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Universal R-Matrix for Quantized (Super)Algebras

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Abstract. For quantum deformations of finite-dimensional contragredient Lie (super)algebras we give an explicit formula for the universal R-matrix. This formula generalizes the analogous formulae for quantized semisimple Lie algebras obtained by M. Rosso, A. N. Kirillov, and N. Reshetikhin, Ya. S. Soibelman, and S. Z. Levendorskii. Our approach is based on careful analysis of quantized rank 1 and 2 (super)algebras, a combinatorial structure of the root systems and algebraic properties of q-exponential functions. We don't use quantum Weyl group.

1. Introduction

Quantum algebras appeared as a generalization of algebraic constructions in quantum spectral transform method [1, 2]. The notion of quantum algebras was systematically developed by Drinfeld [3, 4], Jimbo [5], and by Faddeev's school [6]. Later, Drinfeld [4, 7] defined a class of quasitriangular Hopf algebras possessing the universal solution of Yang-Baxter (YB) equation. Namely, a quasitriangular Hopf algebra is a Hopf algebra $\mathscr{A}$ with an additional element $R \in \mathscr{A} \otimes \mathscr{A}$ such that

$$\Delta'(x) = R\Delta(x)R^{-1}, \quad \forall x \in \mathscr{A}, \tag{1.1}$$

$$(\Delta \otimes \mathrm{id})R = R^{13}R^{23}, \quad (\mathrm{id} \otimes \Delta)R = R^{13}R^{12}. \tag{1.2}$$

The element R satisfies the YB equation and is called "the universal R-matrix." The method of construction of the quasitriangular Hopf algebra is based on the quantum double notion [4]. If $\mathscr{A}$ is any Hopf algebra then the quantum double $W(\mathscr{A})$ is a quasitriangular Hopf algebra ($\sim \mathscr{A} \otimes \mathscr{A}'$ as a vector space) with the canonical R-matrix

$$R = \sum_i e_i \otimes e^i, \tag{1.3}$$

where e_i and e^i are dual bases in $\mathscr{A}$ and $\mathscr{A}'$. For any quantum algebra $U_q(g)$ (the Drinfeld-Jimbo deformation of Kac-Moody algebra g) there exists an epimorphism to $U_q(g)$ from the quantum double of the corresponding Borel subalgebra: $W(U_q(b_+)) \to U_q(g)$. Thus any quantum algebra $U_q(g)$ is a quasitriangular Hopf algebra.

The problem is to obtain an explicit expression for the universal R-matrix directly in terms of $U_q(g)$. The implicit form of such an expression was given by Drinfeld [4, 7]. Rosso [8] obtained the explicit factorized expression of the universal R-matrix for $U_q(sl(n))$ by examining the identification of $U_q(sl(n))$ with the quantum double of $U_q(b_+)$. This formula was generalized in [9, 10] for quantum deformation of semisimple Lie algebras using q-Weyl group.

We deduce the analogous formula for quantum superalgebras (q-deformation of finite-dimensional contragredient Lie superalgebras). Our approach differs from that of [8–10]. We don't use a quantum Weyl group for the superalgebras which do not have good Weyl group. Our method is based on careful analysis of rank 1 and 2 quantum (super)algebras, a combinatorial structure of root systems [11] and algebraic properties of q-exponential functions.

2. (Non)Quantized Kac-Moody (Super)Algebras

Let $g(A, \tau)$ be a Kac-Moody (super)algebra with a symmetrizable Cartan matrix A [i.e. $A = DA^{\mathrm{sym}}$, where $A^{\mathrm{sym}} = (a_{ij}^{\mathrm{sym}})$ is a symmetrical matrix, and $D = \mathrm{diag}(d_1, \ldots, d_n)$, $d_i \neq 0$]. Let, moreover, $\tilde{e}_{\pm\alpha_i}, \tilde{h}_{\alpha_i}, i \in I \equiv \{1, 2, \ldots, n\}$, be the Chevalley basis in $g(A, \tau)$, which is coordinated with the matrix A^{sym}.

The (super)algebra $g := g(A, \tau)$ and its universal enveloping algebra $U(g)$ are completely determined by the following relations [12]:

$$[\tilde{h}_{\alpha_i}, \tilde{h}_{\alpha_j}] = 0, \tag{2.1}$$

$$[\tilde{h}_{\alpha_i}, \tilde{e}_{\pm\alpha_j}] = \pm a_{ij}^{\mathrm{sym}} \tilde{e}_{\pm\alpha_j}, \tag{2.2}$$

$$[\tilde{e}_{\alpha_i}, \tilde{e}_{-\alpha_j}] = \delta_{ij} \tilde{h}_{\alpha_i}, \tag{2.3}$$

$$(\mathrm{ad}\, \tilde{e}_{\pm\alpha_i})^{n_{ij}} \tilde{e}_{\pm\alpha_j} = 0 \quad \text{for} \quad i \neq j, \tag{2.4}$$

$$\deg \tilde{h}_{\alpha_i} = \bar{0}, \ \deg \tilde{e}_{\pm\alpha_i} = \bar{0} \quad \text{for} \quad i \notin \tau \subseteq I, \tag{2.5a}$$

$$\deg \tilde{e}_{\pm\alpha_i} = \bar{1} \quad \text{for} \quad i \in \tau \subseteq I, \tag{2.5b}$$

where

$$n_{ij} = \begin{cases} 1 & \text{if} \quad a_{ii}^{\mathrm{sym}} = a_{ij}^{\mathrm{sym}} = 0, \\ 2 & \text{if} \quad a_{ii}^{\mathrm{sym}} = 0, \ a_{ij}^{\mathrm{sym}} \neq 0, \\ 1 - 2a_{ij}^{\mathrm{sym}} / a_{ii}^{\mathrm{sym}} & \text{if} \quad a_{ii}^{\mathrm{sym}} \neq 0. \end{cases} \tag{2.6}$$

Here the bracket $[\cdot, \cdot]$ and also the symbol ad denote a supercommutator in $U(g)$, i.e.

$$[a, b] := ab - (-1)^{\deg a \deg b} ba, \tag{2.7a}$$

$$(\mathrm{ad}\, a) b := ab - (-1)^{\deg a \deg b} ba. \tag{2.7b}$$

The quantized (super)algebra g, denoted by $U_q(g)$, may be considered as a deformation f (reserving the grading) of the universal enveloping algebra $U(g)$ $U(g) \xrightarrow{f} U_q(g)$, $\tilde{e}_{\pm\alpha_i} \xrightarrow{f} e_{\pm\alpha_i}$, $\tilde{h}_{\alpha_i} \xrightarrow{f} h_{\alpha_i}$, which modifies Eqs. (2.3) and (2.4). More precisely we have [12].

Definition 2.1. The quantized (super)algebra $g:=gl(A,\tau)$ is an unital associative (super)algebra $U_q(g)$ with generators $e_{\pm\alpha_i}$, h_{α_i}, $i\in I$, and defining relations:

$$[h_{\alpha_i}, h_{\alpha_j}]=0, \tag{2.8}$$

$$[h_{\alpha_i}, e_{\pm\alpha_j}]=\pm\alpha_{ij}^{\mathrm{sym}}e_{\pm\alpha_j}, \tag{2.9}$$

$$[e_{\alpha_i}, e_{-\alpha_j}]=\delta_{ij}(k_{\alpha_i}-\bar{k}_{\alpha_i})/(q-\bar{q}), \tag{2.10}$$

$$(\mathrm{ad}_{q'}e_{\pm\alpha_i})^{n_{ij}}e_{\pm\alpha_j}=0 \quad \text{for} \quad i\neq j,\ q'=q,\bar{q}, \tag{2.11}$$

$$\deg h_{\alpha_i}=\bar{0},\ \deg e_{\pm\alpha_i}=\bar{0} \quad \text{for} \quad i\notin\tau\subseteqq I, \tag{2.12a}$$

$$\deg e_{\pm\alpha_i}=\bar{1} \quad \text{for} \quad i\in\tau\subseteqq I. \tag{2.12b}$$

For superalgebras of rank more then two we have to complete the list of relations. See Note added in proof.

Here and everywhere we use the notations $k_\gamma:=q^{h_\gamma}$, $\bar{k}_\gamma:=q^{-h_\gamma}$, $\bar{q}:=q^{-1}$. The bracket $[\cdot,\cdot]_{q'}$ and also the $\mathrm{ad}_{q'}$ denote a deformed supercommutator in $U_q(g)$:

$$(\mathrm{ad}_{q'}e_\alpha)e_\beta\equiv[e_\alpha, e_\beta]_{q'}=e_\alpha e_\beta-(-1)\deg^{e_\alpha\deg e_\beta}q'^{(\alpha,\beta)}e_\beta e_\alpha, \tag{2.13}$$

where $q=e^\hbar$, and (α,β) is a scalar product of the roots α and β. The parameter $\hbar$ is called as a Planck constant. Clearly, $U_q(g)$ reduces to $U(g)$ if $\hbar\to0$ $(q\to1)$.

Anywhere below we shall use the following short notation:

$$\vartheta(\gamma):=\deg e_\gamma. \tag{2.14}$$

Let (*) be an (non-graded) algebra antiautomorphism in a sense of the associative (super)algebra $U_q(g(A,\tau))$, such that

$$(e_\gamma)^*=e_{-\gamma}, \quad (e_{-\gamma})^*=e_\gamma, \quad (h_\gamma)^*=h_\gamma, \quad (\hbar)^*=-\hbar. \tag{2.15}$$

Then it is not difficult to verify that (*) is an antiinvolution in the quantum (super)algebra $U_q(g(A,\tau))$, i.e. the relations (2.8)–(2.12) are invariant with respect to the operation (*). We call this antiinvolution as the Cartan-Planck adjoint. We use also an algebra automorphism ω in $U_q(g(A,\tau))$, where

$$\omega(e_\gamma)=e_{-\gamma}, \quad \omega(e_{-\gamma})=(-1)^{e(\gamma)}e_{-\gamma}, \quad \omega(h_\gamma)=-h_\gamma, \quad \omega(\hbar)=-\hbar. \tag{2.16}$$

It may be shown that $U_q(g)$ is a Hopf (super)algebra with respect to a comultiplication Δ, an antipode S and a counit ε defined as

$$\Delta(h_{\alpha_i})=h_{\alpha_i}\otimes1+1\otimes h_{\alpha_i}, \tag{2.17a}$$

$$\Delta(e_{\alpha_i})=e_{\alpha_i}\otimes1+\bar{k}_{\alpha_i}\otimes e_{\alpha_i}, \tag{2.17b}$$

$$\Delta(e_{-\alpha_i})=e_{-\alpha_i}\otimes k_{\alpha_i}+1\otimes e_{-\alpha_i}, \tag{2.17c}$$

$$S(h_{\alpha_i})=-h_{\alpha_i}, \quad S(e_{\alpha_i})=-k_{\alpha_i}e_{\alpha_i}, \quad S(e_{-\alpha_i})=-e_{-\alpha_i}\bar{k}_{\alpha_i}, \tag{2.18}$$

$$\varepsilon(h_{\alpha_i})=\varepsilon(e_{\alpha_i})=\varepsilon(e_{-\alpha_i})=0, \quad \varepsilon(1)=1. \tag{2.19}$$

The quantum (super)algebra $U_q(g)$ is also a Hopf (super)algebra with respect to a comultiplication $\bar{\Delta}$, an antipode $\bar{S}$ and count $\bar{\varepsilon}$ defined as

$$\bar{\Delta}(h_{\alpha_i})=h_{\alpha_i}\otimes1+1\otimes h_{\alpha_i}, \tag{2.20a}$$

$$\bar{\Delta}(e_{\alpha_i})=e_{\alpha_i}\otimes1+k_{\alpha_i}\otimes e_{\alpha_i}, \tag{2.20b}$$

$$\bar{\Delta}(e_{-\alpha_i})=e_{-\alpha_i}\otimes\bar{k}_{\alpha_i}+1\otimes e_{-\alpha_i}, \tag{2.20c}$$

$$\bar{S}(h_{\alpha_i})=-h_{\alpha_i}, \quad \bar{S}(e_{\alpha_i})=-\bar{k}_{\alpha_i}e_{\alpha_i}, \quad \bar{S}(e_{-\alpha_i})=-e_{-\alpha_i}k_{\alpha_i}, \tag{2.21}$$

$$\bar{\varepsilon}(h_{\alpha_i})=\bar{\varepsilon}(e_{\alpha_i})=\bar{\varepsilon}(e_{-\alpha_i})=0, \quad \bar{\varepsilon}(1)=1. \tag{2.22}$$

Hereafter we shall use also opposite comultiplication Δ' and $\bar{\Delta}'$ defined by relations

$$\Delta' = \sigma\Delta\,, \quad \bar{\Delta}' = \sigma\bar{\Delta}\,, \tag{2.23}$$

where σ is a (super)permutation linear operator in $U_q(g)\otimes U_q(g)$, i.e.

$$\sigma(a\otimes b) = (-1)^{\vartheta(a)\vartheta(b)}(b\otimes a)\,. \tag{2.24}$$

Note that the definition of Hopf superalgebra differs from that of the usual Hopf algebra by the supercommutativity of tensor product, i.e.

$$(a\otimes b)(c\otimes d) = (-1)^{\vartheta(b)\vartheta(c)} ac\otimes bd\,. \tag{2.25}$$

Now we proceed to consider quantized Kac-Moody (super)algebras of finite growth only.

3. The Cartan-Weyl Basis and Its *Q*-Analog

Let $\Pi := \{a_1, \ldots, a_n\}$ be a simple root system of $g := g(A, \tau)$; Δ_+ be the system of all positive roots with respect to Π. We denote by $\underline{\Delta}_+$ *the reduced root system* which is obtained from Δ_+ by removing such *real* roots α for which $\alpha/2$ are roots.

The Cartan-Weyl basis of g ($U(g)$) consists of root vectors $e_{\pm\alpha}$, $\alpha\in\Delta_+$, and some basis in the Cartan subalgebra $\mathscr{H}\subset g$. The basis vectors satisfy the following relations:

$$[h, e_\alpha] = \alpha(h)e_\alpha\,, \quad h\in\mathscr{H}\,, \tag{3.1}$$

$$[e_\alpha, e_{-\alpha}] = h_\alpha\,, \quad \alpha\in\Delta_+\,, \tag{3.2}$$

$$[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}\,, \quad \alpha, \beta\in\Delta\equiv -\Delta_+\cup\Delta_+\,. \tag{3.3}$$

Our task is to construct a q-analog of the Cartan-Weyl basis of $U_q(g)$, such that, first, it coincides with the Cartan-Weyl basis of $U(g)$ if $q\to 1$ and, second, it satisfies the relations similar to (3.1) and (3.2) [i.e. such as (2.8)–(2.10)].

The peculiarity of the construction of the Cartan-Weyl basis in $U(g)$ and in $U_q(g)$ comes from the following observation.

Proposition 3.1. *Consider the (super)algebra g and let α, β, α', β' are positive roots such that $[e_\alpha, e_\beta]\neq 0$, $[e_{\alpha'}, e_{\beta'}]\neq 0$. If a real root $\gamma\in\Delta_+$ can be represented as $\gamma=\alpha+\beta=\alpha'+\beta'$, then $[e_\alpha, e_\beta]\sim[e_{\alpha'}, e_{\beta'}]$.*

Hence for given e_α, e_β and $e_{\alpha'}, e_{\beta'}$ the root vector e_γ is defined up to a multiplication constant by relation

$$e_\gamma = C[e_\alpha, e_\beta] = C'[e_{\alpha'}, e_{\beta'}]\,. \tag{3.4}$$

In the case of the quantum (super)algebra $U_q(g)$ Proposition 1 does not hold in general. It turns out in this case that the procedure for construction of the Cartan-Weyl basis has to be in agreement with the choice of a normal ordering in the reduced root system $\underline{\Delta}_+$. To this end we recall the definition of a normal order in $\underline{\Delta}_+$ [14].

Definition 3.1. It is said that the system $\underline{\Delta}_+$ is in normal ordering, if its roots are written in the following way: (i) all multiple roots follow each other in an arbitrary order; (ii) each nonsimple root $\alpha+\beta\in\Delta_+$, where $\alpha\neq\lambda\beta$, $\alpha,\beta\in\Delta_+$, has to be written between α and β.

The q-analog of the Cartan-Weyl basis is constructed using the following inductive algorithm [13].

Definition 3.2. Fix some normal ordering in Δ_+. Let α, β, $\gamma\in\Delta_+$ be pairwise noncollinear roots, such that $\gamma=\alpha+\beta$. Let, moreover, between α and β (in the considered normal ordering) there are no other roots α' and β' such that $\alpha'+\beta'=\gamma$. Then, if $e_{\pm\alpha}$ and $e_{\pm\beta}$ have already been constructed, we set

$$e_\gamma=[e_\alpha, e_\beta]_q\,, \qquad e_{-\gamma}=[e_{-\beta}, e_{-\alpha}]_{\bar q}\,. \tag{3.5}$$

Remark. Let the root subspace g_α of the root α of the Kac-Moody (super)algebra has a dimension more than 1, $\dim g_\alpha>1$. Then for the quantum (super)algebra $U_q(q)$ we obtain different vectors $e_\alpha^{(s)}$ labelled by index $s=1,2,\ldots,\dim g_\alpha$.

We have the following properties of q-analogs of the Cartan-Weyl generators.

Proposition 3.2. *For any $\gamma\in\Delta_+$ the relations*

$$[h, e_\gamma]=\gamma(h)e_\gamma\,, \tag{3.6}$$

$$[e_\gamma, e_{-\gamma}]=a_\gamma(k_\gamma-\bar{k}_\gamma)/(q-\bar{q})\,, \tag{3.7}$$

are valid where a_γ is a function of q.

We say that $\alpha<\beta$ if α is located on the left side of β in the normal ordering Δ_+.

Proposition 3.3. *Let $\alpha,\beta\in\Delta_+$ and $\alpha<\beta$ in a sense of the normal ordering in Δ_+, then*

$$[e_\alpha, e_\beta]_q=\sum_{\alpha<\gamma_1<\ldots<\gamma_n<\beta} C_{n_i,\gamma_i}e_{\gamma_1}^{n_1}e_{\gamma_2}^{n_2}\ldots e_{\gamma_n}^{n_n}\,, \tag{3.8}$$

where $\sum_i k_i\gamma_i=\alpha+\beta$ and the coefficients $C\ldots$ are functions of q and ones do not depend on the Cartan elements k_{α_i}, $i=1,2,\ldots,n$. Also

$$[e_\beta, e_{-\alpha}]=\sum_{\gamma_1<\ldots<\gamma_r<\alpha<\beta<\gamma'_1<\ldots<\gamma'_s} C'_{n_l,\gamma_l;n'_j,\gamma'_j}e_{-\gamma_1}^{n_1}\ldots e_{-\gamma_r}^{n_r}e_{\gamma'_1}^{n'_1}\ldots e_{\gamma'_s}^{n'_s}\,, \tag{3.9}$$

where $\sum_l(n'_l\gamma'_l-n_l\gamma_l)=\beta-\alpha$ and the coefficients $C'\ldots$ are functions of q and k_α or k_β.

Proof. We propose the following scheme for proof of Propositions 3.1, 3.2. Fix some convenient normal ordering in Δ_+ and construct the q-analog of a Cartan-Weyl basis for this ordering. Then we check (3.7)–(3.9) for this basis. Further we may show that relation (3.7)–(3.9) do not depend on reordering in Δ_+ by using combinatorial properties of the root system Δ_+ [11]. □

Remark. The Poincure-Bircgoff-Witt theorem follows immediately from Proposition 3.3. For quantized simple Lie algebras the PBW theorem can be found in [8, 16, 17].

4. Quantum (Super)Algebras of Rank 1 and 2

The quantized contragredient (super)algebras of rank 1 and 2 play an important role in our approach to the universal R-matrix. Here we give a list of all finite-

Table 1

$g(A,\tau)$	A	Odd roots	Diagram	dim	Δ_+	$\underline{\Delta}_+$
A_1	(2)	$\emptyset$	○	3	α	α
$B(0,1)$	(2)	$\{\alpha\}$	●	5	$\alpha, 2\alpha$	α
$sl(1,1)$	(0)	$\{\alpha\}$	⊗	3	α	α
$g((0),\{1\})$	(0)	$\{\alpha\}$	⊗	4	α	α
$g((0),\emptyset)$	(0)	$\emptyset$	⊙	4	α	α

dimensional contragredient rank 1 and 2 (super)algebras and describe commutation relations for the Cartan-Weyl generators in quantum case.

Basic information about finite-dimensional contragredient (super)algebras of rank 1 and 2 is represented in Tables 1 and 2a, b. In these tables there are listed the standard Cartan matrix A, a symmetrical Cartan matrix A^{sym}, an inverse matrix $(A^{\text{sym}})^{-1}$, the set of odd simple roots (odd roots), the Dynkin diagram (diagram) the (super)algebra dimension (dim), the positive root system Δ_+ and the normal orders $\underline{\Delta}_+$ of the reduced root system Δ_+ for every (super)algebras of rank 1 and 2.

4.1. *The Rank 1 Quantum (Super)-Algebras*

The quantized rank 1 (super)algebras A_1, $B(0,1)$, and $sl(1,1)$ are generated by the elements $\langle e_\alpha, e_{-\alpha}, h_\alpha\rangle$ with relations

$$[h_\alpha, e_{\pm\alpha}] = \pm(\alpha,\alpha)e_{\pm\alpha}, \tag{4.1}$$

$$[e_\alpha, e_{-\alpha}] = (k_\alpha - \bar{k}_\alpha)/(q-\bar{q}). \tag{4.2}$$

The quantized rank 1 (super)algebras $g((0),\{1\})$ and $g((0),\emptyset)$ are generated by elements $\langle e_\alpha, e_{-\alpha}, h_\alpha, h_\beta\rangle$ with relations

$$[h_\alpha, e_{\pm\alpha}] = \pm(\alpha,\alpha)e_{\pm\alpha}, \quad (\alpha,\alpha)=0, \tag{4.3}$$

$$[h_\beta, e_{\pm\alpha}] = \pm(\alpha,\beta)e_{\pm\alpha}, \quad (\alpha,\beta)\neq 0, \tag{4.4}$$

$$[e_\alpha, e_{-\alpha}] = (k_\alpha - k_\alpha)/(q-\bar{q}). \tag{4.5}$$

4.2. *The Rank 2 Quantum (Super)Algebras*

Remark that there are isomorphisms $A(1,0)\approx A'(1,0)$, $B(1,1)\approx B'(1,1)$. The (super)algebras A_1, $A(1,0)$, and $A'(1,0)$ are called the A-type (super)algebras, and (super)algebras B_2, $B(1,1)$, $B'(1,1)$, and $B(0,2)$ are called of B-type.

It should be noted also that the matrix A^{sym} is determined ambiguously and we can take as symmetric Cartan matrix any matrix of a form λA^{sym}, where λ is arbitrary nonzero number. Below we use A^{sym} in a form

$$A^{\text{sym}} = \begin{pmatrix} (\alpha,\alpha) & (\alpha,\beta) \\ (\beta,\alpha) & (\beta,\beta) \end{pmatrix}. \tag{4.6}$$

The rank 2 quantum (super)algebras $U_q(g)$ are generated by elements

$$\langle e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, h_\alpha, h_\beta\rangle$$

Table 2a

$g(A,\tau)$	A	A^{sym}	(A^{sym})	Odd	Diagram roots	dim
A_2	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$\emptyset$	$\overset{\alpha}{\bigcirc}—\overset{\beta}{\bigcirc}$	8
$A(1,0)$	$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha\}$	$\overset{\alpha}{\otimes}—\overset{\beta}{\bigcirc}$	8
$A'(1,0)$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha,\beta\}$	$\overset{\alpha}{\otimes}—\overset{\beta}{\otimes}$	8
B_2	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$\emptyset$	$\overset{\alpha}{\bigcirc}\Rightarrow\overset{\beta}{\bigcirc}$	10
$B(1,1)$	$\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha\}$	$\overset{\alpha}{\otimes}\Rightarrow\overset{\beta}{\bigcirc}$	12
$B'(1,1)$	$\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha,\beta\}$	$\overset{\alpha}{\otimes}\Rightarrow\overset{\beta}{\bullet}$	12
$B(0,2)$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$\{\beta\}$	$\overset{\alpha}{\bigcirc}\Rightarrow\overset{\beta}{\bullet}$	14
G_2	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	$\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$	$\emptyset$	$\overset{\alpha}{\bigcirc}\Rrightarrow\overset{\beta}{\bigcirc}$	14

Table 2b

$g(A,\tau)$	Δ_+	$\underline{\vec{\Delta}}_+$
A_2	$\alpha, \alpha+\beta, \beta$	$\alpha, \alpha+\beta, \beta$ $\beta, \alpha+\beta, \alpha$
$A(1,0)$	$\alpha, \alpha+\beta, \beta$	$\alpha, \alpha+\beta, \beta$ $\beta, \alpha+\beta, \alpha$
$A'(1,0)$	$\alpha, \alpha+\beta, \beta$	$\alpha, \alpha+\beta, \beta$ $\beta, \alpha+\beta, \alpha$
B_2	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
$B(1,1)$	$\alpha, \alpha+\beta, \alpha+2\beta, 2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
$B'(1,1)$	$\alpha, \alpha+\beta, \alpha+2\beta, 2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
$B(0,2)$	$\alpha, \alpha+\beta, 2\alpha+2\beta, \alpha+2\beta, 2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
G_2	$\alpha, \alpha+\beta, 2\alpha+3\beta, \alpha+2\beta, \alpha+3\beta, \beta$	$\alpha, \alpha+\beta, 2\alpha+3\beta, \alpha+2\beta, \alpha+3\beta, \beta$ $\beta, \alpha+3\beta, \alpha+2\beta, 2\alpha+3\beta, \alpha+\beta, \alpha$

with relations

$$[h_\alpha, e_{\pm\alpha}] = \pm(\alpha,\alpha)e_{\pm\alpha}, \quad [h_\alpha, e_{\pm\beta}] = \pm(\alpha,\beta)e_{\pm\beta}, \tag{4.7}$$

$$[h_\beta, e_{\pm\alpha}] = \pm(\beta,\alpha)e_{\pm\alpha}, \quad [h_\beta, e_{\pm\beta}] = \pm(\beta,\beta)e_{\pm\beta}, \tag{4.8}$$

$$[e_\alpha, e_{-\alpha}] = (k_\alpha - \bar{k}_\alpha)/(q-\bar{q}), \quad [e_\alpha, e_{-\beta}] = 0, \tag{4.9}$$

$$[e_\beta, e_{-\beta}] = (k_\beta - \bar{k}_\beta)/(q-\bar{q}), \quad [e_\beta, e_{-\alpha}] = 0, \tag{4.10}$$

$$[e_{\pm\alpha}, [e_{\pm\alpha}, e_{\pm\beta}]_{q'}]_{q'} = 0, \tag{4.11}$$

$$[[e_{\pm\alpha}, e_{\pm\beta}]_{q'}, e_{\pm\beta}]_{q'} = 0 \quad \text{for} \quad g = A\text{-type}, \tag{4.12}$$

$$[[[e_{\pm\alpha}, e_{\pm\beta}]_{q'}, e_{\pm\beta}]_{q'}, e_{\pm\beta}]_{q'} = 0 \quad \text{for} \quad g = B\text{-type}, \tag{4.13}$$

$$[[[[e_{\pm\alpha}, e_{\pm\beta}]_{q'}, e_{\pm\beta}]_{q'}, e_{\pm\beta}]_{q'}, e_{\pm\beta}]_{q'} = 0 \quad \text{for} \quad g = G_2, \tag{4.14}$$

where $q' = q, \bar{q}$.

Proposition 4.1. *If for quantized (superalgebras of A-type we fix the following normal ordering:*

$$\alpha, \alpha+\beta, \beta, \tag{4.15}$$

and in accordance with it we set

$$e_{\alpha+\beta} := [e_\alpha, e_\beta]_q, \tag{4.16}$$

$$e_{-\alpha-\beta} := [e_{-\beta}, e_{-\alpha}]_{\bar{q}}, \tag{4.17}$$

then the root vectors e_γ, $\gamma \in \underline{\Delta}$, satisfy the following relations:

$$[e_\alpha, e_{\alpha+\beta}]_q = [e_{\alpha+\beta}, e_\beta]_q = 0, \tag{4.18}$$

$$[e_{\alpha+\beta}, e_{-\alpha}] = (-1)^{\vartheta(\alpha)\vartheta(\beta)} a e_\beta \bar{k}_\alpha, \tag{4.19}$$

$$[e_\beta, e_{-\alpha-\beta}] = -a e_{-\alpha} \bar{k}_\beta, \tag{4.20}$$

$$[e_{\alpha+\beta}, e_{-\alpha-\beta}] = -a(k_{\alpha+\beta})/(q-\bar{q}), \tag{4.21}$$

where

$$a := (q^{(\alpha,\beta)} - \bar{q}^{(\alpha,\beta)})/(q-\bar{q}). \tag{4.22}$$

The rest of relations is obtained by application of the Cartan-Planck adjoint (*) *to relations* (4.18)–(4.21).

Proposition 4.2. *If for quantized (super)algebras of B-type we fix the following normal ordering:*

$$\alpha, \alpha+\beta, \alpha+2\beta, \beta, \tag{4.23}$$

and in accordance with it we set

$$e_{\alpha+\beta} := [e_\alpha, e_\beta]_q, \quad e_{\alpha+2\beta} := [e_{\alpha+\beta}, e_\beta]_q, \tag{4.24}$$

$$e_{-\alpha-\beta} := [e_{-\beta}, e_{-\alpha}]_{\bar{q}}, \quad e_{-\alpha-2\beta} := [e_{-\beta}, e_{-\alpha-\beta}]_{\bar{q}}, \tag{4.25}$$

then the root vectors e_γ, $\gamma \in \underline{\Delta}$, satisfy the following relations:

$$[e_\alpha, e_{\alpha+\beta}]_q = [e_{\alpha+\beta}, e_{\alpha+2\beta}]_q = [e_{\alpha+2\beta}, e_\beta]_q = 0, \tag{4.26}$$

$$[e_\alpha, e_{2\alpha+\beta}]_q = (-1)^{\vartheta(\beta)\vartheta(\alpha+\beta)}((-1)^{\vartheta(\alpha+\beta)} q^{(\alpha,\alpha+\beta)} - 1) e^2_{\alpha+\beta}, \tag{4.27}$$

$$[e_{\alpha+\beta}, e_{-\alpha}] = (-1)^{\vartheta(\alpha)\vartheta(\beta)} a e_\beta \bar{k}_\alpha, \tag{4.28}$$

$$[e_{\alpha+2\beta}, e_{-\alpha}] = (-1)^{\vartheta(\beta)} a((-1)^{\vartheta(\beta)} q^{-(\alpha,\beta)} - 1) e^2_\beta \bar{k}_\alpha, \tag{4.29}$$

$$[e_{\alpha+2\beta}, e_{-\alpha-\beta}] = (-1)^{\vartheta(\alpha)\vartheta(\beta)} a^2 e_\beta \bar{k}_{\alpha+\beta}, \tag{4.30}$$

$$[e_\beta, e_{-\alpha-\beta}] = -a e_{-\alpha} \bar{k}_\beta, \tag{4.31}$$

$$[e_\beta, e_{-\alpha-2\beta}] = -(-1)^{\vartheta(\beta)} a e_{-\alpha-\beta} \bar{k}_\beta, \tag{4.32}$$

$$[e_{\alpha+\beta}, e_{-\alpha-\beta}] = -a(k_{\alpha+\beta} - \bar{k}_{\alpha+\beta})/(q - \bar{q}), \tag{4.33}$$

$$[e_{\alpha+2\beta}, e_{-\alpha-2\beta}] = (-1)^{\vartheta(\beta)} a^2 (k_{\alpha+2\beta} - \bar{k}_{\alpha+2\beta})/(q - \bar{q}), \tag{4.34}$$

where a is defined by (4.22). *The rest of relations is obtained by applying operation* (*) *to* (4.26)–(4.34).

Proposition 4.3. *If for quantized algebra G_2 we fix the following normal ordering*

$$\alpha, \alpha+\beta, 2\alpha+3\beta, \alpha+2\beta, \alpha+3\beta, \beta, \tag{4.35}$$

and in accordance with it we set

$$e_{\alpha+\beta} := [e_\alpha, e_\beta]_q, \quad e_{\alpha+2\beta} := [e_{\alpha+\beta}, e_\beta]_q, \tag{4.36}$$

$$e_{\alpha+3\beta} := [e_{\alpha+2\beta}, e_\beta]_q, \quad e_{2\alpha+3\beta} := [e_{\alpha+\beta}, e_{\alpha+2\beta}]_q, \tag{4.37}$$

$$e_{-\alpha-\beta} := [e_{-\beta}, e_{-\alpha}]_{\bar{q}}, \quad e_{-\alpha-2\beta} := [e_{-\beta}, e_{-\alpha-\beta}]_{\bar{q}}, \tag{4.38}$$

$$e_{-\alpha-3\beta} := [e_{-\beta}, e_{-\alpha-2\beta}]_{\bar{q}}, \quad e_{-2\alpha-3\beta} := [e_{-\alpha-2\beta}, e_{-\alpha-\beta}]_{\bar{q}}, \tag{4.39}$$

then the root vectors e_γ, $\gamma \in \underline{\Delta}$, satisfy the following relations:

$$\begin{aligned}[e_\alpha, e_{\alpha+\beta}]_q &= [e_{\alpha+\beta}, e_{2\alpha+3\beta}]_q = [e_{2\alpha+3\beta}, e_{\alpha+2\beta}]_q \\ &= [e_{\alpha+2\beta}, e_{\alpha+3\beta}]_q = [e_{\alpha+3\beta}, e_\beta]_q = 0,\end{aligned} \tag{4.40}$$

$$[e_\alpha, e_{2\alpha+3\beta}]_q = \bar{q}^{(\alpha,\beta)} (q - \bar{q})^2 bc e^3_{\alpha+\beta}, \tag{4.41}$$

$$[e_\alpha, e_{\alpha+2\beta}]_q = -\bar{q}^{(\alpha,\beta)/3} (q - \bar{q}) b e^2_{\alpha+\beta}, \tag{4.42}$$

$$[e_\alpha, e_{\alpha+3\beta}]_q = d e_{2\alpha+3\beta} - (q - \bar{q}) a e_{\alpha+\beta} e_{\alpha+2\beta}, \tag{4.43}$$

$$[e_{\alpha+\beta}, e_{\alpha+3\beta}]_q = -\bar{q}^{(\alpha,\beta)/3} (q - \bar{q}) a b^{-1} c e^2_{\alpha+2\beta}, \tag{4.44}$$

$$[e_{2\alpha+3\beta}, e_{\alpha+3\beta}]_q = \bar{q}^{(\alpha,\beta)} (q - \bar{q})^2 a b^{-1} c^2 e^3_{\alpha+2\beta}, \tag{4.45}$$

$$[e_{2\alpha+3\beta}, e_\beta]_q = -\bar{q}^{(\alpha,\beta)/3} (q - \bar{q}) a b^{-1} c e^2_{\alpha+2\beta}, \tag{4.46}$$

$$[e_{\alpha+\beta}, e_{-\alpha}] = a e_\beta \bar{k}_\alpha, \tag{4.47}$$

$$[e_{2\alpha+3\beta}, e_{-\alpha}] = d a e_{\alpha+3\beta} \bar{k}_\alpha - (q - \bar{q}) a^2 e_{\alpha+2\beta} e_\beta \bar{k}_\alpha, \tag{4.48}$$

$$[e_{\alpha+2\beta}, e_{-\alpha}] = -\bar{q}^{(\alpha,\beta)/3} (q - \bar{q}) a b e^2_\beta \bar{k}_\alpha, \tag{4.49}$$

$$[e_{\alpha+3\beta}, e_{-\alpha}] = \bar{q}^{(\alpha,\beta)} (q - \bar{q})^2 a b c e^3_q \bar{k}_\alpha, \tag{4.50}$$

$$[e_{2\alpha+3\beta}, e_{-\alpha-\beta}] = -a^2 e_{\alpha+2\beta} \bar{k}_{\alpha+\beta}, \tag{4.61}$$

$$[e_{\alpha+2\beta}, e_{-\alpha-\beta}] = -ab^2c^{-1}e_\beta \bar{k}_{\alpha+\beta}, \tag{4.52}$$

$$[e_{\alpha+3\beta}, e_{-\alpha-\beta}] = \bar{q}^{(\alpha,\beta)/3}(q-\bar{q})a^2be_\beta^2\bar{k}_{\alpha+\beta}, \tag{4.53}$$

$$[e_\beta, e_{-\alpha-\beta}] = -ae_{-\alpha}\bar{k}_\beta, \tag{4.54}$$

$$[e_{\alpha+2\beta}, e_{-2\alpha-3\beta}] = -a^2b^2c^{-1}e_{-\alpha-\beta}\bar{k}_{\alpha+2\beta}, \tag{4.55}$$

$$\begin{aligned}[e_{\alpha+3\beta}, e_{-2\alpha-3\beta}] &= a^4b^2c^{-2}\bar{q}^{(\alpha,\beta)/3}e_{-\alpha}\bar{k}_{\alpha+3\beta} \\ &\quad + (q-\bar{q})a^3b^2c^{-1}e_{-\alpha-\beta}e_\beta\bar{k}_{\alpha+2\beta},\end{aligned} \tag{4.56}$$

$$[e_\beta, e_{-2\alpha-3\beta}] = -q^{(\alpha,\beta)/3}(q-\bar{q})abe_{-\alpha-\beta}^2\bar{k}_\beta, \tag{4.57}$$

$$[e_{\alpha+3\beta}, e_{-\alpha-2\beta}] = a^2b^2c^{-1}e_\beta\bar{k}_{\alpha+2\beta}, \tag{4.58}$$

$$[e_\beta, e_{-\alpha-2\beta}] = -b^2c^{-1}e_{-\alpha-\beta}\bar{k}_\beta, \tag{4.59}$$

$$[e_\beta, e_{-\alpha-3\beta}] = -ae_{-\alpha-2\beta}\bar{k}_\beta, \tag{4.60}$$

$$[e_{\alpha+\beta}, e_{-\alpha-\beta}] = -a(k_{\alpha+\beta}-\bar{k}_{\alpha+\beta})/(q-\bar{q}), \tag{4.61}$$

$$[e_{\alpha+2\beta}, e_{-\alpha-2\beta}] = ab^2c^{-1}(k_{\alpha+2\beta}-\bar{k}_{\alpha+2\beta})/(q-\bar{q}), \tag{4.62}$$

$$[e_{\alpha+3\beta}, e_{-\alpha-3\beta}] = -a^2b^2c^{-1}(k_{\alpha+3\beta}-\bar{k}_{\alpha+3\beta})/(q-\bar{q}), \tag{4.63}$$

$$[e_{2\alpha+3\beta}, e_{-2\alpha-3\beta}] = a^3b^2c^{-1}(k_{2\alpha+3\beta}-\bar{k}_{2\alpha+3\beta})/(q-\bar{q}), \tag{4.64}$$

where

$$a := (q^{(\alpha,\beta)}-\bar{q}^{(\alpha,\beta)})/(q-\bar{q}), \tag{4.65}$$

$$b := (q^{2(\alpha,\beta)/3}-\bar{q}^{2(\alpha,\beta)/3})/(q-\bar{q}), \tag{4.66}$$

$$c := (q^{(\alpha,\beta)/3}-\bar{q}^{(\alpha,\beta)/3})/(q-\bar{q}), \tag{4.67}$$

$$d := q^{(\alpha,\beta)}-q^{(\alpha,\beta)/3}-\bar{q}^{(\alpha,\beta)/3}. \tag{4.68}$$

All other relations may be obtained by operation (*).

Proofs. All these Propositions 4.1–4.3 can be obtained by direct calculations. □

Remark. Applying the automorphism ω to (4.16)–(4.22), (4.24)–(4.34), (4.36)–(4.68) we obtain root vectors and commutation relations for the opposite normal ordering (Table 2b) of Δ_+ for rank 2 (super)algebras.

5. Disentanglement and Reordering of q-Exponents

To verify the quasitriangular properties (1.2) of the R-matrix we shall use in the next section certain algebraic properties for q-exponents of some elements of a tensor cube of $U_q(g)$. In this section we give these properties which have also original interest because they are connected with a problem of obtaining the Campbell-Hausdorff formula for quantum algebras.

Let us introduce the q-exponent as a formal series of an indeterminate x

$$\exp_q(x) := \sum_{n \geqq 0} x^n/(n)_q!, \tag{5.1a}$$

where we set

$$(n)_q! \equiv (1)_q(2)_q \dots (n)_q, \qquad (k)_q \equiv (1-q^k)/(1-q). \tag{5.1b}$$

It is not difficult to verify that $\exp_{q'}(-x)$ is an inverse to $\exp_q(x)$, i.e.

$$(\exp_q(x))^{-1}=\exp_{\bar{q}}(-x)=\sum_{n\geqq 0}(-x^n)/(n)_{\bar{q}}!\,, \tag{5.2}$$

$$(\exp_q(x))^{-1}\exp_q(x)=exp_q(x)(\exp_q(x))^{-1}=1\,. \tag{5.3}$$

Proposition 5.1. *Let x and y be elements of some unital associative algebra $\mathscr{A}$ over $\mathbb{C}(q)$. There are following addition theorems for q-exponents.*

(i) *If* $[x,y]_q:=xy-qyx=0$, *then*

$$\exp_q(x+y)=\exp_q(y)\exp_q(x)\,. \tag{5.4}$$

(ii) *If* $[x,y]_q=z$, $[x,z]_{q'}=[z,y]_{q'}=0$, *where* $q'=q^2$, *then*

$$\exp_q(x+y)=\exp_q(y)\exp_{q'}(z/(2)_q!)\exp_q(x)\,. \tag{5.5}$$

(iii) *If* $[x,y]_{q'}=(q+1)^{-1}(q'-q)z^2$, $[x,z]_{q'}=[z,y]_{q'}=0$, *where* $q'=q^2$, *then*

$$\exp_{q'}(x+y+z)=\exp_{q'}(y)\exp_q(z)\exp_{q'}(x)\,. \tag{5.6}$$

(iv) *If* $[x,y]_q=z$, $[x,z]_{q'}=v$, $[z,y]_{q'}=w$, $[z,w]_{q''}=[v,z]_{q''}=[x,v]_{q''}=[w,y]_{q''}=0$, $[v,y]_{q''}=az^2$, $[x,w]_{q''}=az^2$, $[v,w]_{q'''}=(1-q)az^2$, *where* $q'=q^2$, $q''=q^3$, $q'''=q^6$, $a=(q^4-q)(q+1)^{-1}$, *then*

$$\exp_q(x+y)=\exp_q(y)\exp_{q''}(w/(3)_q!)\exp_q(z/(2)_q!)\exp_{q''}(v/(3)_q!)\exp_q(x)\,. \tag{5.7}$$

Proof. All Eqs. (5.4)–(5.7) are verified by direct manipulations on a level of formal series. □

Proposition 5.2. *For any x and y of $\mathscr{A}$ there is a following q-analog of Hadamard formula*

$$\begin{aligned}\exp_q(x)y(\exp_q(x))^{-1}&\equiv(\mathrm{Ad}\exp_q(x))(y)\\&=\Big(\sum_{n\geqq 0}(1/(n)_q!)(\overline{\mathrm{ad}}_q x)^n\Big)(y)=(\exp_q(\overline{\mathrm{ad}}_q x))(y)\,,\end{aligned} \tag{5.8}$$

where we set

$$(\overline{\mathrm{ad}}_q x)^0(y)\equiv y\,,\quad (\overline{\mathrm{ad}}_q x)^1(y)\equiv[x,y]\,,\quad (\overline{\mathrm{ad}}_q x)^2(y)\equiv[x,[x,y]]_q\,,$$
$$(\overline{\mathrm{ad}}_q x)^3(y)\equiv[x,[x,[x,y]]_q]_{q^2},\ldots,(\overline{\mathrm{ad}}_q x)^{n+1}(y)=[x,(\overline{\mathrm{ad}}_q x)^n(y)]_{q^n}\,,$$

Proof. The relation (5.8) is verified by direct manipulation on a level of formal series again. □

Proposition 5.3. *Let some elements $x_{n\alpha+m\beta}$ of a tensorial algebra of the rank 2 quantum algebra $U_q(g)$ satisfy the same commutation relations as the elements*

$$x'_{n\alpha+m\beta}:=(-1)^{\theta(n\alpha+m\beta)+mn\theta(\alpha)\theta(\beta)}((e_\beta)^m(\bar{k}_\alpha)^n\otimes(e_\alpha)^n\otimes e_{-n\alpha-m\beta})\,, \tag{5.9}$$

of a tensorial cube $U_q(g)\otimes U_q(g)\otimes U_q(g)$, correspondingly. Then we have:

$$\exp_{q_\beta}(x_\beta)\exp_{q_\alpha}(x_\alpha)=\exp_{q_\alpha}(x_\alpha)\exp_{q_{\alpha+\beta}}(x_{\alpha+\beta})\exp_{q_\beta}(x_\beta) \tag{5.10}$$

for quantized (super)algebras of A-type;

$$\exp_{q_\beta}(x_\beta)\exp_{q_\alpha}(x_\alpha)=\exp_{q_\alpha}(x_\alpha)\exp_{q_{\alpha+\beta}}(x_{\alpha+\beta})\exp_{q_{\alpha+2\beta}}(x_{\alpha+2\beta}/(2)_{q_\beta}!)\exp_{q_\beta}(x_\beta) \tag{5.11}$$

for quantized (super)algebras of B-type;

$$\exp_{q_\beta}(x_\beta)\exp_{q_\alpha}(x_\alpha)=\exp_{q_\alpha}(x_\alpha)\exp_{q_{\alpha+\beta}}(x_{\alpha+\beta})\exp_{q_{2\alpha+3\beta}}/(3)_{q_\beta}!)$$
$$\times\exp_{q_{\alpha+2\beta}}(x_{\alpha+2\beta}/(2)_{q_\beta}!)\exp_{q_{\alpha+3\beta}}(x_{\alpha+3\beta}/(3)_{q_\beta}!)\exp_{q_\beta}(x_\beta)$$

for quantized algebra G_2. (5.12)

Proof. Multiply the left sides of (5.10)–(5.12) on $(\exp_{q_\beta}(x_\beta))^{-1}$ then apply the q-analog of the Hadamard formula and the addition theorems (5.4)–(5.7); we come to (5.10)–(5.12) for each quantized (super)algebras, correspondingly. □

Remark. The algebra with generators (5.9) may be viewed as a formal associative algebra generated by two elements x_α and x_β and defining relations

$$(\overline{\mathrm{ad}}_{q_\beta}e_\beta)^{n_{\beta\alpha}}x_\alpha=0,\qquad(\overline{\mathrm{ad}}_{\bar{q}_\alpha}e_\alpha)^{n_{\alpha\beta}}x_\beta=0,\tag{5.13}$$

where $(\overline{\mathrm{ad}}_q)^n$ is given in Proposition 5.2, and $q_\gamma:=(-1)^{\theta(\gamma)}\bar{q}^{(\gamma,\gamma)}$, the integers $n_{\alpha\beta}$ and $n_{\beta\alpha}$ are determined by (2.5) for every rank 2 (super)algebras. Composed elements $x_{n\alpha+m\beta}$, $n\neq 0$, $m\neq 0$, may be constructed in the standard way. For example if $n_{\alpha\beta}=2$ and $n_{\beta\alpha}=4$ (case G_2-type) we have

$$x_{\alpha+n\beta}:=(\overline{\mathrm{ad}}_{q_\beta}x_\beta)^n x_\alpha,\qquad n=1,2,3,\tag{5.14}$$

$$x_{2\alpha+3\beta}:=x_{\alpha+\beta}x_{\alpha+2\beta}-(\bar{q}_\beta)^2x_{\alpha+2\beta}x_{\alpha+\beta},\tag{5.15}$$

Note that for the supercase we have to complete (5.13) by additional relations $(x_\gamma)^2=0$ if $(\gamma,\gamma)=0$ and $\theta(\gamma)=1$, where $\gamma=n\alpha+m\beta$, in corresponding rank two superalgebra. The relations (5.10)–(5.12) have evident classical analogs with $x_{n\alpha+m\beta}$ to be Cartan-Weyl generators of n_+.

6. The Reduced R-Matrix

Let R be a universal R-matrix for a quantum (super)algebra $U_q(g(A,\tau))$, where $g(A,\tau)$ is a contragredient Lie (super)algebra. We represent the R-matrix R in a form

$$R=\check{R}K,\tag{6.1}$$

where

$$K:=\exp\left(\hbar\sum_{i,j=1}^{n}d_{ij}h_{\alpha_i}\otimes h_{\alpha_j}\right).\tag{6.2}$$

Here (d_{ij}) is an inverse matrix for symmetrical Cartan matrix (a_{ij}^{sym}) if (a_{ij}^{sym}) is non-degenerated. For a degenerated matrix (a_{ij}^{sym}) we extend it up to a non-degenerated matrix $(\tilde{a}_{ij}^{\mathrm{sym}})$ and take an inverse to this extended matrix. To make this we add some element to a Cartan subalgebra and use it in (6.2) (see, for example, below the R-matrix for the quantized rank 1 (super)algebras $U_q(sl(1,1))$, $U_q(gl((0),\{1\}))$, $U_q(g((0),\emptyset)))$.

The element $\check{R}$ in (6.1) is called the reduced R-matrix. Let us consider detailed properties of the reduced R-matrix $\check{R}$ and of element K.

Proposition 6.1. *For any root vector e_γ, $\gamma\in\Delta_+$, the relations*

$$K(e_\gamma\otimes 1)K^{-1}=e_\gamma\otimes k_\gamma,\qquad K(e_{-\gamma}\otimes 1)K^{-1}=e_{-\gamma}\otimes\bar{k}_\gamma,\tag{6.3}$$

$$K(1\otimes e_\gamma)K^{-1}=k_\gamma\otimes e_\gamma,\qquad K(1\otimes e_{-\gamma})K^{-1}=\bar{k}_{-\gamma}\otimes e_\gamma\tag{6.4}$$

are valid.

Proof. It is sufficient to prove (6.3) and (6.4) for the Chevalley generators. We have

$$K(e_{\pm\alpha_i}\otimes 1)=(e_{\pm\alpha_i}\otimes I)\exp\Big(\hbar\sum_{i,j>1}(\pm 1)d_{ij}a_{ij}^{\mathrm{sym}}h_{\alpha_i}\otimes h_{\alpha_j}\Big)K$$
$$=(e_{\pm\alpha_i}\otimes k_i^{\pm 1})K\,. \quad \square$$

Proposition 6.2. *The reduced R-matrix $\check{R}$ is non-degenerated and it is determined by the following relations:*

$$\check{R}(h_{\alpha_i}\otimes 1+1\otimes h_{\alpha_i})=(h_{\alpha_i}\otimes 1+1\otimes h_{\alpha_i})\check{R}\,,\quad i=1,2,\ldots,n\,, \tag{6.5}$$

$$\check{R}(e_{\alpha_i}\otimes k_{\alpha_i}+1\otimes e_{\alpha_i})=(e_{\alpha_i}\otimes \bar{k}_{\alpha_i}+1\otimes e_{\alpha_i})\check{R}\,,\quad i=1,2,\ldots,n\,, \tag{6.6}$$

$$\check{R}(e_{-\alpha_i}\otimes 1+\bar{k}_{\alpha_i}\otimes e_{-\alpha_i})=(e_{-\alpha_i}\otimes 1+k_{\alpha_i}\otimes e_{-\alpha_i})\check{R}\,,\quad i=1,2,\ldots,n\,, \tag{6.7}$$

$$(\Delta\otimes\mathrm{id})\check{R}=\check{R}^{13}K^{13}\check{R}^{23}(K^{13})^{-1}\,, \tag{6.8}$$

$$(\mathrm{id}\otimes\Delta)\check{R}=\check{R}^{13}K^{13}\check{R}^{12}(K^{13})^{-1}\,, \tag{6.9}$$

where we use standard notations

$$H^{12}=\sum a_i\otimes b_i\otimes 1\,,\quad H^{13}=\sum a_i\otimes 1\otimes b_i\,,\quad H^{23}=\sum 1\otimes a_i\otimes b_i\,, \tag{6.10}$$

if H has a form $H=\sum a_i\otimes b_i$.

Proof. The equalities (6.5)–(6.9) follow from (1.1) and (1.2) on account of Proposition 6.1 for a quantum (super)algebra $U_q(g)$. $\square$

Now we should like to establish the uniqueness property of the reduced R-matrix $\check{R}$ in appropriate completion of $U_q(g)\otimes U_q(g)$.

Let $\{e_\alpha, e_{-\alpha}\}$ be root vectors of the q-analog of Cartan-Weyl basis, built with respect to any fixed normal ordering in Δ_+ (Sect. 3). Denote by $\mathrm{Fract}(U_q(\mathscr{H}\otimes\mathscr{H}))$ the fraction field of the algebra $U_q(\mathscr{H}\otimes\mathscr{H})$, where $U_q(\mathscr{H}\otimes\mathscr{H})$ is generated by $1\otimes k_{\alpha_i}^{\pm 1}$, $k_{\alpha_i}^{\pm 1}\otimes 1$. Consider the monomials

$$e_\alpha^{n_\alpha}e_\gamma^{n_\gamma}\ldots e_\beta^{n_\beta}\otimes e_{-\alpha}^{m_\alpha}e_{-\gamma}^{m_\gamma}\ldots e_{-\gamma}^{m_\beta}\ldots e_{-\beta}^{m_\beta}\,, \tag{6.11}$$

where $\alpha<\gamma<\ldots<\beta$ in a sense of chosen normal order, and $n_\alpha, n_\gamma, \ldots, m_\beta$ are nonnegative integers. Denote by $F_q(g\otimes g)$ the vector space of all formal series of monomials (6.11) with coefficients from $\mathrm{Fract}(U_q(\mathscr{H}\otimes\mathscr{H}))$, satisfying the following condition:

$$|n_\alpha-m_\alpha+n_\gamma-m_\gamma+\ldots+n_\beta-m_\beta|\leqq\mathrm{const} \tag{6.12}$$

for each series from $F_q(g\otimes g)$.

The following proposition holds (cf. [13, 14]).

Proposition 6.3. *The linear space $F_q(g\otimes g)$ is an unital associative algebra with respect to the ordinary multiplication of formal series.*

Proof. By means of Eq. (3.8) it is not difficult to verify that a product result of two formal series submitted by the restriction (6.12) satisfies this restriction again and has coefficients from $\mathrm{Fract}(U_q(\mathscr{H}\otimes\mathscr{H}))$, i.e. the product result belongs to $F_q(g\otimes g)$. $\square$

Now we can state slightly modified Drinfeld's assertion about uniqueness of R-matrix [7].

Theorem 6.4. *The reduced R-matrix $\check{R}$ is, up to a factor in* $\mathrm{Fract}(U_q(\mathscr{H}\otimes\mathscr{H}))$, *the unique solution of Eqs.* (6.5)–(6.9) *in the space* $F_q(g\otimes g)$.

Proof. It is same as in [7]. □

We extend the Cartan-Weyl adjoint on a tensor square, cube and so on of $U_q(g)$ by the following way

$$(a\otimes b)^* = b^*\otimes a^*, \tag{6.13}$$

$$(a\otimes b\otimes c)^* = c^*\otimes b^*\otimes a^* \tag{6.14}$$

and so on.

Proposition 6.5. *Let the reduced R-matrix $\check{R}$ is normalized such that its free term is equal to unit. Then*

$$\check{R}^* = \check{R}^{-1}, \tag{6.15}$$

$$\check{R}_{\bar{q}} = \check{R}_q^{-1} \quad \text{if} \quad \check{R}\equiv\check{R}_q. \tag{6.16}$$

Proof. It is a immediate consequence of Proposition 6.2, 6.4. □

7. The Universal *R*-Matrix for the Rank 1 and 2 Quantum (Super)Algebras

In this section we give an explicit expression of the universal R-matrix for quantized rank 1 and 2 (super)algebras listed in Tables 1 and 2. We start from quantized rank 1 (super)algebras.

Theorem 7.1. *The universal R-matrix for the rank* 1 *quantum (super)algebras $U_q(g)$ has the following form:*

$$R = \check{R}K = \check{R}_\alpha K, \tag{7.1}$$

where

$$\check{R}\equiv\check{R}_\alpha = \exp_{q_\alpha}((-1)^{\theta(\alpha)}(q-\bar{q})(e_\alpha\otimes e_{-\alpha}))$$

$$= \sum_{n\geqq 0}(-1)^{\theta(\alpha)n(n+1)/2}\frac{(q-\bar{q})^n}{(n)_{q_\alpha}!}(e_\alpha^n\otimes e_{\alpha}^n), \tag{7.2}$$

$$K = \exp(\hbar(h_\alpha\otimes h_\alpha)/(\alpha,\alpha)) \quad \text{for} \quad g = A_1,\ B(0,1), \tag{7.3}$$

$$K = \exp(\hbar(h_\alpha\otimes h_\beta + h_\beta\otimes h_\alpha)/(\alpha,\beta)) \quad \text{for} \quad g = gl((0),\{1\}), g((0),\emptyset). \tag{7.4}$$

Here and further we use denotation $q_\alpha = (-1)^{\theta(\alpha)}\bar{q}^{(\alpha,\alpha)}$.

Remarks. (i) The formula (7.2) may be simplified for the quantum superalgebra $U_q(gl((0),\{1\}))$

$$\check{R}_\alpha = \exp_{-1}((\bar{q}-q)(e_\alpha\otimes e_{-\alpha})) = 1 + (\bar{q}-q)(e_\alpha\otimes e_{-\alpha}), \tag{7.5}$$

since $e_\alpha^2 = 0$ in this case.
(ii) For the quantum algebra $U_q(g((0),\emptyset))$ the expression (7.2) reduces to the ordinary exponent:

$$\check{R}_\alpha = \exp((q-\bar{q})e_\alpha\otimes e_{-\alpha}) \tag{7.6}$$

since $(\alpha,\alpha)=0$ and $q_\alpha = 1$.

(iii) In the case of $U_q(sl(1,1))$ the element K does not exist in $U_q(\mathcal{H}\otimes\mathcal{H})$ since the Cartan matrix is degenerate in this case. The enlarging of Cartan subalgebra leads to $U_q(gl((0),\{1\}))$. Thus for $U_q(sl(1,1))$ we can use the R-matrix of $U_q(gl((0),\{1\}))$.

Proof of Theorem 7.1. By direct calculations it is easy to verify that the element (7.2) satisfies Eqs. (6.5)–(6.7), i.e.

$$\check{R}_\alpha \bar{\Delta}'(h_\alpha)=\Delta'(h_\alpha)\check{R}_\alpha, \tag{7.7}$$

$$\check{R}_\alpha \bar{\Delta}'(e_{\pm\alpha})=\Delta'(e_{\pm\alpha})\check{R}_\alpha, \tag{7.8}$$

where Δ' and $\bar{\Delta}'$ are opposite coproducts to Δ and $\bar{\Delta}$, correspondingly. The factorization properties (6.8) and (6.9) for $\check{R}_\alpha$ are equivalent to the addition Theorem (5.4) for q-exponents. □

Remark. In fact Eqs. (7.7) and (7.8) define the element $\check{R}_\alpha$ completely, i.e. the factorization properties (6.5)–(6.7) are a consequence of Eqs. (7.7) and (7.8) in this case.

Now we consider quantum (super)algebras of rank 2.

Theorem 7.2. *The universal R-matrix for the rank 2 quantum (super)algebras $U_q(g)$ has the following form:*

$$R=\check{R}_\alpha\check{R}_{\alpha+\beta}\check{R}_\beta K \tag{7.10a}$$

$$=\check{R}_\beta\check{R}'_{\alpha+\beta}\check{R}_\alpha K \quad \textit{for} \quad g=A\textit{-type}, \tag{7.10b}$$

$$R=\check{R}_\alpha\check{R}_{\alpha+\beta}\check{R}_{\alpha+2\beta}\check{R}_\beta K \tag{7.11a}$$

$$=\check{R}_\beta\check{R}'_{\alpha+2\beta}\check{R}'_{\alpha+\beta}\check{R}_\beta K \quad \textit{for} \quad g=B\textit{-type}, \tag{7.11b}$$

$$R=\check{R}_\alpha\check{R}_{\alpha+\beta}\check{R}_{2\alpha+3\beta}\check{R}_{\alpha+2\beta}\check{R}_{\alpha+3\beta}\check{R}_\beta K \tag{7.12a}$$

$$=\check{R}_\beta\check{R}'_{\alpha+3\beta}\check{R}'_{\alpha+2\beta}\check{R}'_{2\alpha+3\beta}\check{R}'_{\alpha+\beta}\check{R}_\alpha K, \quad \textit{for} \quad g=G_2, \tag{7.12b}$$

where the element K is given by (6.2) *and the elements $\check{R}_\gamma$ and $\check{R}'_\gamma$, $\gamma\in\underline{\Delta}_+$, have the form*

$$\check{R}_\gamma:=\exp_q((-1)^{\theta(\gamma)}(a_\gamma)^{-1}(q-\bar{q})(e_\gamma\otimes e_{-\gamma})). \tag{7.13a}$$

$$\check{R}'_\gamma:=\exp_{q_\gamma}((-1)^{\theta(\gamma)}(a'_\gamma)^{-1}(q-\bar{q})(e'_\gamma\otimes e'_{-\gamma})). \tag{7.13b}$$

Here the root vectors e_γ (e'_γ), $\gamma\in\underline{\Delta}_+$, are constructed in accordance with the same normal orders as the products are taken in (7.10)–(7.12). *The coefficients a_γ and a'_γ are defined from the relations*

$$[e_\gamma, e_{-\gamma}]=a_\gamma(k_\gamma-\bar{k}_\gamma)/(q-\bar{q}), \tag{7.14a}$$

$$[e'_\gamma, e'_{-\gamma}]=a'_\gamma(k_\gamma-\bar{k}_\gamma)/(q-\bar{q}). \tag{7.14b}$$

In particular the coefficients a_γ can be taken from Propositions 4.1–4.4. For definition of a'_γ see the Remark to Proposition 4.4.

Proof. By direct calculations it is not difficult to verify that the reduced R-matrices in (7.10)–(7.12) satisfy Eqs. (6.5)–(6.7). The factorization properties (6.8) and (6.9) for $\check{R}$ are equivalent to the addition theorems (5.4)–(5.7) and rearranging identities (5.9)–(5.12) for q-exponents. □

As consequence of Theorem 7.1, 7.2 we have the following statement.

Corollary 7.1. *For quantum (super)algebras $U_q(g)$ of rank 2 there are the following relations:*

$$\check{R}_\alpha \check{R}_\beta = \check{R}_\beta \check{R}_\alpha \quad for \quad g = g_1 \oplus g_1', \tag{7.15}$$

$$\check{R}_\alpha \check{R}_{\alpha+\beta} \check{R}_\beta = \check{R}_\beta \check{R}'_{\alpha+\beta} \check{R}_\alpha \quad for \quad g = A\text{-}type, \tag{7.16}$$

$$\check{R}_\alpha \check{R}_{\alpha+\beta} \check{R}_{\alpha+2\beta} \check{R}_\beta = \check{R}_\beta \check{R}'_{\alpha+2\beta} \check{R}'_{\alpha+\beta} \check{R}_\alpha \quad for \quad g = B\text{-}type, \tag{7.17}$$

$$\check{R}_\alpha \check{R}_{\alpha+\beta} \check{R}_{2\alpha+3\beta} \check{R}_{\alpha+2\beta} \check{R}_{\alpha+3\beta} \check{R}_\beta = \check{R}_\beta \check{R}'_{\alpha+3\beta} \check{R}'_{\alpha+2\beta} \check{R}'_{2\alpha+3\beta} \check{R}_{\alpha+\beta} \check{R}_\alpha \quad for \quad g = G_2 \tag{7.18}$$

on a level of formal series, and also

$$[\Delta'(e_{\pm\alpha}),\ \check{R}_{\alpha+\beta} \check{R}_\beta] = 0, \tag{7.19a}$$

$$[\Delta'(e_{\pm\beta}),\ \check{R}_{\alpha+\beta} \check{R}_\alpha] = 0 \quad for \quad g = A\text{-}type; \tag{7.19b}$$

$$[\Delta'(e_{\pm\alpha}),\ \check{R}_{\alpha+\beta} \check{R}_{\alpha+2\beta} \check{R}_\beta] = 0, \tag{7.20a}$$

$$[\Delta'(e_{\pm\beta}),\ \check{R}'_{\alpha+2\beta} \check{R}'_{\alpha+\beta} \check{R}_\alpha] = 0 \quad for \quad g = B\text{-}type; \tag{7.20b}$$

$$[\Delta'(e_{\pm\alpha}),\ \check{R}_{\alpha+\beta} \check{R}_{2\alpha+3\beta} \check{R}_{\alpha+2\beta} \check{R}_{\alpha+3\beta} \check{R}_\beta] = 0, \tag{7.21a}$$

$$[\Delta'(e_{\pm\beta}),\ \check{R}'_{\alpha+3\beta} \check{R}'_{\alpha+2\beta} \check{R}'_{2\alpha+3\beta} \check{R}'_{\alpha+\beta} \check{R}_\alpha] = 0 \quad for \quad g = G_2. \tag{7.21b}$$

Here in (7.15) g_1 and g_1' are (super)algebras of rank 1.

8. The General Case

Now we consider a case of any quantized finite-dimensional contragredient (super)algebra $g(A, \tau)$.

At first we recall an important combinatorial property of the reduced root system $\underline{\Delta}_+$ for $g(A, \tau)$.

Lemma 8.1 [11, 13, 14]. *Any two normal orderings in $\underline{\Delta}_+$ can be transformed into each other by the following elementary inversions of neighbouring roots:*

$$\alpha, \beta \leftrightarrow \beta, \alpha, \tag{8.1a}$$

$$\alpha, \alpha+\beta, \beta \leftrightarrow \beta, \alpha+\beta, \alpha, \tag{8.1b}$$

$$\alpha, \alpha+\beta, \alpha+2\beta, \beta \leftrightarrow \beta, \alpha+2\beta, \alpha+\beta, \alpha, \tag{8.1c}$$

$$\alpha, \alpha+\beta, 2\alpha+3\beta, \alpha+2\beta, \alpha+3\beta, \beta \leftrightarrow \beta, \alpha+3\beta, \alpha+2\beta, 2\alpha+3\beta, \alpha+\beta, \alpha, \tag{8.1d}$$

where $\alpha-\beta$ is not root.

Proof. See the paper [11]. □

We fix a normal ordering in $\underline{\Delta}_+$ and construct the root q-vectors $e_{\pm\gamma}$, $\gamma \in \underline{\Delta}_+$, with respect to this normal ordering by Definition 3.2. The root q-vectors have the following properties.

Proposition 8.1. *If the positive roots in chosen normal ordering are neighboured as* (8.1a–d), *where $\alpha-\beta$ is not root, then corresponding the root q-vectors generate quantum (super)algebras of rank* 2, *i.e.*

$$[e_\alpha, e_{-\beta}] = 0. \tag{8.2}$$

Proof. It is the same as for Proposition 3.2. □

For any root $\gamma \in \underline{\Delta}_+$ and for a given normal ordering in $\underline{\Delta}_+$ let a_γ be a factor in the relation

$$[e_\gamma, e_{-\gamma}] = a_\gamma(k_\gamma - \bar{k}_\gamma)/(q-\bar{q}), \tag{8.3}$$

and let

$$\check{R}_\gamma := \exp_{q_\gamma}((-1)^{\theta(\gamma)}(a_\gamma)^{-1}(q-\bar{q})(e_\gamma \otimes e_{-\gamma})), \tag{8.4}$$

where $q_\gamma = (-1)^{\theta(\gamma)}\bar{q}^{(\gamma,\gamma)}$.

We put

$$\check{R} := \prod_{\gamma \in \underline{\Delta}_+} \check{R}_\gamma, \tag{8.5}$$

where the order in the product coincides with the chosen normal ordering. The element (8.5) has the following properties.

Proposition 8.2. (i) *The element $\check{R} \in F_q(g \otimes g)$ does not depend on the normal ordering of the root system $\underline{\Delta}_+$, i.e. all elements of the form* (8.5), *constructed for different normal orders, are equal as formal series.*

(ii) *For all simple roots we have*

$$[\bar{\Delta}(e_{\pm\alpha}), \check{R}_\alpha^{-1}\check{R}] = 0 \quad for \quad \forall \alpha \in \Pi, \tag{8.6}$$

or

$$\check{R}\bar{\Delta}'(e_{\pm\alpha}) = \Delta'(e_{\pm\alpha})\check{R}, \quad for \quad \forall \alpha \in \Pi, \tag{8.7}$$

and hence

$$\check{R}\bar{\Delta}'(x) = \Delta'(x)\check{R}, \quad for \quad \forall x \in U_q(g). \tag{8.8}$$

Proof. The assertion (i) follows immediately from Proposition 8.1, Lemma 8.1, and Corollary 7.1. Since the element (8.5) does not depend on the chosen normal ordering in $\underline{\Delta}_+$ we take this element in the following form:

$$\check{R} = \check{R}_\alpha \Big(\prod_{\alpha < \gamma} \check{R}_\gamma\Big), \tag{8.9}$$

where α is any simple root. Using Corollary 7.1 and Lemma 8.1 again we obtain for (8.9),

$$[\bar{\Delta}'(e_{\pm\alpha}), \check{R}_\alpha^{-1}\check{R}] = 0 \quad for \quad \forall \alpha \in \Pi. \tag{8.10}$$

It is evident that the relations (8.7), (8.8) are an immediate consequence of Eq. (8.10). □

Proposition 8.3. *The following identities are valid for any root $\alpha \in \underline{\Delta}_+$:*

$$\Delta'(e_\alpha) = \Big(\prod_{\gamma < \alpha} \check{R}_\gamma\Big)(1 \otimes e_\alpha + e_\alpha \otimes \bar{k}_\alpha)\Big(\prod_{\gamma < \alpha} \check{R}_\gamma\Big)^{-1}, \tag{8.11}$$

$$\Delta(e_\alpha) = \Big(\prod_{\gamma < \alpha} \check{R}_\gamma^{21}\Big)(\bar{k}_\alpha \otimes e_\alpha + e_\alpha \otimes 1)\Big(\prod_{\gamma < \alpha} \check{R}_\gamma^{21}\Big)^{-1}. \tag{8.12}$$

Proof. For rank two case identities (8.11) and (8.12) generalize (8.6) and may be obtained by direct computation. For the general case we use induction on height of root α. □

As a result we have the following basic theorem.

Theorem 8.1. *For any quantized (super)algebra $U_q(g(A,\tau))$, where $g(A,\tau)$ is the finite-dimensional contragredient Lie (super)algebra, and for any normal ordering of its positive root system Δ_+ the universal R-matrix can be written in the following form:*

$$R=\check{R}K=\Big(\prod_{\gamma\in\Delta_+}\check{R}_\gamma\Big)K\,, \tag{8.13}$$

where $\check{R}_\gamma$ is defined by (8.4), K is defined by (6.2) and the order in the product coincides with the chosen normal ordering. Moreover, the expression (8.11) does not depend on the normal ordering of the root system and has the uniqueness property in the space $F_q(g\otimes g)$.

Proof. We have to prove that $\check{R}$ satisfies (6.5)–(6.9). The properties (6.5)–(6.7) are verified in Proposition 8.1. By Proposition 8.3 and (5.4) we have:

$$(\Delta\otimes\mathrm{id})\check{R}_\gamma=\Big(\prod_{\alpha<\gamma}\check{R}_\gamma^{21}\Big)\check{R}_\gamma^{13}\tilde{R}_\gamma^{23}\Big(\prod_{\alpha<\gamma}\check{R}_\gamma^{21}\Big)^{-1}, \tag{8.14}$$

where

$$\tilde{R}_\gamma^{23}=K^{13}\check{R}_\gamma^{23}(K^{13})^{-1}=\exp_{q_\gamma}((-1)^{\theta(\gamma)}(a_\gamma)^{-1}(q-\bar{q})(\bar{k}_\gamma\otimes e_\gamma\otimes e_{-\gamma}))\,. \tag{8.15}$$

Applying formulas (5.10)–(5.12) to (8.14) we express $(\Delta\otimes\mathrm{id})\check{R}_\gamma$ as a product of q-exponents. Then we rearrange both sides of (6.8) or (6.9) by means of the same formulas (5.10)–(5.12) and show that (6.8) and (6.9) are identities. □

Remark. The Cartan matrix A may be degenerated [if $g(A,\tau)=sl(n,n)$, for example]. In this case we have to add some Cartan element h_{-1} to enlarge A up to nondegenerated matrix. Theorem 8.1 holds for this enlarged algebra.

9. Conclusion

We prove that the universal R-matrix for any quantized finite-dimensional (super)algebra Lie can be written in the factorized form (8.9). At first we check this formula for the rank 1 and 2 case by means of solving Eqs. (6.5)–(6.7) for the reduced R-matrix and using the addition Theorems (5.4)–(5.7) and the rearrange (5.10)–(5.12) for Eqs. (6.8), (6.9). In the general case the quasitriangular property directly follows from the rank 1 and 2 case. For other properties (1.2) we use the characteristics of q-exponents again.

We note that our proof looks similar to the proof of properties of extremal projectors [11, 13–15]. Instead of equation $e_\alpha p=pe_{-\alpha}=0$, where p is an extremal projector, we consider Eq. (1.1), instead of $p^2=p$ we have the YB equation, or (1.2).

Moreover we have an analogy in the forms of the extremal projector and the R-matrix, for example, $\lim_{\varrho\to\infty} p(\varrho)=m(\mathrm{id}\otimes S)R$ for $U_q(sl_2)$, where m is multiplication, S is an antipode, $p(\varrho)$ is a "shifted projector." This analogy is likely to be understood. Another connection between extremal projectors and R-matrices was considered in [19].

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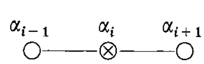
Note added in proof. Kac-Moody superalgebras, just as their quantization, are not defined by the relations (2.1)–(2.5) or (2.8)–(2.11). The additional, forth degree relations looks as follows.

For any subdiagram



of Dynkin diagram, if $(\alpha_i, \alpha_i) = (\alpha_{i+1}, \alpha_{i-1}) = (\alpha_i, \alpha_{i-1} + \alpha_{i+1}) = 0$ we have

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_{i-1}}], [e_{\pm\alpha_i}, e_{\pm\alpha_{i+1}}]] = 0$$

in nonquantized case and

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_{i-1}}]_{q'}, [e_{\pm\alpha_i}, e_{\pm\alpha_{i+1}}]_{q'}]_{q'} = 0, \quad q' = q^{\pm 1}$$

in quantized case.

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