UNIVERSAL RELATIONS AND EFFECTIVE COEFFICIENTS OF MAGNETO-ELECTRO-ELASTIC PERFORATED STRUCTURES

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Summary

Based on the two-scale convergence homogenization method, the limiting equations modelling the behaviour of a three-dimensional magneto-electro-elastic (MEE) composite made of periodically perforated microstructure are rigorously established. The homogenized problem is considered for the particular case of porous materials consisting of identical parallel empty cylinders periodically distributed in a transversely isotropic MEE and homogeneous medium. For these composites, universal relationships involving the MEE effective properties were derived without imposing any restrictions about their global behaviour. For the particular case of transversely isotropic (and tetragonal) effective behaviour, simple analytical expressions for the effective coefficients are explicitly given for two types of empty fibres periodical distribution (square and hexagonal arrays). An analytical formula to estimate the specific volume fraction where the magnetoelectric effective coefficient attains its minimum value is given. These relationships and formulae are used to check the accuracy of a numerical homogenization scheme based on Fourier transform technique. Comparisons with others micromechanical models are also included.

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1. Introduction

Composites made of piezoelectric and piezomagnetic (or piezoelectric and magnetostrictive or pyroelectric and pyromagnetic) materials exhibit a coupling effect between mechanical, electric, magnetic fields and thermal interactions. This is known as the magnetoelectric (ME) effect (**1**). Its magnitude depends on the physical and geometrical properties of the components of the composite (**2**). Predicting global properties of magneto-electro-elastic (MEE) composites in order to obtain new materials with enhanced ME effects is of great interest in the electronic industry. A detailed study of some technical applications, such as power harvesting, current transform, phase filters, magnetic field sensors and filters, using the ME effect, can be found in (**3**). The techniques devoted to determine the global properties of composite materials are called homogenization methods. During the past few years, several homogenization models have been applied to study MEE composites. Based on micromechanical analysis, analytical expressions for effective properties of two-phase continuous cylindrical fibres and layers were explicitly obtained (**4** to **6**). In (**7**, **8**), micromechanical methodologies were applied to determine the effective behaviour of MEE composites with ellipsoidal inclusions. Universal relations between the thermo-magneto-electro-elastic effective properties of two-phase fibrous composites were obtained (**9**). In (**10** to **14**), based on the homogenization theory of multiscale asymptotic analysis, the effective behaviour of laminated, fibrous and particulate MEE multiphase composites was considered.

In view of the great opportunities offered for MEE composites to design more efficient sensors and actuators for smart and intelligent structures, several works have been recently dedicated to the dynamic investigation of composite hollow cylinders and spheres (**15** to **19**). There are also several works dedicated to predict, both theoretically and experimentally, the effective properties of porous piezoelectric ceramics (**14**, **20** to **26**). In (**27**), based on the generalized Eshelby's tensor, the dependence of effective properties of MEE composites on the ellipsoidal void volume fraction and orientation is shown. An early theoretical investigation about the influence of voids in the global behaviour of MEE composites can be found in (**28**).

In this work, based on the periodic homogenization technique, the limiting equations modelling the behaviour of three-dimensional (3D) MEE perforated periodic structures are mathematically established. The local problems and the corresponding effective coefficients are explicitly given. The general homogenization theory is applied to the case of a transversely isotropic MEE matrix with cylindrical holes periodically distributed. No restrictions are supposed for the global behaviour. From the specific local problems and effective coefficients, the procedure to obtain universal relations is described. These relations for anti-plane problems appear to be new. They are an extension of recent results reported in (**14**), where the case of piezoelectric matrix with a transversely isotropic effective behaviour was studied. Also, here, the effect of the dielectric permittivity and the magnetic permeability of the free space is taken into account. An application of the universal relations to derive analytical expressions for effective coefficients is shown. These expressions are very simple and of very easy computational implementation.

This article is structured in four sections. Section 2 is devoted to the statement of the problem and to the mathematical homogenization of 3D perforated MEE structures with rapidly oscillating and periodic material coefficients. The convergence results of the homogenization process are presented based on the two-scale homogenization technique. The general local problems and the homogenized coefficients are derived. Also, a theorem providing strong convergence to the solution of the original problem is given. The main objective of section 3 is the derivation of universal relations for homogeneous and transversely isotropic MEE materials with cylindrical empty fibres (or MEE 3-1 longitudinally porous materials) without restrictions about their global behaviour. The general free boundary local problems and effective coefficients are specified for these kinds of media. In the particular case of a transversely isotropic effective behaviour, Benveniste-type universal relations are derived from the so-called plane local problems. In addition, from the anti-plane local problems, some links between their solutions are found, which allows obtaining universal relationships between the involved effective coefficients. Eventually, a computational scheme based on Fourier transforms is presented to solve numerically periodic boundary value problems for MEE composite materials with an arbitrary geometry of the constituents. Numerical examples are included in order to illustrate the meaning of the found relationships to control numerical codes. Section 5 is dedicated to some concluding comments.

2. Statement of the problem and homogenization

The plan is as follows. In section 2.1, we precisely describe the geometry of the perforated domain. In section 2.2, we recall the classical equilibrium equations of magnetoelectroelasticity. In section 2.3, we introduce the two-scale convergence and give its main properties. The limiting homogeneous model is then derived.

2.1 *Geometry of the structure*

Let $\mathbf{\Omega}$ be a domain (connected bounded subset) in $\mathbb{R}^3$ with Lipschitz boundary $\partial\mathbf{\Omega}$. Let $Y = [0, 1[^3$ be the elementary cell with a hole $\bar{S} \subset Y$; hence, $Y^* = Y \backslash \bar{S}$ represents, in each cell, the domain occupied by the material; the quantity $|Y^*|$ is the volume of the material in each cell Y. We denote by $\varepsilon > 0$ the size of each microstructure (statically, this must be $\varepsilon = a/L$ where a is the microscale and L is the macroscale (size of body)), which is intended to go to zero. The set of holes is obtained by ε-periodicity $S_\varepsilon = \varepsilon(\bar{S} + k) \cap \mathbf{\Omega}$, $k \in \mathbb{Z}^3$. The perforated domain $\mathbf{\Omega}_\varepsilon$ is obtained by removing the holes from the initial domain $\mathbf{\Omega}_\varepsilon = \mathbf{\Omega} \backslash \bar{S}^\varepsilon$ (we assume that $\mathbf{\Omega}_\varepsilon$ is connected and that the holes do not intersect its boundary), see Fig. 1. The equilibrium equations are posed on the perforated domain $\mathbf{\Omega}_\varepsilon$.

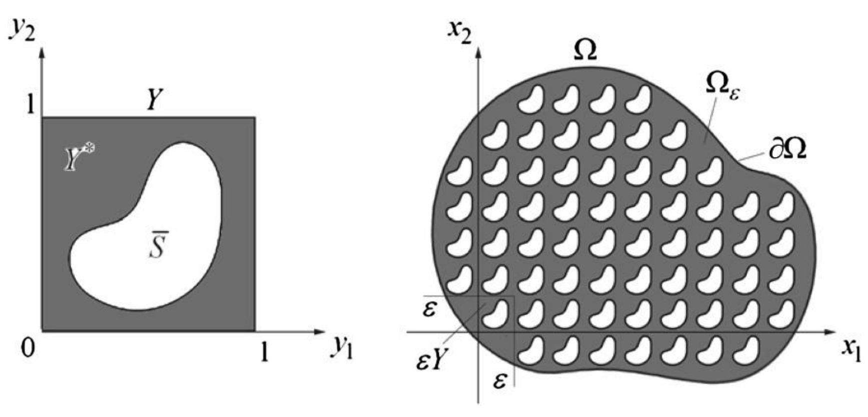

Fig. 1 Description of a perforated periodic domain and the reference cell

2.2 *Constitutive laws and equilibrium equations of MEE perforated structures*

The elastic displacement field[1] $\mathbf{u}^\varepsilon = (u_i^\varepsilon) : \bar{\mathbf{\Omega}}_\varepsilon \to \mathbb{R}^3$, the electric potential $\varphi^\varepsilon : \bar{\mathbf{\Omega}}_\varepsilon \to \mathbb{R}$ and the magnetic potential $\psi^\varepsilon : \bar{\mathbf{\Omega}}_\varepsilon \to \mathbb{R}$, which solve the following equilibrium equations

$$-\mathrm{div}\sigma^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{D}^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon) = 0, \quad \mathrm{div}\,\mathbf{B}^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon) = 0 \quad \text{in } \mathbf{\Omega}_\varepsilon, \tag{2.1}$$

with Dirichlet boundary conditions on $\partial\mathbf{\Omega}$, $(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon) = (\mathbf{0}, 0, 0)$, and Neumann conditions on the hole boundaries ∂S_ε

$$\sigma^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon)\cdot\mathbf{n}_\varepsilon = \mathbf{0}, \quad \mathbf{D}^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon)\cdot\mathbf{n}_\varepsilon = 0, \quad \mathbf{B}^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon)\cdot\mathbf{n}_\varepsilon = 0 \quad \text{on } \partial S_\varepsilon,$$

where $\mathbf{f} \in \mathbf{L}^2(\mathbf{\Omega}_\varepsilon)$, $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)$, $\mathbf{D}^\varepsilon = (D_i^\varepsilon)$, $\mathbf{B}^\varepsilon = (B_i^\varepsilon)$, $(\mathrm{div}\,\sigma^\varepsilon)_i = \partial_j\sigma_{ij}^\varepsilon$, $\mathrm{div}\,\mathbf{D}^\varepsilon = (\partial_i D_i^\varepsilon)$, $\mathrm{div}\,\mathbf{B}^\varepsilon = (\partial_i B_i^\varepsilon)$, $\partial_i = \partial/\partial x_i$ with $\mathbf{x} = (x_i) \in \mathbf{\Omega}$ and $\mathbf{n}_\varepsilon$ is the outer unit normal to ∂S_ε. The stress tensor $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)$, the electric displacement $\mathbf{D}^\varepsilon = (D_i^\varepsilon)$ and the magnetic displacement $\mathbf{B}^\varepsilon = (B_i^\varepsilon)$ are related to the linearized strain, defined as $s_{kl}(\mathbf{u}^\varepsilon) = \frac{1}{2}(\partial_k u_l^\varepsilon + \partial_l u_k^\varepsilon)$, and to the gradients of the electric $(\partial_k\varphi^\varepsilon)$ and magnetic $(\partial_k\psi^\varepsilon)$ potentials through the constitutive laws in $\mathbf{\Omega}_\varepsilon$,

$$\begin{aligned}
\sigma_{ij}^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon) &= c_{ijkl}^\varepsilon s_{kl}(\mathbf{u}^\varepsilon) + e_{kij}^\varepsilon\partial_k\varphi^\varepsilon + q_{kij}^\varepsilon\partial_k\psi^\varepsilon,\\
D_i^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon) &= -e_{ikl}^\varepsilon s_{kl}(\mathbf{u}^\varepsilon) + \kappa_{ik}^\varepsilon\partial_k\varphi^\varepsilon + \alpha_{ik}^\varepsilon\partial_k\psi^\varepsilon,\\
B_i^\varepsilon(\mathbf{u}^\varepsilon, \varphi^\varepsilon, \psi^\varepsilon) &= -q_{ikl}^\varepsilon s_{kl}(\mathbf{u}^\varepsilon) + \alpha_{ik}^\varepsilon\partial_k\varphi^\varepsilon + \mu_{ik}^\varepsilon\partial_k\psi^\varepsilon,
\end{aligned} \tag{2.2}$$

for $1 \leqslant i, j, k, l \leqslant 3$, with

$$c_{ijkl}^\varepsilon = c_{ijkl}(x, x/\varepsilon), \quad e_{kij}^\varepsilon = e_{kij}(x, x/\varepsilon), \quad q_{kij}^\varepsilon = q_{kij}(x, x/\varepsilon),$$

$$\kappa_{ik}^\varepsilon = \kappa_{ik}(x, x/\varepsilon), \quad \alpha_{ik}^\varepsilon = \alpha_{ik}(x, x/\varepsilon), \quad \mu_{ik}^\varepsilon = \mu_{ik}(x, x/\varepsilon).$$

We note that the functions c_{ijkl}^ε, e_{kij}^ε, q_{kij}^ε, κ_{ik}^ε, α_{ik}^ε and μ_{ik}^ε are εY-periodic.

The fourth-order elasticity tensor $c^\varepsilon = (c_{ijkl}^\varepsilon)$ is symmetric, uniformly positive, defined in $\mathbf{\Omega}_\varepsilon$ and satisfies

$$c_{ijkl}^\varepsilon = c_{jikl}^\varepsilon = c_{klij}^\varepsilon = c_{ijlk}^\varepsilon, \quad c_{ijkl}(x, y) \in L^\infty(\mathbf{\Omega}, \mathcal{C}_{\mathrm{per}}(Y)),$$

$$\exists\,\alpha_c > 0 \text{ independent of } \varepsilon : c_{ijkl}^\varepsilon X_{ij}X_{kl} \geqslant \alpha_c X_{ij}X_{ij}, \quad \forall X_{ij} = X_{ji} \in \mathbb{R}^3,$$

where, as usual, $y = x/\varepsilon$ is the local or fast variable and $\mathcal{C}_{\mathrm{per}}(Y)$ denotes the space of Y-periodic continuous functions.

The third-order piezoelectric coupling tensor $e^\varepsilon = (e_{ikl}^\varepsilon)$ satisfies the following properties

$$e_{ikl}^\varepsilon = e_{ilk}^\varepsilon, \quad e_{ikl}(x, y) \in L^\infty(\mathbf{\Omega}, \mathcal{C}_{\mathrm{per}}(Y)).$$

The second-order electric permittivity tensor $\kappa^\varepsilon = (\kappa_{ij}^\varepsilon)$ is symmetric, uniformly positive, defined in $\mathbf{\Omega}_\varepsilon$ and satisfies

$$\kappa_{ij}^\varepsilon = \kappa_{ji}^\varepsilon, \quad \kappa_{ij}(x, y) \in L^\infty(\mathbf{\Omega}, \mathcal{C}_{\mathrm{per}}(Y)),$$

$$\exists\,\alpha_\kappa > 0 \text{ independent of } \varepsilon : \kappa_{ij}^\varepsilon X_i X_j \geqslant \alpha_\kappa X_i X_i, \quad \forall X_i \in \mathbb{R}.$$

[1] Throughout this paper, Latin indices and exponents take their values in the set 1, 2, 3, Greek indices and exponents (except ε) take their values in the set 1, 2 and the summation convention with respect to repeated indices and exponents is used. Boldface letters represent vector-valued functions or spaces.

The second-order magnetic permeability tensor $\mu^{\varepsilon} = (\mu^{\varepsilon}_{ij})$ is symmetric, uniformly positive, defined in ε and satisfies

$$\mu^{\varepsilon}_{ij} = \mu^{\varepsilon}_{ji}, \quad \mu_{ij}(x, y) \in L^{\infty}(\boldsymbol{\Omega}, \mathcal{C}_{\text{per}}(Y)),$$

$$\exists\, \alpha_{\mu} > 0 \text{ independent of } \varepsilon : \mu^{\varepsilon}_{ij} X_i X_j \geqslant \alpha_{\mu} X_i X_i, \quad \forall X_i \in \mathbb{R}.$$

The third-order piezomagnetic coupling tensor $q^{\varepsilon} = (q^{\varepsilon}_{ikl})$ satisfies the following properties

$$q^{\varepsilon}_{ikl} = q^{\varepsilon}_{ilk}, \quad q^{\varepsilon}_{ikl}(x, y) \in L^{\infty}(\boldsymbol{\Omega}, \mathcal{C}_{\text{per}}(Y)).$$

The second-order ME coupling tensor $\alpha^{\varepsilon} = (\alpha^{\varepsilon}_{ij})$ satisfies

$$\alpha^{\varepsilon}_{ij} = \alpha^{\varepsilon}_{ji}, \quad \alpha^{\varepsilon}_{ij}(x, y) \in L^{\infty}(\boldsymbol{\Omega}, \mathcal{C}_{\text{per}}(Y)).$$

2.3 *Two-scale convergence*

Under classical regularity assumptions, problem (2.1) has a unique solution $(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}, \psi^{\varepsilon})$ in $\mathbf{H}^1(\Omega_{\varepsilon}) \times H^1(\Omega_{\varepsilon}) \times H^1(\Omega_{\varepsilon})$, which satisfies the uniform *a priori* estimate

$$\|\mathbf{u}^{\varepsilon}\|^2_{\mathbf{H}^1(\Omega_{\varepsilon})} + \|\varphi^{\varepsilon}\|^2_{H^1(\Omega_{\varepsilon})} + \|\psi^{\varepsilon}\|^2_{H^1(\Omega_{\varepsilon})} \leqslant C, \tag{2.3}$$

with a constant C, which depends upon $\boldsymbol{\Omega}$ and Y^*, but which is independent of ε. This is a consequence of the Korn and Poincaré inequalities for perforated domains (**29**). The aim of this section was to study the convergence of the sequence $\mathbf{u}^{\varepsilon}$, φ^{ε}, ψ^{ε} when ε goes to zero.

We now have sufficient estimates to state the first convergence result. The proof of the homogenization process will be carried out by using *the two-scale convergence* introduced by Nguetseng (**30**) and developed by Allaire (**31**). The basic definition and properties of this concept follow.

THEOREM 2.1. *A sequence of functions (v^{ε}) bounded in $L^2(\boldsymbol{\Omega})$ two-scale converges to a limit $v(x, y)$ belonging to $L^2(\boldsymbol{\Omega} \times Y)$, $v^{\varepsilon} \xrightarrow{2} v$, if*

$$\lim_{\varepsilon \to 0} \int_{\Omega} v^{\varepsilon}(x, t)\, \Psi(x, x/\varepsilon)\, dx = \int_{\Omega}\int_{Y} v(x, y)\, \Psi(x, y)\, dx\, dy,$$

for any test function $\Psi(x, y)$, Y-periodic in the second variable, satisfying

$$\lim_{\varepsilon \to 0} \int_{\Omega} |\Psi(x, x/\varepsilon)|^2\, dx = \int_{\Omega}\int_{Y} |\Psi(x, y)|^2\, dx\, dy. \tag{2.4}$$

(i) From each bounded sequence (v^{ε}) in $L^2(\boldsymbol{\Omega})$, one can extract a subsequence, which two-scale converges.

(ii) Let (v^{ε}) be a bounded sequence in $L^2(\boldsymbol{\Omega})$, which converges weakly to v in $L^2(\boldsymbol{\Omega})$. Then, $v^{\varepsilon} \xrightarrow{2} v$ and there exists a function $\breve{v} \in L^2(\boldsymbol{\Omega}; H^1_{\text{per}}(Y))$ such that, up to a subsequence, $\nabla v^{\varepsilon} \xrightarrow{2} \nabla v(x) + \nabla_y \breve{v}(x, y)$,

where the subscript y represents derivation with respect to the second variable y and the subscript per stands for Y-periodic functions in the last variable.

We denote by $\widetilde{\cdot}$ the extension by zero in the holes $\boldsymbol{\Omega}\backslash\boldsymbol{\Omega}_\varepsilon$ of functions defined on $\boldsymbol{\Omega}_\varepsilon$. By adding the relatively compact property and elementary properties of two-scale convergence, these imply the following result.

LEMMA 2.2. (**29**; **32**)

1. There exists $\mathbf{u}(x) \in \mathbf{H}_0^1(\Omega)$, $\varphi(x) \in H_0^1(\Omega)$ and $\psi(x) \in H_0^1(\Omega)$ such that, the three sequences $(\widetilde{\mathbf{u}^\varepsilon})_\varepsilon$, $(\widetilde{\varphi^\varepsilon})_\varepsilon$ and $(\widetilde{\psi^\varepsilon})_\varepsilon$ two-scale converge to $\chi(y)\mathbf{u}(x)$, $\chi(y)\varphi(x)$ and $\chi(y)\psi(x)$, respectively.
2. There exists $\breve{\mathbf{u}}(x,y) \in \mathbf{L}^2[\Omega; \mathbf{H}^1_{\text{per}}(Y^*)/\mathbb{R}]$, $\breve{\varphi}(x,y) \in L^2[\Omega; H^1_{\text{per}}(Y^*)/\mathbb{R}]$ and $\breve{\psi}(x,y) \in L^2[\Omega; H^1_{\text{per}}(Y^*)/\mathbb{R}]$ such that,

$$\widetilde{\nabla}\, \mathbf{u}^\varepsilon(x) \xrightarrow{2} \chi(y)[\nabla_x \mathbf{u}(x) + \nabla_y \breve{\mathbf{u}}(x,y)],$$

$$\widetilde{\nabla}\, \varphi^\varepsilon(x) \xrightarrow{2} \chi(y)[\nabla_x \varphi(x) + \nabla_y \breve{\varphi}(x,y)],$$

$$\widetilde{\nabla}\, \psi^\varepsilon(x) \xrightarrow{2} \chi(y)[\nabla_x \psi(x) + \nabla_y \breve{\psi}(x,y)].$$

3. We have $\widetilde{s}\,(\mathbf{u}^\varepsilon) \xrightarrow{2} \chi(y)[s_x(\mathbf{u}(x)) + s_y(\breve{\mathbf{u}}(x,y))]$, where χ is the characteristic function of Y^* extended by ε-periodicity to $\mathbb{R}^3$, and the index x or y means that the derivatives are with respect to that variable.

We remark that $(\mathbf{u}, \varphi, \psi)$ and $(\breve{\mathbf{u}}, \breve{\varphi}, \breve{\psi})$ can be interpreted with the following first terms in the asymptotic expansions: $\mathbf{u}^\varepsilon(x) = \mathbf{u}(x) + \varepsilon\breve{\mathbf{u}}(x,y) + \cdots$, $\varphi^\varepsilon(x) = \varphi(x) + \varepsilon\breve{\varphi}(x,y) + \cdots$, $\psi^\varepsilon(x) = \psi(x) + \varepsilon\breve{\psi}(x,y) + \cdots$. Due to the linearity of the original problem, and assuming the regularity in variation of the coefficients, the macroscopic response can be decoupled from the microscopic one; it is straightforward to show that the *correctors* $(\breve{\mathbf{u}}, \breve{\varphi}, \breve{\psi})$ can be written as linear combinations of the form

$$\begin{aligned}
\breve{\mathbf{u}}(x,y) &= s_{rt,x}(\mathbf{u}(x))\mathbf{w}^{rt}(y) + \partial_{m,x}\varphi(x)\mathbf{g}^m(y) + \partial_{n,x}\psi(x)\mathbf{f}^n(y),\\
\breve{\varphi}(x,y) &= s_{rt,x}(\mathbf{u}(x))\zeta^{rt}(y) + \partial_{m,x}\varphi(x)\pi^m(y) + \partial_{n,x}\psi(x)\chi^n(y),\\
\breve{\psi}(x,y) &= s_{rt,x}(\mathbf{u}(x))\eta^{rt}(y) + \partial_{m,x}\varphi(x)\xi^m(y) + \partial_{n,x}\psi(x)\gamma^n(y),
\end{aligned} \tag{2.5}$$

where the local (basis) functions $(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt})$, $(\mathbf{g}^m, \pi^m, \xi^m)$ and $(\mathbf{f}^m, \chi^m, \gamma^m)$ solve local microscopic problems posed in the elementary cell Y or more precisely in Y^*. These local problems are given below in Lemma 2.3.

2.4 *Local problems and evaluation of homogenized coefficients*

We are now able to identify that the basis functions appearing in the correctors $(\breve{\mathbf{u}}, \breve{\varphi}, \breve{\psi})$ are periodic solutions to the following problems similar to (2.1) but with homogeneous Neumann conditions on the hole boundary ∂S of the elementary cell Y.

LEMMA 2.3. (Local functions in Y). There exist unique Y-periodic basis functions (up to additive constants)

$$(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) \in \mathbf{H}^1_{\text{per}}(Y^*/\mathbb{R}) \times H^1_{\text{per}}(Y^*/\mathbb{R}) \times H^1_{\text{per}}(Y^*/\mathbb{R}),$$

$$(\mathbf{g}^m, \pi^m, \xi^m) \in \mathbf{H}^1_{\text{per}}(Y^*/\mathbb{R}) \times H^1_{\text{per}}(Y^*/\mathbb{R}) \times H^1_{\text{per}}(Y^*/\mathbb{R}),$$

$$(\mathbf{f}^m, \chi^m, \gamma^m) \in \mathbf{H}^1_{\text{per}}(Y^*/\mathbb{R}) \times H^1_{\text{per}}(Y^*/\mathbb{R}) \times H^1_{\text{per}}(Y^*/\mathbb{R}),$$

which are weak solutions of the *local* (*microscopic*) *problems* posed in the elementary cell Y^*:

The six triplets $(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt})$ are solutions to problems:

$$-\partial_i \sigma_{ij}(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) = \partial_i c_{ijrt}, \quad -\partial_i D_i(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) = -\partial_i e_{irt},$$
$$-\partial_i B_i(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) = -\partial_i q_{irt} \quad \text{in } Y^*, \tag{2.6}$$

with $\sigma(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) \cdot \mathbf{n} = \mathbf{0}$, $D(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) \cdot \mathbf{n} = 0$ and $B(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) \cdot \mathbf{n} = 0$ on ∂S, where $\mathbf{n}$ is the outer unit normal to ∂S.

The three triplets $(\mathbf{g}^m, \pi^m, \xi^m)$ are solutions to problems:

$$-\partial_i \sigma_{ij}(\mathbf{g}^m, \pi^m, \xi^m) = \partial_i e_{mij}, \quad -\partial_i D_i(\mathbf{g}^m, \pi^m, \xi^m) = -\partial_i \kappa_{im},$$
$$-\partial_i B_i(\mathbf{g}^m, \pi^m, \xi^m) = -\partial_i \alpha_{im} \quad \text{in } Y^*, \tag{2.7}$$

with $\sigma(\mathbf{g}^m, \pi^m, \xi^m) \cdot \mathbf{n} = \mathbf{0}$, $D(\mathbf{g}^m, \pi^m, \xi^m) \cdot \mathbf{n} = 0$ and $B(\mathbf{g}^m, \pi^m, \xi^m) \cdot \mathbf{n} = 0$ on ∂S.

The three triplets $(\mathbf{f}^m, \chi^m, \gamma^m)$ are solutions to problems:

$$-\partial_i \sigma_{ij}(\mathbf{f}^m, \chi^m, \gamma^m) = \partial_i q_{ijm}, \quad -\partial_i D_i(\mathbf{f}^m, \chi^m, \gamma^m) = -\partial_i \alpha_{im},$$
$$-\partial_i B_i(\mathbf{f}^m, \chi^m, \gamma^m) = -\partial_i \mu_{im} \quad \text{in } Y^*, \tag{2.8}$$

with $\sigma(\mathbf{f}^m, \chi^m, \gamma^m) \cdot \mathbf{n} = \mathbf{0}$, $D(\mathbf{f}^m, \chi^m, \gamma^m) \cdot \mathbf{n} = 0$ and $B(\mathbf{f}^m, \chi^m, \gamma^m) \cdot \mathbf{n} = 0$ on ∂S.

Since the data c_{ijrt}, e_{irt}, q_{irt}, κ_{im}, α_{im} and μ_{im} are discontinuous functions, the derivative on the right-hand side of previous problems must be understood in the distributional sense. We note also that, although the initial problem models an insulator (no free electric or magnetic charges), the local problems (2.6)–(2.8) contain fictitious volume electric and magnetic charges.

2.5 *The homogenized problem posed in* Ω

The limit MEE field $(\mathbf{u}, \varphi, \psi)$ solves the same problem as (2.1) but with another definition of the homogeneous stress, electric, magnetic displacement tensors $\bar{\sigma}$, $\bar{\mathbf{D}}$ and $\bar{\mathbf{B}}$:

$$-\text{div}\, \bar{\sigma}(\mathbf{u}, \varphi, \psi) = \theta \mathbf{f}, \quad \text{div}\, \bar{\mathbf{D}}(\mathbf{u}, \varphi, \psi) = 0, \quad \text{div}\, \bar{\mathbf{B}}(\mathbf{u}, \varphi, \psi) = 0 \quad \text{in } \mathbf{\Omega}, \tag{2.9}$$

with the homogeneous boundary condition $\mathbf{u} = \mathbf{0}$, $\varphi = 0$ and $\psi = 0$ on $\partial\mathbf{\Omega}$. The quantity $\theta = |Y^*|/|Y|$ represents the proportion of the material volume in each elementary cell Y. The new constitutive law is of the same kind as (2.2) and is given by

$$\bar{\sigma}(\mathbf{u}, \varphi, \psi) = \bar{c}_{ijkl} s_{kl}(\mathbf{u}) + \bar{e}_{kij}\partial_k \varphi + \bar{q}_{kij}\partial_k \psi,$$
$$\bar{D}_i(\mathbf{u}, \varphi, \psi) = -\bar{e}_{ikl} s_{kl}(\mathbf{u}) + \bar{\kappa}_{ik}\partial_k \varphi + \bar{\alpha}_{ik}\partial_k \psi, \tag{2.10}$$
$$\bar{B}_i(\mathbf{u}, \varphi, \psi) = -\bar{q}_{ikl} s_{kl}(\mathbf{u}) + \bar{\alpha}_{ik}\partial_k \varphi + \bar{\mu}_{ik}\partial_k \psi,$$

where the effective tensors $\bar{c}$, $\bar{e}$, $\bar{q}$, $\bar{\kappa}$, $\bar{\alpha}$ can be calculated from the solutions of the local problems using the following formulas:

$$\begin{aligned}
\overline{c}_{ijrt} &= \left\langle c_{ijkl}[s_{kl,y}(\mathbf{w}^{rt}) + \delta^{rt}_{kl}] + e_{kij}\partial_{k,y}\zeta^{rt} + q_{kij}\partial_{k,y}\eta^{rt} \right\rangle, \\
\overline{e}_{irt} &= \left\langle e_{ikl}[s_{kl,y}(\mathbf{w}^{rt}) + \delta^{rt}_{kl}] - \kappa_{ik}\partial_{k,y}\zeta^{rt} - \alpha_{ik}\partial_{k,y}\eta^{rt} \right\rangle, \\
\overline{e}_{mij} &= \left\langle c_{ijkl}s_{kl,y}(\mathbf{g}^{m}) + e_{sij}[\partial_{s,y}(\pi^{m}) + \delta^{m}_{s}] + q_{sij}\partial_{s,y}\xi^{m} \right\rangle, \\
\overline{q}_{irt} &= \left\langle q_{ikl}[s_{kl,y}(\mathbf{w}^{rt}) + \delta^{rt}_{kl}] - \alpha_{ik}\partial_{k,y}\zeta^{rt} - \mu_{ik}\partial_{k,y}\eta^{rt} \right\rangle, \\
\overline{q}_{mij} &= \left\langle c_{ijkl}s_{kl,y}(\mathbf{f}^{m}) + e_{sij}\partial_{s,y}(\chi^{m}) + q_{sij}[\partial_{s,y}\gamma^{m} + \delta^{m}_{s}] \right\rangle, \\
\overline{\kappa}_{im} &= \left\langle \kappa_{is}[\partial_{s,y}\pi^{m} + \delta^{m}_{s}] - e_{ikl}s_{kl,y}(\mathbf{g}^{m}) + \alpha_{is}\partial_{s,y}\xi^{m} \right\rangle, \\
\overline{\alpha}_{im} &= \left\langle \alpha_{is}[\partial_{s,y}\pi^{m} + \delta^{m}_{s}] - q_{ikl}s_{kl,y}(\mathbf{g}^{m}) + \mu_{is}\partial_{s,y}\xi^{m} \right\rangle, \\
\overline{\alpha}_{im} &= \left\langle \alpha_{is}[\partial_{s,y}\gamma^{m} + \delta^{m}_{s}] - e_{ikl}s_{kl,y}(\mathbf{f}^{m}) + \kappa_{is}\partial_{s,y}\chi^{m} \right\rangle, \\
\overline{\mu}_{im} &= \left\langle \mu_{ik}[\partial_{k,y}\gamma^{m} + \delta^{m}_{k}] - q_{ikl}s_{kl,y}(\mathbf{f}^{m}) + \alpha_{ik}\partial_{k,y}\chi^{m} \right\rangle,
\end{aligned} \tag{2.11}$$

where, for a function $g \in L^1(Y)$, we set $\langle g \rangle = |Y|^{-1}\int_Y g(y)\,dy$ and $\delta^{kl}_{ij} = \frac{1}{2}(\delta^i_j\delta^k_l + \delta^i_k\delta^j_l)$. Note that two definitions for the piezoelectric ($\overline{e}$), piezomagnetic ($\overline{q}$) and ME ($\overline{\alpha}$) effective tensors are given.

The complex procedure of evaluation of the homogenized tensors is detailed in (**13**), and we justify rigorously the limiting equations modelling the homogenized problem.

2.6 *Correctors*

The following theorem complements the two-scale convergence result by providing strong convergence, which is very useful from a theoretical and numerical point of view. It is based on the remark that $s_y(\breve{\mathbf{u}}(x,y))$, $\nabla_y(\breve{\varphi}(x,y))$ and $\nabla_y(\breve{\psi}(x,y))$ are *admissible test functions* in the sense (2.4).

THEOREM 2.4. *The following convergence holds when ε goes to zero:*

$$\mathbf{u}^{\varepsilon}(x) - \mathbf{u}(x) - \varepsilon\breve{\mathbf{u}}(x, x/\varepsilon) \to \mathbf{0} \quad \text{in } \mathbf{H}^1(\boldsymbol{\Omega}),$$

$$\varphi^{\varepsilon} - \varphi(x) - \varepsilon\breve{\varphi}(x, x/\varepsilon) \to 0 \quad \text{in } H^1(\boldsymbol{\Omega}),$$

$$\psi^{\varepsilon}(x) - \varphi\psi(x) - \varepsilon\breve{\psi}(x, x/\varepsilon) \to 0 \quad \text{in } H^1(\boldsymbol{\Omega}).$$

3. Effective properties for MEE 3-1 longitudinally porous materials

In this section, the general local problems (2.6)–(2.8) and the corresponding formulae for effective coefficients (2.11) will be applied to a class of unidirectional empty fibres periodically distributed in a transversely isotropic and homogeneous MEE matrix. The global behaviour of this heterogeneous medium is of any MEE global symmetry. New relationships involving the 18 effective properties are derived from the stated anti-plane shear MEE problems, which are truly unrelated to the geometry of the cross section of the empty fibres. These relations are an extension of recent results reported in (**14**) where the case of transversely isotropic effective behaviour was studied. Moreover, an application to obtain analytical and simple formulae for effective coefficients is given. The long-fibre limits of the Mori–Tanaka formulae are obtained. Finally, an efficient numerical homogenization scheme based on Fourier transform techniques is presented for MEE composite materials with general anisotropy and arbitrarily complex periodic microstructure.

3.1 *Local problems and effective properties*

Porous materials consisting of identical parallel empty cylinders, periodically distributed in a transversely isotropic linear MEE and homogeneous medium, are studied here. The cross section of the empty fibres is assumed smooth enough. This is an example of a two-dimensional homogenization problem. The material properties of the matrix are piecewise constant functions on any orthogonal plane to the axis of the empty fibres (y_3). The local problems (2.6)–(2.8) can be formulated as classical free boundary value problems on the periodic cell as follows:

Local Problems L_1^{rt}: Find the Y-periodic functions $\mathbf{w}^{rt}(y_1, y_2)$, $\zeta^{rt}(y_1, y_2)$ and $\eta^{rt}(y_1, y_2)$ of zero average on Y, such that

$$\begin{aligned}
&\partial_{\delta,y}\sigma_{i\delta}(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) = 0, \quad \partial_{\delta,y} D_\delta(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) = 0\\
&\quad \text{and} \quad \partial_{\delta,y} B_\delta(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt}) = 0 \quad \text{in } Y^*, \\
&\sigma_{i\delta}(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt})n_\delta = -c_{i\delta rt}n_\delta, \quad D_i(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt})n_\delta = -e_{\delta rt}n_\delta\\
&\quad \text{and} \quad B_i(\mathbf{w}^{rt}, \zeta^{rt}, \eta^{rt})n_\delta = -q_{\delta rt}n_\delta \quad \text{on } \partial S,
\end{aligned} \tag{3.1}$$

where $\partial_{\delta,y} = \partial/\partial y_\delta$ and n_δ are the components of the outward unit normal vector to the hole boundary ∂S. $\sigma_{i\delta}$, D_δ and B_δ are, respectively, the $i\delta$-components of the stress tensor, the δ-components of the electric displacement vector and the δ-components of the magnetic induction vector associated with the corresponding local functions.

Local Problems L_2^{m}: Find the Y-periodic functions $\mathbf{g}^{m}(y_1, y_2)$, $\pi^{m}(y_1, y_2)$ and $\xi^{m}(y_1, y_2)$ of zero average on Y, such that

$$\begin{aligned}
&\partial_{\delta,y}\sigma_{i\delta}(\mathbf{g}^{m}, \pi^{m}, \xi^{m}) = 0, \quad \partial_{\delta,y} D_\delta(\mathbf{g}^{m}, \pi^{m}, \xi^{m}) = 0\\
&\quad \text{and} \quad \partial_{\delta,y} B_\delta(\mathbf{g}^{m}, \pi^{m}, \xi^{m}) = 0 \quad \text{in } Y^*, \\
&\sigma_{i\delta}(\mathbf{g}^{m}, \pi^{m}, \xi^{m})n_\delta = -e_{mi\delta}n_\delta, \quad D_\delta(\mathbf{g}^{m}, \pi^{m}, \xi^{m})n_\delta = \kappa_{\delta m}n_\delta\\
&\quad \text{and} \quad B_\delta(\mathbf{g}^{m}, \pi^{m}, \xi^{m})n_\delta = \alpha_{\delta m}n_\delta \quad \text{on } \partial S.
\end{aligned} \tag{3.2}$$

Local Problems L_3^{m}: Find the Y-periodic functions $\mathbf{f}^{m}(y_1, y_2)$, $\chi^{m}(y_1, y_2)$ and $\gamma^{m}(y_1, y_2)$ of zero average on Y, such that

$$\begin{aligned}
&\partial_{\delta,y}\sigma_{i\delta}(\mathbf{f}^{m}, \chi^{m}, \gamma^{m}) = 0, \quad \partial_{\delta,y} D_\delta(\mathbf{f}^{m}, \chi^{m}, \gamma^{m}) = 0\\
&\quad \text{and} \quad \partial_{\delta,y} B_\delta(\mathbf{f}^{m}, \chi^{m}, \gamma^{m}) = 0 \quad \text{in } Y^*,
\end{aligned} \tag{3.3}$$

$$\sigma_{i\delta}(\mathbf{f}^m, \chi^m, \gamma^m)n_\delta = -q_{mi\delta}n_\delta, \quad D_\delta(\mathbf{f}^m, \chi^m, \gamma^m)n_\delta = \alpha_{\delta m}n_\delta$$
$$\text{and} \quad B_\delta(\mathbf{f}^m, \chi^m, \gamma^m)n_\delta = \mu_{\delta m}n_\delta \quad \text{on } \partial S.$$

From now on, the matrix is a homogeneous MEE material with 6-mm symmetry. Then, (2.2) can be written explicitly in terms of 17 independent parameters ($k, m, l, n, p, q, r, e, t, u, q', r', e', \alpha, \beta, \upsilon$ and w), which are given by five elastic constants $2k = c_{1111} + c_{1122}$, $2m = 2c_{1212} = c_{1111} - c_{1122}$, $l = c_{1133} = c_{2233}$, $n = c_{3333}$ and $p = c_{1313} = c_{2323}$; three piezoelectric constants $q = e_{311} = e_{322}$, $r = e_{333}$ and $e = e_{113} = e_{223}$; two dielectric permittivity constants $t = \kappa_{11} = \kappa_{22}$ and $u = \kappa_{33}$; three piezomagnetic constants $q' = q_{311} = q_{322}$, $r' = q_{333}$ and $e' = q_{113} = q_{223}$; two ME constants $\alpha = \alpha_{11} = \alpha_{22}$ and $\beta = \alpha_{33}$ and two magnetic permeability constants $\upsilon = \mu_{11} = \mu_{22}$ and $w = \mu_{33}$. This is the extended Hill's notation (see, for instance, (**10**)). The non-zero terms in these constitutive relations become:

$$\sigma_{11} + \sigma_{22} = 2k(s_{11} + s_{22}) + 2ls_{33} + 2q\partial_3\varphi + 2q'\partial_3\psi,$$
$$\sigma_{33} = l(s_{11} + s_{22}) + ns_{33} + r\partial_3\varphi + r'\partial_3\psi, \quad \sigma_{11} - \sigma_{22} = 2m(s_{11} - s_{22}),$$
$$\sigma_{23} = 2ps_{23} + e\partial_2\varphi + e'\partial_2\psi, \quad \sigma_{13} = 2ps_{13} + e\partial_1\varphi + e'\partial_1\psi, \quad \sigma_{12} = 2ms_{12},$$
$$D_1 = -2es_{12} + t\partial_1\varphi + \alpha\partial_1\psi, \quad D_2 = -2es_{23} + t\partial_2\varphi + \alpha\partial_2\psi,$$
$$D_3 = -q(s_{11} + s_{22}) - rs_{33} + u\partial_3\varphi + \beta\partial_3\psi,$$
$$B_1 = -2e's_{12} + \alpha\partial_1\varphi + \upsilon\partial_1\psi, \quad B_2 = -2e's_{23} + \alpha\partial_2\varphi + \upsilon\partial_2\psi,$$
$$B_3 = -q(s_{11} + s_{22}) - r's_{33} + \beta\partial_3\varphi + w\partial_3\psi.$$

In the subsequent analysis, no restrictions to a particular global behaviour will be assumed. The non-vanishing components of $c_{i\delta rt}$, $e_{\delta rt}$, $q_{\delta rt}$; $e_{mi\delta}$, $\kappa_{\delta m}$, $\alpha_{\delta m}$ and $q_{mi\delta}$, $\alpha_{\delta m}$, $\mu_{\delta m}$ lead, respectively, to the non-homogeneous free boundary problems (3.1); (3.2) and (3.3) that have a non-zero solution. These are six similar uncoupled elastic plane strain problems (L_1^{pp}, L_1^{12}, L_2^3 and L_3^3) and six coupled anti-plane strain and potential systems (L_1^{13}, L_1^{23}, L_2^α and L_3^α). Consequently, the only local functions different to zero are w_α^{pp}, w_α^{12}, g_α^3, f_α^3, w_3^{13}, ζ^{13}, η^{13}, w_3^{23}, ζ^{23}, η^{23}, g_3^α, π^α, ξ^α, f_3^α, χ^α and γ^α.

3.2 *Universal relations involving effective coefficients from L_1^{pp}, L_1^{12}, L_2^3 and L_3^3*

The following non-zero effective properties (2.11) can be derived from the local problems L_1^{pp}, L_1^{12}, L_2^3 and L_3^3:

$$\overline{c}_{1111} = (k+m)\theta + (k+m)\left\langle \partial_{1,y}w_1^{11}\right\rangle_* + (k-m)\left\langle \partial_{2,y}w_2^{11}\right\rangle_*, \tag{3.4}$$

$$\overline{c}_{1122} = (k-m)\theta + (k-m)\left\langle \partial_{1,y}w_1^{11}\right\rangle_* + (k+m)\left\langle \partial_{2,y}w_2^{11}\right\rangle_*$$
$$= (k-m)\theta + (k+m)\left\langle \partial_{1,y}w_1^{22}\right\rangle_* + (k-m)\left\langle \partial_{2,y}w_2^{22}\right\rangle_*, \tag{3.5}$$

$$\overline{c}_{1133} = l\theta + l\left\langle \partial_{\alpha,y} w_\alpha^{11} \right\rangle_* = l\theta + (k+m)\left\langle \partial_{1,y} w_1^{33} \right\rangle_* + (k-m)\left\langle \partial_{2,y} w_2^{33} \right\rangle_*, \tag{3.6}$$

$$\overline{c}_{2222} = (k+m)\theta + (k-m)\left\langle \partial_{1,y} w_1^{22} \right\rangle_* + (k+m)\left\langle \partial_{2,y} w_2^{22} \right\rangle_*, \tag{3.7}$$

$$\overline{c}_{2233} = l\theta + l\left\langle \partial_{\alpha,y} w_\alpha^{22} \right\rangle_* = l\theta + (k-m)\left\langle \partial_{1,y} w_1^{33} \right\rangle_* + (k+m)\left\langle \partial_{2,y} w_2^{33} \right\rangle_*, \tag{3.8}$$

$$\overline{c}_{3333} = n\theta + l\left\langle \partial_{\alpha,y} w_\alpha^{33} \right\rangle_*, \tag{3.9}$$

$$\overline{c}_{1211} = m\left\langle \partial_{2,y} w_1^{11} + \partial_{1,y} w_2^{11} \right\rangle_* = (k+m)\left\langle \partial_{1,y} w_1^{12} \right\rangle_* + (k-m)\left\langle \partial_{2,y} w_2^{12} \right\rangle_*, \tag{3.10}$$

$$\overline{c}_{1222} = m\left\langle \partial_{2,y} w_1^{22} + \partial_{1,y} w_2^{22} \right\rangle_* = (k-m)\left\langle \partial_{1,y} w_1^{12} \right\rangle_* + (k+m)\left\langle \partial_{2,y} w_2^{12} \right\rangle_*, \tag{3.11}$$

$$\overline{c}_{1233} = m\left\langle \partial_{2,y} w_1^{33} + \partial_{1,y} w_2^{33} \right\rangle_* = l\left\langle \partial_{1,y} w_1^{12} + \partial_{2,y} w_2^{12} \right\rangle_*, \tag{3.12}$$

$$\overline{c}_{1212} = m\theta + m\left\langle \partial_{2,y} w_1^{12} + \partial_{1,y} w_2^{12} \right\rangle_*, \tag{3.13}$$

$$\overline{e}_{311} = q\theta + q\left\langle \partial_{\alpha,y} w_\alpha^{11} \right\rangle_* = q\theta + (k+m)\left\langle \partial_{1,y} g_1^{3} \right\rangle_* + (k-m)\left\langle \partial_{2,y} g_2^{3} \right\rangle_*, \tag{3.14}$$

$$\overline{e}_{322} = q\theta + q\left\langle \partial_{\alpha,y} w_\alpha^{22} \right\rangle_* = q\theta + (k-m)\left\langle \partial_{1,y} g_1^{3} \right\rangle_* + (k+m)\left\langle \partial_{2,y} g_2^{3} \right\rangle_*, \tag{3.15}$$

$$\overline{e}_{333} = r\theta + q\left\langle \partial_{\alpha,y} w_\alpha^{33} \right\rangle_* = r\theta + l\left\langle \partial_{\alpha,y} g_\alpha^{3} \right\rangle_*, \tag{3.16}$$

$$\overline{e}_{312} = q\left\langle \partial_{\alpha,y} w_\alpha^{12} \right\rangle_* = m\left\langle \partial_{2,y} g_1^{3} + \partial_{1,y} g_2^{3} \right\rangle_*, \tag{3.17}$$

$$\overline{q}_{311} = q'\theta + q'\left\langle \partial_{\alpha,y} w_\alpha^{11} \right\rangle_* = q'\theta + (k+m)\left\langle \partial_{1,y} f_1^{3} \right\rangle_* + (k-m)\left\langle \partial_{2,y} f_2^{3} \right\rangle_*, \tag{3.18}$$

$$\overline{q}_{322} = q'\theta + q'\left\langle \partial_{\alpha,y} w_\alpha^{22} \right\rangle_* = q'\theta + (k-m)\left\langle \partial_{1,y} f_1^{3} \right\rangle_* + (k+m)\left\langle \partial_{2,y} f_2^{3} \right\rangle_*, \tag{3.19}$$

$$\overline{q}_{333} = r'\theta + q'\left\langle \partial_{\alpha,y} w_\alpha^{33} \right\rangle_* = r'\theta + l\left\langle \partial_{\alpha,y} f_\alpha^{3} \right\rangle_*, \tag{3.20}$$

$$\overline{q}_{312} = q'\left\langle \partial_{\alpha,y} w_\alpha^{12} \right\rangle_* = m\left\langle \partial_{2,y} f_1^{3} + \partial_{1,y} f_2^{3} \right\rangle_*, \tag{3.21}$$

$$\overline{\kappa}_{33} = \langle u \rangle - q\left\langle \partial_{\alpha,y} g_\alpha^{3} \right\rangle_*, \quad \overline{\mu}_{33} = \langle w \rangle - q'\left\langle \partial_{\alpha,y} f_\alpha^{3} \right\rangle_*, \tag{3.22}$$

$$\overline{\alpha}_{33} = \beta\theta - q\left\langle \partial_{\alpha,y} f_\alpha^{3} \right\rangle_* = \beta\theta - q'\left\langle \partial_{\alpha,y} g_\alpha^{3} \right\rangle_*, \tag{3.23}$$

where $\langle g\rangle_* = |Y|^{-1}\int_{Y*} g(y)\,dy$. Moreover, $\langle u\rangle = u\theta + \kappa_0(1-\theta)$ and $\langle w\rangle = w\theta + \mu_0(1-\theta)$ are averages of the corresponding property taking into account the known constants of free space for the dielectric permittivity $\kappa_0 = 8.85\times 10^{-12}\mathrm{C}^2/\mathrm{N\,m}^2$ and for the magnetic permeability $\mu_0 = 4\pi\times 10^{-7}\mathrm{N\,A}^{-2}$. The empty fibre area fraction is denoted by $1-\theta$.

On the other hand, there exist the following useful links among the solutions of the local problems L_1^{11}, L_1^{22}, L_1^{33}, L_2^3 and L_3^3 (see, for instance, (**14**, Section 4.3)):

$$w_\alpha^{11} + w_\alpha^{22} = 2(k/l)w_\alpha^{33} = 2(k/q)g_\alpha^3 = 2(k/q')f_\alpha^3. \tag{3.24}$$

From (3.4) to (3.24), it is straightforward to obtain the following relationships:

$$\begin{aligned}\frac{2k}{l} &= \frac{\overline{c}_{1111}+2\overline{c}_{1122}+\overline{c}_{2222}-4k\theta}{\overline{c}_{1133}+\overline{c}_{2233}-2l\theta} = \frac{\overline{c}_{1133}+\overline{c}_{2233}-2l\theta}{\overline{c}_{3333}-n\theta}\\ &= \frac{\overline{e}_{311}+\overline{e}_{322}-2q\theta}{\overline{e}_{333}-r\theta} = \frac{\overline{q}_{311}+\overline{q}_{322}-2q'\theta}{\overline{q}_{333}-r'\theta},\end{aligned} \tag{3.25}$$

$$\begin{aligned}\frac{2k}{q} &= \frac{\overline{c}_{1111}+2\overline{c}_{1122}+\overline{c}_{2222}-4k\theta}{\overline{e}_{311}+\overline{e}_{322}-2q\theta} = \frac{\overline{c}_{1133}+\overline{c}_{2233}-2l\theta}{\overline{e}_{333}-r\theta}\\ &= -\frac{\overline{e}_{311}+\overline{e}_{322}-2q\theta}{\overline{\kappa}_{33}-\langle u\rangle} = -\frac{\overline{q}_{311}+\overline{q}_{322}-2q'\theta}{\overline{\alpha}_{33}-\beta\theta},\end{aligned} \tag{3.26}$$

$$\begin{aligned}\frac{2k}{q'} &= \frac{\overline{c}_{1111}+2\overline{c}_{1122}+\overline{c}_{2222}-4k\theta}{\overline{q}_{311}+\overline{q}_{322}-2q'\theta} = \frac{\overline{c}_{1133}+\overline{c}_{2233}-2l\theta}{\overline{q}_{333}-r'\theta}\\ &= -\frac{\overline{e}_{311}+\overline{e}_{322}-2q\theta}{\overline{\alpha}_{33}-\beta\theta} = -\frac{\overline{q}_{311}+\overline{q}_{322}-2q'\theta}{\overline{\mu}_{33}-\langle w\rangle},\end{aligned} \tag{3.27}$$

$$\frac{\overline{c}_{1211}+\overline{c}_{1222}}{2k} = \frac{\overline{c}_{3312}}{l} = \frac{\overline{e}_{312}}{q} = \frac{\overline{q}_{312}}{q'} = \left\langle \partial_{\alpha,y} w_\alpha^{12}\right\rangle_*. \tag{3.28}$$

Expressions (3.25)–(3.28) are universal relations for monoclinic MEE empty fibre composites with a 6-mm-class MEE matrix.

For the particular case of a transversely isotropic global behaviour, these formulae coincide with the universal relations reported in (**9**; **33**; **34**) (see also (**11**, Appendix A)) for two-phase fibrous composites as the elastic, piezoelectric, piezomagnetic and ME fibre properties are equal to zero.

According to the extended Hill's notation and assuming transversely isotropic overall behaviour, from (3.25) to (3.27), it is straightforward to obtain the following relationships:

$$\frac{k}{l} = \frac{\overline{k}-k\theta}{\overline{l}-l\theta} = \frac{\overline{l}-l\theta}{\overline{n}-n\theta} = \frac{\overline{q}-q\theta}{\overline{r}-r\theta} = \frac{\overline{q'}-q'\theta}{\overline{r'}-r'\theta}, \tag{3.29}$$

$$\frac{k}{q} = \frac{\overline{k}-k\theta}{\overline{q}-q\theta} = \frac{\overline{l}-l\theta}{\overline{r}-r\theta} = -\frac{\overline{q}-q\theta}{\overline{u}-\langle u\rangle} = -\frac{\overline{q'}-q'\theta}{\overline{\beta}-\beta\theta}, \tag{3.30}$$

$$\frac{k}{q'} = \frac{\overline{k}-k\theta}{\overline{q'}-q'\theta} = \frac{\overline{l}-l\theta}{\overline{r'}-r'\theta} = -\frac{\overline{q'}-q'\theta}{\overline{w}-\langle w\rangle} = -\frac{\overline{q}-q\theta}{\overline{\beta}-\beta\theta}. \tag{3.31}$$

From (3.29) to (3.31), it is clear that the knowledge of 1 of the 10 effective coefficients yields the other 9 effective coefficients.

3.3 *New universal relations involving effective coefficients from* L_1^{13}, L_2^1 *and* L_3^1 (L_1^{23}, L_2^2 *and* L_3^2)

The next step is to describe the derivation of new universal relations involving solutions of the local problems L_1^{13}, L_2^1 and L_3^1 (L_1^{23}, L_2^2 and L_3^2) without solving any local problem.

The problems L_1^{13}, L_2^1 and L_3^1 can be presented in a compact form as one problem L_j, which consists in finding the Y-periodic local functions $U_j(\mathbf{y})$, $\Phi_j(\mathbf{y})$ and $\Psi_j(\mathbf{y})$, harmonic and of zero average on Y, that satisfy the following free boundary conditions on ∂S:

$$\mathbf{A}\mathbf{X}_j = -n_1\mathbf{a}_j, \tag{3.32}$$

where

$$\mathbf{A} = \begin{pmatrix} p & e & e' \\ e & -t & -\alpha \\ e' & -\alpha & -\nu \end{pmatrix} \quad \text{and} \quad \mathbf{X}_j = \begin{pmatrix} \partial_{\delta,y} U_j n_\delta \\ \partial_{\delta,y} \Phi_j n_\delta \\ \partial_{\delta,y} \Psi_j n_\delta \end{pmatrix}.$$

Thus, $\mathbf{A}$ is a symmetric matrix, $\mathbf{X}_j$ is a column matrix involving the normal derivatives to ∂S of the local functions and $\mathbf{a}_j$ is the jth column of the matrix $\mathbf{A}$. From (3.32), the next expression on ∂S can be obtained:

$$\mathbf{X}_j = -n_1\mathbf{e}_j, \tag{3.33}$$

where $\mathbf{e}_j$ are the vectors of the standard orthonormal basis for the real 3D Euclidean space. As a consequence of conditions (3.33), the following relations among the solutions of the local problems L_1^{13}, L_2^1 and L_3^1 are obtained:

$$w_3^{13} = \pi^1 = \gamma^1, \quad \zeta^{13} = \chi^1 = 0, \quad \eta^{13} = \xi^1 = 0, \quad g_3^1 = f_3^1 = 0. \tag{3.34}$$

The problems L_1^{23}, L_2^2 and L_3^2 also can be analyzed as above with the particularity that the free boundary condition (3.33) now follows as $\mathbf{X}_j = -n_2\mathbf{e}_j$. Consequently, the following relations are valid for the solutions of L_1^{23}, L_2^2 and L_3^2:

$$w_3^{23} = \pi^2 = \gamma^2, \quad \zeta^{23} = \chi^2 = 0, \quad \eta^{23} = \xi^2 = 0, \quad g_3^2 = f_3^2 = 0. \tag{3.35}$$

On the other hand, taking into account the material properties of the matrix, expanding (2.11), the following formulae for effective properties can be derived:

$$\bar{c}_{1313} = p\theta + p\left\langle \partial_{1,y} w_3^{13} \right\rangle_* + e\left\langle \partial_{1,y}\zeta^{13} \right\rangle_* + e'\left\langle \partial_{1,y}\eta^{13} \right\rangle_*, \tag{3.36}$$

$$\bar{c}_{2323} = p\theta + p\left\langle \partial_{2,y} w_3^{23} \right\rangle_* + e\left\langle \partial_{2,y}\zeta^{23} \right\rangle_* + e'\left\langle \partial_{2,y}\eta^{23} \right\rangle_*, \tag{3.37}$$

$$\bar{c}_{2313} = p\left\langle \partial_{2,y} w_3^{13} \right\rangle_* = p\left\langle \partial_{1,y} w_3^{23} \right\rangle_*, \tag{3.38}$$

$$\bar{e}_{113} = e\theta + e\left\langle \partial_{1,y} w_3^{13} \right\rangle_* - t\left\langle \partial_{1,y}\zeta^{13} \right\rangle_* - \alpha\left\langle \partial_{1,y}\eta^{13} \right\rangle_*$$

$$= e\theta + p\left\langle \partial_{1,y} g_3^1 \right\rangle_* + e\left\langle \partial_{1,y}\pi^1 \right\rangle_* + e'\left\langle \partial_{1,y}\xi^1 \right\rangle_*, \tag{3.39}$$

$$\bar{e}_{223} = e\theta + p\left\langle \partial_{2,y} w_3^{23} \right\rangle_* - t\left\langle \partial_{2,y}\zeta^{23} \right\rangle_* - \alpha\left\langle \partial_{2,y}\eta^{23} \right\rangle_*$$

$$= e\theta + p\left\langle \partial_{2,y} g_3^2 \right\rangle_* + e\left\langle \partial_{2,y}\pi^2 \right\rangle_* + e'\left\langle \partial_{2,y}\xi^2 \right\rangle_*, \tag{3.40}$$

$$\bar{e}_{213} = p\left\langle \partial_{2,y} w_3^{13} \right\rangle_* - t\left\langle \partial_{2,y}\zeta^{13} \right\rangle_* - \alpha\left\langle \partial_{2,y}\eta^{13} \right\rangle_*$$

$$= p\left\langle \partial_{1,y} g_3^2 \right\rangle_* + e\left\langle \partial_{1,y}\pi^2 \right\rangle_* + e'\left\langle \partial_{1,y}\xi^2 \right\rangle_*, \tag{3.41}$$

$$\bar{e}_{123} = p\left\langle \partial_{1,y} w_3^{23} \right\rangle_* - t\left\langle \partial_{1,y}\zeta^{23} \right\rangle_* - \alpha\left\langle \partial_{1,y}\eta^{23} \right\rangle_*$$

$$= p\left\langle \partial_{2,y} g_3^1 \right\rangle_* + e\left\langle \partial_{2,y}\pi^1 \right\rangle_* + e'\left\langle \partial_{2,y}\xi^1 \right\rangle_*, \tag{3.42}$$

$$\bar{q}_{113} = e'\theta + e'\left\langle \partial_{1,y} w_3^{13} \right\rangle_* - \alpha\left\langle \partial_{1,y}\zeta^{13} \right\rangle_* - \nu\left\langle \partial_{1,y}\eta^{13} \right\rangle_*$$

$$= e'\theta + p\left\langle \partial_{1,y} f_3^1 \right\rangle_* + e\left\langle \partial_{1,y}\chi^1 \right\rangle_* + e'\left\langle \partial_{1,y}\gamma^1 \right\rangle_*, \tag{3.43}$$

$$\bar{q}_{223} = e'\theta + e'\left\langle \partial_{2,y} w_3^{23} \right\rangle_* - \alpha\left\langle \partial_{2,y}\zeta^{23} \right\rangle_* - \nu\left\langle \partial_{2,y}\eta^{23} \right\rangle_*$$

$$= e'\theta + p\left\langle \partial_{2,y} f_3^2 \right\rangle_* + e\left\langle \partial_{2,y}\chi^2 \right\rangle_* + e'\left\langle \partial_{2,y}\gamma^2 \right\rangle_*, \tag{3.44}$$

$$\bar{q}_{213} = e'\left\langle \partial_{2,y} w_3^{13} \right\rangle_* - \alpha\left\langle \partial_{2,y}\zeta^{13} \right\rangle_* - \nu\left\langle \partial_{2,y}\eta^{13} \right\rangle_*$$

$$= p\left\langle \partial_{1,y} f_3^2 \right\rangle_* + e\left\langle \partial_{1,y}\chi^2 \right\rangle_* + e'\left\langle \partial_{1,y}\gamma^2 \right\rangle_*, \tag{3.45}$$

$$\bar{q}_{123} = e'\left\langle \partial_{1,y} w_3^{23} \right\rangle_* - \alpha\left\langle \partial_{1,y}\zeta^{23} \right\rangle_* - \nu\left\langle \partial_{1,y}\eta^{23} \right\rangle_*$$

$$= p\left\langle \partial_{2,y} f_3^1 \right\rangle_* + e\left\langle \partial_{2,y}\chi^1 \right\rangle_* + e'\left\langle \partial_{2,y}\gamma^1 \right\rangle_*, \tag{3.46}$$

$$\bar{\kappa}_{11} = \langle t \rangle - e\left\langle \partial_{1,y} g_3^1 \right\rangle_* + t\left\langle \partial_{1,y}\pi^1 \right\rangle_* + \alpha\left\langle \partial_{1,y}\xi^1 \right\rangle_*, \tag{3.47}$$

$$\bar{\kappa}_{22} = \langle t \rangle - e\left\langle \partial_{2,y} g_3^2 \right\rangle_* + t\left\langle \partial_{2,y}\pi^2 \right\rangle_* + \alpha\left\langle \partial_{2,y}\xi^2 \right\rangle_*, \tag{3.48}$$

$$\bar{\kappa}_{12} = t\left\langle \partial_{2,y}\pi^1 \right\rangle_* - e\left\langle \partial_{2,y} g_3^1 \right\rangle_* + \alpha\left\langle \partial_{2,y}\xi^1 \right\rangle_*$$

$$= t\left\langle \partial_{1,y}\pi^2 \right\rangle_* - e\left\langle \partial_{1,y} g_3^2 \right\rangle_* + \alpha\left\langle \partial_{1,y}\xi^2 \right\rangle_*, \tag{3.49}$$

$$\overline{\mu}_{11} = \langle \nu \rangle + \nu \left\langle \partial_{1,y}\gamma^{1} \right\rangle_{*} - e' \left\langle \partial_{1,y} f_3^1 \right\rangle_{*} + \alpha \left\langle \partial_{1,y}\chi^{1} \right\rangle_{*}, \tag{3.50}$$

$$\overline{\mu}_{22} = \langle \nu \rangle + \nu \left\langle \partial_{2,y}\gamma^{2} \right\rangle_{*} - e' \left\langle \partial_{2,y} f_3^2 \right\rangle_{*} + \alpha \left\langle \partial_{2,y}\chi^{2} \right\rangle_{*}, \tag{3.51}$$

$$\begin{aligned} \overline{\mu}_{12} &= \nu \left\langle \partial_{2,y}\gamma^{1} \right\rangle_{*} - e' \left\langle \partial_{2,y} f_3^1 \right\rangle_{*} + \alpha \left\langle \partial_{2,y}\chi^{1} \right\rangle_{*} \\ &= \nu \left\langle \partial_{1,y}\gamma^{2} \right\rangle_{*} - e' \left\langle \partial_{1,y} f_3^2 \right\rangle_{*} + \alpha \left\langle \partial_{1,y}\chi^{2} \right\rangle_{*}, \end{aligned} \tag{3.52}$$

$$\begin{aligned} \overline{\alpha}_{11} &= \alpha\theta + \alpha \left\langle \partial_{1,y}\pi^{1} \right\rangle_{*} - e' \left\langle \partial_{1,y} g_3^1 \right\rangle_{*} + \nu \left\langle \partial_{1,y}\xi^{1} \right\rangle_{*} \\ &= \alpha\theta + \alpha \left\langle \partial_{1,y}\gamma^{1} \right\rangle_{*} - e \left\langle \partial_{1,y} f_3^1 \right\rangle_{*} + t \left\langle \partial_{1,y}\chi^{1} \right\rangle_{*}, \end{aligned} \tag{3.53}$$

$$\begin{aligned} \overline{\alpha}_{22} &= \alpha\theta + \alpha \left\langle \partial_{2,y}\pi^{2} \right\rangle_{*} - e' \left\langle \partial_{2,y} g_3^2 \right\rangle_{*} + \nu \left\langle \partial_{2,y}\xi^{2} \right\rangle_{*} \\ &= \alpha\theta + \alpha \left\langle \partial_{2,y}\gamma^{2} \right\rangle_{*} - e \left\langle \partial_{2,y} f_3^2 \right\rangle_{*} + t \left\langle \partial_{2,y}\chi^{2} \right\rangle_{*}, \end{aligned} \tag{3.54}$$

$$\begin{aligned} \overline{\alpha}_{12} &= \alpha \left\langle \partial_{2,y}\pi^{1} \right\rangle_{*} - e' \left\langle \partial_{2,y} g_3^1 \right\rangle_{*} + \nu \left\langle \partial_{2,y}\xi^{1} \right\rangle_{*} \\ &= \alpha \left\langle \partial_{1,y}\pi^{2} \right\rangle_{*} - e' \left\langle \partial_{1,y} g_3^2 \right\rangle_{*} + \nu \left\langle \partial_{1,y}\xi^{2} \right\rangle_{*} \\ &= \alpha \left\langle \partial_{1,y}\gamma^{2} \right\rangle_{*} - e \left\langle \partial_{1,y} f_3^2 \right\rangle_{*} + t \left\langle \partial_{1,y}\chi^{2} \right\rangle_{*} \\ &= \alpha \left\langle \partial_{2,y}\gamma^{1} \right\rangle_{*} - e \left\langle \partial_{2,y} f_3^1 \right\rangle_{*} + t \left\langle \partial_{2,y}\chi^{1} \right\rangle_{*}. \end{aligned} \tag{3.55}$$

Substituting (3.34) and (3.35) into (3.36)–(3.55), the following new relations connecting the so-called anti-plane effective coefficients are derived

$$\begin{aligned} &\frac{\overline{c}_{1313} - p\theta}{p} = \frac{\overline{e}_{113} - e\theta}{e} = \frac{\overline{q}_{113} - e'\theta}{e'} = \frac{\overline{\alpha}_{11} - \alpha\theta}{\alpha} = \frac{\overline{\kappa}_{11} - \langle t \rangle}{t} = \frac{\overline{\mu}_{11} - \langle \nu \rangle}{\nu} \\ &\qquad = \left\langle \partial_{1,y} w_3^{13} \right\rangle_{*}, \end{aligned} \tag{3.56}$$

$$\begin{aligned} &\frac{\overline{c}_{2323} - p\theta}{p} = \frac{\overline{e}_{223} - e\theta}{e} = \frac{\overline{q}_{223} - e'\theta}{e'} = \frac{\overline{\alpha}_{22} - \alpha\theta}{\alpha} = \frac{\overline{\kappa}_{22} - \langle t \rangle}{t} = \frac{\overline{\mu}_{22} - \langle \nu \rangle}{\nu} \\ &\qquad = \left\langle \partial_{2,y} w_3^{23} \right\rangle_{*}, \end{aligned} \tag{3.57}$$

$$\frac{\overline{c}_{1323}}{p} = \frac{\overline{e}_{123}}{e} = \frac{\overline{e}_{213}}{e} = \frac{\overline{q}_{123}}{e'} = \frac{\overline{q}_{213}}{e'} = \frac{\overline{\alpha}_{12}}{\alpha} = \frac{\overline{\kappa}_{12}}{t} = \frac{\overline{\mu}_{12}}{\nu} = \left\langle \partial_{2,y} w_3^{13} \right\rangle_{*}, \tag{3.58}$$

where $\langle t \rangle = t\theta + \kappa_0(1-\theta)$ and $\langle \nu \rangle = \nu\theta + \mu_0(1-\theta)$ are averages of the corresponding property taking into account the influence of the free space permittivity κ_0 and permeability μ_0.

Equations (3.56)–(3.58) are true for any shape of the empty fibre cross section. These relations are an extension of recent results reported by (**14**) where the case of piezoelectric matrix with a transversely isotropic effective behaviour was studied.

The problem of computing the anti-plane properties, of the 3-1 longitudinally porous MEE media here considered, for a specific geometry of the empty fibre cross section can be reduced to the calculation of $\left\langle \partial_{1,y} w_3^{13} \right\rangle_*$, $\left\langle \partial_{2,y} w_3^{23} \right\rangle_*$ and $\left\langle \partial_{1,y} w_3^{23} \right\rangle_*$. A brief description on the derivation of a closed-form formula for $\left\langle \partial_{1,y} w_3^{13} \right\rangle_*$ in the case of a circular cross section of ∂S for two types of periodic cells can be found in (**14**, Appendix B). In (**35**), an extension of this procedure for a general parallelogram periodic cell is explained. Also, numerical results from finite element calculations of $\left\langle \partial_{1,y} w_3^{13} \right\rangle_*$ for different geometries of ∂S are shown in (**14**, Section 5). It can be noted that simpler relations between effective coefficients can be derived in an obvious manner from (3.56) to (3.57). They become

$$\frac{\overline{c}_{1313}}{p} = \frac{\overline{e}_{113}}{e} = \frac{\overline{q}_{113}}{e'} = \frac{\overline{\alpha}_{11}}{\alpha}, \tag{3.59}$$

$$\frac{\overline{c}_{2323}}{p} = \frac{\overline{e}_{223}}{e} = \frac{\overline{q}_{223}}{e'} = \frac{\overline{\alpha}_{22}}{\alpha}, \tag{3.60}$$

$$\overline{\kappa}_{11} = t\frac{\overline{c}_{1313}}{p} + \kappa_0(1-\theta), \tag{3.61}$$

$$\overline{\kappa}_{22} = t\frac{\overline{c}_{2323}}{p} + \kappa_0(1-\theta), \tag{3.62}$$

$$\overline{\mu}_{11} = \nu\frac{\overline{c}_{1313}}{p} + \mu_0(1-\theta), \tag{3.63}$$

$$\overline{\mu}_{22} = \nu\frac{\overline{c}_{2323}}{p} + \mu_0(1-\theta). \tag{3.64}$$

It can be observed from (3.61) to (3.64) that the influence of the free space properties could be more noticeable for high values of the empty fibre area fraction $1-\theta$.

In Table 3 of (**35**, p. 81), the variations of the effective longitudinal shear moduli ($\overline{c}_{1323}$, $\overline{c}_{1313}$ and $\overline{c}_{2323}$) are shown for four different parallelogram periodic cells with the small angle equal to $\pi/6$, $\pi/4$ (rhombic cell), $\pi/3$ (hexagonal cell) and $\pi/2$ (square cell). Furthermore, one can see from relations (3.58) to (3.60) that the knowledge of the normalized elastic effective coefficients yields to the other 15 anti-plane effective coefficients. For instance, in Table 1, the normalized effective values ($\overline{c}_{1323}/p$, $\overline{c}_{1313}/p$ and $\overline{c}_{2323}/p$) reported in Table 3 of (**35**) are summarized, with the particularity that now these values are valid for the further involved normalized MEE anti-plane effective coefficients in (3.58)–(3.60).

3.4 *An application to obtain explicit formulae for effective coefficients*

In this section, the derivation of analytical formulae for effective properties of 3-1 longitudinally porous MEE materials is described. The cross section of the empty fibre is considered as a circle of

Table 1 Normalized effective values of the MEE properties involved in relations (3.58)–(3.60) for four different parallelogram periodic cells with the small angle, respectively, equal to $\pi/6$, $\pi/4$ (rhombic cell), $\pi/3$ (hexagonal cell) and $\pi/2$ (square cell)

$1-\theta$	$\pi/6$ (3.58)	$\pi/4$ (3.58)	$\pi/6$ (3.59)	$\pi/4$ (3.59)	$\pi/6$ (3.60)	$\pi/4$ (3.60)	$\pi/3$ (3.60)	$\pi/2$ (3.60)
0.1	−0.00796	−0.00327	0.8030	0.8148	0.8306	0.8213	0.8182	0.8182
0.2	−0.02736	−0.0110	0.6098	0.6548	0.7046	0.6768	0.6667	0.6665
0.3	−0.05522	−0.02122	0.4127	0.5146	0.6040	0.5570	0.5384	0.5376
0.4	−0.09991	−0.03297	0.1691	0.3891	0.5152	0.4551	0.4284	0.4254

radius R. The periodic cells are assumed hexagonal or square. It is known that the global behaviour for the case of a square cell belongs to the tetragonal symmetry 4 mm, whereas the other one has hexagonal symmetry 6 mm. The relationships given by (3.29)–(3.31) and (3.59)–(3.64) are valid for these two types of composites. The results which will be given here are a natural extension of those reported in (**14**).

From expressions (3.30) and (3.31), the following relation can be obtained

$$\frac{\overline{q'} - q'\theta}{q'} = \frac{\overline{q} - q\theta}{q}. \tag{3.65}$$

Formula (36) from (**14**, p. 798) can be written as

$$\frac{\overline{q} - q\theta}{q} = -(1-\theta)\frac{k}{m}K(a), \tag{3.66}$$

where $K(a) = \frac{1}{2}\theta/G + \frac{1}{4}(1+\kappa)G^{-2}V^T M^{-1}\tilde{V}$, $\theta = 1 - \pi R^2/\sin(\pi/a)$ (with $a = 2$ or 3 for square or hexagonal array, respectively), $\kappa = 1 + 2m/k$ and $G = \frac{1}{2} + (1-\theta)/(\kappa - 1)$. The superscripts T and -1 denote transpose vector and inverse matrix, respectively. The components of the infinite order vectors $V(v_s)$, $\tilde{V}(\tilde{v}_t)$ and matrix $M(m_{ts})$ as a function of a can be found in (**14**, Appendix A). However, truncating this product at a second order, one can obtain the following simple analytical expression for $K(a)$

$$K(a) = \frac{\theta}{2G} + \frac{1+\kappa}{4G^2}\,\frac{(2a-1)R^{4a}(S_{2a})^2}{1 + R^{4a-2}[g(a) - r(a) - (2a-1)R^2(S_{2a})^2/G]}, \tag{3.67}$$

where

$$r(a) = c_{4a-1}^{2a-1}c_{4a-1}^{2a+1}R^{4a+2}(S_{4a})^2, \quad g(a) = -(2a-1)(R^2 c_{4a}^{2a} S_{4a} - c_{4a-2}^{2a-1}T_{4a-1}),$$

with $c_k^l = k!/l!(k-l)!$ being the combinatory number and the type of periodicity is determined by the lattice sums S and T ($S_4 = 3.1512120$, $S_8 = 4.2557731$ and $T_7 = 4.5155155$ for $a = 2$ and $S_6 = 5.8630316$, $S_{12} = 6.00096399$ and $T_{11} = 6.0301854$ for $a = 3$).

From (3.65) and (3.66), the piezomagnetic effective coefficient q' is given by the formula

$$\overline{q'} = q'\theta - (1-\theta)q'(k/m)K(a). \tag{3.68}$$

Combining the convenient fractions involved in (3.29)–(3.31) with (3.65) and (3.68), the following analytical expressions for the effective properties can be obtained:

$$\overline{r'} = r'\theta - (1-\theta)\frac{q'l}{m}K(a), \tag{3.69}$$

$$\overline{\beta} = \beta\theta + (1-\theta)\frac{q'q}{m}K(a), \tag{3.70}$$

$$\overline{w} = \langle w\rangle + (1-\theta)\frac{(q')^2}{m}K(a), \tag{3.71}$$

$$\overline{u} = \langle u\rangle + (1-\theta)\frac{q^2}{m}K(a). \tag{3.72}$$

The formula for the axial ME effective coefficient, $\overline{\beta}$, (3.70), reveals the existence of the ME effect in the composite even if none of the individual phases exhibit it.

The meaning of truncation at a second order is the following: the analytical solution of the local problems using elements of the theory of complex functions (see, for instance, (**14**; **35**)) involves the solution of systems of infinite linear algebraic equations involving undetermined constants. These systems require to be truncated. In the present work, if the system is reduced to one equation with one unknown, we say that the corresponding effective properties are 'first-order approximations'. Furthermore, the infinite system can be truncated as a finite system of 2×2, then we call it 'second-order approximation'. For instance, if $K(a) = \theta/2G$, then (3.68)–(3.72) are first-order approximations, and in this particular case, they coincide with the formulae of (**4**) (taking the fibre properties equal to zero).

In a similar way, from formulae (35) and (38) of (**14**, p. 798), one can obtain

$$\overline{p}/p = \overline{e}/e = 1 - 2(1-\theta)P(a), \tag{3.73}$$

where the truncated expression for the second-order approximation of $P(a)$ is

$$P(a) = \left\{2 - \theta - (2a-1)R^{4a}(S_{2a})^2\right\}^{-1}. \tag{3.74}$$

Now, combining (3.73) with (3.59)–(3.63), we obtain the following simple expressions for the anti-plane effective coefficients

$$\overline{p} = p - 2p(1-\theta)P(a), \tag{3.75}$$

$$\overline{e} = e\overline{p}/p, \tag{3.76}$$

$$\overline{e'} = e'\overline{p}/p, \tag{3.77}$$

$$\overline{\alpha} = \alpha\overline{p}/p, \tag{3.78}$$

$$\overline{t} = t\overline{p}/p + \kappa_0(1-\theta), \tag{3.79}$$

$$\overline{\nu} = \nu\overline{p}/p + \mu_0(1-\theta), \tag{3.80}$$

with $\overline{p} = \overline{c}_{1313} = \overline{c}_{2323}$, $\overline{e} = \overline{e}_{113} = \overline{e}_{223}$, $\overline{e'} = \overline{q}_{113} = \overline{q}_{2233}$, $\overline{\alpha} = \overline{\alpha}_{11} = \overline{\alpha}_{22}$, $\overline{t} = \overline{\kappa}_{11} = \overline{\kappa}_{22}$ and $\overline{\nu} = \overline{\mu}_{11} = \overline{\mu}_{22}$. Expressions (3.75)–(3.80) are first-order approximations as $P(a) = 1/(2-\theta)$.

On the other hand, the limit case of the Mori–Tanaka model given by (**36**, Equation (22)) can be obtained from the first-order approximations (3.75) to (3.80) neglecting the influence of free space permittivity and permeability. Indeed, after that, such limit case is

$$\frac{\overline{p}}{p} = \frac{\overline{e}}{e} = \frac{\overline{e'}}{e'} = \frac{\overline{\alpha}}{\alpha} = \frac{\overline{t}}{t} = \frac{\overline{\nu}}{\nu} = \frac{\theta}{2-\theta}. \tag{3.81}$$

3.5 *Two-scale homogenization with a numerical solution of the cell problem*

The previous analytical formulae have been derived for the case of MEE periodically perforated structure with a particular anisotropy of the matrix properties (transverse isotropy) and a simple morphology of the holes (parallel empty cylinders). In the context of microstructural optimization for the enhancement of the ME coupling coefficient, it is of great interest to be able to consider more general situations in terms of anisotropic properties of the constituents as well as their morphology and spatial distribution. In general, the problem is too complex to be solved analytically and it is necessary to resort to a numerical approach. In this context, an attractive alternative to the widely used finite element method is the fast Fourier transform (FFT) homogenization scheme. This approach, originally proposed in (**37**) for elastic-type composites (uncoupled constitutive relation), has been extended to deal with multifield coupling behaviours (**38**; **39**). It relies on an efficient iterative resolution of periodic coupled Lippmann–Schwinger equations. Besides, an attractive feature of the method is the use of the digital image of the microstructure to perform the computations (no additional meshing is required). This allows in practice to consider arbitrary shapes and spatial arrangements of the constituents in the unit cell. In the present context of voided MEE materials, use is made of a constrained minimization approach, which makes use of the augmented Lagrangian. For further details, the reader is referred to (**39**) and references therein. The accuracy of the results obtained with this numerical approach is illustrated in section 4.

4. Numerical examples

A numerical application of the analytical formulae derived in section 3.4 is now given together with the results of the FFT-based numerical homogenization scheme. With this latter procedure, the effective coefficients of interest have been determined by considering successively four electromagnetic loadings, namely, $\partial_1\varphi$, $\partial_3\varphi$, $\partial_1\psi$ and $\partial_3\psi$. The comparison between these two approaches is shown in Figs 2 to 4. The composite considered is a 3-1 longitudinally porous piezoelectric material with a square periodic distributions of empty fibres. The material properties used to calculate the results shown were taken from (**40**, Table 3) and are as follows: (in G Pa) $k = 229.5$, $m = 56.5$, $l = 78$, $n = 269.5$, $p = 43.5$; (in $\mathrm{C/m^2}$) $e = 11.6$, $q = -4.4$, $r = 18.6$; (in m/A) $e' = 550$, $q' = 580.3$, $r' = 699.7$; (in $10^{-10}\mathrm{C^2/N\,m^2}$) $t = 0.8$, $u = 0.93$; (in $10^{-6}\mathrm{N\,s^2/C^2}$) $\upsilon = -590$, $w = 157$ and (in $10^{-12}\mathrm{N\,s/V\,C}$) $\beta = 3$. A very good agreement can be observed, for all the range of empty fibre area fractions, among the analytical and numerical results corresponding to the piezoelectric, piezomagnetic, dielectric, magnetic and ME effective coefficients.

From Fig. 4, we can note the enhancement of the ME effective constant β with the empty fibre area fraction. This is due to the interaction between the product piezoelectric-piezomagnetic property (qq') with area of the hole $(1-\theta)$, which can be seen in (3.70). As a consequence, it is worth noting that the perforated matrix exhibits a ME coefficient, which is three orders of magnitude higher

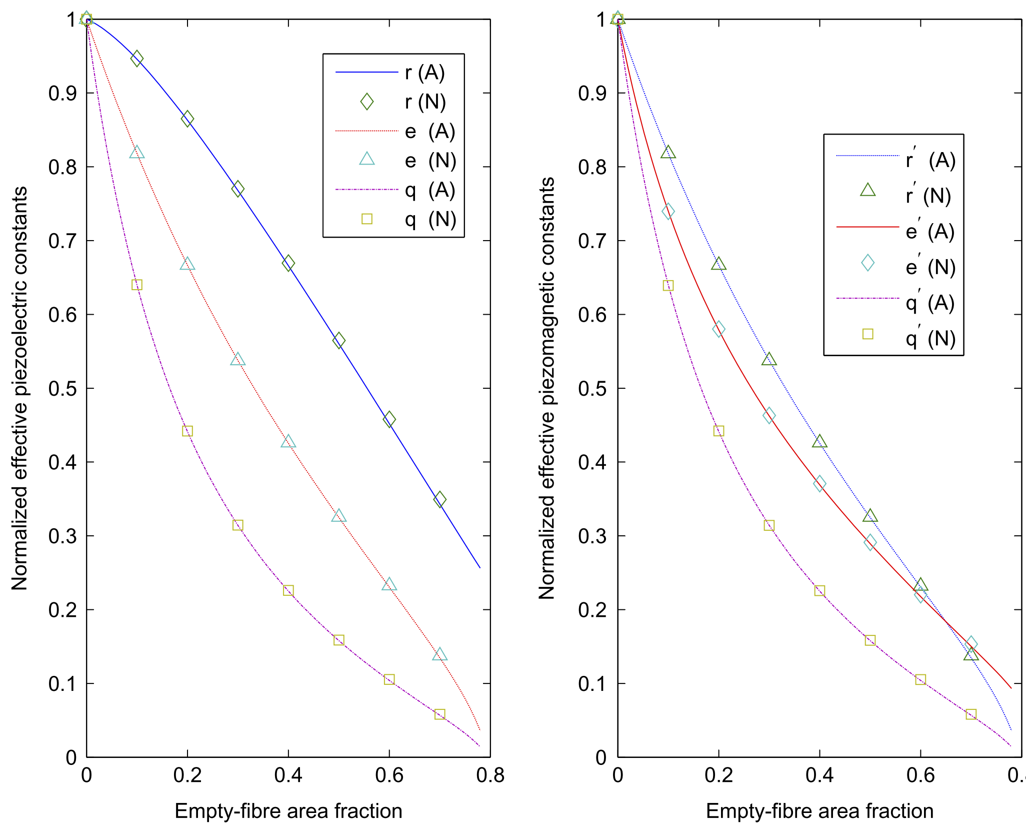

Fig. 2 The normalized piezoelectric and piezomagnetic effective properties for a 3-1 longitudinally porous MEE composite with a square periodic distribution of empty fibres. Comparisons among the results derived from the analytical formulae (A) with those derived by the FFT numerical method (N)

than the matrix itself. As one can observe in Fig. 4, the dotted line (Li and Dunn (**4**)) and the dashed (first order) are indistinguishable. This is because the first-order curve is obtained using (3.70) but neglecting the second term of (3.67) in order to obtain $\overline{\beta} = \theta\beta + qq'(1-\theta)\theta/((1-\theta)k+m)$, which is exactly the formula proposed by Li and Dunn in (**4**) replacing the fibre properties by zero. The agreement between the first- and second-order approximations is excellent for small values of the empty fibre area fraction $1-\theta$. Consequently, after neglecting the first term in the above expression, it is possible to estimate the specific fraction where the absolute minimum of $\overline{\beta}$ is attained by using the expression

$$(1-\theta)_{\min} \approx ((m^2+km)^{\frac{1}{2}} - m)/k. \tag{4.1}$$

In the present numerical example, using (4.1), one obtain $(1-\theta)_{\min} \approx 0.31$, which is in correspondence with Fig. 4.

Another example concerns the application of the universal relations derived in section 3.3. For instance, the values appearing in the second row of Table 2 are the same as those reported in Table 2

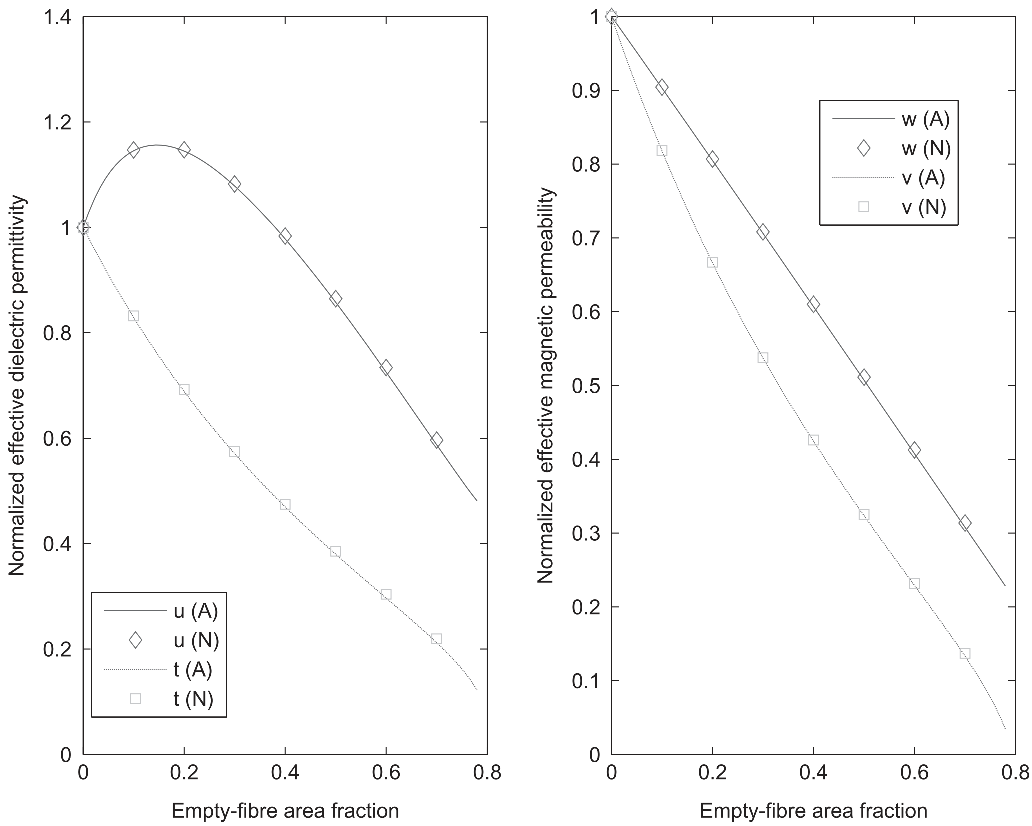

Fig. 3 The normalized (dielectric permittivity and magnetic permeability) effective properties for a 3-1 longitudinally porous MEE composite with a square periodic distribution of empty fibres. Comparisons among the results derived from the analytical formulae (A) with those derived by the FFT numerical method (N)

of (**41**, p. 4777). From these data and using the first relation of proportionality of (3.59), $\overline{e} = e\overline{p}/p$, one can obtain the values of the piezoelectric coefficient listed in the third row. As we can observe, the values on the second row compare well with those of the third row corresponding to results of (**41**). A similar picture is shown for the dielectric effective coefficients from the subsequent rows, where a better agreement is noted when the information involving the property of the free space for dielectric permittivity is taken into account, 0.2 being the higher percentage error.

5. Conclusions

In this paper, we have rigorously established the limiting equation modelling the behaviour of MEE periodically perforated structures: we have explicitly described the homogenized coefficients of the elastic, dielectric, magnetic, piezoelectric, piezomagnetic and ME coupling tensors. Based on the homogenization model, a class of unidirectional perforated composites formed by empty fibres periodically distributed in a homogeneous transversely isotropic MEE material was studied. Without

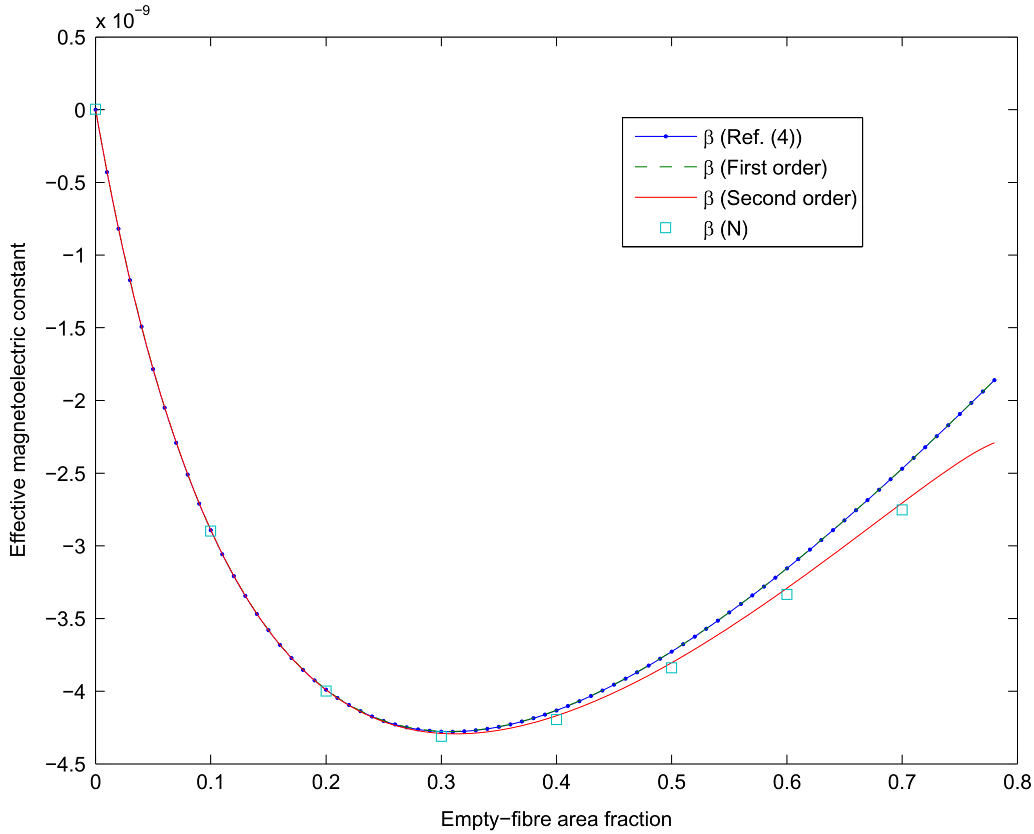

Fig. 4 The ME effective coefficient β for a 3-1 longitudinally porous MEE composite with a square periodic distribution of empty fibres. Comparisons between the present results (first and second order) with those of Li and Dunn (**4**) and those derived from the FFT numerical method (N)

Table 2 Comparisons of effective piezoelectric and dielectric values reported in Table 2 on p. 4792 of (**41**) with those obtained from (3.56)

$1-\theta$	0	0.1	0.3	0.5	0.6	0.8	0.9
$\overline{p}$ (G Pa)	21.10	17.264	11.362	7.033	5.275	2.345	1.111
$\overline{e} = e\overline{p}/p$ (C/m^2)	12.30	10.064	6.623	4.099	3.075	1.367	0.647
$\overline{e}$ (**41**)	12.30	10.064	6.623	4.100	3.075	1.367	0.647
$\overline{t} = t\overline{p}/p$ (nC^2/Nm^2)	8.107	6.633	4.365	2.702	2.028	0.901	0.427
$\overline{t} = t\overline{p}/p + \kappa_0(1-\theta)$	8.107	6.634	4.368	2.707	2.032	0.908	0.435
$\overline{t}$ (**41**)	8.107	6.636	4.372	2.710	2.035	0.910	0.436

the solution of any local problem, universal relations for monoclinic MEE empty fibres composites with a 6-mm-class MEE matrix were found. From the so-called plane local problems, the corresponding universal relations are given by (3.25)–(3.28). From these expressions, the known universal relations (3.29)–(3.31) of Benveniste (**9**; **34**) are obtained for the particular case of transversely isotropic effective behaviour.

From the stated anti-plane shear MEE local problems, new relationships (3.58)–(3.60) involving 18 effective properties were derived. Based on these simple equations, the knowledge, for instance, of the normalized elastic effective values ($\overline{c}_{1323}/p$, $\overline{c}_{1313}/p$ and $\overline{c}_{2323}/p$) allows us to determine the other 15 effective coefficients involved. Such relations are truly unrelated to the geometry of the cross section of the empty fibres. The derived formulae involve the dielectric permittivity and the magnetic permeability of free space, (3.61) and (3.64).

For the particular case of circular cross section of empty fibres and for two types of periodic distribution (hexagonal and square arrays), new unified analytical formulae for the effective properties piezomagnetic (3.68)–(3.69) and (3.77), ME (3.70) and (3.78), and magnetic permeability effective constant (3.71), were obtained. The formulae proposed by Li and Dunn (**4**) (considering null fibre properties) coincide with the first-order approximation (3.68)–(3.72). The accuracy of this approximation for small values (less than 0.4) of the empty fibre area fraction is illustrated in Fig. 4. A simple formula (4.1) is given to estimate the specific empty fibre area fraction where the ME effective coefficient reaches its minimum value. Also, very simple analytical expressions for the anti-plane effective coefficients (3.75)–(3.80) were derived. The first-order approximation of these formulae without the influence of the permittivity and permeability of free space corresponds with the 'long fibre' limit of the Mori–Tanaka model given by formula (22) in (**36**). This result is given by (3.81) which in a compact form is $\overline{\mathbf{A}} = (\theta/[2-\theta])\mathbf{A}$, with $1-\theta$ being the area of the hole and $\mathbf{A}$ the symmetric matrix defined in section 3.3.

The analytical formulae here proposed have been used to check the accuracy of a FFT-based numerical scheme for general multifield coupling behaviours and the comparisons performed have shown a very good agreement for the whole range of porosity values. Besides, our results offer a theoretical fact for obtaining a high ME coefficient in a monolithic MEE material with holes. For future prospects, the proposed numerical scheme is expected to be a useful tool for the study of the enhancement of the ME coefficient in perforated structures by microstructural design.

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