \rightarrow H \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{O}_D^* \rightarrow 0.$$

In particular, if $t = 0$ so that $G = H$, then $o = 0$ if and only if there exists a line bundle \mathcal{L} on \underline{X} such that

$$\mathcal{L} \otimes \mathcal{O}_D \simeq \mathcal{E}xt^1(\Omega_{\underline{X}/\underline{S}}, \mathcal{O}_X).$$

PROOF. 1. The question is local and so we may assume that we have a standard chart

$$\underline{X} \rightarrow \text{Spec}(\mathcal{O}_S[X_1, \dots, X_d]/(X_1 \cdots X_r - t))$$

for \mathcal{M}_X . Let $R = \mathcal{O}_S[X_1, \dots, X_d]$ and let I be the kernel of $R \rightarrow \mathcal{O}_X$. Then the conormal sequence gives an exact sequence

$$(3.4) \quad I/I^2 \rightarrow \Omega_{R/\mathcal{O}_S}^1 \otimes \mathcal{O}_X \rightarrow \Omega_{\underline{X}/\underline{S}}^1 \rightarrow 0.$$

LEMMA 3.17. *The conormal sequence 3.4 is exact on the left.*

PROOF. The module $\Omega_{R/\mathcal{O}_S}^1 \otimes \mathcal{O}_X$ is a free \mathcal{O}_X -module with basis $\{dX_i\}_{i=1}^d$, and I/I^2 is a free \mathcal{O}_X -module with basis $X_1 \cdots X_r - t$. The map $I/I^2 \rightarrow \Omega_{R/\mathcal{O}_S}^1 \otimes \mathcal{O}_X$ is the map induced by

$$X_1 \cdots X_r - t \mapsto \sum_{j=1}^r X_1 \cdots \widehat{X}_j \cdots X_r dX_j,$$

and therefore the kernel can be identified with

$$\{f \in \mathcal{O}_X \mid f X_1 \cdots \widehat{X}_j \cdots X_r = 0 \text{ for all } 1 \leq j \leq r\}.$$

Now the kernel of multiplication by $X_1 \cdots \widehat{X}_j \cdots X_r$ is equal to $K_t^X \cdot (X_i)$, where K_t^X denotes the kernel of multiplication by t on \mathcal{O}_X , and therefore the kernel of $I/I^2 \rightarrow \Omega_{R/\mathcal{O}_S}^1 \otimes \mathcal{O}_X$ is isomorphic to

$$\bigcap_{j=1}^r K_t^X \cdot (X_i).$$

Using the notation of the proof of Lemma 3.15, the ideal $K_t^X \cdot (X_i)$ coincides with the ideal

$$\bigoplus_{\underline{l} \in L_i} K_t X^{\underline{l}},$$

where L_i denotes the subset of I consisting of elements \underline{l} with $l_i \neq 0$. The lemma therefore follows from the statement that

$$\bigcap_{i=1}^r L_i = \emptyset.$$

□

It follows that there is a natural map

$$(3.5) \quad \mathcal{H}om(I/I^2, \mathcal{O}_X) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}^1, \mathcal{O}_X),$$

which sits in an exact sequence

$$0 \rightarrow \mathcal{D}er(\underline{X}/\underline{S}) \rightarrow \mathcal{D}er(R/\mathcal{O}_S) \otimes \mathcal{O}_X \rightarrow \mathcal{H}om(I/I^2, \mathcal{O}_X) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}^1, \mathcal{O}_X) \rightarrow 0.$$

As mentioned above, the sheaf I/I^2 is an invertible sheaf with basis the image of

$$X_1 \cdots X_r - t,$$

and the map

$$\mathcal{D}er(R/\mathcal{O}_S) \otimes \mathcal{O}_X \rightarrow \mathcal{H}om(I/I^2, \mathcal{O}_X)$$

is the map which sends $\partial/\partial X_i$ to the map

$$I/I^2 \rightarrow \mathcal{O}_X, \quad X_1 \cdots X_r - t \mapsto X_1 \cdots \widehat{X}_i \cdots X_r.$$

From this it follows that the map in the equation 3.5 becomes an isomorphism when tensored with \mathcal{O}_D .

2. If we view $H^2(\underline{X}_{et}, H)$ as the group of equivalence classes of gerbes bound by H on \underline{X}_{et} ([7], IV.3.1.1), then $\delta(o')$ can be described as the class of the gerbe on \underline{X}_{et} whose objects are pairs $(\mathcal{L}, \varepsilon)$, where \mathcal{L} is an invertible \mathcal{O}_X -module and

$$\varepsilon : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{O}_D \simeq \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}^1, \mathcal{O}_X)$$

is an isomorphism of \mathcal{O}_D -modules ([7], IV.3.4). Thus to prove the proposition it suffices to exhibit a functor

$$\begin{array}{c} \text{(stack of log structures of embedding type relative to } S) \\ \downarrow F \\ \text{(stack of pairs } (\mathcal{L}, \varepsilon)) \end{array}$$

such that for any two log structures \mathcal{M}_1 and \mathcal{M}_2 of embedding type relative to S , the natural map

$$\underline{\text{Isom}}(\mathcal{M}_1, \mathcal{M}_2) \rightarrow \underline{\text{Isom}}(F(\mathcal{M}_1), F(\mathcal{M}_2))$$

induces an isomorphism

$$\underline{\text{Isom}}(\mathcal{M}_1, \mathcal{M}_2) \times^G H \rightarrow \underline{\text{Isom}}(F(\mathcal{M}_1), F(\mathcal{M}_2)),$$

where $\underline{\text{Isom}}(\cdot, \cdot)$ denotes the sheaf of isomorphisms.

To define F , suppose \mathcal{M}_X is a log structure of embedding type relative to S . We define \mathcal{L} to be the invertible sheaf associated to the torsor of liftings of $\Delta(1) \in \overline{\mathcal{M}}_X$ to \mathcal{M}_X .

To define ε consider first the case when a standard chart is chosen and define R and I as in the proof of the part 1 of the proposition. The choice of chart defines trivializations of I/I^2 and \mathcal{L} and hence an isomorphism $I/I^2 \simeq \mathcal{L}$. From the map 3.5, we therefore get an isomorphism

$$\mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{O}_D \simeq \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}^1, \mathcal{O}_X).$$

By a local calculation (which we leave to the reader), this isomorphism is independent of the choices, and hence we get a global isomorphism ε . □

Define a second sheaf

$$G_2 := \{u \in \mathcal{O}_X^* \mid ut = t\}.$$

Any log structure of embedding type \mathcal{M}_X defines a G_2 -torsor

$$P_{\mathcal{M}_X} := \{m \in \mathcal{M}_X \mid m \text{ maps to } \Delta(1) \text{ in } \overline{\mathcal{M}}_X \text{ and } \alpha(m) = t\}.$$

By definition, a map $f^* \mathcal{M}_S \rightarrow \mathcal{M}_X$, giving \mathcal{M}_X the structure of a semi-stable log structure relative to S , is equivalent to a trivialization of $P_{\mathcal{M}_X}$.

From the proof of 3.14.3 it follows that the action of G on $P_{\mathcal{M}_X}$ induced by the isomorphism $G \simeq \underline{\text{Aut}}(\mathcal{M}_X)$ is simply the action obtained from the inclusion $G \hookrightarrow G_2$. It follows that a semi-stable log structure relative to S admits no automorphisms, and hence the presheaf on \underline{X}_{et}

$$SS^t : \underline{U} \mapsto \{\text{isomorphism classes of semi-stable log structures on } \underline{U}\}$$

is a sheaf. The sheaf G_2 is naturally identified with the sheaf of automorphisms of $f^*\mathcal{M}_S$, and hence G_2 acts naturally on SS^t by

$$(f^*\mathcal{M}_S \longrightarrow \mathcal{M}_X)^\sigma := (f^*\mathcal{M}_S \xrightarrow{\sigma} f^*\mathcal{M}_S \longrightarrow \mathcal{M}_X).$$

Again, because any two log structures of embedding type are locally isomorphic, this action of G_2 is transitive. Moreover, from above we see that this makes SS^t a torsor under the sheaf G_2/G . Thus we obtain:

THEOREM 3.18. *There is a canonical obstruction $o \in H^1(\underline{X}, G_2/G)$ whose vanishing is necessary and sufficient for there to exist a semi-stable log structure on \underline{X} relative to S . If $o = 0$, then the set of isomorphism classes of semi-stable log structures is canonically a torsor under $H^0(\underline{X}, G_2/G)$.*

It will be useful to have a better description of the sheaf G_2/G :

PROPOSITION 3.19. *Let $K_t \subset \mathcal{O}_S$ (resp. $K_t^X \subset \mathcal{O}_X$) be the kernel of multiplication by t on \mathcal{O}_S (resp. \mathcal{O}_X), and let $(t) \subset \mathcal{O}_X$ be the ideal generated by t . Then there is a natural exact sequence*

$$(3.6) \quad 0 \rightarrow (K_t^X \cap (t)) \otimes \mathcal{O}_D \rightarrow G_2/G \rightarrow \mathcal{O}_D^* \rightarrow (\mathcal{O}_D/K_t^X \cdot \mathcal{O}_D)^* \rightarrow 0.$$

Moreover, the image of o under the natural map

$$H^1(\underline{X}_{et}, G_2/G) \rightarrow H^1(\underline{X}_{et}, \mathcal{O}_D^*)$$

is equal to the class of

$$\mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}^1, \mathcal{O}_X), \mathcal{O}_D).$$

PROOF. To see the exactness of 3.6 we may work locally and hence can assume that a standard chart as in 3.6 has been chosen. Moreover, the exactness of

$$G_2/G \rightarrow \mathcal{O}_D^* \rightarrow (\mathcal{O}_D/K_t^X \cdot \mathcal{O}_D)^* \rightarrow 0$$

is by definition of G_2 .

LEMMA 3.20. 1. $K_t^X \cap J = \text{Ker}(\times t : (t) \rightarrow (t)) + \sum_{i=1}^r K_t^X \cdot (X_1 \cdots \widehat{X}_i \cdots X_r)$.
 2. $(t) \cap \sum_{i=1}^r K_t^X \cdot (X_1 \cdots \widehat{X}_i \cdots X_r) = \sum_{i=1}^r (K_t^X \cap (t)) \cdot (X_1 \cdots \widehat{X}_i \cdots X_r)$.

PROOF. Note first of all that since $\underline{X}/\underline{S}$ is flat, $K_t^X = \mathcal{O}_X \otimes_{\mathcal{O}_S} K_t$.

Since all sheaves involved are quasi-coherent, it suffices to consider the ring

$$\mathcal{O}_S[X_1, \dots, X_d]/(X_1 \cdots X_r - t).$$

In terms of the decomposition given in equation 3.3, K_t^X is equal to the ideal

$$\bigoplus_{l \in I} K_t \underline{X}^l,$$

and J has a decomposition

$$\bigoplus_{l \in I} \mathcal{O}_S t^{\delta(l)} \underline{X}^l,$$

where

$$\delta(\underline{l}) = \begin{cases} 0 & \text{if } l \in I_i \text{ for some } i \in [1, r], \\ 1 & \text{otherwise} \end{cases}.$$

From this 1 follows.

To see 2, note that multiplication by X_i on $K_t^X(X_1 \cdots \widehat{X}_i \cdots X_r)$ is the zero map. Hence

$$\sum_{i=1}^r K_t^X(X_1 \cdots \widehat{X}_i \cdots X_r) = \bigoplus_{\underline{l} \in I'} K_t X^{\underline{l}},$$

where $I' \subset I$ is the subset of elements \underline{l} for which there exists exactly one $i \in [1, r]$ such that $l_i = 0$. □

It follows from the first part of the lemma that the kernel of $G_2 \rightarrow \mathcal{O}_D^*$ can be described as units u which can be written as

$$(3.7) \quad 1 + ft + \sum_{i=1}^r a_i X_1 \cdots \widehat{X}_i \cdots X_r,$$

where $t^2 f = ta_i = 0$. Define a map $K_t^X \cap (t) \rightarrow G_2$ by

$$ft \mapsto 1 + ft.$$

Then by the second part of the lemma the map induces an injection $(K_t^X \cap (t) \otimes \mathcal{O}_D) \rightarrow G_2/G$, and since any element as in 3.7 can be written as

$$(1 + ft) \left(1 + \sum_{i=1}^r a_i X_1 \cdots \widehat{X}_i \cdots X_r \right),$$

we get the exactness of the rest of the sequence 3.6.

To get the second statement, recall (proof of Proposition 3.16) that to any log structure \mathcal{M}_X of embedding type relative to S we associated a pair $(\mathcal{L}, \varepsilon)$, where \mathcal{L} is an invertible sheaf and ε is an isomorphism

$$\varepsilon : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{O}_D \simeq \mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}^1, \mathcal{O}_X).$$

Now, from the proof of proposition 3.16, a map $f^* \mathcal{M}_S \rightarrow \mathcal{M}_X$ making \mathcal{M}_X semi-stable relative to S defines a trivialization of \mathcal{L} . Hence we get a map

$$SS^t \rightarrow \underline{\text{Isom}}(\mathcal{O}_D, \mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}xt_{\mathcal{O}_X}^1(\Omega_{\underline{X}/\underline{S}}^1, \mathcal{O}_X), \mathcal{O}_D))$$

compatible with the actions of G_2/G and \mathcal{O}_D^* . □

4. A theorem about effectivity of formal log structures.

THEOREM 4.1. *Suppose A is the completion at a point of a scheme of finite type over an excellent Dedekind ring, and let $f : \underline{X} \rightarrow \text{Spec}(A)$ be a proper morphism of schemes such that all the fibers of \underline{X} are semi-stable varieties in the sense of Definition 1.1. For each $n \geq 0$, let $A_n = A/\mathfrak{m}_A^{n+1}$ and $\underline{X}_n := \underline{X} \times_{\text{Spec}(A)} \text{Spec}(A_n)$ be the reductions and let $f_n : \underline{X}_n \rightarrow \text{Spec}(A_n)$ be the natural map. Fix a log structure \mathcal{M}_A on $\text{Spec}(A)$ and let \mathcal{M}_{A_n}*

be the pullback of \mathcal{M}_A to $\text{Spec}(A_n)$. Then any compatible family of pairs (\mathcal{M}_n, f_n^b) , where \mathcal{M}_n is a log structure on \underline{X}_n and $f_n^b : f_n^* \mathcal{M}_{A_n} \rightarrow \mathcal{M}_n$ is a morphism of log structures making the map

$$(f_n, f_n^b) : (\underline{X}_n, \mathcal{M}_n) \rightarrow (\text{Spec}(A_n), \mathcal{M}_{A_n})$$

log smooth, vertical, and integral, is induced by a unique pair (\mathcal{M}, f^b) , where \mathcal{M} is a log structure on \underline{X} and $f^* \mathcal{M}_A \rightarrow \mathcal{M}$ is a morphism of log structures making

$$(f, f^b) : (\underline{X}, \mathcal{M}) \rightarrow (\text{Spec}(A), \mathcal{M}_A)$$

log smooth, vertical, and integral.

PROOF. By Theorem 2.7, it suffices to consider the case when all the morphisms (f_n, f_n^b) are special in the sense of Definition 2.6. Moreover, by descent theory we may assume that the residue field A/\mathfrak{m}_A is separably closed.

Let Γ denote the set of connected components of the singular locus of the closed fiber \underline{X}_0 . Then by the definition of a special morphism (2.6) there is a canonical isomorphism $N^\Gamma \simeq \overline{\mathcal{M}}_{A_0}$, and we fix a chart $\beta_0 : N^\Gamma \rightarrow \mathcal{M}_{A_0}$ inducing this isomorphism. Since A is strictly henselian and local, there is an isomorphism

$$H^0(\text{Spec}(A), \overline{\mathcal{M}}_A) \simeq \overline{\mathcal{M}}_{A_0}.$$

Moreover, $H^1(\text{Spec}(A), \mathcal{O}_A^*) = 0$, and hence from the exact sequence

$$0 \rightarrow \mathcal{O}_A^* \rightarrow \mathcal{M}_A^{gp} \rightarrow \overline{\mathcal{M}}_A^{gp} \rightarrow 0$$

we see that the map

$$H^0(\text{Spec}(A), \mathcal{M}_A) \rightarrow H^0(\text{Spec}(A), \overline{\mathcal{M}}_A)$$

is surjective. Therefore, there exists a chart $\beta : N^\Gamma \rightarrow \mathcal{M}_A$ inducing β_0 . By the construction in the proof of Theorem 2.7, there is a canonical decomposition for all n

$$\mathcal{M}_n \simeq \bigoplus_{\gamma \in \Gamma, \mathcal{O}_{X_n}^*} \mathcal{M}_n^\gamma.$$

Let t_γ be the image of the standard generator of $N^\gamma \hookrightarrow N^\Gamma$ under the map $N^\Gamma \rightarrow \mathcal{M}_A \rightarrow A$, and let \mathcal{M}_A^γ (resp. $\mathcal{M}_{A_n}^\gamma$) denote the log structure defined by the map $N \rightarrow A$ (resp. $N \rightarrow A_n$) sending 1 to t_γ . Then each \mathcal{M}_n^γ is a semi-stable log structure relative to $(\text{Spec}(A_n), \mathcal{M}_{A_n}^\gamma)$ on any open set of \underline{X}_n not intersecting the components $\Gamma - \{\gamma\}$.

Next, we claim that for any closed point $x_0 \in X$ in the closed fiber, there exists an étale neighborhood \underline{U} of x_0 and an étale morphism

$$(4.1) \quad \underline{U} \rightarrow \text{Spec}(A[X_1, \dots, X_d]/(X_1 \cdots X_r - t_\gamma))$$

for some d, r and γ . To see this, choose an affine étale neighborhood U_0 of x_0 and a standard chart (see 3.6)

$$\underline{U}_0 \rightarrow \text{Spec}(A_0[X_1, \dots, X_d]/(X_1 \cdots X_r))$$

for $\mathcal{M}_{X_0}^\gamma$. For each integer n , there is a unique étale \underline{X}_n -scheme \underline{U}_n reducing to \underline{U}_0 . We inductively lift the standard chart over \underline{U}_0 to a compatible family of standard charts over \underline{U}_n .

For each n , view $\text{Spec}(A_n[X_1, \dots, X_d]/(X_1 \cdots X_r - t_\gamma))$ as a smooth log scheme over $(\text{Spec}(A_n), \mathcal{M}_{A_n}^\gamma)$, and note that giving a lifting to \underline{U}_n of a standard chart over \underline{U}_{n-1} is equivalent to finding a dotted arrow filling in the diagram of log schemes

$$\begin{array}{ccc}
 (\underline{U}_{n-1}, \mathcal{M}_{X_{n-1}}^\gamma) & \longrightarrow & (\text{Spec}(A_n[X_1, \dots, X_d]/(X_1 \cdots X_r - t_\gamma)), \text{standard log str.}) \\
 \downarrow & \nearrow \text{dotted arrow} & \downarrow a \\
 (\underline{U}_n, \mathcal{M}_{X_n}^\gamma) & \longrightarrow & (\text{Spec}(A_n), \mathcal{M}_{A_n}^\gamma).
 \end{array}$$

This is because a map between flat schemes over A_n is étale if and only if its reduction to A_0 is étale. Since the map labelled a in the diagram is log smooth, such a dotted arrow exists. It follows from the Artin approximation theorem ([2], 2.5) that there exists an étale neighborhood \underline{U} of x_0 and a map as in 4.1 approximating our formal solution in $\widehat{\mathcal{O}}_{X, x_0}$ obtained from the charts over the \underline{U}_n . Since the resulting map is formally étale (and hence étale) at x_0 , we obtain the claim.

To prove the theorem, we construct log structures on \underline{X} inducing the \mathcal{M}_n^γ . The fact that \underline{X}/A admits étale locally a log structure of embedding type implies that the singular locus of \underline{X} has a natural scheme structure: define $\underline{X}^{\text{sing}}$ to be the closed subscheme defined by the annihilator of the sheaf

$$\mathcal{E}xt^1(\Omega_{\underline{X}/A}^1, \mathcal{O}_X).$$

If we choose a map as in 4.1, then $\underline{X}^{\text{sing}}$ is by Proposition 3.16 the closed subscheme defined by the ideal

$$(\dots, X_1 \cdots \widehat{X}_i \cdots X_r, \dots)_{i=1}^r.$$

From this local description it also follows that the reduction of $\underline{X}^{\text{sing}}$ to \underline{X}_n is equal to the disjoint union of the D_γ^n , where D_γ^n is the closed subscheme defined in 3.2 applied to any open set of \underline{X}_n not containing the other components $\Gamma - \{\gamma\}$. Now, by the Grothendieck existence theorem ([6], III.5), the functor which sends a closed subscheme of \underline{X} to its reductions induces a bijection

$$\begin{array}{c}
 \text{(closed subschemes of } \underline{X}\text{)} \\
 \downarrow \\
 \text{(compatible families of closed subschemes of the } \underline{X}_n\text{)}.
 \end{array}$$

Therefore, $\underline{X}^{\text{sing}}$ has a canonical decomposition

$$\underline{X}^{\text{sing}} = \coprod_{\gamma \in \Gamma} D_\gamma,$$

where D_γ reduces to D_γ^n over \underline{X}_n . Then to construct a log structure \mathcal{M}^γ inducing the \mathcal{M}_n^γ , we may as well replace \underline{X} by $\underline{V} := \underline{X} - \bigcup_{\gamma' \neq \gamma} D_{\gamma'}$. We apply the results of Section 3 to \underline{V} .

Let G, G_2 and D (resp. G^n, G_2^n and D^n) be as in Section 3 applied to \underline{V} relative to $(\text{Spec}(A), \mathcal{M}_A^\gamma)$ (resp. \underline{V}_n relative to $(\text{Spec}(A_n), \mathcal{M}_{A_n}^\gamma)$). Note that D is proper over $\text{Spec}(A)$, flat over $\text{Spec}(A/t_\gamma)$, and that there is a natural isomorphism

$$D^n \simeq D \times_{\text{Spec}(A)} \text{Spec}(A_n).$$

By Theorem 3.18, it suffices to show that the map

$$(4.2) \quad H^1(\underline{V}, G_2/G) \rightarrow \varprojlim H^1(\underline{V}_n, G_2^n/G^n)$$

is injective, and that the map

$$(4.3) \quad H^0(\underline{V}, G_2/G) \rightarrow \varprojlim H^0(\underline{V}_n, G_2^n/G^n)$$

is bijective.

LEMMA 4.2. *For any $A/(t_\gamma)$ -module M of finite type, the natural map*

$$M \rightarrow H^0(\underline{V}, M \otimes \mathcal{O}_D)$$

is an isomorphism.

PROOF. Since D is proper over \underline{A} , the natural map

$$H^0(\underline{V}, M \otimes \mathcal{O}_D) \rightarrow \varprojlim_n H^0(\underline{V}, M \otimes \mathcal{O}_{D^n}) \simeq \varprojlim_n H^0(\underline{V}, M \otimes A_n \otimes_{A_n} \mathcal{O}_{D^n})$$

is an isomorphism by ([6], III.5.1.2). Therefore, it suffices to consider the case when A is an Artin local ring. Since A is noetherian and artinian, there exists a filtration of M by A/t_γ -modules

$$0 = M_r \subset \dots \subset M_2 \subset M_1 = M$$

such that for each $i, M_i/M_{i+1} \simeq A/\mathfrak{m}_A$. Now, if the results holds for $M = A/\mathfrak{m}_A$ and M_i , then the snake lemma applied to the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M_i & \rightarrow & M_{i+1} & \rightarrow & A/\mathfrak{m}_A & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^0(\underline{V}, M_i \otimes \mathcal{O}_D) & \rightarrow & H^0(\underline{V}, M_{i+1} \otimes \mathcal{O}_D) & \rightarrow & H^0(\underline{V}, \mathcal{O}_{D^0}) & & \end{array}$$

obtained from the exact sequence (D is flat over A/t_γ)

$$0 \rightarrow M_i \otimes \mathcal{O}_D \rightarrow M_{i+1} \otimes \mathcal{O}_D \rightarrow \mathcal{O}_{D^0} \rightarrow 0$$

yields the results for M_{i+1} . Therefore, it suffices to consider the case when $M = A/\mathfrak{m}_A$. In this case we may even base change to an algebraic closure of A/\mathfrak{m}_A , and hence we may assume that A/\mathfrak{m}_A is algebraically closed. But then D^0 is a reduced proper scheme over an algebraically closed field, and hence $H^0(D^0, \mathcal{O}_{D^0}) = A/\mathfrak{m}_A$. \square

Let K_{t_γ} (resp. $K_{t_\gamma}^n$) be the kernel of multiplication by t_γ on A (resp. on A_n), and let \mathcal{F} (resp. \mathcal{F}^n) be the image of the map $G_2 \rightarrow \mathcal{O}_D^*$ (resp. $G_2^n \rightarrow \mathcal{O}_{D^n}^*$). Note that since \underline{V} is flat over A , we have in the notation of Proposition 3.19

$$(K_{t_\gamma} \cap t_\gamma A) \otimes_A \mathcal{O}_D \simeq (K_{t_\gamma}^V \cap t_\gamma \mathcal{O}_{\underline{V}}) \otimes_{\mathcal{O}_{\underline{V}}} \mathcal{O}_D$$

and

$$(K_{t_\gamma}^n \cap t_\gamma A_n) \otimes_{A_n} \mathcal{O}_{D^n} \simeq (K_{t_\gamma}^{V_n} \cap t_\gamma \mathcal{O}_{\underline{V}_n}) \otimes_{\mathcal{O}_{\underline{V}_n}} \mathcal{O}_{D^n}.$$

COROLLARY 4.3.

$$H^0(\underline{V}, \mathcal{F}) = \{u \in A^* \mid ut_\gamma = t_\gamma\} / \{u \in A^* \mid 1 - u \in K_{t_\gamma} \cap t_\gamma A\}$$

and

$$H^0(\underline{V}_n, \mathcal{F}^n) = \{u \in A_n^* \mid ut_\gamma = t_\gamma\} / \{u \in A_n^* \mid 1 - u \in K_{t_\gamma}^n \cap t_\gamma A_n\}.$$

PROOF. Since \underline{X} is flat over A , there are natural isomorphisms

$$\mathcal{O}_D / K_t^X \mathcal{O}_D \simeq \mathcal{O}_D \otimes_{A/t_\gamma} ((A/t_\gamma) / K_{t_\gamma} \cdot (A/t_\gamma)),$$

and

$$\mathcal{O}_{D^n} / K_t^{X_n} \mathcal{O}_{D^n} \simeq \mathcal{O}_{D^n} \otimes_{A_n/t_\gamma} ((A_n/t_\gamma) / K_{t_\gamma}^n \cdot (A_n/t_\gamma)).$$

Hence Lemma 4.2 and the exact sequences

$$(4.4) \quad 0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_D^* \rightarrow (\mathcal{O}_D / K_t^X \mathcal{O}_D)^* \rightarrow 0,$$

$$(4.5) \quad 0 \rightarrow \mathcal{F}^n \rightarrow \mathcal{O}_{D^n}^* \rightarrow (\mathcal{O}_{D^n} / K_t^{X_n} \mathcal{O}_{D^n})^* \rightarrow 0,$$

give exact sequences

$$0 \rightarrow H^0(\underline{V}, \mathcal{F}) \rightarrow (A/t_\gamma)^* \rightarrow ((A/t_\gamma) / K_{t_\gamma} \cdot (A/t_\gamma))^*,$$

$$0 \rightarrow H^0(\underline{V}_n, \mathcal{F}^n) \rightarrow (A_n/t_\gamma)^* \rightarrow ((A_n/t_\gamma) / K_{t_\gamma}^n \cdot (A_n/t_\gamma))^*.$$

From this the corollary follows. \square

The corollary implies that $H^0(\underline{V}, G_2/G) \rightarrow H^0(\underline{V}, \mathcal{F})$ is surjective, and hence there is a commutative diagram

$$(4.6) \quad \begin{array}{ccccccc} 0 & \rightarrow & K_{t_\gamma} \cap t_\gamma A & \rightarrow & H^0(\underline{V}, G_2/G) & \rightarrow & H^0(\underline{V}, \mathcal{F}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \varprojlim K_{t_\gamma}^n \cap t_\gamma A_n & \rightarrow & \varprojlim H^0(\underline{V}_n, G_2^n/G^n) & \rightarrow & \varprojlim H^0(\underline{V}_n, \mathcal{F}^n) & & \end{array}$$

with exact rows. The following lemma implies that the map 4.3 is bijective.

LEMMA 4.4. *Let $L_{t_\gamma}^n := \text{Im}(K_{t_\gamma} \cap t_\gamma A \rightarrow K_{t_\gamma}^n \cap t_\gamma A_n)$. Then there exists integers n_0 and n'_0 such that:*

1. For each n , the map

$$(K_{t_\gamma} \cap t_\gamma A) \otimes_{A_{n+n_0}} \rightarrow (K_{t_\gamma} \cap t_\gamma A) \otimes_{A_n}$$

factors through $L_{t_\gamma}^{n+n_0}$.

2. For every n , the map

$$K_{t_\gamma}^{n+n'_0} \cap t_\gamma A_{n+n'_0} \rightarrow K_{t_\gamma}^n$$

factors through $L_{t_\gamma}^{n+n'_0}$.

3. *The natural map*

$$H^0(\underline{V}, \mathcal{F}) \rightarrow \varprojlim H^0(\underline{V}_n, \mathcal{F}^n)$$

is an isomorphism.

PROOF. By the Artin-Rees lemma there exists an integer n_0 such that

$$K_{t_\gamma} \cap t_\gamma A \cap \mathfrak{m}_A^{n+n_0} \subset \mathfrak{m}_A^n (K_{t_\gamma} \cap t_\gamma A)$$

for every n . From this 1 follows.

Similarly, the Artin-Rees lemma implies that there exists an integer n'_0 such that

$$(t_\gamma)^2 \cap \mathfrak{m}_A^{n+n'_0} \subset \mathfrak{m}_A^n (t_\gamma)^2.$$

Suppose $t_\gamma f \in K_{t_\gamma}^{n+n'_0} \cap t_\gamma A_{n+n'_0}$ and let $\tilde{f} \in A$ be a lifting of f to A . Then $t_\gamma^2 \tilde{f} \in \mathfrak{m}_A^{n+n_0+1}$, and hence $t_\gamma^2 \tilde{f} \in \mathfrak{m}_A^{n+1} (t_\gamma)^2$. Let $g \in \mathfrak{m}_A^{n+1}$ be an element such that $t_\gamma^2 \tilde{f} = t_\gamma^2 g$. Then $t_\gamma(g - f) \in K_{t_\gamma} \cap (t_\gamma)$, and hence the image of

$$K_{t_\gamma}^{n+n'_0} \cap t_\gamma A_{n+n'_0} \rightarrow K_{t_\gamma}^n$$

is contained in $L_{t_\gamma}^{n+n_0}$. This implies 2.

To prove the part 3, let Λ_n be the group

$$\{u \in A_n^* \mid 1 - u \in K_{t_\gamma}^n \cap t_\gamma A_n\}.$$

Then by Corollary 4.3 there are exact sequences

$$0 \rightarrow \Lambda_n \rightarrow \{u \in A_n^* \mid ut_\gamma = t_\gamma\} \rightarrow H^0(\underline{V}_n, \mathcal{F}^n) \rightarrow 0,$$

and by the part 2 the Λ_n satisfy the Mittag-Leffler condition. Hence if Λ denotes the group

$$\{u \in A^* \mid 1 - u \in K_{t_\gamma} \cap t_\gamma A\},$$

there is a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \Lambda & \rightarrow & \{u \in A^* \mid ut_\gamma = t_\gamma\} & \rightarrow & H^0(\underline{V}, \mathcal{F}) \rightarrow 0 \\ & & \downarrow & & \downarrow \simeq & & \downarrow \\ 0 & \rightarrow & \varprojlim \Lambda_n & \rightarrow & \varprojlim \{u \in A_n^* \mid ut_\gamma = t_\gamma\} & \rightarrow & \varprojlim H^0(\underline{V}_n, \mathcal{F}^n) \rightarrow 0. \end{array}$$

Part 3 of the lemma now follows because the part 1 implies that the left vertical arrow is an isomorphism. \square

The preceding lemma also implies that the natural map

$$H^1(\underline{V}, (K_{t_\gamma} \cap t_\gamma A) \otimes \mathcal{O}_D) \rightarrow \varprojlim H^1(\underline{V}_n, (K_{t_\gamma}^n \cap t_\gamma A_n) \otimes \mathcal{O}_{D^n})$$

is an isomorphism. By Corollary 4.3, the maps

$$H^1(\underline{V}, (K_{t_\gamma} \cap t_\gamma A) \otimes \mathcal{O}_D) \rightarrow H^1(\underline{G}_2/G),$$

$$H^1(\underline{V}_n, (K_{t_\gamma}^n \cap t_\gamma A_n) \otimes \mathcal{O}_{D^n}) \rightarrow H^1(\underline{G}_2^n/G^n)$$

are injective. From this and the diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 H^1(\underline{V}, (K_{t_\gamma} \cap t_\gamma A) \otimes \mathcal{O}_D) & \longrightarrow & \varprojlim H^1(\underline{V}_n, (K_{t_\gamma}^n \cap t_\gamma A_n) \otimes \mathcal{O}_{D^n}) \\
 \downarrow & & \downarrow \\
 H^1(\underline{V}, G_2/G) & \longrightarrow & \varprojlim H^1(\underline{V}_n, G_2^n/G^n) \\
 \downarrow & \xrightarrow{\varepsilon} & \downarrow \\
 H^1(\underline{V}, \mathcal{F}) & & \varprojlim H^1(\underline{V}_n, \mathcal{F}^n),
 \end{array}$$

we see that to verify that 4.2 is injective it suffices to show that the map labeled ε in the diagram above is injective.

The exact sequences 4.4 and 4.5 together with Lemma 4.2 imply that there is a commutative diagram

$$\begin{array}{ccc}
 H^1(\underline{V}, \mathcal{F}) & \longrightarrow & H^1(\underline{V}, \mathcal{O}_D^*) \\
 \downarrow & & \downarrow \\
 \varprojlim H^1(\underline{V}_n, \mathcal{F}^n) & \longrightarrow & \varprojlim H^1(\underline{V}_n, \mathcal{O}_{D^n}^*),
 \end{array}$$

where the horizontal arrows are injections. Therefore, to verify that ε is injective it is enough to verify that the map

$$H^1(\underline{V}, \mathcal{O}_D^*) \rightarrow \varprojlim H^1(\underline{V}_n, \mathcal{O}_{D^n}^*)$$

is injective. This follows from ([6], III.5.1.6), and hence the proof of Theorem 4.1 is complete. \square

5. Proof of Theorem 1.2. We now prove Theorem 1.2. Let $\underline{\text{IVLS}}$ be the fibered category over the category of \underline{S} -schemes which to any $\underline{T} \rightarrow \underline{S}$ associates the category of triples $(\mathcal{M}_T, \mathcal{M}_{X_T}, f_T^b)$, where \mathcal{M}_T is a log structure on \underline{T} , \mathcal{M}_{X_T} is a log structure on $\underline{X} \times_{\underline{S}} \underline{T}$, and $f_T^b : \text{pr}_2^* \mathcal{M}_T \rightarrow \mathcal{M}_{X_T}$ is a morphism of log structures making

$$(f, f^b) : (\underline{X}_T, \mathcal{M}_{X_T}) \rightarrow (\underline{T}, \mathcal{M}_T)$$

a special morphism in the sense of Definition 2.6. There is a natural log structure on the fibered category $\underline{\text{IVLS}}$ (see the introduction for the definition of a log structure on a fibered category)

$$\mathcal{M} : \underline{\text{IVLS}} \rightarrow \text{Log}(\underline{S}, \mathcal{O}_{\underline{S}}^*), \quad (\mathcal{M}_T, \mathcal{M}_{X_T}, f_T^b) \mapsto \mathcal{M}_T.$$

Since $\underline{\text{IVLS}}$ classifies sheaves for the étale topology, $\underline{\text{IVLS}}$ is a stack over \underline{S} , and by ([12], 4.18) $\underline{\text{IVLS}}$ is limit preserving. By Theorem 2.7, if $(\underline{T}, \mathcal{M}'_T)$ is a log scheme, then to give an element of $\underline{\text{IVLS}}$ over $(\underline{T}, \mathcal{M}'_T)$ is equivalent to giving an object $(\mathcal{M}_T, \mathcal{M}_{X_T}, f_T^b)$ of $\underline{\text{IVLS}}$ over \underline{T} together with a morphism of log structures $\mathcal{M}_T \rightarrow \mathcal{M}'_T$. Hence, to prove Theorem 1.2, it suffices to show that $\underline{\text{IVLS}}$ is representable by a scheme, and that the map $\underline{\text{IVLS}} \rightarrow \underline{S}$ is a monomorphism of finite type.

Now, any algebraic space which admits a quasi-finite map to a scheme is also a scheme ([4], 3.3), and hence it suffices to show that $\underline{\text{IVLS}}$ is an algebraic space. We proceed in two steps (Theorems 5.1 and 5.4).

THEOREM 5.1. *$\underline{\text{IVLS}}$ is relatively representable and $\underline{\text{IVLS}}$ is equivalent to the stack defined by a sub-functor of $\underline{\mathcal{S}}$.*

PROOF. Suppose $(\mathcal{M}_T, \mathcal{M}_{X_T}, f_T^b)$ and $(\mathcal{M}'_T, \mathcal{M}'_{X_T}, f_T^{b'})$ are two objects of $\underline{\text{IVLS}}$ over some scheme \underline{T} . What has to be shown is that there exists unique isomorphisms $\sigma : \mathcal{M}_T \simeq \mathcal{M}'_T$ and $\phi : \mathcal{M}_{X_T} \simeq \mathcal{M}'_{X_T}$ such that the diagram

$$\begin{array}{ccc} f^* \mathcal{M}_T & \xrightarrow{f^b} & \mathcal{M}_{X_T} \\ \sigma \downarrow & & \downarrow \phi \\ f^* \mathcal{M}'_T & \xrightarrow{f^{b'}} & \mathcal{M}'_{X_T} \end{array}$$

commutes. Let us begin by showing the uniqueness of the pair (σ, ϕ) . For this it suffices to consider the case when $\underline{T} = \text{Spec}(A)$ is a complete local ring.

Let $\text{SS}(\underline{T}, \mathcal{M}_T)$ be the set of isomorphism classes of pairs (\mathcal{M}_X, f^b) , where \mathcal{M}_X is a log structure on \underline{X} and $f^b : f^* \mathcal{M}_T \rightarrow \mathcal{M}_X$ is a morphism of log structures making

$$(\underline{X}, \mathcal{M}_X) \rightarrow (\underline{T}, \mathcal{M}_T)$$

a special morphism (in the sense of Definition 2.6).

Let

$$\rho : \text{Aut}(\mathcal{M}_T) \rightarrow \text{End}(\text{SS}(\underline{T}, \mathcal{M}_T))$$

be the action given by

$$(f^* \mathcal{M}_T \longrightarrow \mathcal{M}_X)^{\rho(\sigma)} := (f^* \mathcal{M}_T \xrightarrow{\sigma} f^* \mathcal{M}_T \longrightarrow \mathcal{M}_X).$$

The following proposition implies the uniqueness of (σ, ϕ) .

PROPOSITION 5.2. 1. *For a pair (\mathcal{M}_X, f^b) as above, there are no automorphisms of \mathcal{M}_X compatible with the map f^b .*

2. *The action ρ on $\text{SS}(\underline{T}, \mathcal{M}_T)$ is simply transitive.*

PROOF. 1. It suffices by descent theory to consider the case when A has separably closed residue field. In this case, any object $(\mathcal{M}_X, f^b) \in \text{SS}(\text{Spec}(A))$ has a canonical decomposition

$$(5.1) \quad \mathcal{M}_X \simeq \bigoplus_{\gamma \in \Gamma} \mathcal{O}_X^* \mathcal{M}_X^\gamma,$$

where Γ denotes the set of connected components of the singular locus of the closed fiber. Moreover, (\mathcal{M}_X, f^b) induces a decomposition

$$(5.2) \quad \mathcal{M}_T \simeq \bigoplus_{\gamma \in \Gamma} \mathcal{O}_T^* \mathcal{M}_T^\gamma.$$

This implies 1, for any automorphism of \mathcal{M}_X must preserve the above decompositions and hence induces automorphisms of the log structures \mathcal{M}_X^γ over \mathcal{M}_T^γ . But by the discussion preceding Theorem 3.18, there are no such automorphisms other than the identity.

2. We first reduce to the case when A is an artin local ring with separably closed residue field. Let \mathcal{M}_{T_n} be the reduction of \mathcal{M}_T to $A_n := A/\mathfrak{m}_A^{n+1}$. By Theorem 4.1 and the fact that the functor $\underline{\text{Aut}}(\mathcal{M}_T)$ is representable, the natural maps

$$\text{Aut}(\mathcal{M}_T) \rightarrow \varprojlim \text{Aut}(\mathcal{M}_{T_n}), \quad \text{SS}(\text{Spec}(A)) \rightarrow \varprojlim \text{SS}(\text{Spec}(A_n))$$

are bijective, and hence it suffices to consider the case when A is an artin local ring. Moreover by 1, both $\underline{\text{Aut}}(\mathcal{M}_T)$ and $\text{SS}(\text{Spec}(A))$ are sheaves on the étale site, so we can by descent theory assume that A has separably closed residue field.

In this case, any object $(\mathcal{M}_X, f^b) \in \text{SS}(\text{Spec}(A))$ has a canonical decomposition as in 5.1 and induces a decomposition of \mathcal{M}_T as in 5.2.

Suppose given two objects

$$(\mathcal{M}_X, f^b), (\mathcal{M}'_X, f^{b'}) \in \text{SS}(\text{Spec}(A))$$

inducing two decompositions

$$\mathcal{M}_T \simeq \bigoplus_{\gamma \in \Gamma} \mathcal{O}_T^* \mathcal{M}_T^\gamma \simeq \bigoplus_{\gamma \in \Gamma} \mathcal{O}_T^* \mathcal{M}_T^{\gamma'}$$

The following lemma, applied to the completion of \mathcal{O}_X at a point in γ , shows that \mathcal{M}_T^γ and $\mathcal{M}_T^{\gamma'}$ are isomorphic as log structures (not as sub-log structures of \mathcal{M}_T):

LEMMA 5.3. *Suppose $t, t' \in A$ are two elements such that there exists $d, d', r, r' \in \mathbf{Z}$ and an isomorphism*

$$\phi : A[[X_1, \dots, X_d]]/(X_1 \cdots X_r - t) \simeq A[[Y_1, \dots, Y_{d'}]]/(Y_1 \cdots Y_{r'} - t').$$

Then there exists a unit $u \in A^$ such that $ut = t'$.*

PROOF. If the result holds with A replaced by $A/(t')$, then $t \in (t')$ and by symmetry $(t) = (t')$. Therefore it suffices to consider the case when $t' = 0$. Moreover, in this case it suffices to consider the reduction modulo t^2 . Write

$$\phi(X_i) = c_i + f(\underline{Y}), \quad i = 1, \dots, r,$$

where $c_i \in A$ and $f(\underline{Y})$ consists of terms of degree > 0 (as in the proof of Proposition 3.8, $A[[Y_1, \dots, Y_{d'}]]/(Y_1 \cdots Y_{r'})$ is graded since $t' = 0$). By Proposition 3.8, $c_i \in (t)$, and hence since we are assuming that $t^2 = 0$ we have $\prod_{i=1}^r c_i = 0$. Thus

$$t = \prod_{i=1}^r \phi(X_i) = \prod c_i + g = g,$$

where $g \in (Y_1, \dots, Y_{d'})$. If $t \neq 0$, this is a contradiction, since

$$A \cap (Y_1, \dots, Y_{d'}) = \{0\}.$$

□

Thus there exists an automorphism σ of \mathcal{M}_T such that the decompositions of \mathcal{M}_T obtained from

$$f^* \mathcal{M}_T \rightarrow \mathcal{M}_X$$

and

$$f^* \mathcal{M}_T \xrightarrow{\sigma} f^* \mathcal{M}'_T \longrightarrow \mathcal{M}'_X$$

are the same. Decompose

$$\mathcal{M}_X \simeq \bigoplus_{\gamma \in \Gamma, \mathcal{O}_X^*} \mathcal{M}_X^\gamma, \quad \mathcal{M}'_X \simeq \bigoplus_{\gamma \in \Gamma, \mathcal{O}_X^*} \mathcal{M}_X^{\gamma'}$$

Then to prove the theorem it suffices to show that there exists a unique automorphism $\sigma : \mathcal{M}_T^\gamma \rightarrow \mathcal{M}'_T^\gamma$ and an isomorphism $\psi : \mathcal{M}_X^\gamma \simeq \mathcal{M}_X^{\gamma'}$ such that the diagram

$$\begin{array}{ccc} f^* \mathcal{M}_T^\gamma & \longrightarrow & \mathcal{M}_X^\gamma \\ \sigma \downarrow & & \downarrow \psi \\ f^* \mathcal{M}'_T^\gamma & \longrightarrow & \mathcal{M}_X^{\gamma'} \end{array}$$

commutes. For this we may, as in the proof of Theorem 4.1, replace X by an open set \underline{V} , which contains the component γ but does not intersect any of the components $\Gamma - \{\gamma\}$, and we can also choose a chart

$$N \rightarrow \mathcal{M}'_T.$$

Let $t_\gamma \in A$ be the image of 1. Then using the notation of the proof of Theorem 4.1, we want to show that the natural map

$$\{u \in A^* \mid ut_\gamma = 1\} \rightarrow H^0(\underline{V}, G_2/G)$$

is bijective. This follows from the exactness of the first line in the equation 4.6 and Corollary 4.3. □

In light of the uniqueness of (σ, ϕ) , to prove Theorem 5.1 it suffices to prove the existence of (σ, ϕ) étale locally. If \underline{T} is the spectrum of an artinian local ring, then it follows from Lemma 5.3 that \mathcal{M}_T and \mathcal{M}'_T are étale locally isomorphic, and hence by Proposition 5.2 a pair (σ, ϕ) exists in this case. By passage to the limit using Theorem 4.1 a pair (σ, ϕ) also exists in the case when \underline{T} is the spectrum of a complete noetherian local ring.

For general T , let $t \in \underline{T}$ be a point. Then from above there exists a pair (σ, ϕ) over the completion $\widehat{\mathcal{O}}_{T,t}$. By the Artin approximation theorem ([3], 1.6), there exists a pointed \underline{T} -scheme $v \in \underline{V}$ together with an isomorphism $\widehat{\mathcal{O}}_{\underline{V},v} \simeq \widehat{\mathcal{O}}_{T,t}$ and a pair (σ_v, ϕ_v) over \underline{V} inducing the pair (σ, ϕ) over $\widehat{\mathcal{O}}_{T,t}$. By construction $\underline{V}/\underline{T}$ is étale at v , and hence after shrinking \underline{V} we can assume that \underline{V} is an étale \underline{T} -scheme.

This concludes the proof of Theorem 5.1. □

THEOREM 5.4. IVLS admits a smooth cover.

PROOF. It suffices to show that for any map

$$(5.3) \quad \underline{T} \rightarrow \text{Log}_{(\underline{\mathcal{S}}, \mathcal{O}_{\underline{\mathcal{S}}}^*)}$$

the product

$$\underline{T} \times_{\text{Log}_{(\underline{\mathcal{S}}, \mathcal{O}_{\underline{\mathcal{S}}}^*)}} \underline{\text{IVLS}}$$

is representable by an algebraic space. If \mathcal{M}_T is the log structure corresponding to the map 5.3, then the product is the functor

$$\text{SS} : (\underline{T} - \text{schemes}) \rightarrow \text{Set},$$

which to any $h : \underline{T}' \rightarrow \underline{T}$ associates the set of special elements in

$$\text{IVLS}((\underline{T}', h^* \mathcal{M}_T)).$$

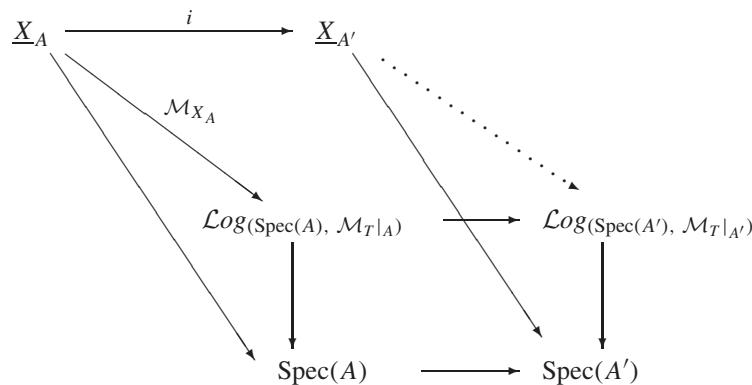
We prove that SS is representable by verifying the conditions of ([5], 5.4). For a \underline{T} -scheme $h : \underline{T}' \rightarrow \underline{T}$ denote by $\mathcal{M}_T|_{\underline{T}'}$ the pullback $h^* \mathcal{M}_T$ and let $\underline{X}_{T'}$ denote the product $\underline{T}' \times_{\underline{T}} \underline{X}$.

By Proposition 5.2, SS is a sheaf for the étale topology, and SS is limit preserving by ([12], 4.18).

LEMMA 5.5. *Suppose $A' \rightarrow A$ is a surjective morphism of rings with square-zero kernel, and suppose $\mathcal{M}_{X_A} \in \text{SS}(\text{Spec}(A))$. Then to give a lifting of \mathcal{M}_{X_A} to $\text{SS}(\text{Spec}(A'))$ is equivalent to giving a 1-morphism*

$$\mathcal{M}_{X_{A'}} : \underline{X}_{A'} \rightarrow \text{Log}(\text{Spec}(A'), \mathcal{M}_T|_{A'})$$

filling in the diagram



(that is, a 1-morphism $\mathcal{M}_{X_{A'}}$ together with an isomorphism $i^* \mathcal{M}_{X_{A'}} \simeq \mathcal{M}_{X_A}$).

PROOF. To give a 1-morphism as in the lemma is equivalent to giving a lifting of \mathcal{M}_{X_A} to $\underline{X}_{A'}$ and a morphism of log schemes $(\underline{X}_{A'}, \mathcal{M}_{X_{A'}}) \rightarrow (\text{Spec}(A'), \mathcal{M}_T|_{A'})$ lifting the map from $(\underline{X}_A, \mathcal{M}_A)$. Thus the lemma is equivalent to the statement that any such lifting is a special morphism. Now by ([11] 4.1 and [17] Theorem 3.13), any such lifting $\mathcal{M}_{A'}$ factors through the maximal open substack $\mathcal{U} \subset \text{Log}(\text{Spec}(A'), \mathcal{M}_T|_{A'})$ flat over $\text{Spec}(A')$. Since $\underline{X}_{A'}/A'$ is flat, this implies that $\mathcal{M}_{X_{A'}}$ is smooth, since its reduction to A is smooth. The other conditions in the definition of a special morphism evidently only depend on the reduced structure of the base. Hence the lemma follows. \square

It follows from the lemma and ([18]) that Schlessinger’s condition (S1’) holds, and that for any A -module I , the module $\text{SS}_{\mathcal{M}_{X_A}}(A[I])$ is isomorphic to

$$\text{Ext}_{\mathcal{O}_X}^0(\mathcal{M}_{X_{A_0}}^* L_{\mathcal{L}og(\text{Spec}(A_0), \mathcal{M}_T|_{A_0})/A_0}^{>-2}, f_T^* I),$$

where $L_{\mathcal{L}og(\text{Spec}(A_0), \mathcal{M}_T|_{A_0})/A_0}^{>-2}$ denotes the truncated cotangent complex of the algebraic stack $\mathcal{L}og(\text{Spec}(A_0), \mathcal{M}_T|_{A_0})$ (see [18] or [12]). Moreover, ([18]) shows that the modules

$$\text{Ext}_{\mathcal{O}_X}^1(\mathcal{M}_{X_{A_0}}^* L_{\mathcal{L}og(\text{Spec}(A_0), \mathcal{M}_T|_{A_0})/A_0}^{>-2}, f_T^* I)$$

define an obstruction theory for SS . To see that the conditions on D and \mathcal{O} of ([5], (S2) and 4.1) are satisfied, use ([6], III.3.1.2 and III.4.5) and ([3], Lemmas 6.8 and 6.9).

Next, we claim that if A_0 is a reduced ring, and if $\mathcal{M}_{X_{A_0}}$ and $\mathcal{M}'_{X_{A_0}}$ are two elements of $\text{SS}(\text{Spec}(A_0))$ which are equal at a dense set of points of finite type, then they are equal on a dense open set. To see this observe that by theorem 5.1 there exists a unique pair of isomorphisms $\sigma : \mathcal{M}_T|_{A_0} \rightarrow \mathcal{M}_T|_{A_0}$ and $\phi : \mathcal{M}_{X_{A_0}} \rightarrow \mathcal{M}'_{X_{A_0}}$ such that the diagram

$$\begin{array}{ccc} f^* \mathcal{M}_T|_{A_0} & \longrightarrow & \mathcal{M}_{X_{A_0}} \\ f^*(\sigma) \downarrow & & \downarrow \phi \\ f^* \mathcal{M}_T|_{A_0} & \longrightarrow & \mathcal{M}'_{X_{A_0}} \end{array}$$

commutes. The condition that $\mathcal{M}_{X_{A_0}}$ and $\mathcal{M}'_{X_{A_0}}$ define the same element of SS is therefore the same as the condition that $\sigma = \text{id}$. Now, since $\mathcal{L}og(\underline{T}, \mathcal{O}_T^*)$ is algebraic, if $\sigma = \text{id}$ for a dense set of points of $\text{Spec}(A_0)$ of finite type, then σ equals the identity on some dense open set by ([5], page 182).

Finally, observe that if \hat{A} is a complete local ring with residue field of finite type over \underline{T} , then the map

$$\text{SS}(\hat{A}) \rightarrow \varprojlim \text{SS}(\hat{A}/\mathfrak{m}_{\hat{A}}^n)$$

is bijective by Theorem 4.1. It therefore follows from ([5], 5.4) that SS is representable by an algebraic space over \underline{T} . This completes the proof of Theorem 5.4. \square

To complete the proof of Theorem 1.2, it remains only to see that IVLS is of finite type over \underline{S} . For this it suffices by the argument given in ([3], page 59) to show the following: Given an integral domain A_0 , and a dense set of points $\mathfrak{S} \subset \text{Spec}(A_0)$ of finite type such that $\text{IVLS}(\text{Spec}(k(p)))$ is non-empty for all $p \in \mathfrak{S}$, there exists a dense open set $\underline{U} \subset \text{Spec}(A_0)$ for which $\text{IVLS}(\underline{U})$ is non-empty. Let $K_0 = \text{Frac}(A_0)$. Then by assumption, X_{K_0} is étale locally isomorphic to

$$K_0[X_1, \dots, X_d]/(X_1 \cdots X_r),$$

and so we can replace A_0 by an affine open set so that X_{A_0} is étale locally isomorphic to

$$A_0[X_1, \dots, X_d]/(X_1 \cdots X_r).$$

Let \mathcal{M}_{A_0} be the log structure on $\mathrm{Spec}(A_0)$ associated to the map $N \rightarrow A_0$, $1 \mapsto 0$. By Theorem 2.7, to prove the IVLS is non-empty over some dense open set of $\mathrm{Spec}(A_0)$, it suffices to prove that there exists a semi-stable log structure on \underline{X}_{A_0} relative to $(\mathrm{Spec}(A_0), \mathcal{M}_{A_0})$.

By Theorem 3.6 there exists such a log structure if and only if the invertible \mathcal{O}_D -module

$$\mathcal{E}xt^1(\Omega_{\underline{X}_{A_0}/A_0}^1, \mathcal{O}_X)$$

is trivial (where D is defined as in Section 3). By ([3], page 70) this holds over some dense open set in $\mathrm{Spec}(A_0)$, and hence there exists a dense open set $U \subset \mathrm{Spec}(A_0)$ for which IVLS(U) is non-empty. This completes the proof of Theorem 1.2. \square

REFERENCES

- [1] A. ALTMAN AND S. KLEIMAN, Introduction to Grothendieck duality theory, Lecture Notes in Math. 146, Springer-Verlag, Berlin, 1970.
- [2] M. ARTIN, Algebraic approximation of structures over complete local rings, Inst. Hautes Études Sci. Publ. Math. 36 (1969), 23–58.
- [3] M. ARTIN, Algebraization of formal moduli. I, Global Analysis (Papers in Honor of K. Kodaira), 21–71, Univ. Tokyo Press, Tokyo, 1969.
- [4] M. ARTIN, The implicit function theorem in algebraic geometry, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), 13–34, Oxford Univ. Press, London, 1969.
- [5] M. ARTIN, Versal deformations and algebraic stacks, Invent. Math. 27 (1974), 165–189.
- [6] A. GROTHENDIECK AND J. DIEUDONNÉ, Éléments de géométrie algébrique I–IV, Inst. Hautes Études Sci. Publ. Math. 4, 8, 11, 17, 20, 24, 28, 32, 1960–1967.
- [7] J. GIRAUD, Cohomologie non abélienne, Die Grundlehren der mathematischen Wissenschaften 179, Springer-Verlag, Berlin, 1971.
- [8] R. HARTSHORNE, Algebraic geometry, Graduate Texts in Math. 52, Springer-Verlag, New York, 1977.
- [9] F. KATO, Log smooth deformation theory, Tôhoku Math. J. 48 (1996), 317–354.
- [10] F. KATO, Log smooth deformation and moduli of log smooth curves, Internat. J. Math. 11 (2000), 215–232.
- [11] K. KATO, Logarithmic structures of Fontaine-Illusie, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.
- [12] G. LAUMON AND L. MORET-BAILLY, Champs algébriques, Ergebnisse der Mathematik 39, Springer-Verlag, Berlin, 2000.
- [13] J. S. MILNE, Étale cohomology, Princeton University Press, Princeton, N. J., 1980.
- [14] Y. NAKAJIMA, Liftings of simple normal crossing log $K3$ and log Enriques surfaces in mixed characteristics, J. Alg. Geom. 9 (2000), 355–393.
- [15] A. OGUS, Logarithmic de Rham cohomology, preprint, 1999.
- [16] A. OGUS, Lectures on logarithmic algebraic geometry, texed notes, 2001.
- [17] M. OLSSON, Logarithmic geometry and algebraic stacks, preprint, 2002.
- [18] M. OLSSON, Deformation theory of 1-morphisms to algebraic stacks, preprint, 2002.

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