UNIVERSAL OPERATORS AND INVARIANT SUBSPACES

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For any Banach space X, let $B(X)$ denote the space of continuous endomorphisms of X. An operator U in $B(X)$ will be called *universal* if, given any T in $B(X)$, then some nonzero multiple of T is similar to a part of U i.e. there exists $\lambda \in \mathbf{C}$, $\lambda \neq 0$, a closed subspace X_0 of X such that $UX_0 \subseteq X_0$ and a linear homeomorphism ϕ of X onto X_0 such that $\lambda T = \phi^{-1}(U|X_0)\phi$. The first example of a universal operator (or *model*) was constructed by G.-C. Rota [1] for the Hilbert space case. In that instance, U is (unitarily equivalent to) the direct sum of countably many copies of the reverse shift $(\xi_1, \xi_2, \xi_3, \cdots) \to (\xi_2, \xi_3, \xi_4, \cdots)$. Such a direct sum obviously defines an operator whose nullspace is infinite-dimensional and whose range is the whole space. In this note, we show that all such operators are universal (when X is a separable Hilbert space) and that, with rather obvious modifications, the arguments extend to arbitrary Banach spaces.

THEOREM. *Let X denote a separable Hilbert space and let U belong to $B(X)$. If U has the following properties*:

(i) *the nullspace $N(U)$ is infinite-dimensional,*

(ii) *the range space $R(U)$ is the space X,*

then U is universal.

PROOF. We begin by constructing operators, V, W in $B(X)$ such that $UV = I$, $UW = 0$, $N(W) = (0)$, $R(W)$ is closed and $R(W) \perp R(V)$. To do this, we write $\tilde{U}$ for the restriction of U to $N(U)^{\perp}$ and define $V = \tilde{U}^{-1}$. We then take an orthonormal basis $\{e_n\}$ for X and an orthonormal basis $\{e_n'\}$ for $N(U)$ and define $We_n = e_n'$. That V and W have the required properties is obvious. Now let T be any operator in $B(X)$. Choose λ so that $|\lambda| \|T\| \|V\| < 1$ and define $\phi = \sum_{k=1}^{\infty} \lambda^k V^k W T^k$, observing that, by choice of λ, this series converges in $B(X)$. It is also evident that

$$U\phi = \lambda \phi T \tag{1}$$

and

$$\phi = \lambda V \phi T + W. \tag{2}$$

We can now deduce from (2) that ϕ is a linear homeomorphism. For suppose $\phi(x) = 0$. Then since $R(W) \perp R(V)$, it is evident that $V\phi Tx = Wx = 0$. But W is invertible so $x = 0$. Secondly, to show $R(\phi)$ is

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closed, consider $\phi(x_n) \to y$. Then from (2), we have $\lambda V\phi T x_n + W x_n \to y$. Hence $W x_n \to P y$ where P is the orthogonal projection onto $R(W)$. Since $R(W)$ is closed, there exists x such that $W x_n \to W x$ and hence $x_n \to x$. Thus $\phi(x_n) \to \phi(x) = y$. Finally, from (1), $R(\phi)$ is U invariant and the result follows.

EXTENSION TO THE GENERAL CASE. It is evident that the above proof is valid whenever we can perform the construction of V and W such that there is a continuous projection onto $R(W)$. In the case of arbitrary Banach space, the theorem is valid if we replace (i) by (i)′: $N(U)$ is a complemented subspace containing a subspace which is linearly homeomorphic to X.

For suppose (i)′ and (ii) hold and X_0 is a closed complement of $N(U)$. Let $\tilde{U} = U | X_0$ and define $V = \tilde{U}^{-1}$. Then for W, take the linear homeomorphism with range in $N(U)$ whose existence is asserted by (i)′.

APPLICATIONS. Let X be a separable Hilbert space. For any $T \in B(X)$, write $\mathcal{J}(T)$ to denote the lattice of closed invariant subspaces of T. If $\mathcal{J}(T) \neq \{0, X\}$, we call $\mathcal{J}(T)$ nontrivial.

(1) Either $\mathcal{J}(T)$ is nontrivial for every T or for every universal operator U, $\mathcal{J}(U)$ has an infinite-dimensional atom.

(2) If U has properties (i) and (ii), then there is a closed subspace X_0 such that $UX_0 = X_0$ and $X_0 \cap N(U) = (0)$.

For take T which is invertible in $B(X)$. Then from equation (1), $UR(\phi) = R(\phi)$ and $N(U) \cap R(\phi) = (0)$.

(3) Suppose U has properties (i) and (ii). Then $\mathcal{J}(U)$ contains a countable family of mutually disjoint nonzero subspaces X_n such that $X_n \cap N(U) = (0)$.

For let $\{W_n\}$ be a sequence of operators with $R(W_n) \cap R(W_m) = (0)$, $(m \neq n)$, $R(W_n)$ closed, $N(W_n) = (0)$ and $R(W_n) \subseteq N(U)$. Such a family can be constructed by the method used for constructing W in the proof of the theorem. Take V as in the proof and any T with $\|T\| < \|V\|^{-1}$ and $N(T) = (0)$. Each W_n defines a linear homeomorphism $\phi_n = \sum_{k=0}^{\infty} V^k W_n T^k$. Moreover, if $m \neq n$, $R(\phi_n) \cap R(\phi_m) = (0)$. For suppose $\phi_n(X) = \phi_m(y)$ then $W_n x + \sum_{k=1}^{\infty} V^k W_n T^k x = W_m y + \sum_{k=1}^{\infty} V^k W_m T^k y$ so that $W_n x - W_m y \in N(U) \cap R(V)$. Thus $W_n x = W_m y = 0$ and therefore $x = y = 0$.

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