UNIVERSAL RELATIONS IN ISOTROPIC NONLINEAR MAGNETOELASTICITY

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Summary

In this paper we first summarize the basic constitutive equations for (nonlinear) magnetoelastic solids capable of large deformations. Equivalent formulations are given using either the magnetic induction vector or the magnetic field vector as the independent magnetic variable in addition to the deformation gradient. The constitutive equations are then specialized to incompressible, isotropic magnetoelastic materials in order to determine universal relations. A universal relation, in this context, is an equation that relates the components of the stress tensor and the components of the magnetic field and/or the components of the magnetic induction that holds independently of the specific choice of constitutive law for the considered class or subclass of materials. As has been shown previously for the case in which the magnetic induction is the independent magnetic variable, in the general case there exists only one possible universal relation. We show that this is also the case if the magnetic field is taken as the independent variable and that the universal relations resulting from the two cases are equivalent. A number of special cases are found for certain specializations of the constitutive equations. These include some connections between the deformation, the magnetic field and magnetic induction that do not involve the components of the stress tensor. Universal relations are then examined for some representative homogeneous and inhomogeneous universal solutions.

1. Introduction

Recently, a number of applications of magneto-sensitive (MS) elastomers have been developed and commercialized. Such materials are capable of large magnetoelastic deformations and change their mechanical response rapidly and significantly on the application of a magnetic field. For a discussion of industrial applications we refer to, for example, the papers by Farshad and Le Roux (**1**) and Jolly *et al.* (**2**). Because of the increasing potential for use of these materials, there is now demand for the development of reliable constitutive equations that can be used in the analysis and solution of representative boundary-value problems and in commercial finite element software.

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Important early publications on the theory of magnetoelasticity are the book of Brown (**3**), the monographs of Truesdell and Toupin (**4**) and Hutter and van de Ven (**5**), and the review article by Pao (**6**). Recently, several alternative formulations of constitutive equations capable of describing the highly nonlinear magnetoelastic interactions have been examined. These are mostly based on the use of a free energy, which is treated as a function of a magnetic field vector (the magnetic field, magnetic induction or magnetization) and the deformation gradient tensor. Also, solutions of a number of representative boundary-value problems for nonlinear magnetoelastic solids have been obtained. Selected references are the papers (**7** to **12**). Valuable background material on electromagnetic fields and their interactions with deformable continua can also be found in (**13**, **14**).

For a particular deformation or class of deformations, a *universal relation* is an equation that relates stress and strain components that holds independently of the specific choice of constitutive law for the considered class of materials. Such relations therefore provide guidelines for the experimenter in the design of tests and loading conditions necessary for practical evaluation of the material response. In this paper we extend the previous brief discussion (**8**) of universal relations in nonlinear magnetoelasticity. For corresponding discussion of universal relations in a purely elastic context, we refer to the review articles by Saccomandi (**15**, **16**) and citations therein. A parallel treatment of universal relations for nonlinear electroelastic solids has been provided recently in (**17**).

In this paper we derive the only independent universal relation possible for an isotropic material in the general case by using either the magnetic field or the magnetic induction field as the independent magnetic variable. Additional universal relations are derived for some cases in which the constitutive law is specialized. The constitutive equations discussed by Dorfmann and Ogden (**7**) are based on a modified (or total) free-energy function. In addition to the deformation gradient, it depends on the magnetic induction vector. The corresponding equations obtained by replacing the magnetic induction vector by the magnetic field are also used. When the Lagrangian form of either of these two vectors is used as the independent magnetic variable the resulting equations have a particularly simple structure, but we point out that if, instead, the magnetization vector is used as the independent magnetic variable then the structure is somewhat less simple.

The structure of the paper is as follows. In section 2 we define the main kinematic quantities necessary to describe large deformations. This is followed by a summary of the magnetic balance equations in both the current and the reference configurations. This section concludes with a review of the corresponding mechanical balance equations and the boundary conditions for the (total) stress and the magnetic field vectors. In section 3 the constitutive equations based on either the magnetic induction or the magnetic field vector as an independent variable are summarized. The corresponding equations obtained by taking the magnetization as an independent variable are also discussed briefly. The constitutive equations are then specialized to the case of an isotropic magnetoelastic material. In section 4 the only universal relation for a general isotropic magnetoelastic solid is derived using first the magnetic induction and then the magnetic field as the independent variable and the equivalence of the two formulations is demonstrated in a simple way. Several additional universal relations, valid for special cases of the constitutive equation, are derived in section 4.1. Examination of the implications of the universal relations for a number of specific deformations is the subject of section 5. Some concluding remarks are provided in section 6.

2. Governing equations and boundary conditions

Consider a magnetoelastic solid occupying the reference configuration $\mathcal{B}_0$. Let a material point in $\mathcal{B}_0$ be identified by its position vector $\mathbf{X}$ relative to an arbitrarily chosen origin. When the body is deformed the point $\mathbf{X}$ occupies a new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where $\boldsymbol{\chi}$ is the deformation mapping. We denote the resulting deformed configuration by $\mathcal{B}$. The deformation gradient relative to $\mathcal{B}_0$, which is denoted by $\mathbf{F}$, and its determinant, denoted J, are given by

$$\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}, \quad J = \det\mathbf{F} > 0, \tag{2.1}$$

respectively, where Grad is the gradient operator with respect to $\mathbf{X}$. The left and right Cauchy–Green deformation tensors, denoted here by $\mathbf{b}$ and $\mathbf{c}$, respectively, are defined by $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, where $^{\mathrm{T}}$ signifies the transpose.

A magnetoelastic material can be deformed by the application of a magnetic field alone (without applied mechanical loads), resulting in the phenomenon of *magnetostriction*, by the action of mechanical loads alone or by the combined action of mechanical loads and a magnetic field.

In the current configuration $\mathcal{B}$ we denote by $\mathbf{H}$ and $\mathbf{B}$, respectively, the magnetic field vector and the magnetic induction vector. These are fundamental quantities which, in the absence of time dependence and distributed currents, satisfy the specializations

$$\operatorname{curl}\mathbf{H} = \mathbf{0}, \quad \operatorname{div}\mathbf{B} = 0 \tag{2.2}$$

of Maxwell's equations both within material and *in vacuo*, where curl and div are the curl and divergence operators with respect to $\mathbf{x}$.

In the absence of material the two fields are related by $\mathbf{B} = \mu_0\mathbf{H}$, where μ_0 is the magnetic permeability *in vacuo*, but inside material an additional vector field, the magnetization $\mathbf{M}$, may be defined by the difference

$$\mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}. \tag{2.3}$$

Equation (2.3) describes the third vector field when one vector is selected as independent and the other is given by a constitutive relation.

Equations (2.2) and (2.3) are expressed in Eulerian form. Corresponding Lagrangian forms, denoted by $\mathbf{H}_l$ and $\mathbf{B}_l$, are defined by pull-back operations from $\mathcal{B}$ to $\mathcal{B}_0$ to give

$$\mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}, \quad \mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}. \tag{2.4}$$

For derivations of these connections we refer to, for example, (**7**, **11**, **13**, **18**). The vectors $\mathbf{H}_l$ and $\mathbf{B}_l$ satisfy the field equations

$$\operatorname{Curl}\mathbf{H}_l = \mathbf{0}, \quad \operatorname{Div}\mathbf{B}_l = 0, \tag{2.5}$$

where, respectively, Curl and Div are the curl and divergence operators with respect to $\mathbf{X}$.

A Lagrangian form of $\mathbf{M}$, denoted by $\mathbf{M}_l$, may be defined similarly to $(2.4)_1$ by

$$\mathbf{M}_l = \mathbf{F}^{\mathrm{T}}\mathbf{M}, \tag{2.6}$$

which enables a Lagrangian form of (2.3) to be obtained, namely

$$\mathbf{M}_l = \mu_0^{-1}J^{-1}\mathbf{c}\mathbf{B}_l - \mathbf{H}_l. \tag{2.7}$$

Turning now to the mechanical balance equations, we denote by ρ_0 and ρ the mass density of the material in $\mathcal{B}_0$ and $\mathcal{B}$, respectively. The conservation of mass equation is then written

$$J\rho = \rho_0, \tag{2.8}$$

with J as defined in (2.1).

In the absence of *mechanical* body forces the equilibrium equation may be written in Eulerian form as

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \tag{2.9}$$

where $\boldsymbol{\tau}$ is the total (Cauchy) stress tensor, which incorporates the effect of the magnetic body forces. According to the balance of angular momentum, $\boldsymbol{\tau}$ is *symmetric*. For details of different stress tensors and magnetic body forces we refer to, for example, (**5**, **7**, **10**, **11**).

The corresponding Lagrangian form of (2.9) is

$$\operatorname{Div} \mathbf{T} = \mathbf{0}, \tag{2.10}$$

in which $\mathbf{T}$ is the total *nominal* stress tensor, analogous to that used in elasticity theory (see, for example, Ogden (**19**)) and obtained from $\boldsymbol{\tau}$ by the operation

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{2.11}$$

To the governing equations for $\mathbf{H}, \mathbf{B}, \boldsymbol{\tau}$ or $\mathbf{H}_l, \mathbf{B}_l, \mathbf{T}$ are appended boundary conditions. In the case of the Eulerian fields, these are

$$\mathbf{n} \times [\mathbf{H}] = \mathbf{0}, \quad \mathbf{n} \cdot [\mathbf{B}] = 0, \quad [\boldsymbol{\tau}]\mathbf{n} = \mathbf{0}, \tag{2.12}$$

where the square brackets indicate a discontinuity across a bounding surface of the body or interface and $\mathbf{n}$ is the unit normal to the surface, defined in the usual sense as the outward pointing normal at the body boundary. In the expression $[\boldsymbol{\tau}]\mathbf{n}$ the traction $\boldsymbol{\tau}\mathbf{n}$ on the boundary includes the contribution from the Maxwell stress exterior to the body and any applied mechanical traction. For corresponding Lagrangian forms of (2.12), see, for example, (**7**).

3. Constitutive equations

The mechanical and magnetic properties of the considered magnetoelastic material are described in terms of constitutive equations, the forms of which depend on the choice of the independent magnetic variable. In this section we summarize the different forms of the constitutive laws for the different choices of independent magnetic variable, first in the general case and then for the isotropic specialization.

3.1 *General forms of the constitutive laws*

3.1.1 *Use of $\mathbf{B}_l$ as the independent magnetic variable.* Following Dorfmann and Ogden (**7**), we first take the variables $\mathbf{F}$ and $\mathbf{B}_l$ to be independent and introduce a free energy function per unit mass, written $\Phi = \Phi(\mathbf{F}, \mathbf{B}_l)$. For an unconstrained material, the total stress tensor is then given by

$$\boldsymbol{\tau} = \rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{F}} + \boldsymbol{\tau}_m, \tag{3.1}$$

where $\boldsymbol{\tau}_m$ is defined by

$$\boldsymbol{\tau}_m = \mu_0^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \tfrac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right], \tag{3.2}$$

with $\mathbf{B} = J^{-1}\mathbf{F}\mathbf{B}_l$ and where $\mathbf{I}$ is the identity tensor. In vacuum, $\Phi \equiv 0$ and the stress $\boldsymbol{\tau}$ reduces to the Maxwell stress $\boldsymbol{\tau}_m$ with $\mathbf{B} = \mu_0\mathbf{H}$.

In terms of Φ we also have

$$\mu_0^{-1}\mathbf{B} - \mathbf{H} = \mathbf{M} = -\rho J \mathbf{F}^{-\mathrm{T}} \frac{\partial \Phi}{\partial \mathbf{B}_l}. \tag{3.3}$$

The structure of the constitutive laws and the governing equations can be simplified by introducing the *total* energy function, which, following (**7**), is defined by

$$\Omega = \Omega(\mathbf{F}, \mathbf{B}_l) = \rho_0 \Phi + \tfrac{1}{2}\mu_0^{-1} J^{-1} \mathbf{B}_l \cdot (\mathbf{c}\mathbf{B}_l) \tag{3.4}$$

per unit reference volume. This yields the simple Lagrangian forms

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l} \tag{3.5}$$

of the required constitutive equations. The corresponding Eulerian forms are

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}} \frac{\partial \Omega}{\partial \mathbf{B}_l}. \tag{3.6}$$

Note that these equations do not involve the magnetization vector, which, if needed, can be calculated from either (2.7) in Lagrangian form or (2.3) in Eulerian form.

3.1.2 *Use of* $\mathbf{H}_l$ *as an independent variable.* If, instead of $\mathbf{B}_l$, we use $\mathbf{H}_l$ as the independent magnetic variable, then the analogues of equations (3.5) and (3.6) are, respectively,

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l} \tag{3.7}$$

and

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B} = -J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{H}_l}, \tag{3.8}$$

where $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{H}_l)$ is the counterpart of Ω. Under suitable invertibility conditions Ω and Ω^* can be related via the Legendre transformation

$$\Omega^*(\mathbf{F}, \mathbf{H}_l) = \Omega(\mathbf{F}, \mathbf{B}_l) - \mathbf{H}_l \cdot \mathbf{B}_l. \tag{3.9}$$

3.1.3 *Use of* $\mathbf{M}_l$ *as an independent variable.* A third option is to use $\mathbf{M}_l$ as the independent magnetic variable. However, this does not lead to quite such a clean structure. The Legendre transformation

$$\Omega^{**}(\mathbf{F}, \mathbf{M}_l) = \rho_0 \Phi(\mathbf{F}, \mathbf{B}_l) + \mathbf{B}_l \cdot \mathbf{M}_l, \tag{3.10}$$

for example, yields

$$\mathbf{T} = \frac{\partial \Omega^{**}}{\partial \mathbf{F}} + J\mathbf{F}^{-1}\boldsymbol{\tau}_m, \quad \mathbf{B}_l = \frac{\partial \Omega^{**}}{\partial \mathbf{M}_l}, \tag{3.11}$$

where $\boldsymbol{\tau}_m$ is given by (3.2) and cannot in general be expressed as a function of $\mathbf{F}$ and $\mathbf{M}_l$ alone. We point out that (3.10) and (3.11) correct the formulae given in (**7**, equation (5.4) and the following line).

Because of the simpler structures we restrict attention in what follows to the formulations in sections 3.1.1 and 3.1.2. Furthermore, it is convenient for our purposes to work in terms of the Eulerian formulation and to consider only incompressible materials.

3.1.4 *Incompressible materials.* The incompressibility constraint

$$\det \mathbf{F} \equiv 1 \tag{3.12}$$

is now adopted and equations (3.6) are therefore amended to

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\frac{\partial \Omega}{\partial \mathbf{B}_l}, \tag{3.13}$$

where $\mathbf{I}$ is again the identity tensor, p is a Lagrange multiplier associated with the constraint (3.12) and $\boldsymbol{\Omega}$ is specialized accordingly.

Similarly, equations (3.8) become

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}, \quad \mathbf{B} = -\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{H}_l}, \tag{3.14}$$

where, instead of p, we have used p^* for the Lagrange multiplier since, in general, it will not equal p.

3.2 *Isotropic magnetoelastic materials*

The application of a magnetic field introduces a preferred direction in the material and the mathematical form of the constitutive law is therefore similar in structure to that for a transversely isotropic elastic solid. Working in terms of the formulation based on $\boldsymbol{\Omega}$ and following the standard analysis for transversely isotropic elastic solids (see, for example, (**20**, **21**)), we define an *isotropic* magnetoelastic material as one for which the total energy function $\boldsymbol{\Omega}$ is an isotropic function of the two tensors $\mathbf{c}$ and $\mathbf{B}_l \otimes \mathbf{B}_l$. For an incompressible material, the form of $\boldsymbol{\Omega}$ reduces to dependence on five independent invariants, denoted by I_1, I_2, I_4, I_5, I_6. These are the principal invariants of $\mathbf{c}$, given by

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \tfrac{1}{2}[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,(\mathbf{c}^2)], \tag{3.15}$$

where tr denotes the trace of a second-order tensor, and the invariants involving $\mathbf{B}_l$, which are defined as

$$I_4 = |\mathbf{B}_l|^2, \quad I_5 = (\mathbf{c}\mathbf{B}_l)\cdot\mathbf{B}_l, \quad I_6 = (\mathbf{c}^2\mathbf{B}_l)\cdot\mathbf{B}_l. \tag{3.16}$$

From (3.13) and the definitions of the invariants in (3.15) and (3.16) the explicit forms of $\boldsymbol{\tau}$ and $\mathbf{H}$ are obtained as

$$\boldsymbol{\tau} = 2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5\mathbf{B}\otimes\mathbf{B} + 2\Omega_6(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B}) \tag{3.17}$$

and

$$\mathbf{H} = 2(\Omega_4\mathbf{b}^{-1}\mathbf{B} + \Omega_5\mathbf{B} + \Omega_6\mathbf{b}\mathbf{B}), \tag{3.18}$$

respectively, where the subscripts 1, 2, 4, 5, 6 signify partial differentiation with respect to I_1, I_2, I_4, I_5, I_6, respectively, and $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy–Green deformation tensor.

Alternatively, if the magnetic field $\mathbf{H}_l$ is the independent variable, the invariants defined by (3.16) need to be replaced by invariants involving $\mathbf{H}_l$. For these we use the notation K_4, K_5, K_6, defined by

$$K_4 = |\mathbf{H}_l|^2, \quad K_5 = (\mathbf{c}\mathbf{H}_l)\cdot\mathbf{H}_l, \quad K_6 = (\mathbf{c}^2\mathbf{H}_l)\cdot\mathbf{H}_l. \tag{3.19}$$

Use of the energy function $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^*(I_1, I_2, K_4, K_5, K_6)$ in (3.14) enables $\boldsymbol{\tau}$ and $\mathbf{B}$ to be written as

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p^*\mathbf{I} + 2\Omega_5^*\mathbf{bH}\otimes\mathbf{bH} + 2\Omega_6^*(\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H}\otimes\mathbf{bH}) \tag{3.20}$$

and

$$\mathbf{B} = -2(\Omega_4^*\mathbf{bH} + \Omega_5^*\mathbf{b}^2\mathbf{H} + \Omega_6^*\mathbf{b}^3\mathbf{H}), \tag{3.21}$$

respectively, where Ω_i^* is defined as $\partial\Omega^*/\partial I_i$ for $i = 1, 2$, and $\partial\Omega^*/\partial K_i$ for $i = 4, 5, 6$.

4. Universal relations

To derive universal relations for an incompressible magnetoelastic solid we consider the constitutive laws summarized in the previous section. First, using (3.17), it is convenient to introduce the notation

$$\gamma_1 = 2(\Omega_1 + \Omega_2 I_1), \quad \gamma_2 = -2\Omega_2, \quad \gamma_4 = 2\Omega_4, \quad \gamma_5 = 2\Omega_5, \quad \gamma_6 = 2\Omega_6, \tag{4.1}$$

and to rewrite (3.17) in the more compact form

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1\mathbf{b} + \gamma_2\mathbf{b}^2 + \gamma_5\mathbf{B}\otimes\mathbf{B} + \gamma_6(\mathbf{B}\otimes\mathbf{bB} + \mathbf{bB}\otimes\mathbf{B}). \tag{4.2}$$

Following Dorfmann *et al.* (**8**) and the parallel development for electroelastic solids by Bustamante and Ogden (**17**), we form the antisymmetric tensor

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \gamma_5(\mathbf{B}\otimes\mathbf{bB} - \mathbf{bB}\otimes\mathbf{B}) + \gamma_6(\mathbf{B}\otimes\mathbf{b}^2\mathbf{B} - \mathbf{b}^2\mathbf{B}\otimes\mathbf{B}), \tag{4.3}$$

noting that this vanishes when $\mathbf{B}$ is an eigenvector of $\mathbf{b}$.

Next, we recall that for any antisymmetric second-order tensor an associated axial vector can be defined. For example, for the tensor $\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u}$, where $\mathbf{u}$ and $\mathbf{v}$ are two vectors, the axial vector is $\mathbf{v}\times\mathbf{u}$. More generally, if $\mathbf{W}$ is an antisymmetric second-order tensor, we denote by $(\mathbf{W})_\times$ its axial vector.

Therefore, the axial vector corresponding to the expression in (4.3) has the form

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times = (\gamma_5\mathbf{bB} + \gamma_6\mathbf{b}^2\mathbf{B})\times\mathbf{B}, \tag{4.4}$$

from which the universal relation

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times\cdot\mathbf{B} = 0 \tag{4.5}$$

follows immediately. This is identical to the universal relation found in (**8**).

For the alternative formulation (3.20) we use the notation

$$\gamma_1^* = 2(\Omega_1^* + \Omega_2^* I_1), \quad \gamma_2^* = -2\Omega_2^*, \quad \gamma_4^* = 2\Omega_4^*, \quad \gamma_5^* = 2\Omega_5^*, \quad \gamma_6^* = 2\Omega_6^*, \tag{4.6}$$

to rewrite $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = -p\mathbf{I} + \gamma_1^*\mathbf{b} + \gamma_2^*\mathbf{b}^2 + \gamma_5^*(\mathbf{bH}\otimes\mathbf{bH}) + \gamma_6^*(\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H}\otimes\mathbf{bH}), \tag{4.7}$$

from which we obtain

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \gamma_5^*(\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H} - \mathbf{b}^2\mathbf{H}\otimes\mathbf{bH}) + \gamma_6^*(\mathbf{bH}\otimes\mathbf{b}^3\mathbf{H} - \mathbf{b}^3\mathbf{H}\otimes\mathbf{bH}), \tag{4.8}$$

similarly to (4.4), and hence the axial vector

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_{\times} = (\gamma_5 \mathbf{b}^2\mathbf{H} + \gamma_6 \mathbf{b}^3\mathbf{H}) \times \mathbf{bH}. \tag{4.9}$$

We then obtain the universal relation

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_{\times} \cdot (\mathbf{bH}) = 0. \tag{4.10}$$

Equivalence of the two relations (4.5) and (4.10) can be shown as follows. Using the definitions of γ_4, γ_5 and γ_6 and the constitutive equation (3.18) we have

$$\mathbf{H} = \gamma_4 \mathbf{b}^{-1}\mathbf{B} + \gamma_5 \mathbf{B} + \gamma_6 \mathbf{bB}, \tag{4.11}$$

from which we obtain

$$\mathbf{B} \otimes \mathbf{bH} - \mathbf{bH} \otimes \mathbf{B} = \gamma_5(\mathbf{B} \otimes \mathbf{bB} - \mathbf{bB} \otimes \mathbf{B}) + \gamma_6(\mathbf{B} \otimes \mathbf{b}^2\mathbf{B} - \mathbf{b}^2\mathbf{B} \otimes \mathbf{B}), \tag{4.12}$$

which is equal to the right-hand side of (4.3).

Similarly, the constitutive equation (3.21) can be rewritten as

$$\mathbf{B} = -\gamma_4^* \mathbf{bH} - \gamma_5^* \mathbf{b}^2\mathbf{H} - \gamma_6^* \mathbf{b}^3\mathbf{H}, \tag{4.13}$$

which leads to

$$\mathbf{B} \otimes \mathbf{bH} - \mathbf{bH} \otimes \mathbf{B} = \gamma_5^*(\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} - \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH}) + \gamma_6^*(\mathbf{bH} \otimes \mathbf{b}^3\mathbf{H} - \mathbf{b}^3\mathbf{H} \otimes \mathbf{bH}). \tag{4.14}$$

This is equal to the right-hand side of (4.8).

It follows immediately that

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \mathbf{B} \otimes \mathbf{bH} - \mathbf{bH} \otimes \mathbf{B}, \tag{4.15}$$

from which the expressions for the universal relations given by equations (4.5) and (4.10) follow. These two universal relations are therefore equivalent. This is consistent with the finding by Dorfmann *et al.* (**8**) that only one general universal relation exists for the considered class of magnetoelastic solids.

4.1 *Special cases*

The special cases considered in this subsection are based on the energy Ω and the corresponding constitutive laws given by (3.17) and (3.18). Similar cases can be considered starting from the alternative formulation based on Ω^*, but we omit the details here.

Case 1: $\Omega = \Omega(I_1, I_2, I_4, I_5)$. This is the case where Ω does not depend on I_6, so that $\gamma_6 = 0$. Equation (4.3) reduces to

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \gamma_5(\mathbf{B} \otimes \mathbf{bB} - \mathbf{bB} \otimes \mathbf{B}), \tag{4.16}$$

from which we obtain the two universal relations

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_{\times} \cdot \mathbf{B} = 0, \quad (\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_{\times} \cdot (\mathbf{bB}) = 0. \tag{4.17}$$

Similarly, from the corresponding specialization of (4.11), we obtain

$$(\mathbf{H} \times \mathbf{b}^{-1}\mathbf{B}) \cdot \mathbf{B} = 0, \tag{4.18}$$

which does not involve the stress $\boldsymbol{\tau}$. This can be regarded as an additional universal relation.

Case 2: $\boldsymbol{\Omega} = \boldsymbol{\Omega}(I_1, I_2, I_4, I_6)$. We consider the special case for which $\boldsymbol{\Omega}$ is independent of I_5. From (4.3) we obtain

$$\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau} = \gamma_6(\mathbf{B} \otimes \mathbf{b}^2\mathbf{B} - \mathbf{b}^2\mathbf{B} \otimes \mathbf{B}), \tag{4.19}$$

and the corresponding axial vector

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times = \gamma_6(\mathbf{b}^2\mathbf{B}) \times \mathbf{B}. \tag{4.20}$$

Two universal relations follow, namely

$$(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot \mathbf{B} = 0, \quad (\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times \cdot (\mathbf{b}^2\mathbf{B}) = 0, \tag{4.21}$$

which is consistent with the findings of Pucci and Saccomandi (**22**).

For this special case, the magnetic field $\mathbf{H}$ is given by (4.11) with $\gamma_5 = 0$ and we obtain another universal relation not involving the stress, namely

$$(\mathbf{H} \times \mathbf{b}^{-1}\mathbf{B}) \cdot (\mathbf{b}\mathbf{B}) = 0. \tag{4.22}$$

Case 3: $\boldsymbol{\Omega} = \boldsymbol{\Omega}(I_1, I_2, I_5, I_6)$. Here, $\boldsymbol{\Omega}$ is independent of I_4, which implies that the constitutive equation (4.11) yields universal relation

$$(\mathbf{H} \times \mathbf{B}) \cdot (\mathbf{b}\mathbf{B}) = 0. \tag{4.23}$$

No additional relation is found involving the stress components as (4.2) does not involve γ_4.

Case 4: $\boldsymbol{\Omega} = \boldsymbol{\Omega}(I_1, I_4, I_5, I_6)$. When $\boldsymbol{\Omega}$ is independent of I_2, no additional universal relation can be obtained by starting from (4.3). However, consideration of the antisymmetric tensor

$$\boldsymbol{\tau}\mathbf{B} \otimes \mathbf{B} - \mathbf{B} \otimes \boldsymbol{\tau}\mathbf{B} = \gamma_1(\mathbf{b}\mathbf{B} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{b}\mathbf{B}) + \gamma_6(\mathbf{B} \cdot \mathbf{B})(\mathbf{b}\mathbf{B} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{b}\mathbf{B}), \tag{4.24}$$

leads to the corresponding axial vector

$$(\boldsymbol{\tau}\mathbf{B} \otimes \mathbf{B} - \mathbf{B} \otimes \boldsymbol{\tau}\mathbf{B})_\times = \boldsymbol{\tau}\mathbf{B} \times \mathbf{B} = [\gamma_1 + \gamma_6(\mathbf{B} \cdot \mathbf{B})](\mathbf{b}\mathbf{B} \times \mathbf{B}), \tag{4.25}$$

and hence to the universal relation

$$(\boldsymbol{\tau}\mathbf{B} \times \mathbf{B}) \cdot (\mathbf{b}\mathbf{B}) = 0. \tag{4.26}$$

Case 5: $\boldsymbol{\Omega} = \boldsymbol{\Omega}(I_2, I_4, I_5, I_6)$. This case is similar to Case 4. For convenience, we write the reduced form of the total stress tensor $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = -p\mathbf{I} + \tilde{\gamma}_2(I_1\mathbf{b} - \mathbf{b}^2) + \gamma_5\mathbf{B} \otimes \mathbf{B} + \gamma_6(\mathbf{B} \otimes \mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B} \otimes \mathbf{B}), \tag{4.27}$$

where we have defined $\tilde{\gamma}_2 = 2\Omega_2$. Then, by first forming the expression $\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau}$, we obtain the universal relation shown previously in (4.5). An additional universal relation can be found by considering the expression $\boldsymbol{\tau}\mathbf{b}\mathbf{B} \otimes \mathbf{b}\mathbf{B} - \mathbf{b}\mathbf{B} \otimes \boldsymbol{\tau}\mathbf{b}\mathbf{B}$. On use of the Cayley–Hamilton theorem in the form

$$\mathbf{b}^3 - I_1\mathbf{b}^2 + I_2\mathbf{b} - \mathbf{I} = \mathbf{0}, \tag{4.28}$$

we obtain the axial vector

$$\boldsymbol{\tau}\mathbf{b}\mathbf{B} \times \mathbf{b}\mathbf{B} = [\gamma_5(\mathbf{B} \cdot \mathbf{b}\mathbf{B}) + \gamma_6\mathbf{B} \cdot (\mathbf{b}^2\mathbf{B}) - \tilde{\gamma}_2](\mathbf{B} \times \mathbf{b}\mathbf{B}), \tag{4.29}$$

from which follows the additional universal relation

$$(\boldsymbol{\tau}\mathbf{b}\mathbf{B} \times \mathbf{b}\mathbf{B}) \cdot \mathbf{B} = 0. \tag{4.30}$$

5. Applications

5.1 *Homogeneous deformation in a uniform field*

Consider a slab of uniform thickness bounded by two faces normal to the X_3 direction and unbounded in the X_1 and X_2 directions, where (X_1, X_2, X_3) are rectangular Cartesian coordinates of the point $\mathbf{X}$ in the reference configuration $\mathcal{B}_0$. The universal relation given by (4.5) has been applied to a similar geometry subject to triaxial stretch and a single simple shear in (**8**). We also recall that the boundary-value problems for the separate special cases of pure homogeneous deformation and simple shear with an applied magnetic field normal to the top and bottom faces of the slab was examined in (**9**).

Here we assume that the slab is subjected to a uniform magnetic field and stretched along the three coordinate axes with stretch ratios μ_1, μ_2, μ_3, and then sheared with shears κ_1 and κ_2 in the two in-plane directions. The combined triaxial stretch and shear deformation is given by

$$x_1 = \mu_1 X_1 + \kappa_1 \mu_3 X_3, \quad x_2 = \mu_2 X_2 + \kappa_2 \mu_3 X_3, \quad x_3 = \mu_3 X_3, \tag{5.1}$$

where (x_1, x_2, x_3) are the rectangular Cartesian coordinates in the deformed configuration of the material point initially located at $\mathbf{X}$, and μ_1, μ_2, μ_3 and κ_1, κ_2 are constants. For this homogeneous deformation and uniform applied magnetic field all strain components are constant and the field equations (2.2) and (2.9) are satisfied automatically.

The matrix of the Cartesian components F of the deformation gradient tensor $\mathbf{F}$ is

$$\mathsf{F} = \begin{pmatrix} \mu_1 & 0 & \kappa_1\mu_3 \\ 0 & \mu_2 & \kappa_2\mu_3 \\ 0 & 0 & \mu_3 \end{pmatrix}. \tag{5.2}$$

The components of the tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ needed to evaluate the total stress components in (4.2) can now be obtained, but they are not written explicitly here.

The uniform magnetic field vector $\mathbf{B}$ in the deformed configuration $\mathcal{B}$, related through equation $(2.4)_1$ to its Lagrangian counterpart, has the Cartesian component matrix $\mathsf{B} = [B_1, B_2, B_3]^{\mathrm{T}}$. The components of $\mathbf{B} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B}$ can now be calculated for use in (4.2), but their expressions are not given explicitly here.

The universal relation (4.5) is given in terms of the components by

$$\begin{aligned}
&[\tau_{13}\kappa_1\kappa_2\mu_3^2 + \tau_{23}(\mu_2^2 + \mu_3^2(\kappa_2^2 - 1)) + (\tau_{33} - \tau_{22})\kappa_2\mu_3^2 - \tau_{12}\kappa_1\mu_3^2]B_1 \\
&\quad + [(\tau_{11} - \tau_{33})\kappa_1\mu_3^2 + \tau_{12}\kappa_2\mu_3^2 - \tau_{13}(\mu_1^2 + \mu_3^2(\kappa_1^2 - 1)) - \tau_{23}\kappa_1\kappa_2\mu_3^2]B_2 \\
&\quad + [\tau_{12}(\mu_1^2 - \mu_2^2 + \mu_3^2(\kappa_1^2 - \kappa_2^2)) + \mu_3^2((\tau_{22} - \tau_{11})\kappa_1\kappa_2 + \tau_{23}\kappa_1 - \tau_{13}\kappa_2)]B_3 = 0,
\end{aligned} \tag{5.3}$$

where the stress components are τ_{ij} with subscripts i and j assuming the values 1, 2, 3.

For illustration, consider the special case of simple shear in the x_1 direction only ($\kappa_2 \equiv 0$). Suppose further that the applied magnetic field vector is oriented along the x_2 direction with components $(0, B_2, 0)$, $B_2 \neq 0$. Then, (5.3) reduces to the universal relation

$$\kappa_1\mu_3^2(\tau_{11} - \tau_{33}) = \tau_{13}[\mu_1^2 + \mu_3^2(\kappa_1^2 - 1)], \tag{5.4}$$

which is independent of B_2 and for $\mu_1 = \mu_3 = 1$ reduces to the well-known universal relation $\tau_{11} - \tau_{33} = \kappa_1\tau_{13}$ in the purely elastic case.

Another special case is obtained by considering a triaxial stretch with no shear ($\kappa_1 = \kappa_2 \equiv 0$) for which we obtain from (5.1) the universal relation

$$\tau_{23}(\mu_2^2 - \mu_3^2)\, B_1 + \tau_{13}(\mu_3^2 - \mu_1^2)\, B_2 + \tau_{12}(\mu_1^2 - \mu_2^2)\, B_3 = 0. \tag{5.5}$$

We conclude this section by considering the special case of the energy function $\mathbf{\Omega}$, as outlined in section 4.1, for which $\mathbf{\Omega}$ is independent of I_6, so that $\gamma_6 = 0$. This furnishes the additional universal relation $(4.17)_2$. To obtain an explicit expression for the universal relation $(4.17)_2$ requires computation of the vector $\mathbf{bB}$ in addition to $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times$, the components of the latter being the coefficients of B_1, B_2 and B_3 in (5.3). From the components of the left Cauchy–Green tensor $\mathbf{b}$ corresponding to the deformation (5.1) and the components of the magnetic field $\mathbf{B}$, the vector $\mathbf{bB}$ is found to have the form

$$[(\mu_1^2 + \kappa_1^2\mu_3^2)B_1 + \kappa_1\kappa_2\mu_3^2 B_2 + \kappa_1\mu_3^2 B_3]\mathbf{i}_1 + [\kappa_1\kappa_2\mu_3^2 B_1 + (\mu_2^2 + \kappa_2^2\mu_3^2)B_2 + \kappa_2\mu_3^2 B_3]\mathbf{i}_2 + [\kappa_1\mu_3^2 B_1 + \kappa_2\mu_3^2 B_2 + \mu_3^2 B_3]\mathbf{i}_3, \tag{5.6}$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the unit base vectors in the x_1, x_2, x_3 directions. The universal relation in $(4.17)_2$ is obtained by taking the scalar product of the axial vector $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times$ with $\mathbf{bB}$, which yields

$$\begin{aligned}
&[-\tau_{12}\kappa_1\mu_2^2\mu_3^2 + \tau_{23}(\mu_1^2(\mu_2^2 - \mu_3^2) + \mu_3^2(\kappa_1^2\mu_2^2 + \kappa_2^2\mu_1^2)) + (\tau_{33} - \tau_{22})\kappa_2\mu_1^2\mu_3^2]B_1 \\
&\quad + [\tau_{13}(\mu_2^2(\mu_3^2 - \mu_1^2) - \mu_3^2(\kappa_1^2\mu_2^2 + \kappa_2^2\mu_3^2)) + \tau_{12}\kappa_2\mu_1^2\mu_3^2 + (\tau_{11} - \tau_{33})\kappa_1\mu_2^2\mu_3^2]B_2 \\
&\quad + [-\tau_{13}\kappa_2\mu_1^2\mu_3^2 + \tau_{12}\mu_3^2(\mu_1^2 - \mu_2^2) + \tau_{23}\kappa_1\mu_2^2\mu_3^2]B_3 = 0.
\end{aligned} \tag{5.7}$$

If we now assume, for simplicity, that $\kappa_2 = 0$ and $B_1 = B_3 \equiv 0$, then this universal relation reduces to the same form as that given by (5.4).

For this example, the two universal relations given by $(4.17)_1$ and $(4.17)_2$ assume the forms shown in (5.3) and (5.7), respectively. If we eliminate B_2 from these equations and set $\kappa_2 = 0$ (with $\mu_2 \neq 0$), the resulting equation simplifies to

$$(\tau_{23}B_1 - \tau_{12}B_3)(-\mu_2^4 + \mu_2^2\mu_3^2 + \mu_2^2\kappa_1^2\mu_3^2 + \mu_2^2\mu_1^2 - \mu_3^2\mu_1^2) = 0. \tag{5.8}$$

Since this condition must be satisfied for all deformations, we deduce the connection

$$\tau_{23}B_1 - \tau_{12}B_3 = 0, \tag{5.9}$$

which can also be obtained directly from the component form of $\boldsymbol{\tau}$. Similarly, if we eliminate B_1 from (5.3) and (5.7) and take $\kappa_1 = 0$ (with $\kappa_2 \neq 0$), we obtain the connection

$$\tau_{12}B_3 - \tau_{13}B_2 = 0. \tag{5.10}$$

5.2 *Extension and torsion of a circular cylinder*

Consider an infinitely long solid circular cylinder whose reference geometry is described by

$$0 \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad -\infty \leqslant Z \leqslant \infty \tag{5.11}$$

in terms of cylindrical polar coordinates (R, Θ, Z). Combined torsion and axial extension is defined by

$$r = \lambda_z^{-1/2} R, \quad \theta = \Theta + \lambda_z \tau Z, \quad z = \lambda_z Z, \tag{5.12}$$

where τ is the (angular) torsional twist per unit deformed length, λ_z is the axial stretch (constant) and (r, θ, z) are cylindrical polar coordinates in the deformed configuration. For details of the solution of this boundary-value problem in the context of magnetoelasticity, we refer to (**7**).

The components of the deformation gradient $\mathbf{F}$, referred to the two sets of cylindrical polar coordinate axes, and those of the left Cauchy–Green tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ are represented by the matrices F and b, respectively, and given by

$$\mathsf{F} = \begin{pmatrix} \lambda_z^{-1/2} & 0 & 0 \\ 0 & \lambda_z^{-1/2} & \lambda_z\kappa \\ 0 & 0 & \lambda_z \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} + \lambda_z^2\kappa^2 & \lambda_z^2\kappa \\ 0 & \lambda_z^2\kappa & \lambda_z^2 \end{pmatrix}, \tag{5.13}$$

wherein the notation $\kappa = \tau r$ has been adopted.

The corresponding matrix of $\mathbf{b}^2$ needed to evaluate the stress components has the component form

$$\mathsf{b}^2 = \begin{pmatrix} \lambda_z^{-2} & 0 & 0 \\ 0 & (\lambda_z^{-1} + \lambda_z^2\kappa^2)^2 + \lambda_z^4\kappa^2 & \lambda_z^2\kappa(\lambda_z^{-1} + \lambda_z^2\kappa^2 + \lambda_z^2) \\ 0 & \lambda_z^2\kappa(\lambda_z^{-1} + \lambda_z^2\kappa^2 + \lambda_z^2) & \lambda_z^4(1+\kappa^2) \end{pmatrix}. \tag{5.14}$$

Depending on the selection of the independent magnetic field variable, the total stress tensor $\boldsymbol{\tau}$ is given by (4.2) or (4.7). The consequences of using one or the other formulation have been discussed in detail in (**7**) and will not be repeated here. The invariants I_1, I_2 are given by

$$I_1 = 2\lambda_z^{-1} + \lambda_z^2(1+\kappa^2), \quad I_2 = 2\lambda_z + \lambda_z^{-2} + \lambda_z\kappa^2. \tag{5.15}$$

5.2.1 *Axial magnetic field.* Following the development by Dorfmann and Ogden (**7**), it is convenient to select the formulation based on $\boldsymbol{\Omega}^*$, with the constitutive equations (4.7) and (4.13) for the total stress $\boldsymbol{\tau}$ and the magnetic induction $\mathbf{B}$, respectively.

Consider, for example, an axial magnetic field, which, in the deformed configuration, has component H_z. From the field equation $(2.2)_1$ we conclude that H_z is constant. In the reference configuration, the magnetic field is given by $\mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}$, the only non-zero component of which is

$$H_{lZ} = \lambda_z H_z. \tag{5.16}$$

From (3.19) the corresponding invariants are

$$K_4 = H_{lZ}^2, \quad K_5 = (1+\kappa^2)\lambda_z^2 K_4, \quad K_6 = [\kappa^2\lambda_z + (1+\kappa^2)^2\lambda_z^4]K_4. \tag{5.17}$$

The non-zero components of the total stress $\boldsymbol{\tau}$ are obtained from (4.7). Thus,

$$\begin{aligned} \tau_{rr} &= -p + \gamma_1^*\lambda_z^{-1} + \gamma_2^*\lambda_z^{-2}, \\ \tau_{\theta\theta} &= -p + \gamma_1^*(\lambda_z^{-1} + \lambda_z^2\kappa^2) + \gamma_2^*(I_1\lambda_z^2\kappa^2 + \lambda_z^{-2}) + \gamma_5^*\kappa^2\lambda_z^2K_4 \\ &\quad + 2\gamma_6^*\kappa^2\lambda_z^2[\lambda_z^{-1} + \lambda_z^2(1+\kappa^2)]K_4, \\ \tau_{zz} &= -p + \gamma_1^*\lambda_z^2 + \gamma_2^*\lambda_z^4(1+\kappa^2) + \gamma_5^*\lambda_z^2K_4 + 2\gamma_6^*\lambda_z^4(1+\kappa^2)K_4, \\ \tau_{\theta z} &= \gamma_1^*\lambda_z^2\kappa + \gamma_2^*\lambda_z\kappa(1 + \lambda_z^3 + \lambda_z^3\kappa^2) + \gamma_5^*\kappa\lambda_z^2K_4 + \gamma_6^*\kappa\lambda_z[1 + 2\lambda_z^3(1+\kappa^2)]K_4, \end{aligned} \tag{5.18}$$

and, from (4.13), the components of **B** are found to be $B_r = 0$ and

$$B_\theta = -\{\gamma_4^* \lambda_z^2 + \gamma_5^*[\lambda_z + \lambda_z^4(1+\kappa^2)] + \gamma_6^*[1 + \lambda_z^3(1+2\kappa^2) + \lambda_z^6(1+\kappa^2)^2]\}\kappa H_z,$$
$$B_z = -\{\gamma_4^* + \gamma_5^* \lambda_z^2(1+\kappa^2) + \gamma_6^*[\kappa^2 + \lambda_z^3(1+\kappa^2)^2]\}\lambda_z^3 H_z. \tag{5.19}$$

It then follows from (4.9) that the components of the axial vector $(\boldsymbol{\tau}\mathbf{b} - \mathbf{b}\boldsymbol{\tau})_\times$ are

$$[\kappa \lambda_z^2 H_z^2\{\gamma_6^*[1 + \lambda_z^3(1+\kappa^2)] + \gamma_5^* \lambda_z\},\ 0,\ 0], \tag{5.20}$$

where use has been made of the components in (5.18) and $(5.13)_2$.

The components of the vector **bH** are $\lambda_z^2 H_z(0,\ \kappa,\ 1)$. Referring to (4.10) it follows that, for the considered combination of deformation and magnetic field, the universal relation (4.10) is satisfied identically. This example is included to illustrate that non-trivial universal relations do not always arise. This can also be shown to be the situation if the formulation based on $\boldsymbol{\Omega}$ is used with an axial magnetic induction vector **B**. Equally, if one considers a circular cylindrical tube subject to a cylindrically symmetric deformation in the presence of a circumferential magnetic field the general universal relation is again satisfied trivially. We do not give the details for these cases.

5.3 *Helical shear*

In this section we consider the problem of helical shear for a right circular cylindrical tube with internal and external radii A and B, respectively, in the reference configuration, the material being confined within the annular region $A \leqslant R \leqslant B$. This deformation reveals a different type of universal relation, one that is cubic rather than linear in the components of $\boldsymbol{\tau}$. We summarize briefly the relevant equations given by Dorfmann and Ogden (**7**). Helical shear is defined by the equations

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R) \tag{5.21}$$

and

$$0 < A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad -\infty \leqslant Z \leqslant \infty, \tag{5.22}$$

where $g(R)$ and $w(R)$ are deformation functions to be determined by solution of the governing equations together with the boundary conditions. In the context of elasticity theory a cubic universal relation was derived in (**23**) and examined from a more general perspective in (**24**). Here we show that when the constitutive law is specialized this same universal relation holds in the present situation with either an axial or a circumferential magnetic field.

The components of the deformation gradient **F** and the left and right Cauchy–Green tensors **b** and **c**, referred to the relevant cylindrical polar coordinate axes, are represented by the matrices F, b and c, which are given, respectively, by

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1+\kappa_\theta^2 & \kappa_\theta\kappa_z \\ \kappa_z & \kappa_\theta\kappa_z & 1+\kappa_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1+\kappa^2 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}. \tag{5.23}$$

Here and in what follows we use the notation

$$\kappa_\theta = r g'(r), \quad \kappa_z = w'(r), \quad \kappa = \sqrt{\kappa_\theta^2 + \kappa_z^2}, \tag{5.24}$$

and we treat g and ω as functions of $r\ (= R)$ and set $a = A, b = B$. For further details of the kinematics, we refer to (**7**).

Now, since $\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{H}_l$, it follows that if we take the radial component of $\mathbf{H}_l$ to vanish and write its other two components as $H_{l\Theta}, H_{lZ}$ then, for the considered deformation, $\mathbf{H}$ has components

$$H_r = -\bar{H}, \quad H_\theta = H_{l\Theta}, \quad H_z = H_{lZ}, \tag{5.25}$$

where, for convenience, we have defined $\bar{H} = \kappa_\theta H_{l\Theta} + \kappa_z H_{lZ}$.

It follows that the principal invariants (3.15) and (3.19) are given by

$$I_1 = I_2 = 3 + \kappa^2, \quad K_4 = H_{l\Theta}^2 + H_{lZ}^2, \quad K_5 = K_4, \quad K_6 = K_4 + \bar{H}^2. \tag{5.26}$$

The resulting components of $\boldsymbol{\tau}$ are calculated from (3.20) as

$$\tau_{rr} = -p + 2(\Omega_1^* + 2\Omega_2^*), \tag{5.27}$$

$$\tau_{\theta\theta} = -p + 2\Omega_1^*(1 + \kappa_\theta^2) + 2\Omega_2^*(2 + \kappa^2) + 2\Omega_5^* H_\theta^2 + 4\Omega_6^* H_\theta(H_\theta + \kappa_\theta \bar{H}), \tag{5.28}$$

$$\tau_{zz} = -p + 2\Omega_1^*(1 + \kappa_z^2) + 2\Omega_2^*(2 + \kappa^2) + 2\Omega_5^* H_z^2 + 4\Omega_6^* H_z(H_z + \kappa_z \bar{H}), \tag{5.29}$$

$$\tau_{r\theta} = 2(\Omega_1^* + \Omega_2^*)\kappa_\theta + 2\Omega_6^* H_\theta \bar{H}, \tag{5.30}$$

$$\tau_{rz} = 2(\Omega_1^* + \Omega_2^*)\kappa_z + 2\Omega_6^* H_z \bar{H}, \tag{5.31}$$

$$\tau_{\theta z} = 2\Omega_1^* \kappa_\theta \kappa_z + 2\Omega_5^* H_\theta H_z + 2\Omega_6^*[(2 + \kappa^2) H_\theta H_z + K_4 \kappa_\theta \kappa_z], \tag{5.32}$$

while the corresponding components of $\mathbf{B}$ are obtained from (3.21) as

$$B_r = -2[\Omega_5^* + \Omega_6^*(2 + \kappa^2)]\bar{H}, \tag{5.33}$$

$$B_\theta = -2[(\Omega_4^* + \Omega_5^* + \Omega_6^*)H_\theta + \{\Omega_5^* + (3 + \kappa^2)\Omega_6^*\}\kappa_\theta \bar{H}], \tag{5.34}$$

$$B_z = -2[(\Omega_4^* + \Omega_5^* + \Omega_6^*)H_z + \{\Omega_5^* + (3 + \kappa^2)\Omega_6^*\}\kappa_z \bar{H}]. \tag{5.35}$$

Now, as discussed in (**7**), we must have $B_r = 0$ to avoid a singularity on $r = 0$, and this requires that either

$$\Omega_5^* + \Omega_6^*(2 + \kappa^2) = 0 \tag{5.36}$$

or $\bar{H} = 0$. Here we consider the case in which $\bar{H} \neq 0$, so that (5.36) must hold. This places restrictions on the admissible class of constitutive laws (see (**7**) for details). Here, for illustration, we consider a special case of this restriction for which $\Omega_5^* = \Omega_6^* = 0$, so that Ω^* depends on the magnetic field only through K_4 and the components (5.27) to (5.35) specialize accordingly.

Most of the counterparts for Ω^* of the special cases considered in section 4.1 for Ω are then satisfied trivially, but it is easy to show from (5.28) to (5.32) that the nonlinear universal relation

$$\tau_{\theta z}(\tau_{r\theta}^2 - \tau_{rz}^2) = \tau_{r\theta}\tau_{rz}(\tau_{\theta\theta} - \tau_{zz}) \tag{5.37}$$

holds. Thus, the universal relation (5.37) found for the purely (incompressible, isotropic) elastic case (**23**) holds also for an incompressible isotropic magnetoelastic solid under the considered restrictions. There are several variants of this result that can be considered but the present one suffices for illustration.

Examples of solutions for helical shear boundary-value problems are given in (**7**) and are not examined further here.

6. Closing remarks

In this paper we have shown that for the constitutive equations of an isotropic magnetoelastic solid only one universal relation exists in the general case, while specialization of the constitutive law allows additional universal relations, not included in the previous paper by Dorfmann *et al.* (**8**), to be formulated. Such universal relations can in principle be used by the experimenter to determine whether the particular class or subclass of materials can be described by a given model or if a wider class of models needs to be considered. In this context it should be noted that exact analytical solutions can be found only for particular geometries such as an infinite tube or a slab of infinite extent in the two in-plane directions. If, on the other hand, we consider, for example, a tube of finite length then the jump conditions given by (2.12) cannot all be satisfied on both the lateral and end boundaries of the tube for the considered uniform axial field, so that edge effects are present and an exact solution cannot be determined for the whole tube and its exterior. Numerical methods must therefore be used to obtain the distribution of the magnetic field and magnetic induction, inside and outside the body. A detailed description and application of numerical solutions to the boundary-value problem for nonlinear magnetoelastic solids is under development and will form a forthcoming publication.

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