Universality of Local Eigenvalue Statistics for Some Sample Covariance Matrices

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Abstract

We consider random, complex sample covariance matrices $\frac{1}{N}X^*X$, where X is a $p \times N$ random matrix with i.i.d. entries of distribution μ. It has been conjectured that both the distribution of the distance between nearest neighbor eigenvalues in the *bulk* and that of the smallest eigenvalues become, in the limit $N \to \infty$, $\frac{p}{N} \to 1$, the same as that identified for a complex Gaussian distribution μ. We prove these conjectures for a certain class of probability distributions μ.

1 Introduction

We address here the problem of universality of local eigenvalue statistics for some complex, random sample covariance matrices. Consider large random matrices $\frac{1}{N}X^*X$, X being a $p \times N$ random matrix with centered i.i.d. complex entries X_{ij} of distribution μ with variance σ^2. We will restrict ourselves to the case where $p = N + \nu$ and ν is a fixed integer. We will be interested in universal features of local properties of the spectrum in the large N limit, that is, features that do not depend on the precise details of the probability distribution μ on $\mathbb{C}$.

Before stating results about local properties of the spectrum, it is important to recall that for such random matrix ensembles, the global behavior of the spectrum has been known for a long time. Let $\lambda_1 \le \cdots \le \lambda_N$ be the ordered eigenvalues of $\frac{1}{N}X^*X$, and define $\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$ to be its spectral measure. Then, it is the first fundamental result due to [26] (see also [32]) that, as N grows to infinity, μ_N converges to the Marchenko-Pastur law with density ρ_σ, depending only on the variance σ^2 given by

$$\rho_\sigma(x) = \begin{cases} \dfrac{1}{2\pi\sqrt{x}\sigma^2}\sqrt{4\sigma^2 - x} & \text{if } 0 \le x \le 4\sigma^2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

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Beyond this global result about the whole spectrum, one is also interested in local properties of the spectrum. We will first consider local properties in the *bulk* of the spectrum for some particular ensembles of such random sample covariance matrices, that is, properties of eigenvalues in any interval $[\epsilon, 4\sigma^2(1-\epsilon)]$. Then we will turn to properties of the lower edge of the spectrum, the so-called *hard edge*, that is, properties of eigenvalues close to 0.

To study such local characteristics of the spectrum, it is convenient to define the so-called local eigenvalue statistics. Given a symmetric function $f \in L^\infty(\mathbb{R}^m)$, m fixed, with compact support, a point $u \in [0, 4\sigma^2]$, and a scaling factor ρ_N, we define the local eigenvalue statistics $S_N^m(f, u)$ by

$$S_N^m(f) = \sum_{i_1,\dots,i_m} f(\rho_N(\lambda_{i_1} - u), \dots, \rho_N(\lambda_{i_m} - u)), \tag{1.2}$$

where the sum is over all distinct indices from $\{1, \dots, N\}$. When u is in the bulk of the spectrum, the natural choice for the scaling factor is $\rho_N = N\rho_\sigma(u)$, while for the bottom edge, this factor is then given by $4N^2/\sigma^2$.

The computation of these local eigenvalue statistics is not an easy task in general. In the well-known case where the distribution μ is Gaussian with variance $\sigma^2 = 1$ (which defines the so-called Laguerre unitary ensemble (LUE) in mathematical physics or the Wishart distribution in the statistical literature), the behavior of these local eigenvalue statistics is well understood. More precisely, for $u \in [\epsilon, 4(1-\epsilon)]$, $\rho_N = N\rho_1(u)$, the following bulk asymptotics were proven by [27]: for fixed ν,

$$\lim_{N\to\infty} \mathbb{E}S_N^m(f) = \int_{\mathbb{R}^m} f(t_1, \dots, t_m) \det\left(K_{\sin}(t_i, t_j)\right)_{i,j=1}^m dt_1 \cdots dt_m,$$

where $K_{\sin}(x_i, x_j) = \frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)}$ is the so-called sine kernel.

For $u = 0$ and $\rho_N = 4N^2$, Forrester proved the *hard edge* asymptotics [12]:

$$\lim_{N\to\infty} \mathbb{E}S_N^m(f) = \int_{\mathbb{R}^m} f(t_1, \dots, t_m) \det\left(K_{\mathrm{Bes}}(t_i, t_j)\right)_{i,j=1}^m dt_1 \cdots dt_m,$$

where the *Bessel kernel* is defined by

$$K_{\mathrm{Bes}}^\nu(x_i, x_j) = \frac{J_\nu(x_i^{1/2}) x_j^{1/2} J_{\nu+1}(x_j^{1/2}) - J_\nu(x_j^{1/2}) x_i^{1/2} J_{\nu+1}(x_i^{1/2})}{2(x_i - x_j)}, \tag{1.3}$$

where J_ν is the usual Bessel function of index ν. Our goal in this work is to extend this result to the case where the distribution μ of the entries of X is not Gaussian but belongs to the class introduced by [21], which we call *Gaussian divisible*.

DEFINITION 1.1 A probability measure μ on $\mathbb{C}$ is said to be *Gaussian divisible* if it can be written

$$\mu = P \star G$$

for some probability measure P such that $\int x\, dP(x) = 0$, $\int |x|^2\, dP(x) < \infty$, and a complex-centered Gaussian law G with positive finite variance.

We will call σ^2 the variance of the probability distribution μ; in particular, we know that the limiting spectral measure of $\frac{1}{N} X^* X$ converges to the Marchenko-Pastur distribution with parameter σ.

In Section 2 we give precise statements of our results and sketch the strategy of the proof, inspired by [21]. The proof will mainly rely on the study of the so-called deformed Laguerre ensemble, which is the law of the random matrix $\frac{1}{N}(W + aB)^*(W + aB)$, where B is a $p \times N$ random matrix with i.i.d. complex centered Gaussian entries, and W is a fixed matrix with positive pairwise distinct singular values. In Section 3, we obtain an integral representation of the correlation kernel of the deformed Laguerre ensemble. We then allow the matrix W to be random and give conditions under which we can determine the limiting behavior of local eigenvalue statistics: this is the object of Section 4. The limiting eigenvalue statistics are then identified through a saddle point analysis (Section 5). The last sections deal with the same study for the hard edge.

2 Universality in the Bulk of the Spectrum and at the Hard Edge

For any integer m, define $k(m) = 4(m + 2)$.

2.1 Universality in the Bulk

THEOREM 2.1 *Assume the $p \times N$ random matrix X has i.i.d. entries with a Gaussian divisible law of variance σ^2 and that $\nu = p - N$ is fixed. Then for a given integer $m > 2$, if the probability distribution μ admits moments up to order $k(m)$, then $\forall \delta > 0$, $u \in [\delta, 4\sigma^2(1 - \delta)]$, and $\rho_N = N\rho_\sigma(u)$ in the local eigenvalue statistics* (1.2),

$$\lim_{N\to\infty} \mathbb{E}S_N^m(f) = \int_{\mathbb{R}^m} f(t_1, \dots, t_m) \det\left(\frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)}\right)_{i,j=1}^m dt_1 \cdots dt_m. \tag{2.1}$$

Remark 2.2. The condition on ν can be weakened to $\nu = O(N^{43/48})$.

Remark 2.3. Theorem 2.1 is an integrated version of universality. Indeed, if one considers the so-called m-point correlation functions R_N^m (see below for the definition), then $\mathbb{E}S_N^m(f) = \int f(t_1, \dots, t_m) R_N^m(t_1, \dots, t_m) dt_1 \cdots dt_m$. A strong universality result would have been to state that the m-point correlation functions converge a.s. as N grows to infinity to the determinant $\det(K_{\sin}(t_i, t_j))_{i,j=1}^m$.

We can also prove that the spacing distribution, close to a point u in the interior of the support of Marchenko-Pastor's law, is universal.

Let $s \geq 0$, $\lambda \in \mathbb{R}_+^N$, and (t_N) be a sequence such that $\lim_{N\to\infty} t_N = +\infty$, $\lim_{N\to\infty} t_N/N = 0$. Define then the "spacing function" by

(2.2)
$$S_N(s, \lambda, u) = \frac{1}{2t_N} \sharp \left\{ 1 \leq j \leq N-1,\ \lambda_{j+1} - \lambda_j \leq \frac{s}{N\rho(u)},\ |\lambda_j - u| \leq \frac{t_N}{N\rho(u)} \right\}.$$

Intuitively the expectation of the spacing function is the probability, knowing that there exists an eigenvalue in an interval I_N of length $2t_N$ centered at u, of finding its nearest neighbor within a distance $\frac{s}{N\rho(u)}$. Finally, for K being the operator in $L^2(0, s)$ with kernel $K_{\sin}(t, s)$, we define

$$p(s) = \frac{d^2}{ds^2} \det(I - K)_{L^2(0,s)}. \tag{2.3}$$

THEOREM 2.4 *Assume that the Gaussian divisible law μ admits moments up to order $16 + \epsilon$, $\epsilon > 0$. Let $S_N(s, \lambda, u)$ be defined by* (2.2) *for a point u in the bulk of the spectrum. Then, for any $s \geq 0$,*

$$\lim_{N\to\infty} \mathbb{E} S_N(s, \lambda(X^*X), u) = \int_0^s p(w)dw, \tag{2.4}$$

where $p(w)$ is given by (2.3)*.*

Remark 2.5. We refer the reader to [21] for the proof that Theorem 2.4 is an easy consequence of Theorem 2.1.

Another consequence of Theorem 2.1 deals with the fluctuations of the number of eigenvalues of the random sample covariance matrices $\frac{1}{N} X^*X$ in an interval centered around a point u in the bulk. Define, for such a point u,

$$\nu_N(L) = \frac{1}{N} \sharp \left\{ \lambda_i \in \left[u, u + \frac{L}{N\rho(u)} \right] \right\},$$

and let $\phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) dt$.

PROPOSITION 2.6

$$\lim_{L\to\infty} \lim_{N\to\infty} P_N \left(\frac{\nu_N(L) - L}{\sqrt{\frac{1}{\pi^2} \log L}} \leq x \right) = \phi(x).$$

Remark 2.7. The proof follows the same steps as those used by [7] in a similar study of the Gaussian unitary ensemble. See also, for example, [34].

2.2 Convergence of the Eigenvalue Statistics at the Hard Edge

We first show the universality of the Bessel kernel at the *hard edge*.

THEOREM 2.8 *Let X be a random matrix with a Gaussian divisible law μ admitting moments up to order $k > k(m)$. For $u = 0$ and $\rho_N = 4N^2/\sigma^2$ in the local* *eigenvalue statistics* (1.2)

$$\lim_{N\to\infty} \mathbb{E}S_N^m(f) = \int\limits_{\mathbb{R}_+^m} f(x_1, \ldots, x_m) \det(K_{\text{Bes}}^\nu(x_i, x_j))_{i,j=1}^m \prod_{i=1}^m dx_i, \tag{2.5}$$

where $K_{\text{Bes}}^\nu(x, y)$ *is the Bessel kernel defined in* (1.3).

A second result is that the limiting distribution of the smallest eigenvalue is also universal. We need a preliminary definition.

DEFINITION 2.9 The *gap probability of the Bessel kernel* is

$$E_{\text{Bes}}^\nu(0, s) = \det(I - K_{\text{Bes}}^\nu)_{L^2(0,s)} = \sum_{m=0}^\infty \int_0^s \det\left(K_{\text{Bes}}^\nu(x_i, x_j)\right)_{i,j=1}^m \prod_{i=1}^m dx_i. \tag{2.6}$$

Let also $E(n, J)$ be the probability that the random matrix $\frac{1}{N}X^*X$ has precisely n eigenvalues in the interval J.

THEOREM 2.10 *Assume that the Gaussian divisible law* μ *admits moments up to any order. Then*

$$\lim_{N\to\infty} E\left(0, \frac{\sigma^2 s}{4N^2}\right) = E_{\text{Bes}}^\nu(0, s) \tag{2.7}$$

where $E_{\text{Bes}}^\nu(0, s)$ *is the gap probability of the Bessel kernel.*

Remark 2.11. In particular, for $\nu = 0$, the gap probability is given by

$$E_{\text{Bes}}^0(0, s) = \exp\{-s\},$$

as first noticed by Forrester [12] (see also [10]).

Remark 2.12. In the proof, it will become apparent that the "hard edge" is actually not really an edge. Indeed, concentration results we will use are those of eigenvalues of Hermitian matrices around 0. Of course, this is essentially because the limiting Marchenko-Pastur law is, when $\frac{p}{N} \to 1$, the law of a squared Wigner variable.

Remark 2.13. Theorem 2.8 implies that fluctuations of the number of eigenvalues in an interval close to 0 are the same as for the LUE. We refer to [34] for more details.

At this point, we would like to point out that the limiting behavior of largest eigenvalues could also be studied by our method. Yet it has already been investigated: Soshnikov has obtained that the largest eigenvalue exhibits universal fluctuations provided the probability measure μ is symmetric and admits sub-Gaussian tails [35]. Our approach would give the same type of results under very different hypotheses, namely, a Gaussian divisible distribution and a finite number of moments.

2.3 Sketch of the Proof of Theorems 2.1 to 2.10

The method used here will follow the same steps as Johansson's proof of the universality for the local statistics of eigenvalues of Hermitian matrices (see [21]).

The first part of the proof deals with the study of the deformed Laguerre ensemble, which is the distribution of the random matrix

$$\frac{1}{N}(W + aB)^*(W + aB),$$

where B is a matrix with i.i.d. complex centered Gaussian entries and for a given $p \times N$ matrix W. We will call H the rescaled matrix $(1/\sqrt{N})W$ and assume that its singular values $\sqrt{y_1}, \ldots, \sqrt{y_N}$ are pairwise distinct. We will denote by Q_N^H the joint eigenvalue distribution induced by the deformed Laguerre ensemble. Then one can first compute the density

$$\frac{dQ_N^H(x_1, \ldots, x_N)}{dx_1 \cdots dx_N}$$

of the eigenvalue distribution Q_N^H. This will imply that the deformed Laguerre ensemble induces a so-called determinantal random point field, as we now explain.

The m-point correlation functions of the joint eigenvalue density induced by the deformed Laguerre ensemble are defined by

$$R_N^m(x_1, \ldots, x_m; y) = \frac{N!}{(N-m)!} \int_{\mathbb{R}^{N-m}} \frac{dQ_N^H(x_1, \ldots, x_N)}{dx_1 \cdots dx_N} dx_{m+1} \cdots dx_N.$$

They give the marginal distribution of m unordered eigenvalues. For the distribution Q_N^H, we show that these correlation functions are given by a determinant $R_N^m(x_1, \ldots, x_m; y) = \det K_N(x_i, x_j)_{i,j=1}^m$ involving a so-called correlation kernel K_N. This defines the determinantal random point field structure. We further obtain an explicit integral representation of the correlation kernel, using an approach due to [23] and previously used in [4] (Section 3). This integral representation further depends on H^*H only through its spectral measure $\frac{1}{N}\sum_{i=1}^N \delta_{y_i}$. The following is heavily technical and deals with various rewritings of the correlation kernel (Section 3.4), which are necessary for the later asymptotic analysis of the correlation kernel.

The second part of the proof deals with the case that the matrix H is random. Denote by Q_N the distribution of the spectrum of the random matrix

$$\frac{1}{N}X^*X = \frac{1}{N}(W + aB)^*(W + aB).$$

Here, the eigenvalue distribution Q_N defined by

$$Q_N = \int_{M_{p,N}(\mathbb{C})} Q_N^H \, dP_N(H)$$

defines a determinantal random point field for general probability measure $P_N(H)$ of the random matrix H. Let $R_N^m(u, v; y(H))$ be the m-point correlation function of the deformed Laguerre ensemble (for a fixed matrix H), and $R_N^m(u, v)$ that of the sample covariance matrix $\frac{1}{N}X^*X$.

PROPOSITION 2.14

$$R_N^m(u, v) = \int_{M_{p,N}(\mathbb{C})} dP_N(H) R_N^m(u, v; y(H)).$$

This proposition and the detailed study of the deformed Laguerre ensemble are the basis of the proof. Indeed, the results we need to analyze the limiting behavior of the correlation kernel is the following:

$$\frac{1}{N}\sum_{i=1}^{N} \log(z - y_i) = \int \log(z - y) d\rho(y) + o(N^{-\xi})$$

where ρ is Marchenko-Pastur's distribution and $\xi > 0$.

In Section 4, we establish the condition under which one can replace P_N by a probability measure $\bar{P}_N$ satisfying the above conditions without affecting the limiting behavior of local eigenvalue statistics. This will be obtained using concentration results for the spectral measure of large random matrices due to [14]. Then, in Section 5, we proceed to the saddle point analysis of the correlation kernel under $\bar{P}_N$.

3 Integral Representation of the Correlation Kernel of the Deformed Laguerre Ensemble

3.1 Known Results for the Deformed Laguerre Ensemble

The computation of the joint eigenvalue density of the deformed Laguerre ensemble has been obtained by Jackson, Sener, and Verbaarschot for a matrix of arbitrary dimensions [17] and Guhr and Wettig for square matrices [13]. We here assume that the entries of B are Gaussian with variance $4\sigma_1^2$ for some parameter σ_1^2 that will later be chosen to be the variance of the law P.

Then, setting $t = s/2 = 4\sigma_1^2 a^2/(2N)$, $H = W/\sqrt{N}$, and $\mu_{N,p}^H$, the law of $\frac{1}{N}X^*X$ given H, one obtains the following:

PROPOSITION 3.1 *The symmetrized eigenvalue probability distribution Q_N^H on $\mathbb{R}_+^N$ has a density given by*

$$\frac{dQ_N^H(x_1, \ldots, x_N)}{dx^N} = \frac{V(x)}{V(y)} \det\left(\frac{1}{2t}\exp\left(-\frac{y_i + x_j}{2t}\right) I_\nu\left(\frac{\sqrt{y_i x_j}}{t}\right)\left(\frac{x_j}{y_i}\right)^{\frac{\nu}{2}}\right)_{i,j=1}^{N}, \tag{3.1}$$

where $y_1, \dots, y_N$ stand for the positive eigenvalues of H^*H, I_ν is the usual modified Bessel function, $V(x) = \prod_{i<j}(x_i - x_j)$, and $t = 4\sigma_1^2 a^2/(2N)$.

The joint eigenvalue distribution of the deformed Laguerre ensemble offers a nice interpretation in terms of squared Bessel processes conditioned, in the sense of Doob, never to collide. Let Q_N^H be the probability measure defined in (3.1), and $P_t(y, x)$ the transition probability density at time t of N squared Bessel processes X_t^i of dimension $d = 2(p - N + 1)$ conditioned, in the sense of Doob, never to intersect pairwise, starting at y_i.

THEOREM 3.2 [24] *For any $y \in W_+^N = \{y \in \mathbb{R}_+^N, y_1 < \cdots < y_N\}$,*

$$\frac{dQ_N^H(x_1, \dots, x_N)}{dx^N} = P_t(y, x). \tag{3.2}$$

Remark 3.3. $P_t(y, x)$ is equivalently the transition probability of the h-process obtained from an N-dimensional squared Bessel process, where h is defined to be $h(x) = |\prod_{i<j}(x_i - x_j)|$.

Let

$$p_t(x, y) = \frac{1}{2t} \exp\left(-\frac{y + x}{2t}\right) I_\nu\left(\frac{\sqrt{yx}}{t}\right)\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \tag{3.3}$$

be the transition density of a squared Bessel process of dimension $2(\nu + 1)$. Given a point $z = (z_1, \dots, z_N)$ with $z_1 < \cdots < z_N$, define a probability measure by the density

$$P_t^T(x; y, z) = \frac{\det(p_t(y_j, x_i))_{i,j=1}^N \det(p_T(x_i, z_j))_{i,j=1}^N}{\det(p_{t+T}(y_i, z_j))_{i,j=1}^N}. \tag{3.4}$$

This is just the probability density of N squared Bessel processes starting at y conditioned not to collide up to time $T + t$ and to end at this time at the point z. Then it was also proven in [24] that, for any point z,

$$P_t(y, x) = \lim_{T \to \infty} P_t^T(x; y, z). \tag{3.5}$$

This approach follows [21] and uses the famous Karlin-McGregor theorem [22].

3.2 A First Step Toward the Correlation Function of the Deformed Laguerre Ensemble

In this section, we start from formula (3.5). We will first consider the probability measure defined in (3.4). This probability will prove to define a determinantal random point field for which we can compute the correlation kernel. This will then be used to derive an explicit representation for the correlation kernel of the deformed Laguerre ensemble by letting T grow to infinity.

PROPOSITION 3.4 *Let $R_N^{m,T}(x_1,\dots,x_m;y)$ denote the m-point correlation function of the probability measure with density function $P_t^T(x;y,z)$ given by* (3.4). *Then*

$$R_N^{m,T}(x_1,\dots,x_m;y)=\det\left(K_N^T(x_i,x_j)\right)_{i,j=1}^m, \tag{3.6}$$

where

$$K_N^T(u,v;y)=\sum_{k,j=1}^{N}p_t(y_k,u)p_T(v,z_j)(p_{T+t}(y_l,z_p))_{j,k}^{-1} \tag{3.7}$$

with $p_t(x,y)$ as defined in (3.3).

PROOF OF PROPOSITION 3.4: Such a result seems to have been first noticed by [3] for some particular ensembles of random matrices. Let us first consider $P_t^T(x,y)$ given by (3.4). Now set $\phi_i(x)=p_t(y_i,x)$ and $\psi_j(x)=p_T(x,z_j)$. Because of the semigroup property, one easily shows that

$$\int_{\mathbb{R}_+}\phi_j(x)\psi_k(x)dx=p_{T+t}(y_j,z_k).$$

Thus, with corollary 1.5 in [31], for instance, the m-point correlation function of P_t^T is given by

$$R_N^{m,T}(x_1,\dots,x_m;y)=\det\left(K_N^T(x_i,x_j;y)\right)_{i,j=1}^m.$$

□

At this stage, we are not able to determine the limiting kernel $K_N(u,v;y)$ of $K_N^T(u,v;y)$ as $T\to\infty$; this will be done in the next section by using an integral representation of the kernel.

3.3 Kazakov's Type Formula

Here we will express the correlation kernel of the deformed Laguerre ensemble $K_N(u,v;y)$ as a double integral over some contours in the complex plane. From the unitary invariance of the Gaussian law, we know that the correlation function depends only on H^*H through its empirical spectral measure. "Kazakov's formula," which was first used in [4], is the trick to explicitly bring out the spectral measure $\frac{1}{N}\sum_{i=1}^N\delta_{y_i}$.

Let $s=2t=4\sigma_1^2a^2/N$.

THEOREM 3.5 *The correlation kernel of the deformed Laguerre ensemble is given by*

$$\begin{aligned} K_N(u,v;y) = \frac{e^{\frac{v-u}{s}}}{i\pi s^2} \int_\Gamma \int_\gamma \exp\left(\frac{z^2-w^2}{s}\right) J_\nu\left(2\frac{zu^{\frac12}}{s}\right) J_\nu\left(2\frac{wv^{\frac12}}{s}\right) \\ \times \prod_{i=1}^N \frac{w^2+y_i}{z^2+y_i}\left(\frac{w}{z}\right)^\nu \frac{-2zw}{w^2-z^2} dz\, dw, \end{aligned} \tag{3.8}$$

where $\gamma = \mathbb{R}^+$ and Γ is a contour encircling the $i\sqrt{y}_j$, $j=1,\dots,N$ (but not the $-i\sqrt{y_j}, 1,\dots,N$), not crossing γ.

PROOF OF THEOREM 3.5: We can first rewrite, using Cramer's formula,

$$K_N^T(u,v) = \sum_{j=1}^N p_t(y_j,u)\frac{\det A_j(v)}{\det A} \tag{3.9}$$

where $A_{i,j} = p_{t+T}(y_j,z_i)$ and $A_j(v)$ is the matrix obtained from A by replacing the column j by $(p_T(v,z_1),\dots,p_T(v,z_N))^{\mathsf{T}}$. This can also be written, by multilinearity of the determinant, as

$$K_N^T(u,v;y) = \left(\frac{u}{v}\right)^{\frac{\nu}{2}} \sum_{j=1}^N p_t(y_j,u)\left(\frac{y_j}{u}\right)^{\frac{\nu}{2}} \frac{\det B(v)}{\det B}, \tag{3.10}$$

where

$$B_{i,j} = I_\nu\left(\frac{\sqrt{y_i z_j}}{T+t}\right)\exp\left(-\frac{y_i+z_j}{2(T+t)}\right)$$

and $B(v)$ is obtained from B by replacing $T+t$ with T and y_j with v.

The next step will be achieved in the following proposition. In this proposition we rewrite the ratio of determinants in (3.10) and then let T grow to infinity to obtain an expression for the correlation kernel of the deformed Laguerre ensemble.

PROPOSITION 3.6

$$\begin{aligned} &K_N(u,v;y) = \\ &\sum_{j=1}^N \frac{2}{s^2} e^{\left(\frac{v-u}{s}\right)}\left(\frac{u}{v}\right)^{\frac{\nu}{2}} e^{i\frac{\nu\pi}{2}} \exp(-y_j) I_\nu\left(\frac{2\sqrt{y}_j u^{\frac12}}{s}\right) \\ &\quad\times \int_{\mathbb{R}^+} \exp\left(-\frac{w^2}{s}\right) J_\nu\left(\frac{2v^{\frac12}w}{s}\right) \prod_{i\neq j}\frac{-w^2-y_i}{y_j-y_i}\left(\frac{iw}{\sqrt{y}_j}\right)^\nu w\, dw. \end{aligned} \tag{3.11}$$

Remark 3.7. Afterwards we will not consider $(\frac{u}{v})^{\nu/2}$ any more since it will not play a role in the asymptotic

$$\det(K_N(x_i,x_j;y))_{i,j=1}^m = \det\left(\left(\frac{x_j}{x_i}\right)^{\frac{\nu}{2}} K_N(x_i,x_j;y)\right)_{i,j=1}^m.$$

PROOF OF PROPOSITION 3.6: Because the two matrices under consideration in (3.10) differ by the j^{th} column only, we will find an integral transform expressing this column in $B(v)$ in terms of that of B. This will make use of some kind of time inversion for the semigroup with transition density $p_t(y,x)$. Eventually we will let $T \to \infty$ to obtain the correlation kernel of the deformed Laguerre ensemble.

LEMMA 3.8

$$\begin{aligned}(3.12)\quad \frac{1}{p^2} I_\nu\left(\frac{\sqrt{v}\sqrt{z}}{T}\right) \exp\left(\frac{-v(T+t)}{2tT}\right) \exp\left(\frac{-zt}{2(T+t)^2}\right) = \\ \frac{1}{t}\int_{i\mathbb{R}^-} \exp\left(\frac{x^2(T+t)}{2tT}\right) I_\nu\left(\frac{(T+t)\sqrt{v}x}{tT}\right) I_\nu\left(\frac{\sqrt{z}x}{T+t}\right) x\,dx.\end{aligned}$$

PROOF: We start from formula [28, p. 108], valid for any a, b:

$$\int_{\mathbb{R}^+} \exp(-p^2x^2) x J_\nu(ax) J_\nu(bx) dx = \frac{1}{2p^2} I_\nu\left(\frac{ab}{2p^2}\right) \exp\left(\frac{-a^2-b^2}{4p^2}\right).$$

The left-hand side can be rewritten as

$$\frac{1}{p^2}\int_{\mathbb{R}^+} \exp(-x^2) J_\nu\left(\frac{ax}{p}\right) J_\nu\left(\frac{bx}{p}\right) x\,dx. \tag{3.13}$$

We first make the change of variables $x = iy$, obtaining that (3.13) can be rewritten as

$$\begin{aligned}(3.13) &= \frac{1}{p^2}\int_{i\mathbb{R}^-} \exp(y^2) J_\nu\left(\frac{aiy}{p}\right) J_\nu\left(\frac{biy}{p}\right) y\,dy \\ &= e^{(\nu i\pi)} \frac{1}{p^2}\int_{i\mathbb{R}^-} \exp(y^2) I_\nu\left(\frac{ay}{p}\right) I_\nu\left(\frac{by}{p}\right) y\,dy,\end{aligned}$$

where we have used in the last equality that $I_\nu(z) = J_\nu(iz)\exp\frac{\nu i\pi}{2}$.

For $p = \sqrt{t/(T(t+T))}$ and making the change of variables $y = x/\sqrt{2t}$, we obtain

$$\begin{aligned}(3.14)\quad \int_{i\mathbb{R}^-} e^{y^2} I_\nu\left(\frac{ay}{p}\right) I_\nu\left(\frac{by}{p}\right) y\,dy = \\ \frac{1}{2t}\int_{i\mathbb{R}^-} e^{\frac{x^2}{2t}} I_\nu\left(\frac{ax\sqrt{T(t+T)}}{\sqrt{2}t}\right) I_\nu\left(\frac{bx\sqrt{T(t+T)}}{\sqrt{2}t}\right) x\,dx.\end{aligned}$$

Setting then

$$a = \frac{T+t}{T}\frac{\sqrt{2}\sqrt{v}}{\sqrt{T(t+T)}}, \qquad b = \frac{T}{T+t}\frac{\sqrt{2}t\sqrt{z}}{T\sqrt{T(t+T)}},$$

in (3.14), we obtain

$$(3.14) = \frac{1}{2t}\int_{i\mathbb{R}^-} \exp\left(\frac{x^2(T+t)}{2tT}\right) I_\nu\left(\frac{(T+t)\sqrt{v}x}{tT}\right) I_\nu\left(\frac{\sqrt{z}x}{T+t}\right) x\,dx, \tag{3.15}$$

and the left-hand side is then given by

$$\frac{1}{2p^2} I_\nu\left(\frac{\sqrt{v}\sqrt{z}}{T}\right) \exp\left(\frac{-v(T+t)}{2tT}\right) \exp\left(\frac{-zt}{2(T+t)^2}\right), \tag{3.16}$$

which finishes the proof of the lemma. □

We come back to the proof of Proposition 3.6. Then developing the determinant along the j^{th} column, we obtain the representation

$$\frac{\det B(v)}{\det B} = (-1)^\nu \int_{i\mathbb{R}^-} \frac{1}{t} \exp\left(\frac{u^2(T+t)}{2tT}\right) I_\nu\left(\frac{\sqrt{v}u(T+t)}{tT}\right) \frac{\det \tilde{B}(u)}{\det B} u\,du,$$

where the matrix $\tilde{B}(u)$ has been obtained from B by changing y_j to u. We can now pass to the limit $T \to \infty$, thanks to the dominated convergence theorem and to the fact (proven in [24]) that $\prod_{i<j}(x_i - x_j)$ is a minimal harmonic function for squared Bessel processes on the Weyl chamber $W = \{x_1 < \cdots < x_N\}$. We obtain that

$$\lim_{T\to\infty} \frac{\det \tilde{B}(u)}{\det B} = \prod_{i\neq j} \frac{u^2 - y_i}{y_j - y_i}.$$

This then gives that

$$\frac{\det B(v)}{\det B} = \frac{(-1)^\nu}{t} \int_{i\mathbb{R}^-} \exp\left(\frac{u^2}{2t}\right) I_\nu\left(\frac{u\sqrt{v}}{t}\right) \prod_{i\neq j} \frac{u^2 - y_i}{y_j - y_i} \left(\frac{u}{\sqrt{y_j}}\right)^\nu u\,du.$$

We then change $u \to iw$ using that $I_\nu(z) = J_\nu(iz) \exp \frac{\nu i\pi}{2}$ and then change t to $\frac{s}{2}$; we thus obtain the result. □

We can now turn to the proof of Theorem 3.5. The sum over y_j occurring in Proposition 3.6 can be written as a residue integral. This is Kazakov's formula [23], which seems to have been used first by Brézin and Hikami [4, 5]. We eventually make the change of variables $z \mapsto iz$. □

3.4 Rewriting the Kernel

The formula for the correlation kernel, obtained in the preceding subsection, is not yet satisfactory. Indeed, we will see that the critical points for the z- and w-integrals are equal, and thus the term $1/(w^2 - z^2)$ is singular at the critical points. In order to remove this singularity, we will first find a new expression that will allow us to use the well-known behavior of Bessel functions of large arguments. This is the object of the next proposition and claim. We will then remove the singularity.

Let H^1_ν be the modified Bessel function of the third kind, also known as the Hankel function.

PROPOSITION 3.9 *The correlation kernel is also given by*

$$K_N(u,v;y) = \frac{-1}{4i\pi s^2}\int_{iA+\mathbb{R}} dw \int_{\Gamma} dz\, \exp\left(\frac{z^2-w^2}{s}\right) H^1_\nu\left(\frac{2w\sqrt{v}}{s}\right) \times J_\nu\left(\frac{2z\sqrt{u}}{s}\right)\prod_{i=1}^N \frac{w^2+y_i}{z^2+y_i}\left(\frac{w}{z}\right)^\nu \frac{w+z}{w-z}$$

where Γ *encircles the* $\pm i y_j$, $j = 1,\dots,N$, *and* A *is large enough so that* $\gamma = iA+\mathbb{R}$ *does not cross* Γ.

PROOF: We begin with Proposition 3.6. The main step is the following lemma:

LEMMA 3.10

$$\int_{\mathbb{R}_+} \exp\left(-\frac{w^2}{s}\right) J_\nu\left(\frac{2w\sqrt{v}}{s}\right) w^\nu \prod_{i\neq j}(w^2+y_i) w\, dw = \frac{1}{2}\int_{\mathbb{R}} \exp\left(-\frac{w^2}{s}\right) H^1_\nu\left(\frac{2w\sqrt{v}}{s}\right) w^\nu \prod_{i\neq j}(w^2+y_i) w\, dw. \tag{3.17}$$

PROOF: The proof of this formula is given in [40, p. 211] in the case $\nu = 0$ and can easily be extended to arbitrary ν [40, exercise 15]. $\square$

We now come back to the proof of Proposition 3.9. The use of formula (3.17) can now be explained: we can move the contour from $\mathbb{R}$ to $\gamma := \mathbb{R} + iA$ with $A > 0$ thanks to Cauchy's formula. We then choose A large enough so that, when applying Kazakov's formula, the contour Γ (symmetric around the origin) for the z-integral encircles the y_i, $i = 1,\dots,N$, and does not cross $i\gamma$. At this point, we do not yet make the change of variables $iz \mapsto z$ as in Section 3.3.

Next, we rewrite

$$\frac{wz}{w^2+z^2} = \frac{1}{4i}\left(\frac{w+iz}{w-iz} - \frac{w-iz}{w+iz}\right)$$

and obtain in an obvious way a rewriting of the correlation kernel $K_N(u,v;y) = K^1_N(u,v;y) - K^2_N(u,v;y)$ where K^1_N is given by

$$K^1_N = \frac{1}{8i^2\pi s^2}\int_{iA+\mathbb{R}} dw \int_{i\Gamma} dz \times \exp\left(\frac{-z^2-w^2}{s}\right) H^1_\nu\left(\frac{2w\sqrt{v}}{s}\right) I_\nu\left(\frac{2z\sqrt{u}}{s}\right)\prod_{i=1}^N \frac{w^2+y_i}{-z^2+y_i}\left(\frac{w}{z}\right)^\nu \frac{w+iz}{w-iz}$$

and K_N^2 is the kernel

$$K_N^2 = \frac{1}{8i^2\pi s^2} \int_{iA+\mathbb{R}} dw \int_{i\Gamma} dz \times \exp\left(\frac{-z^2 - w^2}{s}\right) H_\nu^1\left(\frac{2w\sqrt{v}}{s}\right) I_\nu\left(\frac{2z\sqrt{u}}{s}\right) \prod_{i=1}^{N} \frac{w^2 + y_i}{-z^2 + y_i} \left(\frac{w}{z}\right)^\nu \frac{w - iz}{w + iz}.$$

Then we can modify the contour Γ so that it encircles the $\pm y_i$ and joins $-\infty \pm i\eta$ for some positive η. We then change iz to z, changing $i\Gamma$ to Γ. Using the symmetries when making the change of variables $z \mapsto -z$, we can see that $K_N^1 = -K_N^2$, and we can then consider K_N^1 only. This proves Proposition 3.9. $\square$

We will now use the asymptotic expansion of Bessel functions of large arguments.

Claim 3.1.

$$K_N(u, v; y) = \left(1 + O\left(\frac{1}{N^{1/4}}\right)\right) \times \frac{-1}{8i\pi s(uv)^{1/4}} \left(K_{1,N}(u, v; y) + (-1)^\nu e^{-\frac{i\pi}{2}} K_{2,N}(u, v; y)\right)$$

with

$$K_{1,N} = \int_\Gamma \int_\gamma dz\, dw \exp\left(\frac{z^2 - w^2 + 2iw\sqrt{v} - 2iz\sqrt{u}}{s}\right) \times \left(\frac{w}{z}\right)^\nu \frac{w + z}{w - z} \frac{1}{\sqrt{w}\sqrt{z}} \prod_{i=1}^{N} \frac{w^2 + y_i}{z^2 + y_i}, \tag{3.18}$$

$$K_{2,N} = \int_\Gamma \int_\gamma dz\, dw \exp\left(\frac{z^2 - w^2 + 2iw\sqrt{v} + 2iz\sqrt{u}}{s}\right) \times \left(\frac{w}{z}\right)^\nu \frac{w - z}{w + z} \frac{1}{\sqrt{w}\sqrt{z}} \prod_{i=1}^{N} \frac{w^2 + y_i}{z^2 + y_i}, \tag{3.19}$$

where Γ is a new contour that has been cut on a small neighborhood of the negative real axis. We note x_0 and $x_1 = \bar{x}_0$, the two points where we cut Γ. They will be fixed later.

The proof of this claim is postponed to the end of Section 5.

From now on, we will only consider the kernel $K_{1,N}$, since the analysis for $K_{2,N}$ is exactly the same. In this paragraph, we are going to remove the singularity $\frac{1}{w-z}$.

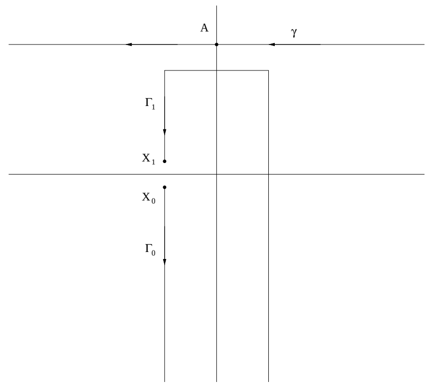

FIGURE 3.1. Preliminary contours.

Set

(3.20)

$$
\begin{aligned}
S_{1,N} = {} & \frac{1}{2i(\sqrt{v}-\sqrt{u})} \int_{\Gamma} dz \int_{\gamma} dw \Big(-1 + e^{\left(-i\frac{2(\sqrt{v}-\sqrt{u})z}{s}\right)} \Big) \prod_{i=1}^{N} \frac{w^2+y_i}{z^2+y_i} \left(\frac{w}{z}\right)^{\nu} \\
& \times \exp\left(\frac{z^2 - w^2 - 2zi\sqrt{u} + 2wi\sqrt{v}}{s} \right) \frac{w+z}{\sqrt{w}\sqrt{z}} \frac{1}{z} \\
& \times \left(2(z+w) - 2s(z+w) \sum_{i=1}^{N} \frac{y_i}{(w^2+y_i)(z^2+y_i)} - 2i\sqrt{v} \right)
\end{aligned}
$$

and

(3.21)

$$
\begin{aligned}
R'_{1,N}(x_0) = {} & -\frac{1}{2i(\sqrt{v}-\sqrt{u})} \int_{\gamma} \Big(-1 + e^{\left(-i\frac{2(\sqrt{v}-\sqrt{u})x_0}{s}\right)} \Big) \prod_{i=1}^{N} \frac{w^2+y_i}{x_0^2+y_i} \left(\frac{w}{x_0}\right)^{\nu} \\
& \times \exp\left(\frac{x_0^2 - w^2 - 2x_0 i\sqrt{u} + 2wi\sqrt{v}}{s} \right) \frac{w+x_0}{\sqrt{w}\sqrt{x_0}} \frac{1}{x_0} \\
& \times \left(2(x_0+w) - 2s(x_0+w) \sum_{i=1}^{N} \frac{y_i}{(w^2+y_i)(x_0^2+y_i)} - 2i\sqrt{v} \right).
\end{aligned}
$$

PROPOSITION 3.11 *We find that the kernel $K_{1,N}$ can be rewritten*

$$
K_{1,N}(u,v;y) = S_{1,N} + R'_{1,N}(x_0) - R'_{1,N}(x_1). \tag{3.22}
$$

Remark 3.12. The two contours Γ and γ in (3.20) can now cross each other since we have removed the singularity $\frac{1}{w-z}$.

PROOF OF PROPOSITION 3.11: In the expression of the kernel $K_{1,N}$, we make the change of variables $z \mapsto \beta z$, $w \mapsto \beta w$, for β real close to 1. We then obtain for the two "half-contours" Γ_0 and Γ_1 defining Γ and for

$$E(z,w) = \exp\left(\frac{z^2 - w^2 - 2ziu + 2iwv}{s}\right)$$

the following:

$$\int_{\Gamma_j} dz \int_{\gamma} dw E(z,w) \frac{1}{w-z} \prod_{i=1}^{N} \frac{w^2 + y_i}{z^2 + y_i} \frac{w+z}{\sqrt{w}\sqrt{z}}$$

$$- \int_{\Gamma_j} dz \int_{\gamma} dw \frac{\beta}{w-z} E(\beta z, \beta w) \prod_{i=1}^{N} \frac{\beta^2 w^2 + y_i}{\beta^2 z^2 + y_i} \frac{w+z}{\sqrt{w}\sqrt{z}} \tag{3.23}$$

$$= \int_{\frac{1}{\beta}x_j}^{x_j} dz \int_{\gamma} dw \frac{\beta}{w-z} E(\beta z, \beta w) \prod_{i=1}^{N} \frac{w^2 + y_i}{z^2 + y_i} \frac{w+z}{\sqrt{w}\sqrt{z}} \tag{3.24}$$

$$:= R_{1,N,j}.$$

Consider the kernel $R_{1,N,j}$ defined in (3.24). This kernel admits a derivative with respect to β taken at $\beta = 1$, which, as we will see later, can be analyzed by a saddle point method. We set

$$R_{1,N} = \frac{d}{d\beta}(R_{1,N,0} + R_{1,N,1})\Big|_{\beta=1}. \tag{3.25}$$

Then in (3.23) we make the change of variables $z \longrightarrow bz$, $w \longrightarrow bw$, with b very close to 1 and differentiate with respect to b. This modifies the contour but by Cauchy's theorem we can deform back to Γ and γ. This gives

$$\begin{aligned} &- K_{1,N}(u, v; y) = \\ &R_{1,N} + \Bigg\{ \int_{\Gamma} \int_{\gamma} \prod_{i=1}^{N} \frac{w^2 + y_i}{z^2 + y_i} \left(\frac{w}{z}\right)^{\nu} \frac{w+z}{\sqrt{w}\sqrt{z}} \frac{1}{s(w-z)} E(z,w) \\ &\times \left(-2w^2 + 2z^2 - 2i\sqrt{u}z + 2i\sqrt{v}w + 2s \sum_{i=1}^{N} \left(\frac{w^2}{w^2 + y_i} - \frac{z^2}{z^2 + y_i} \right) \right) dw\, dz \Bigg\}. \end{aligned} \tag{3.26}$$

Using that

$$\frac{\partial K_{1,N}}{\partial \sqrt{u}} = \int_{\Gamma} \int_{\gamma} dz\, dw \left(-\frac{2iz}{s}\right) \left(\frac{w}{z}\right)^{\nu} \frac{w+z}{w-z} \frac{1}{\sqrt{w}\sqrt{z}} \prod_{i=1}^{N} \frac{w^2 + y_i}{z^2 + y_i} E(z,w),$$

(3.26) can be rewritten

$$(3.27)\quad K_{1,N}+(-\sqrt{u}+\sqrt{v})\frac{\partial}{\partial\sqrt{u}}K_{1,N}=$$
$$\Bigg\{\frac{1}{s}\int_{\gamma_2}\int_{\Gamma_2}\prod_{i=1}^{N}\frac{w^2+y_i}{z^2+y_i}\frac{w+z}{\sqrt{w}\sqrt{z}}\left(\frac{w}{z}\right)^{\nu}\exp\left(\frac{z^2-w^2-2zi\sqrt{u}+2wi\sqrt{v}}{s}\right)$$
$$\times\left(2(z+w)-2s(w+z)\sum_{i=1}^{N}\frac{y_i}{(w^2+y_i)(z^2+y_i)}-2i\sqrt{v}\right)dw\,dz\Bigg\}+R_{1,N}.$$

Solving the ordinary differential equation (3.27) then gives (3.22). Because we have now removed the singularity the two contours, Γ and γ, can cross each other. Proposition 3.11 is now proven. □

We will here write the kernel $S_{1,N}$ in a way more suitable for the saddle point analysis. The same rewriting can be done for the kernels $R'_{1,N}$.

Define

$$G(w,z;y)=$$
$$\prod_{i=1}^{N}\frac{w^2+y_i}{z^2+y_i}\left(\frac{w}{z}\right)^{\nu}\frac{w+z}{\sqrt{w}\sqrt{z}}\exp\left(\frac{z^2-w^2}{s}\right)\exp\left(\frac{2i\sqrt{v}w-2i\sqrt{u}z}{s}\right)$$

and

$$(3.28)\qquad h_N(z)=\left(\frac{1-\exp\left\{\frac{2iz(\sqrt{v}-\sqrt{u})}{s}\right\}}{2i(\sqrt{u}-\sqrt{v})}\right).$$

PROPOSITION 3.13 *The kernel $S_{1,N}$ can be written*

$$(3.29)\qquad S_{1,N}=\int_{\Gamma}\int_{\gamma}g_N(z,w)\exp\left(\frac{\hat{G}_{N,\sqrt{v}}(z)-\hat{G}_{N,\sqrt{v}}(w)}{s}\right)\times\left(\frac{w}{z}\right)^{\nu}\frac{w+z}{\sqrt{w}\sqrt{z}}h_N(z)dw\,dz$$

where $\hat{G}_{N,u}(z)=z^2-2izu-s\sum_{i=1}^{N}\log(z^2+y_i)$ and

$$(3.30)\qquad g_N(z,w)=\frac{\hat{G}'_{N,\sqrt{v}}(z)-\hat{G}'_{N,\sqrt{v}}(w)}{z-w}+\frac{\hat{G}'_{N,\sqrt{v}}(w)}{z}.$$

PROOF OF PROPOSITION 3.13: The exponential term intervening in the correlation kernel $S_{1,N}$ defined in (3.20) (and in the kernels $R'_{1,N}$) can be written

$$G(w,z;y)\frac{1-e^{\frac{-2iz(\sqrt{v}-\sqrt{u})}{s}}}{2i(\sqrt{v}-\sqrt{u})} = \exp\left\{s^{-1}\left(-\hat{G}_{N,\sqrt{v}}(w)+\hat{G}_{N,\sqrt{v}}(z)\right)\right\}h_N(z). \tag{3.31}$$

Now defining

$$g_N^1(z,w)=\frac{2}{z}\left(z+w-(z+w)s\sum_{i=1}^{N}\frac{y_i}{(w^2+y_i)(z^2+y_i)}-i\sqrt{v}\right), \tag{3.32}$$

it is easily checked that, as a direct consequence of the method used to get rid of $\frac{1}{w-z}$,

$$g_N^1(z,w)=g_N(z,w)=\frac{\hat{G}'_{N,\sqrt{v}}(z)-\hat{G}'_{N,\sqrt{v}}(w)}{z-w}+\frac{\hat{G}'_{N,\sqrt{v}}(w)}{z}.$$

□

4 Concentration of Measure

In this section, we prove some results that will be needed to localize the critical points in the exponential terms of the preceding kernels, when the matrix H is random. To this end, we need to prove the uniform convergence (in w) of the random term $\prod_i(w^2-y_i)$ towards its a.s. limit. To obtain this uniform convergence, we will need to replace P_N by a new probability distribution: the aim of this section is to prove that such a replacement does not affect the limiting behavior of local eigenvalue statistics.

4.1 Preliminaries

We choose the principal branch on the complex plane cut along R_- for the logarithm branch. Provided we choose γ_l far from the eigenvalues, we can write

$$\prod_{i=1}^{N}(w^2+y_i)=\exp\int\log(w^2+y)d\mu_N(y) \tag{4.1}$$

with μ_N the spectral measure of H^*H.

From now on, we will consider a random matrix W and assume that P, the law of the entries of W, satisfies

$$\int z\,dP(z)=0,\qquad \int|zz^*|dP(z)=\sigma_1^2=\frac{1}{4}. \tag{4.2}$$

Such a condition on the variance of P is not restrictive. It can indeed be achieved by rescaling the entries of the random matrix X. In particular, conditions (4.2) ensure convergence of the spectral measure of H^*H to the Marchenko-Pastur law with density

$$\rho(x) = \frac{2}{\pi}\frac{\sqrt{1-x}}{\sqrt{x}}. \tag{4.3}$$

Condition (4.2) also implies that the limiting spectral measure of $\frac{1}{N}X^*X$ is then given by Marchenko-Pastur's law with parameter $1/4 + a^2$; we denote $\rho = \rho_a$ the density of this probability measure. We also reset $t = s/2 = a^2/N$.

The idea is then, roughly speaking, to replace the random terms intervening in the correlation kernel by their almost sure limit expressed in terms of Marchenko-Pastur's distribution. This will be done using concentration results we will now establish. We have made the assumptions to ensure the convergence of $d\mu_N$ to the law ρ of a squared Wigner random variable. Yet we need to prove some uniform convergence of (4.1) towards $\int_0^1 \log -((iz)^2 - y)d\rho(y)$. This will now be proven with the results of [14] on concentration of measure. For convenience, noticing that $\prod_{i=1}^N (w^2 + y_i) = H(iw)$ for $H(w) = \prod_{i=1}^N (y_i - w^2)$, we will prove concentration results for F, and then just a rotation in the complex plane will be enough.

Let us recall the results of Guionnet-Zeitouni and those of Bai, which will be the basis for what follows.

PROPOSITION 4.1 ([14]) *Let Y be an $N \times M$ matrix, $N \le M$, with independent entries $Y_{k,l}$ of law $P_{k,l}$. Set $P_{N,M} = \prod_{k=1}^N \prod_{l=1}^M P_{k,l}$, $Z = YY^*$.*

If the $P_{k,l}$ are supported in a compact set K, for any function f so that $g(x) = f(x^2)$ is convex and has finite Lipschitz norm $|g|_l$, for any $\delta > \delta_0(N+M) = 4|K|\sqrt{\pi}|g|_l/(N+M)$,

$$P_{N,M}\left(|\mathrm{tr}_N(f(Z)) - E(\mathrm{tr}_N(f(Z)))| > \delta\frac{M+N}{N}\right) \le 2\exp\left(-\frac{(\delta-\delta_0)^2(N+M)^2}{4|K|^2|g|_l^2}\right)$$

where tr_N denotes the normalized trace.

Remark 4.2. In fact, g is to be seen as a function of the $(p+N)\times(p+N)$ Hermitian matrix $RW(Y)$

$$\begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix}.$$

From the discussion before corollary 1.8 in [14], this is the Lipschitz norm of this function that we have to consider. In particular, if H has entries bounded by 1, then RW^2 has a spectral radius of order $N + p$.

PROPOSITION 4.3 ([1]) *Let Y be an $N \times M$ matrix, $N \le M$, with independent entries $Y_{k,l}$ of law $P_{k,l}$. We assume the entries are centered of variance* 1 *and admit* *moments up to order* 4*. Let F_N be the empirical spectral distribution of $\frac{1}{N}Y^*Y$; then we have*

$$\|EF_N - F_{p/N}\|_\infty = O(N^{-\frac{5}{48}}).$$

To ensure that $\|F_{p/N} - F_1\| = O(N^{-5/48})$, where $F_1 = F$ is the limiting Marchenko-Pastur distribution function, one needs to assume $\nu = O(N^{43/48})$. This is the reason for the restriction we have made on ν.

4.2 Concentration of Measure

Set

$$\Omega_{R,\eta} = \{z : \mathrm{Im}(z) \in [\eta, R], \mathrm{Re}(z) < \mathbb{R}\}. \tag{4.4}$$

Let $\lambda(H) = (\lambda_1(H), \ldots, \lambda_N(H))$ be the spectrum of H^*H. We here prove concentration results assuming that the entries $W_{j,k}$ are random variables of law $P_{j,k}$. In particular, they do not need to be identically distributed, yet of the same variance. We also establish concentration results for an arbitrary parameter

$$\gamma := \lim_{N\to\infty} \frac{p}{N} \geq 1.$$

THEOREM 4.4 *Assume the entries of H admit moments up to order q where q is strictly greater than* 8 *and let $R, \eta > 0$ be given. There exists $0 < \xi < \frac{1}{4} - \frac{2}{q}$ and a probability measure with compact support on $M_{p,N}(\mathbb{C})$, denoted $d\overline{P}_N$, such that for any symmetric function $F \in L^\infty(\mathbb{R}_+^N)$ and for $\alpha = -2 + q(\frac{1}{4} - \xi)$,*

$$\left| \int_{M_{p,N}(\mathbb{C})} F(\lambda(H))dP_N(H) - \int_{M_{p,N}(\mathbb{C})} F(\lambda(H))d\overline{P}_N(H) \right| \leq \frac{1}{N^\alpha}\|F\|_\infty \tag{4.5}$$

and

$$\sup_{z\in\Omega_{R,\eta}} \left| \frac{1}{N} \operatorname{tr}\log - (z^2 - M) - \int_0^1 \log - (z^2 - y)d\rho(y) \right| \leq CN^{-\xi}, \tag{4.6}$$

$\overline{P}_N$ almost surely.

PROOF: In view of Proposition 4.1, we first need to replace P_N by a measure with compact support. This is the object of the following lemma.

LEMMA 4.5 *Let $F \in L^\infty(\mathbb{R}^N)$ be a symmetric function. There exists a probability measure P_{L_N} on $M_{p,N}(\mathbb{C})$ with support in $[-L_N, L_N]^{Np}$ such that*

$$\left| \int_{M_{p,N}(\mathbb{C})} F(\lambda(H))dP_N(H) - \int_{M_{p,N}(\mathbb{C})} F(\lambda(H))dP_{L_N}(H) \right| \leq \gamma\|F\|_\infty \frac{N^2}{L_N^q}. \tag{4.7}$$

PROOF: Set
$$dP_{j,k}^{(L)} := \frac{1}{Z_{j,k}^L}\chi_{[-L_N,L_N]}dP_{j,k}$$
with $Z_{j,k}^L$ a normalization constant, and $Z_N^L = \prod_{j=1}^p \prod_{k=1}^N Z_{j,k}^L$. For $F \in L^\infty$, one can easily show that

$$\left| \int_{M_{p,N}(C)} F(M)dP_N(M)dM - \int_{M_{p,N}(C)} F(M)dP_{L_N}(M)dM \right| \leq \|F\|_\infty(1-Z_N^L)\left(1+\frac{1}{Z_N^L}\right), \tag{4.8}$$

and

$$\begin{aligned} 1 - Z_N^L &= P(\exists\, 1 \leq j \leq p, 1 \leq k \leq N \text{ s.t. } |W_{j,k}| \geq L_N) \\ &\leq Np \sup_{j,k} \frac{E|W_{j,k}|^q}{L_N^q} \leq C\frac{Np}{L_N^q}\|F\|_\infty. \end{aligned} \tag{4.9}$$

L_N will be fixed afterwards. □

Now set

$$D_N = \Omega_{R,\eta} \cap \frac{1}{N}\mathbb{Z}^2. \tag{4.10}$$

Note that $\sharp D_N \leq CN^2$ for some constant C depending only on R and η.

Set

$$A_N(f,\delta) = \left\{ H \in M_N(C) : \left| \frac{1}{N} \operatorname{tr} f_z(HH^*) - \int f_z(x)d\rho(x) \right| \geq \delta \right\} \tag{4.11}$$

and $A_N(\delta) = \bigcap_{z \in D_N} A_N(f_z,\delta)$ where $f_z(x) = \log -(z^2 - x)$.

To use concentration results, from the remark following Proposition 4.1, we need to control the Lipschitz norm of $y \longrightarrow \log -(z^2 - y^2) := f_z(y^2)$. Here y is to be seen as an eigenvalue of $RW(H)$ that is of spectral radius smaller than $L_N\sqrt{N}$, since we have truncated the entries of H. Then, combining Proposition 4.1 and Proposition 4.3, we obtain that for $|f|_l = |f|_{\text{Lip}} + \|F\|_\infty$, there exist some constants C_1 and C_2 such that if we set

$$\delta_0(N) = \frac{C_1 L|f|_l}{N} + C_2|f|_{\text{Lip}'}N^{-\frac{5}{48}}, \tag{4.12}$$

where this time $|f|_{\text{Lip}'}$ is the Lipschitz norm of $y \mapsto \log -(z^2 - y)$, then for all $z \in A_N(\delta)$, we have

$$P_{L_N}\left[\left| \frac{1}{N} \operatorname{tr} f_z(HH^*) - \int f_z(x)d\rho(x) \right| \geq \delta \right] \leq 4e^{-\frac{(\delta-\delta_O)^2 4N^2}{CL^2|f|_l^2}}. \tag{4.13}$$

Then, provided $\operatorname{Im}(z^2) > \delta$, we can prove as in [21] that

$$|f|_l = |f|_{\text{Lip}} + \|F\|_\infty \leq C(L_N\sqrt{N} + \log(L_N\sqrt{N})). \tag{4.14}$$

Otherwise for $\operatorname{Im}(z) > \delta$ and z close to the imaginary axis, we can bound this norm thanks to the bounded variation theorem, to find that

$$|f|_l \leq C(\delta) L_N \sqrt{N}. \tag{4.15}$$

To sum up, one needs

(1) $\frac{Np}{L_N^q} \ll 1$ and
(2) $\frac{L_N^2 \sqrt{N}}{N} \leq 1$.

Set $L_N = N^{\frac{1}{4}-\xi}$.

(2) is obviously satisfied, and (1) is true for $q > 9$ provided ξ is small enough. Eventually, one shows that $\forall z \in \Omega_{R,\eta}$, one has $\delta_0(N) < CN^{-\xi}$, and

$$P_{L_N}\left[\left|\frac{1}{N} \operatorname{tr} f_z(H^*H) - \int f_z(x) d\rho(x)\right| \geq N^{-\xi}\right] \leq Ce^{-CN^{2\xi}}$$

for some nonnegative constant C.

We can now estimate the probability of the complement of the event $A_N(N^{-\xi})$ using that $p \sim \gamma N$:

$$P_{L_N}[A_N(N^{-\xi})^{\mathrm{c}}] \leq CN^2 \exp\{-CN^{2\xi}\}. \tag{4.16}$$

Set

$$d\overline{P}_N(H) = \frac{1}{P_{L_N(A_N(N^{-\xi}))}} dP_{L_N} \mathbb{1}(A_N(N^{-\xi})). \tag{4.17}$$

Then Theorem 4.4 is proven for $z \in D_N$, and we can extend it to all $z \in \Omega_{R,\eta}$ by a straightforward approximation argument. □

From now on we replace dP_N by $d\overline{P}_N$ and define for any integer N and $C > 0$

(4.18) $Y_{R,\eta} =$

$$\left\{y \in \mathbb{R}_+^N : \sup_{z \in \Omega_{R,\eta}} \left| \int \log -(z^2 - y) d\rho(y) - \frac{1}{N} \sum \log -(z^2 - y_i)\right| \leq CN^{-\xi}\right\}.$$

Let us recall that, by (4.6), $\bar{P}_N(\operatorname{sp}(H^*H) \in Y_{R,\eta}) = 1$ for all $N \geq 1$, where "sp" stands for "spectrum."

5 Saddle Point for the Bulk Correlation Functions under $\bar{P}_N$

In this section, we show that under $\bar{P}_N$, a saddle point analysis of the local eigenvalue statistics can be achieved.

5.1 Preliminaries

In this subsection we consider the kernel $S_{1,N}$ as written in Proposition 3.13 (the other kernels will be analyzed similarly). We can now use the concentration results established in the preceding section to say that, uniformly for z varying in a compact set,

$$\frac{1}{N}\sum_{i=1}^{N}\log(z^2+y_i)$$

is well approximated by its nonrandom limit, given in terms of Marchenko-Pastur's law. Because the variance of the entries of W has been set equal to $\frac{1}{4}$, the spectral measure of H^*H converges to Marchenko-Pastur's law with density

$$\frac{2}{\pi\sqrt{x}}\sqrt{1-x},$$

and recall that $s=a^2/N$.

Now let u_N be a sequence converging to $u=\sqrt{u_0}$, a point in the bulk. Here, we introduce u_N because $u=u_0+\frac{x}{N}$ and $v=u_0+\frac{y}{N}$ vary around the point in the bulk u_0, which can be first ignored in the search of saddle points.

Define

$$\hat{G}_u(z)=z^2-2uiz-a^2\int\log(z^2+x)d\rho(x), \tag{5.1}$$

where ρ is the Marchenko-Pastur distribution with parameter $\sigma^2=\frac{1}{4}+a^2$. Here $\hat{G}_u(z)$ is the almost sure limit of the exponential term $\hat{G}_{N,u}(z)=z^2-2izu-s\sum_{i=1}^{N}\log(z^2+y_i)$. First from (4.18) and Cauchy's formula, we know there exists a constant C such that for all $N\geq 1$, u in a compact set K, and $z\in\Omega_{R/2,2\eta}$, $\bar{P}_N$ a.s.,

$$\begin{aligned}|\hat{G}_N'(z)-\hat{G}_u'(z)|&\leq C(N^{-\xi}+|u-u_N|),\\ |\hat{G}_N''(z)-\hat{G}_u''(z)|&\leq C(N^{-\xi}+|u-u_N|).\end{aligned}\tag{5.2}$$

This will enable us to replace u_N by its limit in the integrand.

Comparison with the Exponential Term for the Hermitian Case

Let u be a fixed point, $u\in\left[\sqrt{1+4a^2}\delta_0,\sqrt{1+4a^2}(1-\delta_0)\right]$, $0<\delta_0<1$. Introduce

$$G_{N,-u}(iz)=-\hat{G}_{N,u}(z),$$

$$G_{N,u}(z)=z^2-2uz+a^2\int\log(y-z^2)d\mu_N(y),$$

$$G_u(z)=z^2-2uz+a^2\int\log(y-z^2)d\rho(y), \tag{5.3}$$

$$F_u(z)=\frac{z^2-2uz}{2}+a^2\int\log(z-t)\sigma(t)dt, \tag{5.4}$$

σ being the underlying density function of Wigner's semicircle law.

The reason for introducing these functions is that the eigenvalues we consider have asymptotically the same law as that of the squares of the eigenvalues of a Hermitian matrix. To be more precise, we will show that the exponential term is of the same form as the Johansson-Wigner ensemble.

Then we find from (5.3), (5.4), and the definition of ρ and σ that

$$G_u^{'}(z) = 2F_u^{'}(z), \tag{5.5}$$

where F_u is the exponential term for the Hermitian case studied in [21]. We now make a saddle point for $S_{1,N}$, noticing that the exponential term can be written $\hat{G}_{N,u}(w) = -2F_{N,-u}(iw)$ for F_N being the same exponential term as [21]. It will thus be enough to prove the results for F_N. Therefore we will refer to Johansson's saddle point analysis of the kernel of Hermitian Brownian motion [21].

Critical Points

Set $\sqrt{u} = \sqrt{1+4a^2}\cos\theta_c$. Since we will constantly refer to Johansson's results [21], we will do a saddle point for F_N, and then a rotation of $-\frac{\pi}{2}$ will give the result. Johansson's computations show that the critical points for F_u are given by the equation

$$F_u^{'}(z) = (z - \sqrt{u}) + 2a^2(z - \sqrt{z^2-1}) = 0,$$

and the critical points $\hat{z}_c$ of $\hat{G}_u$ are just $\frac{1}{i}z_c$, where z_c are those of F_u. Setting

$$z = S(x) = \frac{1}{2}\left(x + \frac{1}{x}\right), \tag{5.6}$$

one obtains that the critical points for F_u are given by $z_c = S(\pm x_c)$ with

$$x_c = \sqrt{1+4a^2}\exp \pm i\theta_c = 2\sigma \exp \pm i\theta_c. \tag{5.7}$$

5.2 Contours for an Arbitrary Point in the Bulk

Let us note $r_o = \sqrt{1+4a^2}$ and $r_o e^{\pm i\theta_o}$ the critical points defined in (5.6)–(5.7).

Consider $\delta > 0$, $\epsilon > 0$ small enough. We now define the contours we will use in the saddle point analysis. Let the contours for z be defined by

$$\begin{aligned}
\Gamma_1^+(t) &= \frac{1}{i}S(r_o e^{i\delta} - t), && -\infty < t \le 0;\\
\Gamma_2^+(t) &= \frac{1}{i}S(r_o e^{it}), && \Delta < t \le \Delta - \epsilon;\\
\Gamma_3^+(t) &= \frac{1}{i}S(r_o e^{it}), && \Delta - \epsilon < t \le \Delta + \epsilon;\\
\Gamma_4^+(t) &= \frac{1}{i}S(r_o e^{it}), && \Delta + \epsilon < t \le \pi - \delta;\\
\Gamma_5^+(t) &= \frac{1}{i}S(r_o e^{i(\pi-\delta)} - t), && \infty > t \ge 0,
\end{aligned}$$

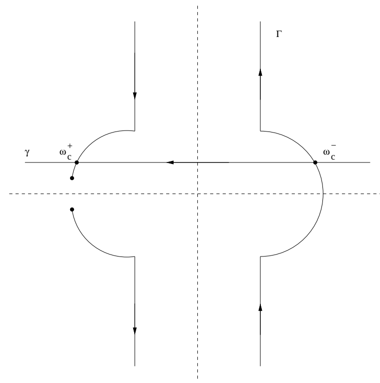

FIGURE 5.1. Contours for the bulk.

$$\Gamma_j^-(t) = -\overline{\Gamma_j^+(t)}.$$

We cut this contour so that it does not cross $\mathbb{R}_-$, as in Section 3.4. Then let t_o be such that $\mathrm{Im}(S(t_0 w_\Theta)) = \eta$ and $\tau = \mathrm{Re}(S(t_0 w_\Theta))$. η will be fixed afterwards. One then defines the contour for w, in the case $\mathrm{Re}(F(w_c(u_o))) > \lim_{\eta\to 0} \mathrm{Re}(F(i\eta))$:

$$\begin{aligned}
\gamma_1^+(t) &= \frac{1}{i}(\tau + it), && 0 \le t \le \eta,\\
\gamma_2^+(t) &= \frac{1}{i}S(tw_c), && t_o \le t \le 1-\epsilon,\\
\gamma_3^+(t) &= \frac{1}{i}S(tw_c), && 1-\epsilon \le t \le 1+\epsilon,\\
\gamma_4^+(t) &= \frac{1}{i}S(tw_c), && 1+\epsilon \le t \le +\infty,
\end{aligned}$$

and

$$\gamma_j^-(t) = -\overline{\gamma_j^+(t)}. \tag{5.8}$$

Or if $\mathrm{Re}(x_c^2) \ge 0$, we define the contour γ as $\gamma^+ = \frac{1}{i}(w_c + it)$, $t \in \mathbb{R}^+$, and $\gamma^- = \frac{1}{i}(w_c - it)$, $t \in \mathbb{R}^+$. This is the modification of the contour needed to obtain the asymptotics of the correlation kernel in the whole bulk of the spectrum (see Appendix A for an explanation).

Finally, set $\Gamma = \sum_{j=1}^5(\Gamma_j^+ - \Gamma_j^-)$ and $\gamma = \sum_{j=1}^4(\gamma_j^+ - \gamma_j^-)$. A plot of the contours is given by Figure 5.1.

Define

$$L_N(\tau) = \frac{1}{s}\int_{\gamma_3}\int_{\Gamma_3}\left(\frac{w}{z}\right)^{\nu}\frac{w+z}{\sqrt{w}\sqrt{z}}g_N(z,w) \times \exp\left(\frac{\hat{G}_{N,\sqrt{v}}(z)-\hat{G}_{N,\sqrt{v}}(w)}{s}\right)h_N(z)dw\,dz.$$

LEMMA 5.1 *We can find R_o, η_o, ϵ, and δ positive such that the contours $\gamma_3^{\pm}$ and $\Gamma_3^{\pm}$ belong to some neighborhood of $\hat{z}_c^{\pm}$ included in $\Omega_{R_o/2,2\eta_o}$ and such that $\forall N \geq 1$, u, v in a compact set K, $\bar{P}_N$ a.s.,*

$$|S_{1,N} - L_N(\tau)| \leq Ce^{-cN} \tag{5.9}$$

for some constant $c > 0$.

PROOF: The proof is exactly the same as the one given by Johansson [21]. □

5.3 Saddle Point Analysis

By (5.2), the critical points z_N for F_N satisfy

$$|\hat{z}_N^{\pm} - \hat{z}_c^{\pm}| \leq C(N^{-\xi} + |u - u_N|). \tag{5.10}$$

One can then deform the contours γ_3 (respectively, Γ_3) into contours γ_N (respectively, Γ_N) within a C^1 distance at most $C(N^{-\xi} + |u - u_N|)$ leaving the endpoints unchanged and such that $\gamma_N(0)^{\pm} = \hat{w}_{N,c}^{\pm}$ and $\Gamma_N(0)^{\pm} = \hat{z}_{N,c}^{\pm}$. We can also choose these contours such that, for $|t| < \epsilon$,

$$\Gamma_N^{\pm}(t) = \frac{1}{i}S(w_{N,c}^{\pm}\exp \pm it) \quad \text{and} \quad \gamma_N^{\pm}(t) = \frac{1}{i}S(z_{N,c}^{\pm}(1+t)).$$

Using the contours defined in the preceding subsection, we then obtain by a standard saddle point argument

$$\begin{aligned}\frac{L_N^{b,d}}{2\pi N\rho(u)} = &\left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)\frac{\left(\hat{w}_{N,c}^d + \hat{z}_{N,c}^b\right)h_N\left(\hat{z}_{N,c}^b\right)}{sN\rho(u)\sqrt{\hat{w}_{N,c}^d}\sqrt{\hat{z}_{N,c}^b}} \\ &\times g_N\left(\hat{w}_{N,c}^d, \hat{z}_{N,c}^b\right)\left(\frac{\hat{w}_{N,c}^d}{\hat{z}_{N,c}^b}\right)^{\nu} \\ &\times \frac{\left(\left(\Gamma_N^d\right)'(0)\right)\left(\gamma_N^b\right)'(0)\exp\left\{s^{-1}\left(\hat{G}_{N,\sqrt{v}}\left(\hat{z}_{N,c}^b\right) - \hat{G}_{N,\sqrt{v}}\left(\hat{w}_{N,c}^d\right)\right)\right\}}{\sqrt{-\hat{G}''_{N,\sqrt{v}}\left(\hat{z}_{N,c}^b\right)\left(\Gamma_N^b\right)'(0)^2}\sqrt{\hat{G}''_{N,\sqrt{v}}(\hat{w}_{N,c}^d)\left(\gamma_N^d\right)'(0)^2}}.\end{aligned} \tag{5.11}$$

where b and d stand for $\pm$ according to the contour $\Gamma_3^{\pm}$ (respectively, $\gamma_3^{\pm}$) along which the integration is performed.

As already observed, the two critical points are the same, denoted $\hat{z}_c$, and one can check from (3.30) and the definitions of the contours Γ_N and γ_N that

$$g_N\left(\hat{z}_c^+, \hat{z}_c^-\right) = g_N\left(\hat{z}_c^-, \hat{z}_c^+\right) = 0,$$
$$g_N\left(\hat{z}_c^+, \hat{z}_c^+\right) = \hat{G}''_{N,\sqrt{v}}\left(\hat{z}_c^+\right), \quad g_N\left(\hat{z}_c^-, \hat{z}_c^-\right) = \hat{G}''_{N,\sqrt{v}}\left(\hat{z}_c^-\right)$$
$$\left(\gamma_N^b\right)'(0) = bS'\left(w_N^b\right)\hat{w}_{N,c}^b, \quad \left(\Gamma_N^b\right)'(0) = \hat{w}_{N,\Delta}^b S'\left(w_N^b\right).$$

Because $\hat{G}_{N,\sqrt{v}}(\hat{w}_c^+) = \overline{\hat{G}_{N,\sqrt{v}}(\hat{w}_c^-)}$, we can consider just the joint contribution of equal critical points. That is, we consider only L_N^{++} and L_N^{--}. We obtain that

$$\frac{S_{1,N}}{N\rho(u)} = \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \times \sum_{b=\pm 1} \frac{-bi}{sN\rho(u)} \left(\frac{\hat{w}_c^b}{\hat{z}_c^b}\right)^{v} \exp\left(\frac{\hat{G}_{N,\sqrt{v}}\left(\hat{w}_c^b\right) - \hat{G}_{N,\sqrt{v}}\left(\hat{z}_c^b\right)}{s}\right) h_N\left(\hat{z}_c^b\right) \frac{\hat{w}_c^b + \hat{z}_c^b}{\sqrt{\hat{w}_c^b}\sqrt{\hat{z}_c^b}}. \tag{5.12}$$

Now, for $w_c = z_c = w^+$ with positive real and imaginary parts, for instance, the large exponential terms cancel each other, and we only have to examine the contribution of h_N. Define then h as

$$h(w) = \frac{1}{N\rho(u)(uv)^{1/4}} h_N(w). \tag{5.13}$$

Then, taking (5.10) into account and assuming $\xi < \frac{5}{96}$, we have for some constant C

$$\left| \frac{1}{2i\pi N\rho(u)} L_N^{++}(\tau) + h(\hat{z}_c^+) \right| \le C(|u - u_N| + N^{-\xi}). \tag{5.14}$$

We have the same kind of formula for L_N^{--}.

We have that

$$\frac{2\frac{N}{a^2}\hat{z}_c^{\pm}\left(u_N^{1/2} - \left(u_N + \frac{\tau}{N\rho(u)}\right)^{1/2}\right)}{a^2\rho(u)} = \tau(w_o \pm i\pi). \tag{5.15}$$

So at the end, adding the contribution of w_c^+ and of w_c^- (which gives the conjugate), we obtain

$$\frac{1}{N\rho(u)}\left(-h\left(\hat{z}_c^+\right) + h\left(\hat{z}_c^-\right)\right) = -2\frac{\sin \pi\tau}{\pi\tau} \exp\{-w_o\tau\}. \tag{5.16}$$

Thus

$$\frac{1}{2i\pi N\rho(u)} S_{1,N} = -2\frac{\sin \pi\tau}{\pi\tau} \exp\{-w_o\tau\}\left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right).$$

We can then drop the constant term w_0 by multilinearity of the determinant insofar as we are interested in correlation functions.

Remark 5.2. We would like to insist here on the fact that this particular form of the exponential term also justifies the use of a saddle point method. Indeed, concentration results are not strong enough simply to replace $\int \log -(z^2 - y) d\mu_N(y)$ by its almost sure limit; yet because at the critical points the large exponential terms $\exp(s^{-1}\hat{G}_{N,\sqrt{v}}(\hat{z}_c))$ and $\exp(s^{-1}\hat{G}_{N,\sqrt{v}}(\hat{w}_c))$ cancel each other, one just needs to know where these critical points lie.

5.4 Asymptotic of the Correlation Kernel under $\bar{P}_N$

Admitting for a while that the contribution from the kernel $R'_{1,N}$ is negligible, we obtain that

$$\frac{1}{N\rho(u)} \frac{-1}{8i\pi s(uv)^{1/4}} K_{N,1} = \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \frac{1}{2} \frac{\sin(\pi\tau)}{\pi\tau},$$

and the similarly rescaled $K_{N,2}$ gives the same contribution.

Now, adding all these contributions, we finally get

$$\left| \frac{1}{N\rho(u)} K_N\left(u_N, u_N + \frac{\tau}{N\rho(u)}\right) - \frac{\sin(\pi\tau)}{\pi\tau} \right| \leq C(N^{-\xi} + |u - u_N|) + Ce^{-cN}. \tag{5.17}$$

5.5 Proof of Claim 3.1 and of $R'_{1,N}$ Being Negligible

We prove in this section the claim stated at the beginning of Section 3.4 and that the kernel $R'_{1,N}$ is negligible.

Bessel's Approximation. We will next use the well-known asymptotic behavior of Bessel functions [30]. First, for large w, one has

$$H^1_\nu(w) = \sqrt{\frac{2}{\pi w}} \exp\left(i\left(w - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right)\left(1 + O\left(\frac{1}{\sqrt{w}}\right)\right).$$

This asymptotic expansion is valid as long as $\mathrm{Im}(w) > 0$. We can also make use of the well-known behavior of the Bessel function J_ν:

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{\sqrt{z}}\right)\right),$$

which is valid for $|\arg(z)| < \pi - \delta$.

Remark 5.3. There might be some problem when the contour Γ crosses $\mathbb{R}_-$, yet we will see that the contribution from this part of the contour can be neglected.

Now, we choose the contour for the z-integral given as before except that on the top side the contour Γ crosses the imaginary axis to encircle the $i\sqrt{y_i}$, $j = 1, \dots, N$. Then we can apply both the approximations of Bessel functions: indeed, the two curves γ and Γ remain far away from 0, and γ is located in the positive imaginary half-plane.

We can first analyze the w-integral (3.17) by an a priori saddle point analysis: the large exponential term is still a function of the spectral measure $\frac{1}{N}\sum \delta_{y_i}$. Thus we can replace H^1_ν by its approximation with an error of at most of order $1/N^{1/4}$.

Next, when γ has been moved back to $\gamma = iA + t$, keeping an error of at most $1/N^{1/4}$, we can choose Γ to follow the same curve as in the preceding saddle point analysis except that it crosses the imaginary axis at a distance of at least 1 from $i\sqrt{y_1}$. Then we obtain a similar error. The sole point is the justification of Bessel's approximation close to the negative real axis. Yet we know, a priori, that the contribution for z describing the interval $(-r_o - i\epsilon, -r_o + i\epsilon) := (x_0, x_1)$ and γ (we can consider the contribution of such contours since they do not cross) is exponentially small. Thus we can replace the contour Γ by two contours Γ_1 and Γ_2 obtained from Γ by making a cut around the negative real axis. We obtain an expression of the correlation kernel with an error of order $1/N^{1/4}$ as long as we move the two remaining contours far away from 0.

The Kernel $R'_{1,N}$. The kernel (3.24) admits a derivative with respect to β that can be analyzed by a saddle point approximation: its contribution will be of order $\exp(-c_0 N)$ for some positive c_0. Once more, this is because the relevant critical points of the exponential terms do not lie close to the real axis, since u is a point in the bulk of the spectrum. We can thus deform $\gamma = iA + \mathbb{R}$ to the contour defined for the saddle point analysis, that is, $\gamma = i(w_c + it)$, $t \in \mathbb{R}$. We then use the same arguments as for the analysis of $S_{1,N}$.

The Contribution of (x_0, x_1). It is clear that, as the curve (x_0, x_1) lies far away from w_c, one can deform the contour $\gamma = A + it$ so that it is as in the saddle point analysis. Then the contribution from this part of the contour is negligible.

So far we have obtained universality of correlation functions under $\bar{P}_N$, which we state in the following proposition:

PROPOSITION 5.4 *Uniformly for t_i and t_j varying in a compact set, $\exists \xi$, $0 < \xi < \frac{1}{2}$, such that*

$$\left| \frac{1}{N\rho(u)} K_N\left(u + \frac{t_i}{N\rho(u)}, u + \frac{t_j}{N\rho(u)}\right) - \frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)} \right| \leq C N^{-\xi}, \tag{5.18}$$

$\overline{P}_N$ *almost surely.*

6 Proof of Universality of Local Eigenvalue Statistics

We will only give the proof of weak universality of correlation functions, since the proof of the spacing distribution can easily be deduced from [21]. Recall that we consider $f \in L^\infty(\mathbb{R}^m)$ with compact support and that $S_N^m(f)$ is defined by (1.2) with $\rho_N = N\rho(u)$. Given an integer m, we set $dt^m = \prod_{i=1}^m dt_i$. We also set, for $t \in \mathbb{R}$ and $u(t) = u + t/N\rho(u)$,

$$\lim_{N\to\infty} \mathbb{E} S_N^m(f) = \lim_{N\to\infty} \int_{\mathbb{R}^m} f(t_1, \dots, t_m) \frac{1}{(N\rho(u))^m} R_N^m(u(t_1), \dots, u(t_m)) dt^m. \tag{6.1}$$

PROOF OF THEOREM 2.1: We have seen that

$$\int_{\mathbb{R}^m} \frac{f(t_1,\dots,t_m)}{(N\rho(u))^m} R_N^m(u(t_1),\dots,u(t_m))dt^m = \int_{M_{p,N}(\mathbb{C})} dP_N(H) \int_{\mathbb{R}_+^N} \rho_N(x,y(H)) S_N^m(f)[x]dx^N$$

where $\rho_N(x,\cdot)$ is the density function of the eigenvalues of $\frac{1}{N}X^*X$ knowing H. Then, using Theorem 4.4 and the remark following it,

$$\begin{aligned} &\Bigg| \int_{M_{p,N}(\mathbb{C})} dP_N(H) \int_{\mathbb{R}_+^N} \rho_N(x,y(H)) S_N^m(f)[x]dx^N \\ &\quad - \int_{M_{p,N}(\mathbb{C})} d\overline{P}_N(H) \int_{\mathbb{R}_+^N} \rho_N(x,y(H)) S_N^m(f)[x]dx^N \Bigg| \\ &\quad \leq CN^m \|f\|_\infty N^{2-p(\frac{1}{4}-\xi)} = o(1) \end{aligned} \tag{6.2}$$

for $p > 4(m+2)$ and ξ small enough.

Moreover,

$$\begin{aligned} &\int_{M_{p,N}(\mathbb{C})} d\overline{P}_N(H) \int_{\mathbb{R}_+^N} \rho_N(x,y(H)) S_N^m(f)[x] = \\ &\int_{M_{p,N}(\mathbb{C})} d\overline{P}_N(H) \int_{\mathbb{R}_+^m} f(t_1,\dots,t_m) \\ &\qquad \times \det\left(\frac{1}{N\rho(u)} K_N(u(t_i),u(t_j);y(H))\right)_{i,j=1}^m dt^m. \end{aligned} \tag{6.3}$$

Since f is compactly supported, all the t_i's vary in a compact set, so that we can use Proposition 5.4. We can then pass to the limit $N \to \infty$, thus proving Theorem 2.1. □

Remark 6.1. The class of universality of the sine kernel should also include the sample covariance matrices with parameter $\gamma \neq 1$. In this case, the main difficulty is probably the analysis of Bessel functions of large degree and large parameter. We have not found the suitable integral representation of Bessel functions to achieve the analysis of correlation functions.

7 Correlation Functions for the Hard Edge

We first show that the correlation kernel of the deformed Laguerre ensemble can be conveniently rewritten in terms of the Bessel kernel.

THEOREM 7.1 *The correlation kernel of the deformed Laguerre ensemble is also given by*

$$K_N(u^2, v^2; y) = \frac{4}{4i\pi s^3} \exp \frac{2\nu i\pi}{2} \int_\Gamma \int_\gamma dw\, dz\, wz\, K_B\left(\frac{2zu}{s}, \frac{2wv}{s}\right) \times \prod_{i=1}^{N} \frac{w^2 - y_i}{z^2 - y_i} \left(\frac{w}{z}\right)^\nu \exp\left(\frac{w^2 - z^2}{s}\right) \left(1 - s \sum_{i=1}^{N} \frac{y_i}{(w^2 - y_i)(z^2 - y_i)}\right) \tag{7.1}$$

where the contour Γ*, symmetric around* 0*, encircles the* $\pm y_i$*'s,* γ *is the imaginary axis oriented positively* $0 \longrightarrow +\infty$*,* $0 \longrightarrow -\infty$*, and* K_B *is the kernel defined by*

$$K_B(x, y) = \frac{x I_\nu'(x) I_\nu(y) - y I_\nu'(y) I_\nu(x)}{x^2 - y^2}. \tag{7.2}$$

Remark 7.2. The kernel K_B is related to the Bessel kernel K_{Bes}^ν by

$$\exp\left(\frac{2\nu i\pi}{2}\right) K_B(2ir_o u, i2r_o v) = K_{\text{Bes}}^\nu((r_o u)^2, (r_o v)^2).$$

PROOF OF THEOREM 7.1: The first step is the following lemma:

LEMMA 7.3 *Let* $K_B(x, y)$ *be the kernel defined in* (7.2)*. Then*

$$\beta I_\nu(\beta y) I_\nu(\beta x) = \frac{d}{d\beta}\left[\beta^2 K_B(\beta x, \beta y)\right]. \tag{7.3}$$

PROOF: One has

$$\beta^2 K_B(\beta x, \beta y) = \frac{1}{x^2 - y^2}\left[\beta x I_\nu'(\beta x) I_\nu(\beta y) - \beta y I_\nu'(\beta y) I_\nu(\beta x)\right].$$

Thus

$$\begin{aligned}
\frac{d}{d\beta}&\left[\beta^2 K_B(\beta x, \beta y)\right] \\
&= \frac{1}{\beta(x^2 - y^2)}\Big\{\beta x I_\nu'(\beta x) I_\nu(\beta y) - \beta y I_\nu'(\beta y) I_\nu(\beta x) \\
&\qquad + \beta^2 x^2 I_\nu''(\beta x) I_\nu(\beta y) - \beta^2 y^2 I_\nu''(\beta y) I_\nu(\beta x) \\
&\qquad + \beta^2 xy I_\nu'(\beta x) I_\nu'(\beta y) - \beta^2 xy I_\nu'(\beta x) I_\nu'(\beta y)\Big\} \\
&= \frac{1}{\beta(x^2 - y^2)}\Big\{\left[\beta^2 x^2 I_\nu''(\beta x) + \beta x I_\nu'(\beta x)\right] I_\nu(\beta y) \\
&\qquad - \left[\beta^2 y^2 I_\nu''(\beta y) + \beta y I_\nu'(\beta y)\right] I_\nu(\beta x)\Big\}.
\end{aligned} \tag{7.4}$$

Now, using the fact that I_ν is a solution of the differential equation [8]

$$z^2 w^{''}(z) + z w^{'}(z) - (z^2 + \nu^2) w(z) = 0, \tag{7.5}$$

one obtains from (7.4) that

$$\frac{d}{d\beta}\big[\beta^2 K_B(\beta x, \beta y)\big] = \frac{1}{\beta(x^2 - y^2)}\big[\beta^2 (x^2 - y^2) I_\nu(\beta x) I_\nu(\beta y)\big],$$

which is the desired result. □

We come back to the proof of Theorem 7.1 and rewrite the kernel in a different way. We consider the kernel

$$R_N(u, v; y) = \\ \int_\Gamma \int_{i\mathbb{R}^+} dw\, dz \frac{wz}{w^2 - z^2} I_\nu\left(\frac{2wv}{s}\right) I_\nu\left(\frac{2zu}{s}\right) \exp\left(\frac{w^2 - z^2}{s}\right)\left(\frac{w}{z}\right)^\nu \prod_{i=1}^N \frac{w^2 - y_i}{z^2 - y_i}.$$

We first make the change of variables $z \mapsto \beta z$, $w \mapsto \beta w$, and define

$$h(w, z) = \left(\frac{w}{z}\right)^\nu \prod_{i=1}^N \frac{w^2 - y_i}{z^2 - y_i}, \tag{7.6}$$

obtaining

$$\begin{aligned} R_N(u, v; y) &= \int_{\Gamma'} \int_{i\mathbb{R}^+} dw\, dz \frac{\beta^2 wz}{w^2 - z^2} \exp\left(\frac{\beta^2 w^2 - \beta^2 z^2}{s}\right) \\ &\qquad \times h(\beta w, \beta z) I_\nu\left(\frac{2\beta wv}{s}\right) I_\nu\left(\frac{2\beta zu}{s}\right) \\ &= \int_{\Gamma'} \int_{i\mathbb{R}^+} dw\, dz \frac{wz}{w^2 - z^2} \exp\left(\frac{\beta^2 w^2 - \beta^2 z^2}{s}\right) \\ &\qquad \times h(\beta w, \beta z) \beta \frac{d}{d\beta} \beta^2 K_B\left(\beta \frac{2wv}{s}, \beta \frac{2zu}{s}\right). \end{aligned}$$

Now, thanks to Cauchy's theorem, for the z-integral we can move $\Gamma^{'}$ back to Γ. We then integrate over β, varying from a to 1 ($a > 0$, which we'll make tend to 0).

Thus,

$$\int_a^1 R_N(u,v;y)d\beta =$$
$$\int_a^1 d\beta \int_\Gamma \int_{i\mathbb{R}^+} dw\,dz \frac{wz}{w^2-z^2} h(\beta w, \beta z)\beta$$
$$\times \frac{d}{d\beta}\left[\beta^2 K_B\left(\beta\frac{2wv}{s}, \beta\frac{2zu}{s}\right)\right]\exp\left(\frac{\beta^2w^2-\beta^2z^2}{s}\right).$$

We can integrate this by parts over β obtaining

$$(1-a)R_N(u,v;y) =$$
$$\int dz \int dw K_B\left(\frac{2wv}{s}, \frac{2zu}{s}\right)\frac{wz}{w^2-z^2}\exp\left(\frac{w^2-z^2}{s}\right)h(w,z) \tag{7.7}$$
$$-\int dz\int dw\, a^3 K_B\left(a^2\frac{2wv}{s}, a^2\frac{2zu}{s}\right)\frac{wz}{w^2-z^2} \times \exp\left(a^2\frac{w^2-z^2}{s}\right)h(aw,az) \tag{7.8}$$
$$-\int dz\int dw\int d\beta\beta^2 K_B\left(\beta\frac{2wv}{s}, \beta\frac{2zu}{s}\right)\frac{wz}{w^2-z^2} \times \exp\left(\frac{\beta^2w^2-\beta^2z^2}{s}\right)h(\beta w,\beta z) \tag{7.9}$$
$$-\int dz\int dw\int d\beta\beta^2 K_B\left(\beta\frac{2wv}{s}, \beta\frac{2zu}{s}\right)\beta\frac{wz}{w^2-z^2} \times 2\beta\left(w^2-z^2-\sum_{i=1}^N\frac{(w^2-z^2)y_i}{(\beta^2w^2-y_i)(\beta^2z^2-y_i)}\right) \times \exp\left(\frac{\beta^2w^2-\beta^2z^2}{s}\right)h(\beta w,\beta z). \tag{7.10}$$

Now, make the change of variables $\beta z \mapsto z$, $\beta w \mapsto w$, in the reverse order to obtain that

$$(7.9) + (7.7) = 0, \qquad (7.8) = aR_N^2,$$

where R_N^2 is a well-defined integral independent of a and that (7.10) is precisely the kernel we want.

We can now pass to the limit $a \longrightarrow 0$. Now, we know that the Bessel kernel has only an artificial singularity at $x = y$; thus the contours can cross and we can also extend the contour for z to a contour symmetric with respect to the imaginary axis. Up to a multiplication by a factor $\frac{1}{4}$, we may replace Γ with two axes parallel to the real axis and replace γ with the imaginary axis oriented positively from 0 to

$+\infty$ and from 0 to $-\infty$, since the integrand is an odd function of w. Theorem 7.1 is now proven. $\square$

8 Saddle Point Analysis for the Hard Edge

We are now able to proceed to the study of the large N-asymptotic of the kernel. We will use the results of the saddle point analysis made for the bulk, since the exponential term is of the same nature. We keep the hypothesis $\sigma_1^2 = \frac{1}{4}$ and the notation $s = a^2/N$. All along this section, we consider $d\overline{P}_N(H)$ defined in Theorem 4.4 instead of $dP_N(H)$ as before.

Set $G_N(z) = z^2 + \frac{1}{N}\sum_{i=1}^N \log(-(z^2 - y_i))$ and define

$$g_N(z, w) = 1 - s\sum_{i=1}^N \frac{y_i}{(w^2 - y_i)(z^2 - y_i)} \tag{8.1}$$

and

$$r_o = \frac{1}{2}\left(\sqrt{1+4a^2} - \frac{1}{\sqrt{1+4a^2}}\right) = \frac{1}{2}\left(2\sigma - \frac{1}{2\sigma}\right). \tag{8.2}$$

LEMMA 8.1 *For r_o given by* (8.2) *and setting*

$$u = \frac{xa^2}{N2r_o}, \qquad v = \frac{x'a^2}{N2r_o},$$

one has

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{2\pi s}\int_\Gamma dz\int_\gamma dw\, K_B\left(\frac{2zu}{s}, \frac{2wv}{s}\right) wzg_N(z, w) \\ \times e^{\{s^{-1}(G_N(w) - G_N(z))\}}\end{aligned} \tag{8.3}$$

$$= r_o^2 K_{\text{Bes}}^\nu(x^2, x'^2).$$

PROOF OF LEMMA 8.1: This will be the object of the rest of this section. The core of the argument will again be a saddle point method.

8.1 Contours for the Saddle Point Analysis

Because we have chosen $u = \frac{x}{N}$ for some fixed x, the argument of Bessel functions is of order 1. This means that the Bessel functions do not contribute to the leading exponential term to be considered in the saddle point analysis.

Thus, the exponential term to be considered for the equation for the critical points is given by

$$G_o(z) := z^2 + s\log(-(z^2 - y))d\rho(y) = 2F_o(z), \tag{8.4}$$

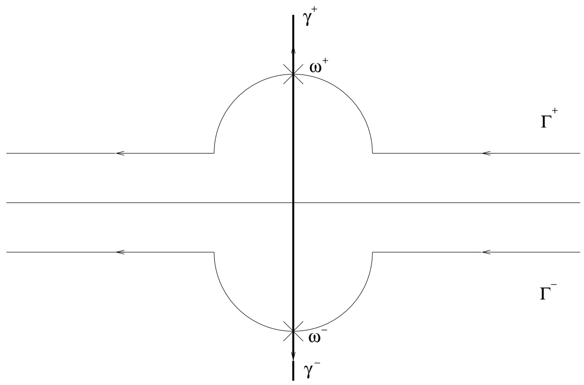

FIGURE 8.1. Contours for the hard edge.

where F_o is the exponential term for the Johansson ensemble [21] defined in (5.4). This is actually the specification $u = 0$ in the exponential term of the bulk kernel, and recall that the negative sign has been introduced to have a definite logarithm.

The critical points are known: they are given by $z_c = \pm i r_o$, where we again will be able to use concentration of measure's results proven in Section 4. We thus choose the same contours (unrotated) as [21] for eigenvalues of Hermitian matrices close to 0. This is exactly where the interpretation of eigenvalues of the deformed Laguerre ensemble as squares of eigenvalues of Hermitian matrices plays a role.

Now we will just indicate what changes from the saddle point analysis of Section 5: the contours are chosen so that they correspond to the critical points defined above, and they are oriented in the same way for the contours for z, whereas the orientation is inverse for the contour for w lying in the negative half-plane.

8.2 Contribution of g_N at the Critical Points

We now turn to the function g_N, defined in (8.1), at the critical points.

Recall the link between G_N and F_N observed for the bulk: the exponential term is now exactly twice that of the Hermitian case.

One has first

$$g_N(w_c, w_c) = g_N(-w_c, w_c) = g_N(w_c, -w_c) = g_N(-w_c, -w_c).$$

Consider now

$$\begin{aligned} G_N''(z) &= 2 - s \sum_{i=1}^{N} \left(\frac{2y_i}{(z^2 - y_i)^2} - \frac{2z^2}{(z^2 - y_i)^2} \right) \\ &= 2 - s \sum_{i=1}^{N} \left(\frac{4y_i}{(z^2 - y_i)^2} + \frac{2}{(z^2 - y_i)} \right). \end{aligned} \tag{8.5}$$

In the large N-limit, using the results of concentration of measure and the fact that the critical points satisfy the equation

$$G_N^{'}(z) = 2z + \sum_{i=1}^{N} \frac{2zs}{z^2 - y_i} = 0, \tag{8.6}$$

we find

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z^2 - y_i} = \int_0^1 \frac{1}{z^2 - y}\, d\rho(y), \tag{8.7}$$

where the Marchenko-Pastur distribution ρ is the law of the square of a Wigner variable, still denoted σ. From (8.1), (8.5), and (8.7), we thus obtain that

$$g_N(\pm w_c, \pm w_c) = \frac{1}{4} G_N^{''}(w_c). \tag{8.8}$$

8.3 Saddle Point Asymptotics

We can now perform the saddle point analysis of the correlation kernel as in [21]. We obtain that

$$\frac{G_N^{''}(z_c^b)(\gamma_N^b)^{'}(0)(\Gamma_N^b)^{'}(0)}{\sqrt{G_N^{''}(w_c^b)(\gamma_N^b)^{'}(0)}\sqrt{-G_N^{''}(z_c^b)(\Gamma_N^b)^{'}(0)}} = -bi, \tag{8.9}$$

where b stands for $\pm$. We now obtain the exponential convergence towards the signed sum of the four contributions depending on the different combinations of $\pm z_c$ and $\pm w_c$: here the contribution of opposite critical points has to be taken into account.

Of course, for $w = z = +ir_o$ we obtain the Bessel kernel:

$$\exp\left(\frac{2\nu i\pi}{2}\right) K_B(ir_o u, ir_o v) = 2K_{\text{Bes}}^{\nu}((r_o u)^2, (r_o v)^2) \tag{8.10}$$

Since the w-integrand is an odd function of w, for $z = -w = ir_o$ we get a second Bessel kernel taking into account the orientation for the contours. The other cases ($z = w = -ir_o$ and $z = -w = -ir_o$) are similar.

Eventually, by the same saddle point argument as for the bulk, using (8.8), (8.9), and (8.10), we obtain that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{a^4}{4N^2 r_o^2} K_N\left(\left(\frac{xa^2}{N2r_o}\right)^2, \left(\frac{x'a^2}{2Nr_o}\right)^2; y\right) \\
&= \lim_{N\to\infty} \frac{\sigma^2}{4N^2} K_N\left(\frac{\sigma^2 x^2}{4N^2}, \frac{\sigma^2 x'^2}{4N^2}; y\right) \\
&= K_{\text{Bes}}^{\nu}(x^2, x'^2)
\end{aligned} \tag{8.11}$$

where K_{Bes}^{ν} is the limiting kernel defined in (1.3).

Note that the error term in the asymptotic expansion is of order $1/\sqrt{N}$ as for the bulk correlation under the same conditions of moments. □

Remark 8.2. Such a computation has been made for $x \neq x'$, but it can be extended to the equality case. One just needs to take the derivative at $x' = x$ of the right-hand side of (7.3) to obtain a similar expression for $K_N(x, x; y)$. The scaling is the same and the limit is, of course, the derivative at x of $y \mapsto K^{\nu}_{\text{Bes}}(x, y)$.

To sum up, we have obtained, as for the bulk, the following result:

PROPOSITION 8.3 *Let the x_i vary in a compact set $[0, L]$; then, $\bar{P}_N$ almost surely,*

$$\frac{\sigma^2}{4N^2} K_N\left(\frac{x_i\sigma^2}{4N^2}, \frac{x_j\sigma^2}{4N^2}; y\right) = K^{\nu}_{\text{Bes}}(x_i, x_j)\left(1 + O\left(\frac{1}{\sqrt{N}}\right) + O\left(\frac{L}{N^2}\right)\right). \tag{8.12}$$

9 Universality of Eigenvalue Statistics at the Hard Edge

9.1 Proof of Theorem 2.8

Here we give only a sketch of the proof. For the scaling factor $\rho_N = 4N^2/\sigma^2$, one has

$$\begin{aligned} &\Bigg| \int_{M_{p,N}(\mathbb{C})} dP_N(H) \int_{\mathbb{R}_+^N} \rho_N(x, y(H)) S_N^m(f)[x] dx^N \\ &\quad - \int_{M_{p,N}(\mathbb{C})} d\overline{P}_N(H) \int_{\mathbb{R}_+^N} \rho_N(x, y(H)) S_N^m(f)[x] dx^N \Bigg| \\ &\quad \le C N^m \|f\|_\infty N^{2-p(\frac{1}{4}-\xi)} = o(1) \end{aligned} \tag{9.1}$$

for $p > 4(m+2)$ and ξ small enough. And

$$\begin{aligned} &\int_{M_{p,N}(\mathbb{C})} d\overline{P}_N(H) \int_{\mathbb{R}_+^N} \rho_N(x, y(H)) S_N^m(f)[x] = \\ &\int_{M_{p,N}(\mathbb{C})} d\overline{P}_N(H) \int_{\mathbb{R}_+^m} f(t_1, \dots, t_m) \det\left(\frac{a^4}{4N^2 r_o^2} K_N\left(\frac{a^4}{4N^2 r_o^2} x_i, \frac{a^4}{4N^2 r_o^2} x_j; y\right)\right)_{i,j=1}^m \\ &\qquad \times \prod_{i=1}^m dx_i . \end{aligned} \tag{9.2}$$

Because f has compact support K, all x_i belong to K; thus we have

$$\begin{aligned} &\left| \det\left(\frac{a^4}{4N^2 r_o^2} K_N\left(\frac{a^4 x_i}{4N^2 r_o^2}, \frac{a^4 x_j}{4N^2 r_o^2}; y\right)\right)_{i,j=1}^m - \det\left(K^{\nu}_{\text{Bes}}(x_i, x_j)\right)_{i,j=1}^m \right| \le \\ &\quad C\left(N^{-\xi} + \frac{K}{N}\right) + Ce^{-cN}, \end{aligned} \tag{9.3}$$

which holds $\overline{P}_N$ a.s. We can then pass to the limit $N \to \infty$.

9.2 Fluctuations of the Smallest Eigenvalues: Proof of Theorem 2.10

Let $E(0, s)$ be the probability that no eigenvalue of $\frac{1}{N}X^*X$ lies in the interval $[0, s]$. Then, by the inclusion-exclusion formula, one has for $\alpha = \sigma^2/4 = a^4/4r_o^2$

$$E\left(0, \frac{\alpha s}{N^2}\right) = \int\limits_{M_{p,N}(\mathbb{C})} dP_N(H) \det(I - K_N)_{L^2(0,\alpha s/N^2)}. \tag{9.4}$$

We next develop the Fredholm determinant and obtain

$$E\left(0, \frac{\alpha s}{N^2}\right) = \int\limits_{M_{p,N}(\mathbb{C})} dP_N(H) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^{\frac{\alpha s}{N^2}} \det(K_N(x_i, x_j; y))_{i,j=1}^m \prod_{i=1}^m dx_i. \tag{9.5}$$

We can first replace P_N by $\bar{P}_N$ as in (9.1). Given an integer M, for $m < M$, using Proposition 8.3,

$$\begin{aligned} &\int_0^{\frac{\alpha s}{N^2}} \det(K_N(x_i, x_j; y))_{i,j=1}^m \prod_{i=1}^m dx_i \\ &= \int_0^s \det\left(\frac{\alpha}{N^2} K_N\left(\frac{\alpha x_i}{N^2}, \frac{\alpha x_j}{N^2}; y\right)\right)_{i,j=1}^m \prod_{i=1}^m dx_i, \\ &= \int_0^s \det\left(K_{\mathrm{Bes}}^{\nu}(x_i, x_j)\right)_{i,j=1}^m \prod_{i=1}^m dx_i \left(1 + O\left(\frac{1}{\sqrt{N}}\right) + O\left(\frac{s}{N^2}\right)\right). \end{aligned} \tag{9.6}$$

Then we eventually obtain, given a fixed $\epsilon > 0$, that

$$\begin{aligned} &\int_0^s \sum_{m \ge M} \frac{1}{m!} \left| \det\left(\frac{\alpha}{N^2} K_N\left(\frac{\alpha x_i}{N^2}, \frac{\alpha x_i}{N^2}; y\right)\right)_{i,j=1}^m \right| \le \\ &\qquad \int_0^s \frac{1}{M!} \left| \det\left(\frac{\alpha}{N^2} K_N\left(\frac{\alpha x_i}{N^2}, \frac{\alpha x_i}{N^2}; y\right)\right)_{i,j=1}^M \right| < \epsilon \end{aligned} \tag{9.7}$$

for M large enough, since the sum over m is actually finite for fixed N and the determinant intervening in the right-hand side of (9.7) can be compared to that of the Bessel kernel for which the result is known. Adding (9.6), (9.7), and the error term due to the replacement of P_N by $\bar{P}_N$,

$$\begin{aligned} &\left|\det(I - K_N)_{L^2(0,s)} - \det(I - K_{\mathrm{Bes}})_{L^2(0,s)}\right| \\ &\le \sum_{m<M} \frac{1}{m!} C \frac{1}{\sqrt{N}} + 2\epsilon + N^{-\xi} \\ &< \left(C\left(N^{-\xi} + \frac{1}{\sqrt{N}}\right)\right) + 2\epsilon. \end{aligned} \tag{9.8}$$

Thus

$$\lim_{N\to\infty} \det(I - K_N)_{L^2(0,s)} = \det(I - K^{\nu}_{\mathrm{Bes}})_{L^2(0,s)} \tag{9.9}$$

for s in a compact set K, $\bar{P}_N$ almost surely.

We can then use the well-known fluctuation results obtained in [37] for the smallest particle of a determinantal random point field with correlation kernel being the Bessel kernel.

Appendix: Extension of the Domain of Universality

In this subsection, we will prove that Johansson's proof of universality of local eigenvalue statistics for Hermitian matrices [21] can be extended to the whole bulk of the spectrum, giving in turn the same interval of universality for our model.

The model studied in [21] deals with local eigenvalue statistics of random Hermitian matrices. Let $\mathcal{H}_N$ be the space of $N \times N$ hermitian matrices.

DEFINITION A.1 A *Johansson-type random matrix M on $\mathcal{H}_N$* is a random matrix that can be written $M = (1/\sqrt{N})\hat{M}$, where the entries of $\hat{M}$ satisfy the following conditions: The entries $\hat{M}_{ij}$, $i \geq j$, are independent random variables with a Gaussian divisible law μ with parameters a and σ_1 for $i > j$ and of parameters $\frac{a}{2}$ and σ_1 for $\hat{P}_{ii}$.

We note σ^2, the variance of the Gaussian divisible law μ, and $\lambda_1 \leq \cdots \leq \lambda_N$, the ordered eigenvalues of M. In particular, the limiting spectral measure of M, $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$, is given by the semicircle law defined by the density

$$\tilde{\sigma}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma,2\sigma]}(x).$$

PROPOSITION A.2 *Let $u \in [\delta, 4\sigma^2 - \delta]$ be a point in the bulk of the spectrum. Then, with $\rho_N = N\tilde{\sigma}(u)$ in* (1.2),

$$\lim_{N\to\infty} \mathbb{E} S_N^m(f) = \int_{\mathbb{R}^m} f(t_1, \ldots, t_m) \det\left(K_{\sin}(t_i, t_j)\right)_{i,j=1}^{m} \prod_{i=1}^{m} dt_i .$$

SKETCH OF PROOF: We refer to Johansson's paper [21] and only indicate the main changes from his proof. The correlation kernel that corresponds to the correlation kernel of the deformed Laguerre ensemble is given by the following double integral in the complex plane:

$$
\text{(A.1)}\quad K_N(u,v;y) = \frac{e^{\frac{v^2-u^2}{2t}}}{(2i\pi)^2 t(u-v)} \int_\Gamma \int_{i\mathbb{R}+A} dz\, dw \frac{1}{z}\bigl(1-e^{\{(v-u)\frac{z}{t}\}}\bigr) \prod_{j=1}^N \frac{w-x_j}{z-x_j}
$$

$$
\times \exp\left\{\frac{w^2-2wv-z^2+2uz}{2t}\right\}\left(w+z-v-t\sum_{j=1}^N \frac{y_j}{(w-y_j)(z-y_j)}\right),
$$

for some arbitrary A.

Here we assume that $\sigma_1^2 = \frac{1}{4}$ so that, in view of [21],

$$
\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^N \delta_{x_j} = \sigma_{\mathrm{sc}}
$$

where σ_{sc} is the semicircle law with density $\frac{2}{\pi}\sqrt{1-x^2}$. The only step in Johansson's proof that is no longer valid is the choice of the contour γ for the w-integral. We have to find a contour in the complex plane satisfying the saddle point requirements. We consider $u = \sqrt{1+4a^2}\cos(\theta_o)$, for some $\theta_o \in [\delta, \pi-\delta]$, where $\delta > 0$ is given.

We recall that the exponential term $F_u(w)$ to be analyzed satisfies

$$
F_u'(w) := \frac{w-u}{a^2} + \int_{-1}^1 \frac{1}{w-y}\, d\sigma_{\mathrm{sc}}(y)
$$

where $u = \sqrt{1+4a^2}\cos\theta_o$. Let us assume $\cos\theta_o > 0$ so that the critical points (proof of lemma 3.2 in [21]) w_c and $\bar{w}_c$ have positive real part. We note $w_c = a' + ib$. For the z-integral, we keep the same contour as chosen in [21]. Consider the contour

$$
\gamma(t) := a' + it, \quad t \in \mathbb{R}.
$$

We then have to show that this contour can be chosen to complete a saddle point analysis of the kernel. Then, along γ,

$$
\mathrm{Re}\left(\frac{dF_u(w_c+it)}{dt}\right) = -(b+t)\left(\frac{1}{a^2} - \int_{-1}^1 \frac{d\sigma(y)}{|w_c+it-y|^2}\right) := G(t)(b+t).
$$

We restrict ourselves here to t such that $b+t > \eta$ so that concentration results hold. The function $G(t)$ is a monotonic function of t^2 that anneals at $t = \pm b$. This is enough to obtain that $\mathrm{Re}\,(F_u)$ achieves its maximum at w_c and $\bar{w}_c$ along this part of the contour γ.

We then have to show that the contribution from γ when $|b+t| < \eta$ is negligible. Using that w_c is a critical point and the expression of $(b+t)G(t)$ as above, one obtains that

$$
\mathrm{Re}(F(a'+i\eta)) - \mathrm{Re}\,F(w_c) \leq -\frac{\eta b^2}{(1+b^2)^2}.
$$

This is enough to obtain that, for η small enough and $|t| \le \eta$,

$$\mathrm{Re}(F(a' + it)) - \mathrm{Re}\, F(w_c) \le -c_o$$

for some positive c_o. In the last step, we use that $\mathrm{Re}(F(a' + it)) \le \mathrm{Re}(F(a' + i\eta)) + \eta^2$. The rest of the proof then follows exactly the same steps as in [21]. $\square$

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