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UNIVERSALITY AND SELF-SIMILARITY IN THE BIFURCATIONS OF CIRCLE MAPS

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The bifurcation structure in a two-parameter family of circle maps is considered. These maps have a (topological) degree that may be different from one. A generalization of the rotation number is given and symmetries of the bifurcations in parameter space are described. Continuity arguments are used to establish the existence of periodic orbits. By plotting the locus of parameter values associated with superstable cycles, self-similar bifurcations are found. These bifurcations are a generalization of the familiar period-doubling cascade in maps with one extrema, to two-parameter maps with two extrema. Finally, a scheme for the global organization of bifurcations in these maps is proposed.

1. Introduction

Our interest in the bifurcations of circle maps arises from studies on the effects of brief periodic electrical stimulation on spontaneously beating aggregates of embryonic chick heart cells [1–4]. In these experiments there are two independent parameters, the frequency and amplitude of the periodic stimulus. The dynamics are determined as a function of the frequency and amplitude of the periodic stimulation. If the autonomous oscillation is described by a strongly attracting limit cycle which is rapidly reestablished following a stimulus, then the effects of periodic stimulation are well described by a circle map, i.e. a map of the unit circle S^1 into itself, which can be experimentally measured,

$$x_{t+1} = g(x_t; b) + \tau = f(x_t; b, \tau). \qquad (1)$$

In eq. (1), x_t represents the phase in the cycle of a stimulus, τ represents the time between stimuli, and b is a parameter which depends on stimulus strength. Theoretical computations of predicted dynamics using eq. (1) show close agreement with experiment [1, 3, 4].

There have been numerous studies in which dynamics in physical and biological systems are analyzed using one-dimensional circle maps [5–29, 31–34]. A particularly simple functional form (often called the "canonical" example) for circle maps which arises in diverse contexts is

$$x_{t+1} = Mx_t + b\sin 2\pi x_t + \tau \pmod 1, \qquad (2)$$

where M is an integer which gives the (topological) degree of f, and b and τ are real parameters. This function with $M = 0$ has been proposed as a model for hybrid optical devices [10–13] and periodically stimulated biological cells [14], and with $M = 1$ as a model for a periodically stimulated nonlinear oscillator [9], periodically forced

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Josephson junction [15, 16], and periodically stimulated biological oscillators [3, 17–19]. The canonical example with $M = 1$ also arises as a special case of the standard map, and has been analyzed as a general model of the quasiperiodic route to chaos [20–29]. Several recent studies of eq. (1) with $M = 1$ use renormalization techniques at the value $b = 1/2\pi$ [25–28]. We know of no application which treats cases with $M > 1$. However, the map with $M = 2$, $b = 0$ is often given as a textbook example of a map which displays ergodic dynamics [30, 31]. In this paper the canonical map is used in numerical studies to exemplify many of the more general mathematical results. Other functional forms for circle maps have also been proposed [2, 4, 7, 8, 33, 34].

Starting from an initial condition, x_0, one can iterate eq. (1) to generate a sequence $x_0, x_1 = f(x_0), \ldots, x_n = f^n(x_0)$. A periodic orbit will arise if $x_{i+n}^* = x_i^*$, $x_{i+j}^* \neq x_i^*$ for $1 \leq j < n$. Let

$$\lambda = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=x_i^*} f^{(n)}(x) = \prod_{i=1}^{n}\left(\frac{\partial f}{\partial x}\right)\bigg|_{x=x_i^*}. \tag{3}$$

The stability of the periodic orbit is determined by the value of λ (as seen using the mean value theorem): for $|\lambda| < 1$, the orbit is stable, and if $|\lambda| \neq 1$, it is called hyperbolic. When $|\lambda| = 1$, there is a bifurcation of the periodic orbit [35]. In generic situations, there are two definite possible bifurcations, depending upon the sign of λ. Suppose the map f depends on a parameter μ, and that at $\mu = \mu_0$, $|\lambda| = 1$. When $\lambda = 1$, for values of μ close to μ_0 there is either no periodic orbit (say for $\mu < \mu_0$) or two orbits of period n, one stable and the other unstable (say for $\mu > \mu_0$). There is a tangent, or saddle-node, bifurcation at $\mu = \mu_0$.

When $\lambda = -1$, depending on the sign of $\mu - \mu_0$, for μ small, there is either a stable orbit of period n, or a stable orbit of period $2n$ and an unstable orbit of period n. The point $\mu = \mu_0$, where $\lambda = -1$, is a period-doubling, or flip, or subharmonic, bifurcation point. As stated before, these two types of bifurcation cover the generic case, but other possibilities do exist [35]. Clearly, if an extremal point, at which $(\partial f/\partial x) = 0$, is a periodic point on a cycle, the cycle will be stable and is then called a superstable cycle.

Studying the iterates of extremal points has played an important role in the analysis of one dimensional maps [36, 37]. In maps with two parameters, we call the locus of superstable periodic points as a function of those parameters the skeleton. The main justification for the use of the skeleton is a nondegeneracy assumption, that all stable periodic orbits are "created" through a generic bifurcation, either flip or saddle-node, and lose their stability through a flip bifurcation. For an orbit of period n, the derivative of f^n must therefore go from 1 to -1 in the process and will cross the origin. Stated otherwise, we assume that close to any stable periodic orbit is a superstable orbit. Several of the figures in this paper give the skeleton for members of the canonical family.

Another concept which has played an important role in the study of circle maps is the rotation number, designated ρ [38, 39]. The rotation number counts the average increment in x per iteration of the map. In our studies of the periodically stimulated heart cells, the rotation number is approximately the number of cardiac contractions divided by the number of electrical stimuli. If there is a stable periodic pattern consisting of N stimuli and M contractions we say there is $N{:}M$ phase locking with $\rho = M/N$ (i.e. ρ is rational).

A problem of general interest in the analysis of circle maps is to characterize the dynamics for the canonical family eq. (2) for fixed M as a function of x_0, b, τ. At the present time a complete analysis of this difficult problem has not yet been achieved. However, in the following we show that by using comparatively elementary techniques, several properties of the global bifurcations of circle maps can be found, and other properties can be conjectured. In section 2 properties of rotation numbers are summarized and symmetries are described. In section 3 we state several results concerning the existence of fixed points of circle maps. A main result of this section holds for all circle maps of degree one, and monotonic maps of degree $M > 1$:

for *any* initial condition $x \in S^1$, there exists at least one value of τ such that x is a fixed point of rotation number ρ, for any value of ρ. Stable dynamics in physical systems correspond to existence of stable fixed points in these maps. In the case that g is a circle map of degree one with two extrema, then for each rational rotation number there are at least two values of τ giving rise to superstable orbits. We also give a new construction for the Arnol'd tongues [40, 41] in degree 1 monotonic circle maps. In section 4 we describe the skeleton computed numerically in the "canonical" example. The skeleton for degree 0 circle maps shows self-similarity in the two-parameter space (fig. 3). The skeleton of the degree 0 circle map, is apparently repeated in an orderly way in circle maps of higher topological degree (fig. 6). The implications of these results for experiments are discussed in section 5.

2. Rotation numbers and symmetries in circle maps

2.1. *Rotation numbers*

Nonlinear oscillators perturbed by periodic pulsatile stimuli can in some limits be described by circle maps of topological degree 0 [2]. Therefore, it is useful to have a definition for the rotation number for circle maps of topological degree different from 1. In [2] a definition was proposed which is equivalent to the original definition used by Poincaré [42], but is different from definitions currently used. In this section we define the rotation number for maps of degree one and extend the definition to more general maps.

Consider the continuous map $F: \mathrm{R} \to \mathrm{R}$ with the symmetry $F(x+1) = F(x) + M$ for all $x \in \mathrm{R}$. By considering $f = F \pmod 1$ we restrict the function F to the circle S^1 and thus define a map $f: S^1 \to S^1$. The function F is called the lift of f, and M is the degree. The iterates of the lift are continuous. When f is a monotonic map of degree 1 (an orientation preserving homeomorphism) the rotation number can be defined

$$\rho(f, x) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}.$$

This limit exists and is independent of x. The rotation number is rational for a periodic orbit. If the rotation number is irrational the dynamics are called quasiperiodic.

When f has degree one but is not monotonic (i.e. it is not an invertible map and not a homeomorphism), a supremum must be used in the limit, and each map gives rise to a rotation set $R_f = \{\omega \in \mathrm{R} \mid \rho(f, x) = \omega$ for some $x \in S^1\}$ [29, 43–45]. Here the rotation number may depend on the initial condition.

When the degree of f is not necessarily one, an auxiliary function is defined: for any $x \in S^1$, $x_0 = y_0$ and, for $i \geq 1$,

$$\begin{aligned} x_i &= f(x_{i-1}) = F(x_{i-1}) \pmod 1, \\ \Delta_{i-1}(x_{i-1}) &= F(x_{i-1}) - x_{i-1}, \\ y_i(x_0) &= \sum_{j=0}^{i-1} \Delta_j(x_j) = y_{i-1}(x_0) + \Delta_{i-1}(x_{i-1}). \end{aligned} \tag{4}$$

We then define the rotation number

$$\rho(f, x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} \Delta_j(x_j).$$

This definition is equivalent to standard definitions when f has degree one. However, when f has degree different from one the definitions are not equivalent. To illustrate, consider the map $f(x) = 2x \pmod 1$ which has a cycle of period 3, $1/7 \to 2/7 \to 4/7$. If we use the lift to compute ρ we have

$$\frac{F^3(1/7) - 1/7}{3} = \frac{1}{3}, \quad \frac{F^3(2/7) - 2/7}{3} = \frac{2}{3}$$

$$\text{and} \quad \frac{F^3(4/7) - 4/7}{3} = \frac{4}{3}.$$

Clearly, the lift cannot be used to define the rotation number since all points of a cycle must be

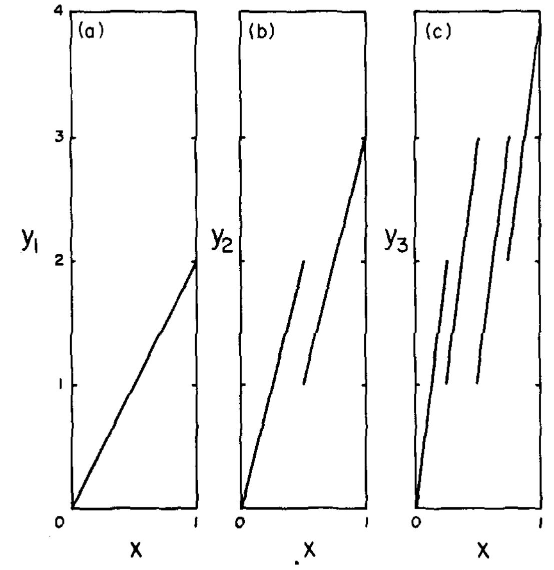

Fig. 1. Graph of the auxiliary function $y_i(x)$ defined by eq. (4) for the special case $f(x)=2x$; a) $y_1(x)$; b) $y_2(x)$; c) $y_3(x)$.

associated with the same rotation number. However, applying the definition used here we see that $\rho(1/7) = \rho(2/7) = \rho(4/7) = 1/3$. One consequence of the definition is that the rotation number is no longer a continuous function of x. This can be seen in fig. 1 which shows $y_i(x)$ for $f(x)=2x$. As well, in non-invertible maps cycles of different periods and chaotic dynamics can all have the same rotation number. Such behavior was observed in the bifurcations of degree 0 and non-monotonic degree-1 maps modelling periodically forced oscillations [1, 3, 4]. Despite its limitations in the analysis of non-invertible maps, the rotation number often has a simple physical interpretation and consideration of its properties is of interest.

2.2. *Symmetries*

There are symmetries in the periodic points and their rotation numbers. First consider any circle map of degree M, eq. (1). Using the definition for the rotation number we find that if x^* is a periodic point of period N with $\rho = K/N$ for $\tau=\tau^*$, then x^* will be a periodic point of period N with $\rho=(K+N)/N$ for $\tau=\tau^*+1$. Thus when considering the bifurcations of eq. (1) we need only consider values of $0 \le \tau \le 1$ since there is a translational symmetry in τ.

Now consider the canonical map eq. (2). This map shows the symmetry

$$g(1-x; b) = M - g(x; b).$$

Assume that x^* is a periodic point of period N with $\rho = K/N$ at $\tau=\tau^*$ $(0 \le \tau \le 1)$. Then $1-x^*$ will be a periodic point of period N with rotation number $\rho=(MN-K)/N$ at $\tau = 1-\tau^*$. Proof of this result follows along the lines sketched out in [2].

3. Periodic points

In this section we discuss results concerning the existence and stability of periodic points in circle maps. We first consider linear maps for which a complete description of periodic points is possible. Then we establish the existence of periodic points using the intermediate value theorem. Finally, we use the implicit function theorem to demonstrate continuation of the locus of periodic points in parameter space.

3.1. *Linear maps*

The map defined by eq. (2) reduces when $b=0$ and $\tau=0$, to the well-studied map $f(x)=Mx$ (mod 1). A general analyzis of its statistical properties has been performed by Rényi [46], who allowed M to be any positive real number. Shub [47] has also studied expanding maps of the circle which included linear maps of degree greater than 1 as a special case. None of these earlier studies considered the rotation number for maps with degree greater than one. A detailed description of the periodic points in linear maps can be developed. We briefly recount results for the linear map of degree 2, $f(x)=2x+\tau$. Results for linear maps of other degrees can be readily derived.

First take $\tau = 0$ and associate each point $x \in [0, 1)$ with its binary expansion. The action of the map on x is the same as the shift map on the set of sequences of two symbols. Counting the number of periodic orbits is a classic problem of combinatorics [48] and is equivalent to the counting of periodic orbits in the map $\lambda x(1-x)$ when it is onto [49]. For $\tau = 0$, all points having eventually periodic binary expansions are periodic and their rotation number is the arithmetic mean of the elements in one basic period of the periodic part of their expansion. For example, the points $x = 1/15$ and $x = 3/7$ have binary expansions 0.0001 and 0.011 and rotation numbers 1/4 and 2/3, respectively.

All fixed points of the map with $\tau = 0$ continue in the x, τ plane in a simple way. In particular, the locus of fixed points in the x, τ plane fall on the family of straight lines $x = q/(2^N - 1) - \tau$ where q and N are integers.

3.2. *Periodic points in nonlinear circle maps*

The continuity of the lift of circle maps can be used to show the existence of a minimal number of periodic points. Consider the circle map $f(x; \tau) = g(x) + \tau$ of degree M with lift $F(x; \tau)$. By iteration of the lift we compute

$$F^n(x + p_1; \tau + p_2) = F^n(x; \tau) + p_1 M^n + p_2 \sum_{j=0}^{n-1} M^j, \tag{5}$$

where p_1 and p_2 are integers.

Now fix $x = x_0$ and consider the fixed points as τ varies from τ^* to $\tau^* + 1$. Each value of τ for which $(F^n(x_0; \tau) - x_0)$ takes an integer value gives a periodic orbit. Hence from the intermediate value theorem there must be at least $\Sigma_{j=0}^{n-1} M^{j-1}$ different values of τ between τ^* and $\tau^* + 1$ for which x_0 is a periodic point of the map f^n. The rotation number of these periodic orbits is only accessible through the auxiliary function y_n defined in eq. (4). Its discontinuity when $M > 1$ restricts its use in straightforward applications of the intermediate value theorem.

However the following result has been obtained for the map $f(x; \tau) = g(x) + \tau$ with $g(x)$ a map of degree 1 or a monotonic map of degree $M > 1$:

Theorem. For any rational number k/n and any point x_0 on the circle, there is a value of τ for which x_0 is a periodic point of $f(x; \tau)$ with period n and rotation number k/n.

The proof of this result for $M = 1$ follows from the continuity of the lift, the intermediate value theorem and the translational symmetry of the rotation numbers [19]. The proof of the result for monotonic maps of higher degree follows since the only discontinuities of the function y_n, considered as a function of τ are "decreasing" jumps. From numerical studies we conjecture that the theorem holds more generally for all circle maps with degree $M > 1$. Stability of the periodic points identified in the above results is guaranteed if the initial condition x_0 is an extremum of $f(x; \tau)$. In this case the periodic points are on superstable cycles. When $f(x; \tau)$ has multiple extrema, two different extrema may be periodic points for the same value τ. For example when $f(x; \tau) = x + b \sin 4\pi x + \tau$, a case considered as a model for the periodically forced Van der Pol oscillator [50], such degeneracy occurs and the skeleton is identical to the skeleton of eq. (2) with $M = 1$.

Now fix $\tau = \tau_0$ and consider fixed points as x varies from x^* to $x^* + 1$. Each value of x for which $(F^n(x, \tau_0) - x)$ takes an integer value gives a periodic orbit. From eq. (5), there must be a minimum of $M^n - 1$ different values of x in S^1 which are periodic points of the map f^n.

3.3. *Continuation of periodic orbits*

The results in the preceding section treat fixed points in the map $f(x; \tau) = g(x) + \tau$. We now let g depend on a parameter b, and consider the case in which g is monotonic.

Consider the function

$$H(x, b, \tau) = F^n(x; b, \tau) - x - J, \tag{6}$$

where J, n are integers and n is positive, and F is a lift of the map defined by eq. (1); then a point in 3 space $(x; b, \tau)$ where $H = 0$ corresponds to a periodic orbit of period n for f.

Suppose that a periodic orbit exists for a fixed period n at a point x^*, and parameter values (b_0, τ_0); both $\partial H/\partial b$ and $\partial H/\partial \tau$ are easily computed, and, since the latter derivative is non-zero whenever g is monotonic, the implicit function theorem yields the existence of a curve $\tau = \tau(b)$ in the parameter space (τ, b) along which the point x^* is a periodic point of period n. The monotonicity of g is a sufficient but not a necessary condition for a periodic point to be continuable.

These observations lead to a new construction of the structure identified by Arnol'd, the so-called "Arnol'd tongues", describing the rotation number distribution in a two-parameter family of monotonic degree-1 maps [40, 41].

The map $f(x; b, \tau) = x + \tau + b \sin 2\pi x$ is readily seen to have rotation number p/q, when $b \in [0, 1/2\pi]$ and $\tau \in [0, 1]$, if and only if there is a point on the circle corresponding to a value of the parameters (τ, b) lying on one of the continuation lines described above. Let $\tau = \tau(b)$ be a line for which a given point x^* is a periodic point with rotation number p/q. The projection of the family of such continuation lines with the same rotation number for all points $x \in S^1$ on the plane $x = 0$ is called the Arnol'd tongue. The order of these tongues along the τ-axis must remain the same for all values of b between 0 and $1/2\pi$ in view of the uniqueness of the rotation number of f for these parameter values. The compactness of the tongues also follows from this uniqueness and the periodicity of $f(x; b, \tau)$ in the variable τ. If the periodic orbit existing at $(b, \tilde{\tau})$ is hyperbolic, then its structural stability implies that there is an interval of values of τ about $\tilde{\tau}$ for which this orbit is preserved.

We thus see that properties of the dependence of the rotation number on parameters, which have been considered before [41], can be obtained by very simple arguments using the implicit function theorem. These arguments, however, cannot be used to detect the appearance of new periodic points via saddle-node bifurcations where the map is non-monotonic.

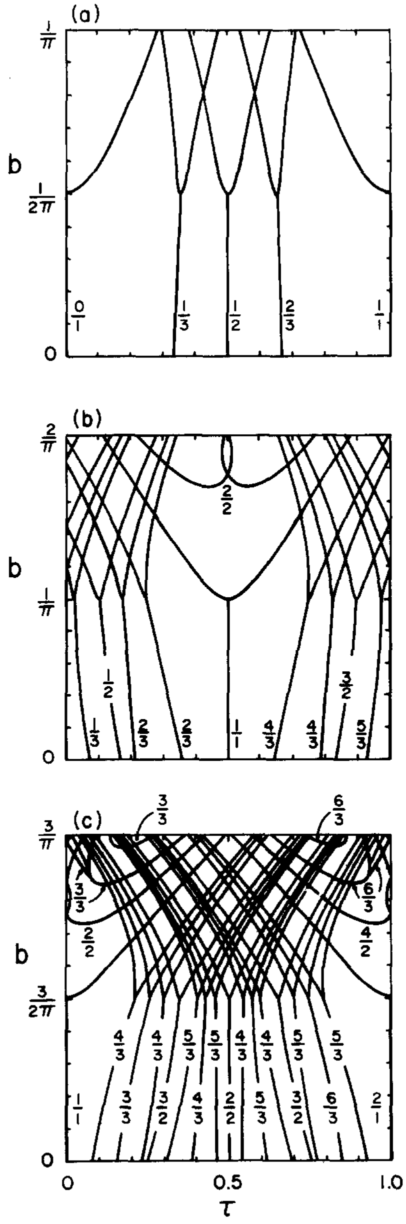

Fig. 2. Skeleton lines $(b > M/2\pi)$ and branches for which $x = 0.5$ is a periodic point $(b < M/2\pi)$ for the canonical map (eq. (2)). The rotation number of the cycle at the bottom of the stem is indicated. a) $M = 1$; b) $M = 2$; c) $M = 3$.

We conclude this section by presenting a numerical computation for the canonical family (eq. (2)), with $M = 1, 2, 3$ which illustrates some of our main results (fig. 2). For $0 \leq b \leq M/2\pi$, the canonical family is monotonic. For this region we plot the locus of points for which $x = 0.5$ is a fixed point of period N, $N = 1, 2, 3$. Since the map is monotonic the value of τ for which $x = 0.5$ is a fixed point for given period at $b = 0$ is uniquely continuable in the (b, τ) plane. For $b > M/2\pi$ there are two critical points which bifurcate from $x = 0.5$ as b increases. For $b > M/2\pi$ we plot the skeleton up to period 3. Note that the two branches of the skeleton of each period are required by the result of section 3 when $M = 1$, and consistent with our conjecture when $M > 1$. As well the symmetries described in section 2.2 are found. The description thus far has not considered the higher order bifurcations. (Beginnings of these higher order bifurcations can already be seen in fig. 2 for the degree-2 and -3 maps.)

4. Skeletons

We now turn to a numerical study of the superstable orbits and bifurcations of the families defined by eq. (2). We first consider degree 0 circle maps and then maps of higher degree.

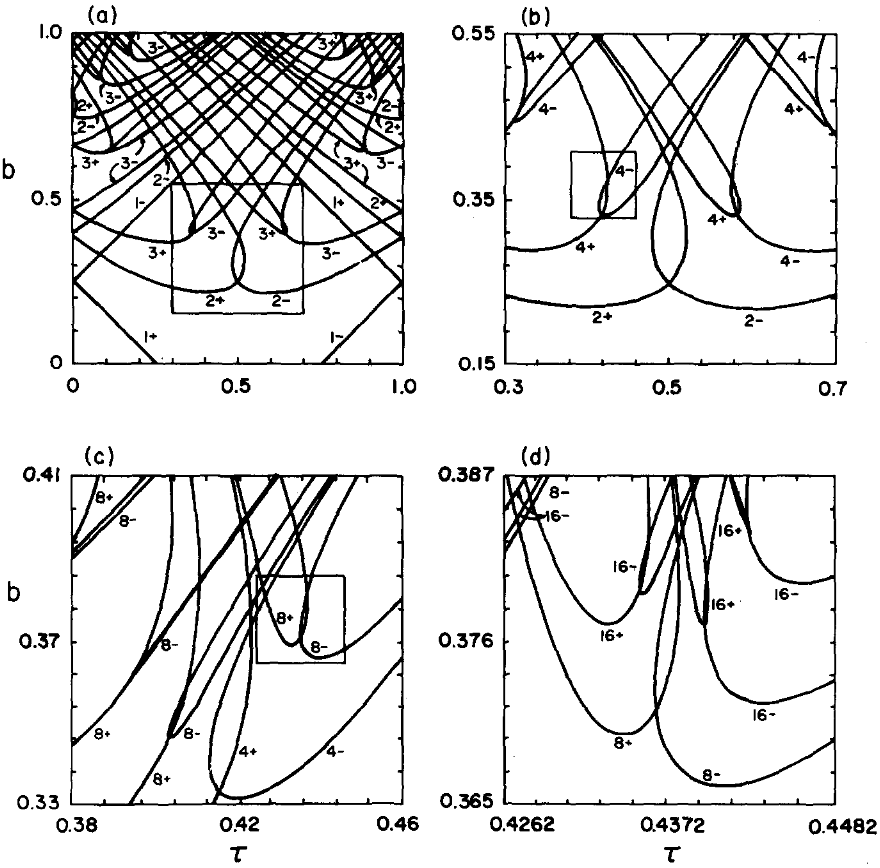

Fig. 3. Skeleton for the map of eq. (2) with $M = 0$; $k+$ represents a cycle of period k associated with the maximum and $k-$ a cycle of period k associated with the minimum. a) Locus of superstable cycles of periods 1,2,3; not all cycles of period 3 are labelled; b) Enlargement of the square in (a); periods 2 and 4 are now shown; c) Enlargement of the square in (b); periods 4 and 8 are shown; d) Enlargement of the square in (c), showing periods 8 and 16. Notice the topological equivalence, especially between (b) and (d).

4.1. *Maps of degree zero*

We consider first the maps of degree zero $f(x, b, \tau) = \tau + b \sin 2\pi x$. In fig. 3, we show the lines of the skeleton for chosen regions of the parameter plane (b, τ), and for appropriate periods. In fig. 3a, the skeleton is shown for periods up to 3. Branches of the skeleton associated with the minimum are denoted by a $-$ sign, those associated with the maximum by a $+$ sign. In fig. 3b, we show magnification of the square enclosed in fig. 3a, and the lines of the skeleton associated with periods 2 and 4; this process of magnification is repeated from fig. 3b to fig. 3c, and from fig. 3c to fig. 3d, for periods 4 and 8, and 8 and 16 respectively.

The same computations have previously been performed on a piecewise quadratic degree-0 circle map [14], leading to essentially the same results. This further supports our observation that the structure we describe is topological in nature, and independent of the precise functional nature of the maps. This self-similarity, repetition of the same geometric features on different length scales, has also been found for other period-doubling sequences (e.g. $3, 6, 12, \ldots$).

The skeleton does not give direct information about the bifurcations in the map. However, there is a close connection between the skeleton and the lines of bifurcations which is revealed from numerical studies. In fig. 4 we show the regions of stable period-2 orbits and the associated bifurcation lines and superstable cycles. Note the cusp shaped border formed by the lines of tangent bifurcation. Just as the skeleton has a self-similar structure in parameter space, so do the zones of stable orbits. In fig. 5, we give a schematic diagram of the zones of stable periodic orbits which emphasizes the self-similar nature of these zones. This structure has been previously described in degree-1 circle maps [22–24]. A self-similar structure has also been observed in one parameter families of maps with one critical point [51]. Figs. 3 and 5 generalize the familiar period-doubling sequence to a map with two extrema and two parameters.

On the line $\tau = 0, 0 \leq b \leq 0.5$, two distinct one-parameter maps with one critical point are defined. Therefore, on this line, the bifurcations

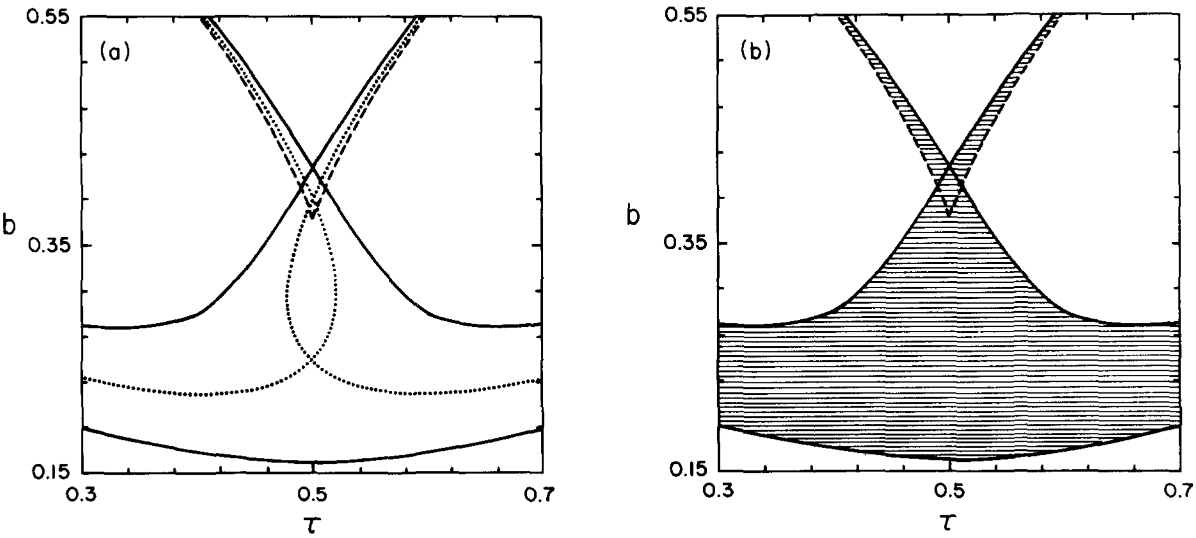

Fig. 4. Lines of the skeleton and locus of bifurcations for the map of eq. (2) with $M = 0$; a) Solid lines show flip bifurcations, dashed lines show saddle-node bifurcations, and dotted lines show period 2 superstable cycles; b) the zone of existence of a period 2 stable cycle is hatched.

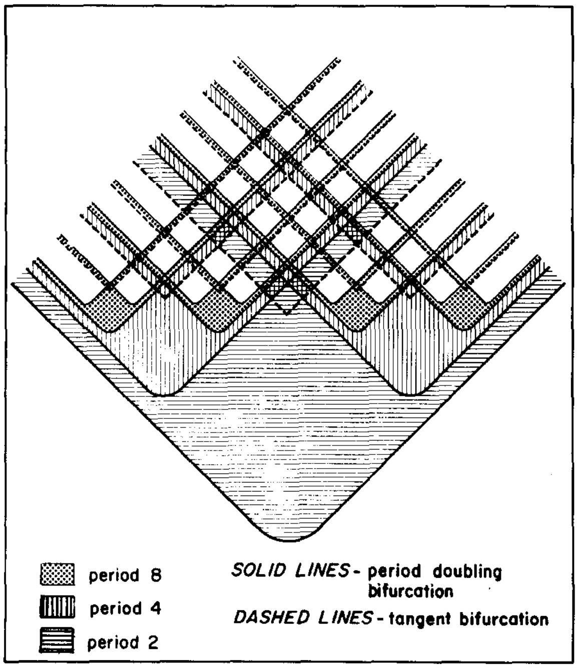

Fig. 5. Schematic (conjectured) organization of the self-similar period-doubling bifurcations in two-parameter maps with two extrema. Bifurcation lines and locus of superstable cycles are shown. (Drawn after the style of Mondrian and Mandelbrot). See also the drawing in [23].

observed are the same as in interval maps with one maximum. For the region $0 \le \tau + b \le 0.5$, the interval $(0, 0.5)$ is invariant and the branches of the skeleton extend from the line at $\tau = 0$. In this region, the behavior is thus a straightforward extension of the results about interval maps with one maximum; in particular, for a fixed τ, as b increases, the order of appearance of periodic orbits is described by the U-sequence [36]. Further discussion of bifurcations in this map follows the same arguments in [14].

There are two types of superstable cycles possible: "doubly" superstable cycles, when both critical points are on a unique cycle, and "bistable" cycles, due to co-existence of two periodic orbits, each one having only one critical point [22–24, 52].

4.2. *Maps of degree different from zero*

The skeleton of the map of degree zero recurs in the bifurcation diagrams of maps of other degrees.

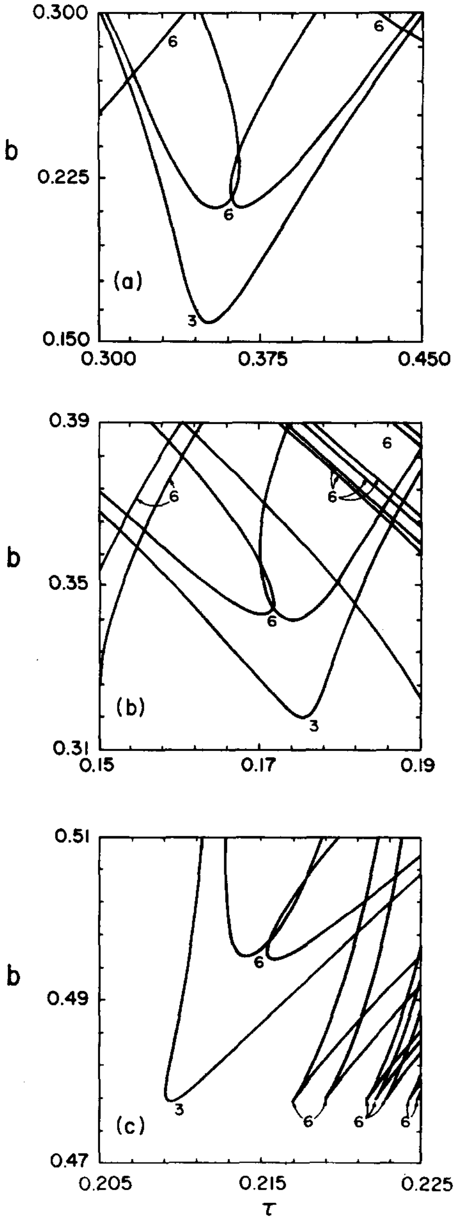

Fig. 6. Skeleton lines near the values of b where the map defined in eq. (2) loses its monotonicity; only cycles of periods 3 and 6 are represented. a) $M = 1$; b) $M = 2$; c) $M = 3$.

Specifically, consider first the map of degree 1, $f(x; b, \tau) = x + \tau + b \sin 2\pi x$, for values of b where the map is not monotonic, $b > 1/2\pi$.

Within each of the phase locking zones described in the last section, we observe a global arrangement of the skeleton lines that is topologically the one present for the degree zero map. We show in fig. 6a the skeleton inside the tongue based at $\tau = 1/3$. Tangent to the line $b = 1/2\pi$ is a branch associated with both extrema. Inside this basic branch, associated with period 3, are branches corresponding to period 6, and the latter parts of the skeleton are observed to be the first lines in a self-similar diagram essentially identical to the one of fig. 3. In general, inside a zone of $p:q$ phase-locked dynamics, there will be a main branch of the skeleton, tangent to the line $b = 1/2\pi$ [18, 19], associated with orbits of period q. This branch will enclose a structure of skeleton branches, representing periods a multiple of q (e.g. $2q$, $4q$, etc.) organized in a network topologically similar to the one of fig. 3.

As illustrated in figs. 6b and 6c, the same arrangement is present in maps of other degrees, based at the "stems" illustrated in fig. 2. It is apparent from fig. 6c that the skeleton structure becomes "denser" as the degree of the map becomes larger. Yet the skeleton maintains a remarkable crystalline beauty and simplicity. Thus, the skeleton in maps of degree greater than 1 appears to be a straightforward extension of the structure previously described in degree-1 circle maps [19].

Most results in this section are based on numerical evidence. However, as mentioned, partial analytical results for degree 1 maps have been found by Boyland [29]. His analysis mainly relies on the rotation sets of maps and as such does not give information about higher order bifurcations. Recent results using renormalization in degree-1 maps further support the numerical results presented here [24].

5. Discussion

In the preceding we have described the global bifurcations of circle maps. The theory for monotonic circle maps is straightforward. In non-monotonic maps the bifurcations are more complex. However, they nevertheless follow a simple logic and order which has been partially described.

A number of further mathematical studies are suggested by our results. Problems still to be worked out include:

1) To determine the complete skeleton of the degree-0 map. A construction analogous to the one given by Metropolis, Stein and Stein [36] for the U-sequence in the interval map is needed.

2) To determine the class of circle maps which display the same bifurcations as the "canonical" maps.

3) To determine the "interference" between the skeletons in neighboring p/q and p'/q' zones in circle maps with degree ≥ 1.

4) To determine if the results extend to maps in higher dimensions (see, for example, [49, 53]).

Despite these problems, amongst others of a mathematical nature which have not yet been resolved, we believe that the current computations provide a firm basis for experimental studies of bifurcations in diverse systems. The current computations clearly show that in a 2-parameter space the bifurcations in systems described by circle maps show an extremely delicate but regular structure. One's ability to resolve this structure would depend on intrinsic characteristics such as noise and stability of the system, as well as the care and experimental precision with which the system was investigated. In experiments on periodically stimulated cardiac cells, the experimentally observed bifurcations appear to be consistent with those theoretically predicted from iteration of an experimentally measured circle map [1, 3, 4]. However, in this biological system, there is intrinsic "noise" and it is impossible to resolve the fine details of the structure theoretically predicted.

Much greater precision should be obtainable in non-living systems. We believe that it is likely that some of the experimental systems currently being investigated will show the same complex bifurcations in a space of two parameters such as those described in this manuscript. We cite several examples which bear closer study:

1) Mackay [54] showed bifurcations in a two-parameter phase space of an electronic network having the same splitting of Arnol'd tongues observed in the degree-1 circle maps.

2) Hybrid optical systems described by degree-0 circle maps [10–13] should follow the bifurcations in fig. 4.

3) In general, a physical or biological system described by a circle map will be expected to display bistability, hysteresis, quasiperiodic dynamics, period-doubling, and intermittency. There are many examples of chemical, electronic and hydrodynamic systems [5] where these diverse dynamics are *all found in a single system* as parameters describing the system are varied. More systematic investigation of the dynamics as two or more parameters are varied over very narrow scales will be needed in order to demonstrate the structures shown in fig. 5, if they are indeed present.

4) Circle maps have been shown to occur in electronic systems [33]. It should be possible to implement an electrical circuit which is described by the canonical example with degree 1 (see the construction in [18]).

Previous workers have shown that diverse physical and biological systems display the same "universal" bifurcations found in one-parameter, one-dimensional maps. We believe that careful studies of dynamics as a function of two parameters will reveal that the global bifurcations described in this manuscript are much more common than is currently recognized.

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