UNIVERSAL QUANTUM GROUPS

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Universal Quantum Groups

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Abstract

For each invertible $m \times m$ matrix Q a compact matrix quantum group $A_u(Q)$ is constructed. These quantum groups are shown to be universal in the sense that any compact matrix quantum group is a quantum subgroup of some of them. Their orthogonal version $A_o(Q)$ is also constructed. Finally, we discuss related constructions in the literature.

1 Construction of $A_u(Q)$ and $A_o(Q)$

In the theory of groups, the complex $m \times m$ unitary groups $U(m)$ are the universal compact Lie groups in the sense that any compact Lie group is a subgroup of some of them, and the symmetric groups S_m on m generators are the universal finite groups in the same sense. It is an open problem whether similar objects exist in the theory of quantum groups.

As an initial attempt, the second author constructed [12, 13] two classes of compact matrix quantum groups $A_u(m)$ and $A_o(m)$ in relation to free products and the study of the universal noncommutative unitary C^*-algebras $U_{nc}(m)$ [1]. The quantum groups $A_u(m)$ are universal only among the unimodular compact matrix quantum groups, as for instance, the quantum groups $SU_{-1}(m)$ and the compact quantum groups constructed by Rieffel [9] are their quantum subgroups (see [15]); the other quantum groups $SU_q(m)$ for $q \neq \pm 1$ are not their quantum subgroups.

In this section, we solve the problem mentioned above for compact matrix quantum groups based on the fundamental work of Woronowicz [16]. For each invertible $m \times m$ matrix Q we construct two compact matrix quantum groups $A_u(Q)$ and $A_o(Q)$, which in a certain sense are deformations of the quantum groups $A_u(m)$ and $A_o(m)$. The $A_u(Q)$'s are the universal compact matrix quantum groups.

In the following, unexplained results and notation can be found in [16, 17, 18, 12, 13]. For each matrix $u = (u_{ij})$ with entries in a *-algebra B, denote by u^t the transpose of u, $\bar{u}$ the matrix (u_{ij}^*), and $u^* = \bar{u}^t$ the usual conjugation of u. The usual leg numbering notation will be used: For any *-algebra B, there are *-homomorphisms from $M_m(\mathbf{C}) \otimes B$ to $M_m(\mathbf{C}) \otimes B \otimes B$ defined by

$$(x \otimes b)_{12} = x \otimes b \otimes 1 \quad \text{and} \quad (x \otimes b)_{13} = x \otimes 1 \otimes b$$

for $x \in M_m(\mathbf{C})$ and $b \in B$. There is one possible confusion: for a matrix u with entries in a *-algebra, u_{12} and u_{13} are the leg numbering notation, while u_{ij} are the matrix entries of u. This will not cause confusion if the reader keeps the context in mind.

For convenience of the reader, we recall the following equivalent definition of compact matrix quantum groups given by Woronowicz [17].

1.1. Definition. We call $G = (A, u)$ a **compact matrix quantum group** if A is a unital C^*-algebra which has a system of generators of the form u_{ij} $(i, j = 1, \cdots, m)$ fulfilling the following two axioms:

(1). There is a unital C^*-homomorphism $\Phi : A \longrightarrow A \otimes A$ such that $\Phi(u_{ij}) = \sum_{k=1}^{m} u_{ik} \otimes u_{kj}$ for each i, j;

(2). The matrices $u = (u_{ij})$ and u^t are invertible in $M_m(\mathbf{C}) \otimes A$.

The dense *-subalgebra generated by the u_{ij}'s is a Hopf *-algebra, and is denoted by $\mathcal{A}$.

Replacing the generators u_{ij} in the above definition by a system of generators of the form u^{α}_{ij} ($\alpha \in \mathsf{N}$, $i, j = 1, \cdots, d_\alpha$, where N is any index set, d_α a natural number), and retaining (with obvious modifications) the rest of the axioms, one obtains the notion of a **compact quantum group**, see [12, 13] (compare also [18, 11]).

By abuse of terminology, we also call A a compact (matrix) quantum group, referring to the underlying geometric object G.

1.2. Lemma. *Let A be any compact quantum group. Let u be any finite dimensional representation of the quantum group A. Then u^t is invertible and the inverse $(u^t)^{-1}$ is a representation of the quantum group A equivalent to the conjugate representation $\bar{u}$.*

Proof. By considering the C^*-subalgebra of A generated by the matrix entries of u if necessary, we may assume that A is a compact matrix quantum group.

Let h be the Haar measure on (A, u). Put

$$E = (1 \otimes h)u^t\bar{u}.$$

Since u^t is invertible, and $(u^t)^* = \bar{u}$, so $u^t\bar{u}$ is invertible and positive. Thus there is a positive number δ such that $u^t\bar{u} \geq \delta 1$, so $E \geq \delta 1$. From this we see that E is an invertible matrix.

Using $(h \otimes 1)\Phi(\cdot) = h(\cdot)1$, and applying $(1 \otimes h \otimes 1)$ to both sides of the identity

$$(1 \otimes \Phi)u^t\bar{u} = u^t_{13}u^t_{12}\bar{u}_{12}\bar{u}_{13},$$

one obtains $E = u^t E \bar{u}$. This is the same as

$$(u^t)^{-1} = E\bar{u}E^{-1},$$

which is what we wanted to show. Q.E.D.

Now let $G = (A, u)$ be a compact matrix quantum group. From the proof of the preceding lemma, the following relations are fulfilled:

$$u^t E \bar{u} E^{-1} = I_m = E \bar{u} E^{-1} u^t,$$

where $I_m = (\delta_{ij})$, with δ_{ij} identified with $\delta_{ij} 1_A$, 1_A being the identity of A. This is one set of the relations for the formulation of the universal compact matrix quantum groups $A_u(Q)$.

1.3. Theorem. *Let Q be any invertible $m \times m$ matrix. Let $A_u(Q)$ be the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \cdots, m)$, subject to the following two sets of relations:*

$$uu^* = I_m = u^* u, \tag{1}$$

$$u^t Q \bar{u} Q^{-1} = I_m = Q \bar{u} Q^{-1} u^t. \tag{2}$$

Then

(1). $(A_u(Q), u)$ is a compact matrix quantum group;

(2). The $(A_u(Q), u)$'s are the universal compact matrix quantum groups in the sense that any compact matrix quantum group is a quantum subgroup of some of these.

Proof. (1). Put $v = (v_{ij}) = u_{12} u_{13}$, so that $v_{ij} = \sum_{k=1}^m u_{ik} \otimes u_{kj}$. The matrix v is clearly unitary. Since the matrix entries of u_{12} commute with those of u_{13}, one has $(u_{12} u_{13})^t = u_{13}^t u_{12}^t$. Thus from the commutation relations for the u_{ij}'s, one obtains

$$\begin{aligned} v^t Q \bar{v} Q^{-1} &= u_{13}^t u_{12}^t Q \bar{u}_{12} \bar{u}_{13} Q^{-1} \\ &= u_{13}^t (u^t Q \bar{u} Q^{-1})_{12} Q \bar{u}_{13} Q^{-1} = u_{13}^t Q \bar{u}_{13} Q^{-1} \\ &= (u^t Q \bar{u} Q^{-1})_{13} = I_m, \end{aligned}$$

Similarly, ones has

$$Q \bar{v} Q^{-1} v^t = I_m.$$

Thus the elements v_{ij} of the C^*-algebra $A_u(Q) \otimes A_u(Q)$ satisfy the two sets of relations for the u_{ij}'s. Therefore there is a well-defined morphism Φ of unital C^*-algebras

$$\Phi : \ A_u(Q) \longrightarrow A_u(Q) \otimes A_u(Q)$$

such that,

$$\Phi(u_{ij}) = \sum_{k=1}^m u_{ik} \otimes u_{kj}.$$

The second set of relations for the u_{ij}'s implies in particular that the matrix u^t is invertible. Thus $(A_u(Q), u)$ is a compact matrix quantum group.

(2). Recall [12, 13] that a compact quantum group B is called a quantum subgroup of another A if there is a surjective morphism of C^*-algebras from A to B that preserves the coproducts.

If $G = (A, u)$ is any compact matrix quantum group, then the reprensentation u of G is equivalent to a unitary one. Therefore we may assume u is unitary. The rest is now clear from the discussions preceding this theorem. Q.E.D.

The next theorem gives the orthogonal version of $A_u(Q)$, whose proof is analogous to that of the previous one:

1.4. Theorem. *Let Q be any invertible $m \times m$ matrix. Let $A_o(Q)$ be the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \cdots, m)$, subject to the following three sets of relations:*

$$u = u, \tag{3}$$

$$uu^t = I_m = u^t u, \tag{4}$$

$$u^t Q u Q^{-1} = I_m = Q u Q^{-1} u^t, \tag{5}$$

Then $(A_o(Q), u)$ is a compact matrix quantum group.

1.5. Remarks. (1). Taking the *-operation of the second set of defining relations for $A_u(Q)$, one finds immediately $A_u(Q) = A_u(Q^*)$. Similarly $A_o(Q) = A_o(Q^*)$.

(2). It is clear that the compact matrix quantum groups $A_u(m)$ and $A_o(m)$ constructed in [12, 13] are precisely the algebras $A_u(I_m)$ and $A_o(I_m)$, respectively, as defined above. One sees from this that the quantum groups $A_u(Q)$ and $A_o(Q)$ are in some sense deformations of $A_u(m)$ and $A_o(m)$, with deformation parameter the matrix Q.

(3). We give the formulas for the antipode and counit of $A_u(Q)$ (those for $A_o(Q)$ are similar).

$$(1 \otimes \kappa)(u) = u^* = \bar{u}^t, \quad (1 \otimes \kappa)\bar{u} = Q^{-1} u^t Q,$$
$$(1 \otimes \kappa^2)(u) = Q^t u (Q^t)^{-1}, \quad (1 \otimes \kappa^2)(\bar{u}) = Q^{-1} \bar{u} Q,$$
$$(1 \otimes \epsilon)(u) = I_m = (1 \otimes \epsilon)(\bar{u}).$$

The counit is continous for all the compact matrix quantum groups defined above. It would be of interest to characterize when the map κ is bounded (or unbounded).

In the following, for either $A_u(Q)$ or $A_o(Q)$ we call the matrix entries u_{ij} their **standard generators.**

2 Non-nuclearity of $A_u(Q)$ and $A_o(Q)$

Using the method in [12, 13] for $A_u(m)$ and $A_o(m)$, we now show that $A_u(Q)$ is non-nuclear for each $m \geq 2$ and any invertible normal $m \times m$ complex matrix Q and that $A_o(Q)$ is non-nuclear for each $m \geq 3$ and any invertible symmetric $m \times m$ real matrix Q.

2.1. Lemma. *(1). Let V be any unitary matrix. Assume that two invertible complex matrices Q_1 and Q_2 are related by $Q_1 = VQ_2V^{-1}$. Then the quantum groups $A_u(Q_1)$ and $A_u(Q_2)$ are similar in the sense of [16] (see p618 there), so in particular they are isomorphic as quantum groups (in the sense of [12, 13]).*

(2). Let V be any orthorgonal matrix. Assume that two invertible real matrices Q_1 and Q_2 are related by $Q_1 = VQ_2V^{-1}$. Then $A_o(Q_1)$ and $A_o(Q_2)$ are similar, so in particular they

are isomorphic as quantum groups.

Proof. We prove (1). The method for the proof of (2) is the same. Let $u_{ij}^{(k)}$ be the standard generators of $A_u(Q_k)$ $(k = 1, 2)$. Put $u = V^t u^{(1)} \overline{V}$. Then the matrix entries of u (as elements of the algebra $A_u(Q_1)$) satisfy the defining relations for $A_u(Q_2)$. Therefore by universality, there is a well-defined morphism of unital C^*-algebras sending the matrix entries of $u^{(2)}$ to those of u. It is easy to see that this morphism is an isomorphism of C^*-algebras (see p618 in [16]). Q.E.D.

2.2. Proposition. *(1). The C^*-algebra $A_u(Q)$ is non-nuclear for each $m \geq 2$ and each invertible normal $m \times m$ complex matrix Q;*

(2). The C^*-algebra $A_o(Q)$ is non-nuclear for each $m \geq 3$ and each invertible symmetric $m \times m$ real matrix Q.

Proof. (1). Because of lemma 2.1.(1), we can assume that Q is a diagonal matrix $Q = diag(q_1, \cdots, q_m)$. Consider the group C^*-algebra $C^*(F_m)$ of the free group on m generators $z_1, \cdots, z_m$. Put $w = (z_i \delta_{ij})$. Then one sees that w satisfies the defining relations for $A_u(Q)$. Therefore by universality, the non-nuclear C^*-algebra $C^*(F_m)$ is a homomorphic image of $A_u(Q)$. Hence, $A_u(Q)$ is non-nuclear.

(2). Because of lemma 2.1.(2), we can assume that Q is a diagonal matrix $Q = diag(q_1, \cdots, q_m)$. Consider the group C^*-algebra $C^*(Z_2 * \cdots * Z_2)$ of the free product of m copies of the two element group with generators $z_1, \cdots, z_m$. Put $w = (z_i \delta_{ij})$. Then one sees that w satisfies the defining relations for $A_o(Q)$. Therefore by universality, the non-nuclear C^*-algebra $C^*(Z_2 * \cdots * Z_2)$ (it is non-nuclear because $Z_2 * \cdots * Z_2$ contains F_2) is a homomorphic image of $A_o(Q)$. Q.E.D.

Remarks: For matrices Q other than the ones in proposition 2.2, it is not clear if the algebras $A_u(Q)$ and $A_o(Q)$ have dimensions more than one, though the one dimensional algebra is always a quotient of them.

3 Related constructions

Kirchberg's algebras

We first discuss the relation between the algebras $A_u(Q)$ constructed above and Kirchberg's algebras $A(\gamma)$ that he presented at the Oberwolfach conference [7]. For each anti-homomorphism γ of the matrix algebra $M_m(\mathbf{C})$, the algebra $A(\gamma)$ is the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \cdots, m)$ such that both u and $(\gamma \otimes 1)u$ are unitary in $M_n(\mathbf{C}) \otimes A(\gamma)$.

We claim that $A(\gamma)$ is a compact matrix quantum group by showing that $A(\gamma)$ corresponds precisely to the special case $A_u(Q)$ with Q an invertible positive matrix. The general form of γ is given by $\gamma(x) = B x^t B^{-1}$ for $x \in M_m(\mathbf{C})$, where B is an invertible matrix in $M_m(\mathbf{C})$. From this we see that the relation $(\gamma \otimes 1)u$ is unitary means precisely

$$u^t (B^* B)^{-1} \bar{u} B^* B = I_m = (B^* B)^{-1} \bar{u} B^* B u^t.$$

Letting Q be the positive matrix $(B^*B)^{-1}$, we see that $A(\gamma) = A_u(Q)$. Conversely, if Q is an invertible positive matrix, then it is easy to see that $A_u(Q)$ is a Kirchberg algebra.

Applying the proof of lemma 1.2 to the quantum group $G = (A_u(Q), u)$, one has

$$u^t E \bar{u} E^{-1} = I_m = E \bar{u} E^{-1} u^t.$$

Therefore the C^*-algebra $A_u(Q)$ is a homomorphic image of the C^*-algebra $A_u(E)$ (which is a Kirchberg algebra, note that the matrix E defined there is in fact positive) with the homomorphism that sends the standard generators of $A_u(E)$ to those of $A_u(Q)$. It is clear that this homomorphism preserves the coproducts. From this one sees that the compact matrix quantum groups $A_u(E)$ (for E any invertible positive matrix) already form a family of universal compact matrix quantum groups. Note that the family $A_u(Q)$ properly contains Kirchberg's algebras: If Q is a diagonal matrix whose scalar multiples do not contain a positive matrix, one sees from theorem 5.4 of [16] and remark 1.5.(3) that $A_u(Q)$ is not a Kirchberg algebra.

Algebras of Dubois-Violette et al and other constructions

In the paper of Dubois-Violette and Launer [4], the single relation (5) in our theorem 1.4 is used to define a Hopf algebra (no C^*-structure) for an arbitrary invertible matrix Q, which is denoted by $\mathcal{A}(Q)$ in their paper. The dense Hopf subalgebra of $A_o(Q)$ (in the sense of [16]) is a quotient of their Hopf algebra $\mathcal{A}(Q)$. It is clear that both our algebras $A_u(Q)$ and $A_o(Q)$ are different from their algebra $\mathcal{A}(Q)$.

Relation (5) also appeared in [5] earlier (see relation (1.13) there) for *very restrictive* matrices Q (denoted by C in (1.13) of [5]) for the definition of quantum $SO(N)$ groups. It is also clear that our algebras are different from the ones constructed in [5] (see also the remarks in 3.5).

In fact, both relation (2) in theorem 1.3 and relation (5) in theorem 1.4 were already used in section X.2 of Helgason [6] to construct all the classical Lie groups. This indicates that our construction will probably give rise to all classical quantum Lie groups.

Relation with Manin's construction

We now construct a Hopf *-algebra $H(m)$ for each natural number m (For the notion of Hopf *-algebra, the reader can see e.g. [10]). The relation between the notation of Woronowicz and the notation in Hopf algebra theory is:

$$(\Phi, \epsilon, \kappa) = (\Delta, \epsilon, S).$$

We now follow the notation in the theory of Hopf algebras.

3.1. Theorem. *Let $H(m)$ be the universal unital algebra generated by elements $u_{ij}^{(n)}$ ($n = 0, \pm 1, \pm 2, \cdots,\ i, j = 1, \cdots, m$) satisfying the following infinite set of relations:*

$$(u^{(n)})^t u^{(n+1)} = I_m = u^{(n+1)} (u^{(n)})^t, \tag{6}$$

where $u^{(n)}$ is the $m \times m$ matrix $(u_{ij}^{(n)})$. Then $H(m)$ is a Hopf algebra with coproduct, counit, *and antipode defined respectively by*

$$\Delta(u_{ij}^{(n)}) = \sum_{k=1}^{m} u_{ik}^{(n)} \otimes u_{kj}^{(n)},$$

$$\epsilon(u_{ij}^{(n)}) = \delta_{ij}, \quad S(u_{ij}^{(n)}) = u_{ji}^{(n-1)}.$$

The operation

$$(u^{(n)})^* = (u^{(-n)})^{-1} = (u^{(-n-1)})^t \tag{7}$$

*defines an involution on the algebra $H(m)$ making it into a Hopf *-algebra.*

Proof. Straightforward computation.

Remarks. (1). Note that for each integer k, the subalgebra $H_k(m)$ of $H(m)$ generated by the elements $u_{ij}^{(n)}$ ($n = -k, -k-1, -k-2, \cdots, \ \ i, j = 1, \cdots, m$) are Hopf subalgebras of $H(m)$ but not *-subalgebras. The Hopf algebras $H_k(m)$ for different k's are clearly isomorphic to each other. Thus we have actually constructed three algebras: The Hopf algebra $H_0(m)$, the Hopf algebra $H(m)$, and the Hopf *-algebra $H(m)$.

(2). The algebras $H(m)$ and $H_0(m)$ are examples of Manin's construction in his paper [8]. By taking R_0 the zero ideal in 7.2 of [8], then the Hopf algebra in 7.3 of [8] is the same as our algebra $H_0(m)$. Also by taking R_0 the zero ideal and using remark 7.4.a) and proposition 10.2 of [8], the resulting Hopf *-algebra is the same as our algebra $H(m)$. However our description of $H_0(m)$ and $H(m)$ is simpler than that of Manin.

3.3. Theorem. *The Hopf *-algebra $H(m)$ has the following universal properties:*

*(1). The dense Hopf *-subalgebra $\mathcal{A}$ for every compact matrix quantum group $G = (A, (u_{ij}))$ $(i, j = 1, \cdots, m)$ is a homomorphic image of $H(m)$.*

(2). The Hopf algebras of algebraic functions on quantum simple groups constructed by FRT [5] are homomorphic images of $H(m)$ (including those corresponding to roots of unity).

Proof. We only indicate the method of proof. For (1), one only needs to treat the case $G = (A_u(Q), u)$. For (2), one treats case by case. For series A_{n-1}, use theorem 4 of [5] (see also 4.7 of [10]); for series B_n, C_n, D_n, use formula (1.13) in [5]. Q.E.D.

Note that the universal property in the theorem above is different from the universal property in 7.3 of Manin [8].

The following result shows that in general a Hopf *-algebra cannot be endowed with a Woronowicz Hopf C^*-structure (however compare [14, 12, 2]).

3.4. Proposition. *The Hopf *-algebra $H(m)$ has no C^*-norms under which the coproduct is bounded.*

Proof. Assume on the contrary there is a C^*-norm $|| \cdot ||$ on $H(m)$ for which the coproduct is bounded. Let $A(m)$ be the corresponding C^*-completion. Then $A(m)$ is a compact quantum group and each $u^{(n)}$ is a represention of this quantum group. By lemma 1.2, there are invertible complex matrices Q_n such that

$$((u^{(n)})^t)^{-1} = Q_n \overline{u^{(n)}} Q_n^{-1}.$$

According to the definition of $H(m)$,

$$((u^{(n)})^t)^{-1} = u^{(n+1)}, \quad \overline{u^{(n)}} = u^{(-n-1)}.$$

From these one sees that

$$u^{(n+1)} = Q_n u^{(-n-1)} Q_n^{-1}.$$

This contradicts the definition of $H(m)$. Q.E.D.

3.5. Remarks. From the above discussions, we see that $H(m)$ are very large quantum groups in some sense. The quantum groups obtained by QISM (note that one can recover Drinfeld-Jimbo quantum groups [3] from those of FRT [5] and vice versa), and ones constructed by Manin, as well as those by Woronowicz are all inside them. The quantum groups $H(m)$ will pobably play a role in quantum group theory analogous to that of the general linear groups $GL(m)$ in Lie group theory. The quantum groups obtained by QISM correspond to the Hopf ideals in $H(m)$ defined by

$$\hat{R}_q(u^{(0)} \otimes u^{(0)}) = (u^{(0)} \otimes u^{(0)})\hat{R}_q$$

together with some other relations, where $\hat{R}_q$ is a Yang-Baxter operator.

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