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Universality of the REM for Dynamics of Mean-Field Spin Glasses

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Abstract: We consider a version of Glauber dynamics for a p-spin Sherrington–Kirkpatrick model of a spin glass that can be seen as a time change of simple random walk on the N-dimensional hypercube. We show that, for all $p \geq 3$ and all inverse temperatures $\beta > 0$, there exists a constant $\gamma_{\beta,p} > 0$, such that for all exponential time scales, $\exp(\gamma N)$, with $\gamma < \gamma_{\beta,p}$, the properly rescaled *clock process* (time-change process) converges to an α-stable subordinator where $\alpha = \gamma/\beta^2 < 1$. Moreover, the dynamics exhibits aging at these time scales with a time-time correlation function converging to the arcsine law of this α-stable subordinator. In other words, up to rescaling, on these time scales (that are shorter than the equilibration time of the system) the dynamics of p-spin models ages in the same way as the REM, and by extension Bouchaud's REM-like trap model, confirming the latter as a universal aging mechanism for a wide range of systems. The SK model (the case $p = 2$) seems to belong to a different universality class.

1. Introduction and Results

Aging has become one of the main paradigms to describe the long-time behavior of complex and/or disordered systems. Systems that have strongly motivated this research are *spin glasses*, where aging was first observed experimentally in the anomalous relaxation patterns of the magnetization [LSNB83, Cha84]. To capture the features of activated dynamics, early on people introduced effective dynamics where the state space is reduced to the configurations with lowest energy [DDOL85, KH89]. The theoretical modeling of aging phenomena took a major leap with the introduction of so-called *trap models* by Bouchaud and Dean in the early 1990'ies [Bou92, BD95] (see [BCKM98] for a review). These models reproduce the characteristic power law behavior seen experimentally, while being sufficiently simple to allow for a detailed analytical treatment. While trap models are heuristically motivated to capture the behavior of the dynamics of spin glass models, there is no clear theoretical, let alone mathematical derivation of these from an underlying spin-glass dynamics. The first attempt to establish such a connection was made in [BBG02,BBG03a,BBG03b] where it was shown that, starting from a particular Glauber dynamics of the Random Energy Model (REM), at low temperatures and at a time scale slightly shorter than the equilibration time of the dynamics, an appropriate time-time correlation function of the dynamics converges to that given by Bouchaud's REM-like trap model.

On the other hand, in a series of papers [BČ05,BČM06,BČ08,BČ07] a systematic investigation of a variety of trap models was initiated. In this process, it emerged that there appears to be an almost universal aging mechanism based on α-stable subordinators that governs aging in most trap models. It was also shown that the same feature holds for the dynamics of the REM at shorter time scales than those considered in [BBG03a,BBG03b], and that this also happens at high temperatures, provided appropriate time scales are considered [BČ08]. For a general review on trap models see [BČ06].

However, both in the REM and in the trap models that were analyzed so far, the random variables describing the quenched disorder were considered to be independent. Aging in correlated spin glass models was investigated rigorously only in some cases of spherical SK models and at very short time scales [BDG01]. In the present paper we show for the first time that the same type of aging mechanism is also relevant in correlated spin glasses, at least on time scales that are short compared to the equilibration time (but exponentially large in the volume of the system).

Let us first describe the class of models we are considering. Our state spaces will be the N-dimensional hypercube, $\mathcal{S}_N \equiv \{-1,1\}^N$. $R_N : \mathcal{S}_N \times \mathcal{S}_N \to [-1,1]$ denotes as usual the normalized overlap, $R_N(\sigma,\tau) \equiv N^{-1}\sum_{i=1}^N \sigma_i\tau_i$. The Hamiltonian of the p-spin SK-model is defined as $\sqrt{N}H_N$, where $H_N : \mathcal{S}_N \to \mathbb{R}$ is a centered Gaussian process indexed by $\mathcal{S}_N$ with covariance

$$\mathbb{E}[H_N(\sigma)H_N(\tau)] = R_N(\sigma,\tau)^p, \tag{1.1}$$

for $3 \le p \in \mathbb{N}$. We will denote by $\mathcal{H}$ the σ-algebra generated by the random variables $\{H_N(\sigma), \sigma \in \mathcal{S}_N, N \in \mathbb{N}\}$. The corresponding Gibbs measure is given by

$$\mu_{\beta,N}(\sigma) \equiv Z_{\beta,N}^{-1} e^{\beta\sqrt{N}H_N(\sigma)}, \tag{1.2}$$

where $Z_{\beta,N}$ denotes the normalizing partition function.

We define the dynamics as a nearest neighbor continuous time Markov chain $\sigma_N(\cdot)$ on $\mathcal{S}_N$ with transition rates

$$w_N(\sigma,\tau) = \begin{cases} N^{-1}e^{-\beta\sqrt{N}H_N(\sigma)}, & \text{if } \mathrm{dist}(\sigma,\tau)=1,\\ 0, & \text{otherwise;}\end{cases} \tag{1.3}$$

here $\mathrm{dist}(\cdot,\cdot)$ is the graph distance on the hypercube,

$$\mathrm{dist}(\sigma,\tau) = \frac{1}{2}\sum_{i=1}^N |\sigma_i - \tau_i|. \tag{1.4}$$

A simple way to construct this dynamics is as a time change of a simple random walk on $\mathcal{S}_N$: We denote by $Y_N(k) \in \mathcal{S}_N$, $k \in \mathbb{N}$, the simple unbiased random walk (SRW) on $\mathcal{S}_N$ started at some fixed point of $\mathcal{S}_N$, say at $(1,\ldots,1)$. For $\beta > 0$ we define the *clock-process* by

$$S_N(k) = \sum_{i=0}^{k-1} e_i \exp\left\{\beta\sqrt{N} H_N\left(Y_N(i)\right)\right\}, \tag{1.5}$$

where $\{e_i, i \in \mathbb{N}\}$ is a sequence of mean-one i.i.d. exponential random variables. We denote by $\mathcal{Y}$ the σ-algebra generated by the SRW random variables $\{Y_N(k), k \in \mathbb{N}, N \in \mathbb{N}\}$. The σ-algebra generated by the random variables $\{e_i, i \in \mathbb{N}\}$, will be denoted by $\mathcal{E}$. For non-integer $t \geq 0$ we define $S_N(t) = S_N(\lfloor t \rfloor)$ and we write S_N^{-1} for the generalized right-continuous inverse of S_N. Then the process $\sigma_N(\cdot)$ can be written as

$$\sigma_N(t) \equiv Y_N(S_N^{-1}(t)). \tag{1.6}$$

Obviously, σ_N is reversible with respect to the Gibbs measure $\mu_{\beta,N}$, and $S_N(k)$ is the instant of the k^{th} jump of σ_N. We will consider all random processes to be defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that the three σ-algebras, $\mathcal{H}$, $\mathcal{Y}$, and $\mathcal{E}$, are all independent under $\mathbb{P}$.

We will systematically exploit the construction of the dynamics given by (1.3) or (1.6). The same dynamics was used in the analysis of the REM and in most work on trap models. It differs substantially from more popular dynamics such as the Metropolis or the heat-bath algorithm. The main difference is that in these dynamics the trajectories are not independent of the environment and are biased against going up in energy. This may have a substantial effect, and we do not know whether our results will apply, at least qualitatively, in these cases. The fact is that we currently do not have the tools to analyze these dynamics even in the case of the REM!

Let $V_\alpha(t)$ be an α-stable subordinator with the Laplace transform given by

$$\mathbb{E}[e^{-\lambda V_\alpha(t)}] = \exp(-t\lambda^\alpha). \tag{1.7}$$

Our main technical result is the following theorem that describes the asymptotic behavior of the clock process.

Theorem 1.1. *There exists a function, $\zeta(p)$, such that, for all $p \geq 3$, and γ satisfying*

$$0 < \gamma < \min\left(\beta^2, \zeta(p)\beta\right), \tag{1.8}$$

under the conditional distribution $\mathbb{P}[\cdot|\mathcal{Y}]$ the law of the stochastic process

$$\bar{S}_N(t) = e^{-\gamma N} S_N\left(\left\lfloor t N^{1/2} e^{N\gamma^2/2\beta^2} \right\rfloor\right), \qquad t \geq 0, \tag{1.9}$$

defined on the space of càdlàg functions equipped with the Skorokhod M_1-topology, converges, $\mathcal{Y}$-a.s., to the law of the γ/β^2-stable subordinator, $V_{\gamma/\beta^2}(Kt)$, $t \geq 0$, where K is a positive constant depending on γ, β, and p.

Moreover, the function $\zeta(p)$ is increasing and satisfies

$$\zeta(3) \simeq 1.0291 \qquad \textit{and} \qquad \lim_{p\to\infty} \zeta(p) = \sqrt{2\ln 2}. \tag{1.10}$$

We will explain in Sect. 5 what the M_1-topology is. Roughly, it is a weak topology that does not convey much information at the jumps of the limiting process: e.g., it allows for several jumps of the approximating processes at rather short distances in time to merge to one big jump of the limit process. This will actually occur in our models for $p < \infty$, while it does not happen in the REM. Therefore, we cannot replace the M_1-topology with the stronger J_1-topology in Theorem 1.1.

To control the behavior of spin-spin correlation functions that are commonly used to characterize aging, we need to know more on how these jumps occur at finite N. What we will show is that if we slightly coarse-grain the process $\bar{S}_N$ over blocks of size $o(N)$, the rescaled process does converge in the J_1-topology. What this says, is that the jumps of the limiting process are compounded by smaller jumps that are made over $\le o(N)$ steps of the SRW. In other words, the jumps of the limiting process come from waiting times accumulated in one slightly extended trap, and during this entire time only a negligible fraction of the spins are flipped. This will imply the following aging result.

Theorem 1.2. *Let $A_N^\varepsilon(t,s)$ be the event defined by*

$$A_N^\varepsilon(t,s) = \left\{R_N\left(\sigma_N\left(te^{\gamma N}\right), \sigma_N\left((t+s)e^{\gamma N}\right)\right) \ge 1-\varepsilon\right\}. \tag{1.11}$$

Then, under the hypothesis of Theorem 1.1, for all $\varepsilon \in (0,1)$, $t > 0$, $s > 0$, and $\alpha = \gamma/\beta^2$,

$$\lim_{N\to\infty} \mathbb{P}[A_N^\varepsilon(t,s)] = \frac{\sin\alpha\pi}{\pi}\int_0^{t/(t+s)} u^{\alpha-1}(1-u)^{-\alpha}\,\mathrm{d}u. \tag{1.12}$$

Remark. We will in fact prove the stronger statement that aging in the above sense occurs along almost every random walk trajectory, that is

$$\lim_{N\to\infty} \mathbb{P}[A_N^\varepsilon(t,s)|\mathcal{Y}] = \frac{\sin\alpha\pi}{\pi}\int_0^{t/(t+s)} u^{\alpha-1}(1-u)^{-\alpha}\,\mathrm{d}u, \qquad \mathcal{Y}\text{-a.s.} \tag{1.13}$$

Let us discuss the meaning of these results. $e^{\gamma N}$ is the time-scale at which we want to observe the process. According to Theorem 1.1, at this time the random walk will make about $N^{1/2}e^{N\gamma^2/2\beta^2} \ll e^{\gamma N}$ steps. Since this number is also much smaller than 2^N (as follows from (1.10)), the random walk will essentially visit that number of sites.

If the random process H_N were i.i.d., then the maximum of H_N along the trajectory up to time $N^{1/2}e^{N\gamma^2/\beta^2}$ would be $\left(2\ln(N^{1/2}e^{N\gamma^2/2\beta^2})\right)^{1/2} \sim N^{1/2}\gamma/\beta$, and the time spent in the site with maximal H_N would be of order $e^{\gamma N}$. Since Theorem 1.1 holds also in the i.i.d. case, that is in the REM (see [BČ08]), the time spent in the maximum is comparable to the total time and the convergence to the α-stable subordinator implies that the total accumulated time is composed of pieces of order $e^{\gamma N}$ that are collected along the trajectory. In fact, each jump of the subordinator corresponds to one visit to a site that has waiting time of that order. In a common metaphor, the sites are referred to as traps and the mean waiting times as their depths.

The theorem in the general case states that the same is essentially true in the p-spin model. The difference is that now the traps do not consist of a single site, but of a deep valley (along the trajectory) whose bottom has approximately the same energy as in the i.i.d. case and whose shape and width we will describe quite precisely. Remarkably, the number of sites contributing significantly to the residence time in the valley is essentially finite, and different valleys are statistically independent.

The fact that traps are finite may appear quite surprising to those familiar with the statics of p-spin models. From the results there (see [Tal03,Bov06]), it is known that the Gibbs measure concentrates on "lumps" whose diameter is of order $N\varepsilon_p$, with $\varepsilon_p > 0$. The mystery is however solved easily: the process $H_N(\sigma)$ does indeed decrease essentially linearly with speed $N^{-1/2}$ from a local maximum. Thus, the residence times in such sites decrease geometrically, so that the contributions of a neighborhood of size K of a local maximum amounts to a fraction of $(1 - c^{-K})$ of the total time spent in that valley; for the support of the Gibbs measure, one needs however to take into account the entropy, that is the fact that the volume of the balls of radius r increases like N^r. For the dynamics, at least at our time-scales, this is, however, irrelevant, since the SRW leaves a local minimum essentially ballistically.

The proof of Theorem 1.1 relies on the combination of detailed information on the properties of the SRW on the hypercube, which is provided in Sect. 4 (but see also [Mat89,BG06,ČG08]), and comparison of the process H_N on the trajectory of the SRW to a simpler Gaussian process using interpolation techniques à la Slepian, familiar from extreme value theory of Gaussian processes.

Let us explain this in more detail. On the time scales we are considering, the SRW makes $tN^{1/2}\exp(N\gamma^2/2\beta^2) \ll tN^{1/2}\exp(N\zeta(p)^2/2) \ll 2^N$ steps. In this regime the SRW is extremely "transient", in the sense that (i) starting from a given point x, for times $t \le \nu \sim N^{\omega}$, $\omega < 1$, the distance from x grows essentially linearly with speed one, that is there are no backtrackings with high probability; (ii) the SRW will *never* return to a neighborhood of size ν of the starting point x, with high probability. The upshot is that we can think of the trajectory of the SRW essentially as of a straight line.

Next we consider the Gaussian process restricted to the SRW trajectory. We expect that the main contributions to the sums $S_N(k)$ come from places where H_N is maximal (on the trajectory). We expect that the distribution of these extremes does not feel the correlations between points farther than ν apart. On the other hand, for points closer than ν, the correlation function $R_N(Y_N(i), Y_N(j))^p$ can be well approximated by a linear function $1 - 2pN^{-1}|i - j|$ (using that $R_N(Y_N(i), Y_N(j)) \sim 1 - 2N^{-1}|i - j|$). This is convenient since this process has an explicit representation in terms of i.i.d. random variables, which allows for explicit computations (in fact, this is one of the famous processes for which the extremal distribution can be computed explicitly [Sle61,She71]). Thus the idea is to cut the SRW trajectory into blocks of length ν and to replace the original process $H_N(Y_N(i))$ by a new one U_i, where U_i and U_j are independent, if i, j are not in the same block, and $\mathbb{E}[U_i U_j] = 1 - 2pN^{-1}|i - j|$ if they are. For the new process, Theorem 1.1 is relatively straightforward. The main step is the computation of Laplace transforms in Sect. 2. Comparing the real process with the auxiliary one is the bulk of the work and is done in Sect. 3. The properties of SRW needed are established in Sect. 4. In Sect. 5 we present the proofs of the main theorems.

Our results exhibit considerable universality of the REM for dynamics of p-spin models with $p \ge 3$. This dynamic universality is close to the static universality of the REM, which shows that various features of the landscape of energies (that is of the Hamiltonian H_N) are insensitive to correlations. This static universality in a microcanonical context has been introduced by [BM04] (see [BK06a,BK06b] for rigorous results in the context of spin-glasses). The static results closest to our dynamics question are given in [BGK06,BK07], where it is shown that the statistics of extreme values for the restriction of H_N to a random set $X_N \subset \mathcal{S}_N$ are universal, for $p \ge 3$ and $|X_N| = e^{cN}$, for c small enough.

2. Behavior of the One-Block Sums

In this section we analyze the distribution of the block-sums, $\sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N}U_i}$, where $\{e_i, i \in \mathbb{N}\}$ are mean-one i.i.d. exponential random variables, and $\{U_i, i = 1, \dots, \nu\}$ is a centered Gaussian process with the covariance $\mathbb{E}U_iU_j = 1 - 2pN^{-1}|i-j|$; $\nu = \nu_N$ is a function of N of the form

$$\nu = \lfloor N^{\omega} \rfloor, \qquad \text{with } \omega \in (1/2, 1). \tag{2.1}$$

As explained in the introduction, this process will serve as a local approximation of the corresponding block sums along a SRW trajectory. We characterize the distribution of the block-sums in terms of its Laplace transform

$$\mathcal{F}_N(u) = \mathbb{E}\left[\exp\left\{-ue^{-\gamma N}\sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N}U_i}\right\}\right]. \tag{2.2}$$

Proposition 2.1. *For all γ such that $\gamma/\beta^2 \in (0,1)$ there exists a constant, $K = K(\gamma, \beta, p)$, such that, uniformly for u in compact subsets of* $[0,\infty)$,

$$\lim_{N\to\infty} N^{1/2}\nu^{-1}e^{N\gamma^2/2\beta^2}\,[1 - \mathcal{F}_N(u)] = Ku^{\gamma/\beta^2}. \tag{2.3}$$

Proof. For all N the argument of the limit on the left-hand side of (2.3) is continuous and increasing in $u \in [0,\infty)$. The same is true for the right-hand side of (2.3). Therefore, the uniform convergence claimed in the proposition is a direct consequence of the point-wise convergence for $u \in (0,\infty)$, which we will prove in the following.

We first compute the conditional expectation in (2.2) given the σ-algebra, $\mathcal{U}$, generated by the Gaussian process U,

$$\begin{aligned}\mathbb{E}\left[\exp\left\{-ue^{-\gamma N}\sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N}U_i}\right\}\Big|\mathcal{U}\right] &= \prod_{i=1}^{\nu}\frac{1}{1+ue^{-\gamma N}e^{\beta\sqrt{N}U_i}} \\ &= \exp\left\{-\sum_{i=1}^{\nu} g\left(ue^{-\gamma N}e^{\beta\sqrt{N}U_i}\right)\right\},\end{aligned} \tag{2.4}$$

where

$$g(x) \equiv \ln(1+x). \tag{2.5}$$

Note that $g(x)$ is monotone increasing and non-negative for $x \in \mathbb{R}_+$. We use the well-known fact (see e.g. [Sle61]) that the random variables U_i can be expressed using a sequence of i.i.d. standard normal variables, Z_i, as follows. Set $Z_1 = (U_1 + U_\nu)/(4 - 4p(\nu-1)/N)^{1/2}$ and $Z_k = (U_k - U_{k-1})/(4p/N)^{1/2}$, $k = 2, \dots, \nu$. Then Z_i are i.i.d. standard normal variables and

$$U_i = \Gamma_1 Z_1 + \cdots + \Gamma_i Z_i - \Gamma_{i+1}Z_{i+1} - \cdots - \Gamma_\nu Z_\nu, \tag{2.6}$$

where

$$\Gamma_1 = \sqrt{1 - \frac{p}{N}(\nu-1)} \qquad \text{and} \qquad \Gamma_2 = \cdots = \Gamma_\nu = \sqrt{\frac{p}{N}}. \tag{2.7}$$

Observe that $\sum_{i=1}^{\nu} \Gamma_i^2 = 1$. Let us define $G_i(z) = G_i(z_1, \dots, z_\nu)$ as

$$G_i(z) = \Gamma_1 z_1 + \cdots + \Gamma_i z_i - \Gamma_{i+1} z_{i+1} - \cdots - \Gamma_\nu z_\nu. \tag{2.8}$$

Using this notation we get

$$1 - \mathcal{F}_N(u) = \int_{\mathbb{R}^\nu} \frac{\mathrm{d}z}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} z_i^2} \left\{ 1 - \exp\left[-\sum_{i=1}^{\nu} g\left(u e^{-\gamma N} e^{\beta\sqrt{N} G_i(z)} \right) \right] \right\}. \tag{2.9}$$

We divide the domain of integration into several parts according to which of the $G_i(z)$ is maximal. Define $D_k = \{z \in \mathbb{R}^\nu : G_k(z) \geq G_i(z) \forall i \neq k\}$. On D_k we use the substitution

$$\begin{aligned} z_i &= b_i + \Gamma_i(\gamma N - \ln u)/(\beta\sqrt{N}), \qquad \text{if } i \leq k, \\ z_i &= b_i - \Gamma_i(\gamma N - \ln u)/(\beta\sqrt{N}), \qquad \text{if } i > k. \end{aligned} \tag{2.10}$$

It will be useful to define $\sum_{j=i+1}^{k} a_j$ as $\sum_{j=1}^{k} a_j - \sum_{j=1}^{i} a_j$, which is meaningful also for $k < i + 1$. Using this definition

$$G_k(b) - G_i(b) = 2 \sum_{j=i+1}^{k} \Gamma_\nu b_j. \tag{2.11}$$

Set $\theta = -\ln(u)/(\gamma N)$ and define

$$D'_k = \left\{ b \in \mathbb{R}^\nu : \sum_{j=i+1}^{k} b_j + \frac{\gamma\sqrt{p}}{\beta} |k - i|(1 + \theta) \geq 0 \quad \forall i \neq k \right\}. \tag{2.12}$$

After a straightforward computation we find that (2.9) equals (up to a multiplicative correction converging to one as $N \to \infty$)

$$\begin{aligned} & e^{-N\gamma^2/2\beta^2} u^{\gamma/\beta^2} \sum_{k=1}^{\nu} \int_{D'_k} \frac{\mathrm{d}b}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} b_i^2} e^{-\frac{\gamma}{\beta}\sqrt{N} G_k(b)(1+\theta)} \\ & \times \left\{ 1 - \exp\left(-\sum_{i=1}^{\nu} g\left(e^{\beta\sqrt{N} G_k(b) - 2\beta\sqrt{p}\sum_{j=i+1}^{k} b_j - 2p\gamma|k-i|(1+\theta)} \right) \right) \right\}. \end{aligned} \tag{2.13}$$

To finish the proof we have to show that asymptotically the only dependence in (2.13) on u (or θ) is through the factor u^{γ/β^2}, and that the sum is of order $\nu N^{-1/2}$. We change variables once more to $a_j = b_j/(1 + \theta)$ in order to remove the dependence of the integration domains D'_k on u. Then the sum (without the prefactor) in (2.13) can be expressed as

$$\begin{aligned} & \sum_{k=1}^{\nu} \int_{D''_k} \frac{(1+\theta)^\nu \mathrm{d}a}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}(1+\theta)^2 \sum_{i=1}^{\nu} a_i^2} \left[e^{-\frac{\gamma}{\beta}\sqrt{N} G_k(a)(1+\theta)^2} \right. \\ & \left. \times \left\{ 1 - \exp\left(-\sum_{i=1}^{\nu} g\left(e^{(\beta\sqrt{N} G_k(a) - 2\beta\sqrt{p}\sum_{j=i+1}^{k} a_j - 2p\gamma|k-i|)(1+\theta)} \right) \right) \right\} \right], \end{aligned} \tag{2.14}$$

where $D''_k = \left\{ a \in \mathbb{R}^\nu : \sum_{j=i+1}^{k} a_j + \frac{\gamma\sqrt{p}}{\beta}|k - i| \geq 0 \quad \forall i \neq k \right\}$.

Let $\delta > 0$ be such that $(1+\delta)\gamma/\beta^2 < 1$, and let $N > |\ln u|/(\gamma\delta)$, so that $|\theta| \le \delta$. We first examine the square bracket in the above expression for a fixed k. On D_k''

$$\exp\left\{-\sum_{i=1}^{\nu} g\left(e^{(\beta\sqrt{N}G_k(a)-2\beta\sqrt{p}\sum_{j=i+1}^{k}a_j-2p\gamma|k-i|)(1+\theta)}\right)\right\} \\ \ge \exp\left\{-\nu g\left(e^{\beta\sqrt{N}G_k(a)(1+\theta)}\right)\right\}. \tag{2.15}$$

Write $G_k(a)$ as (recall (2.1))

$$G_k(a) = \frac{\xi - \omega \ln N}{(1+\theta)\beta\sqrt{N}}. \tag{2.16}$$

The square bracket of (2.14) is then smaller than

$$e^{-\frac{\gamma}{\beta^2}(\xi-\omega\ln N)(1+\theta)}\left\{1-\exp\left(-\nu g\left(e^{\xi-\omega\ln N}\right)\right)\right\} \\ \le N^{\frac{\gamma\omega(1+\theta)}{\beta^2}} e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)}\left\{1-\exp\left(-\nu g\left(e^{\xi}/\nu\right)\right)\right\}. \tag{2.17}$$

If $(1+\theta)\gamma/\beta^2 < 1$, then the function $e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)}\left\{1-\exp\left(-\nu g\left(e^{\xi}/\nu\right)\right)\right\}$ is uniformly bounded in $\xi \in \mathbb{R}$ and ν. Indeed, if $\xi \ge 0$, then

$$e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)}\left\{1-\exp\left(-\nu g\left(e^{\xi}/\nu\right)\right)\right\} \le e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)} \le 1. \tag{2.18}$$

If $\xi < 0$, then, since $g(x) \le x$,

$$\left\{1-\exp\left(-\nu g\left(e^{\xi}/\nu\right)\right)\right\} \le \left\{1-\exp\left(-e^{\xi}\right)\right\}, \tag{2.19}$$

which behaves like e^{ξ}, as $\xi \to -\infty$. This compensates the exponentially growing prefactor if $(1+\theta)\gamma/\beta^2 < 1$. Thus, under this condition, the bracket of (2.14) increases at most polynomially with N. Therefore, there exists $\delta > 0$ small, such that the domain of integration in (2.14) may be restricted to a_i's satisfying

$$\nu^{-1}\sum_{i=1}^{\nu} a_i^2 \in (1-\delta, 1+\delta), \quad |a_1| \le N^{1/4}, \quad \sum_{i=1}^{\nu}|a_i| \le \nu^{1+\delta}. \tag{2.20}$$

The integral over the remaining a_i's decays at least as $e^{-N^{\delta'}}$, for some $\delta' > 0$ (by a simple large deviation argument). For all a satisfying (2.20), $|G_k(a)| \le N^{1/4} + N^{-1/2}\nu^{1+\delta'} \ll N^{1/2}$ and thus, for any fixed u, uniformly in such a's, we have

$$\frac{e^{-\frac{\gamma}{\beta}\sqrt{N}G_k(a)(1+\theta)^2}}{e^{-\frac{\gamma}{\beta}\sqrt{N}G_k(a)}} \xrightarrow{N\to\infty} 1, \quad \text{and} \quad \frac{e^{-\frac{1}{2}(1+\theta)^2\sum_{i=1}^{\nu}a_i^2}}{e^{-\frac{1}{2}\sum_{i=1}^{\nu}a_i^2}} \xrightarrow{N\to\infty} 1. \tag{2.21}$$

Also, $(1+\theta)^{\nu} \xrightarrow{N\to\infty} 1$. Hence, up to a small error, we can remove all but the last occurrence of θ in (2.14).

Finally, taking $x_i = a_i$ for $i \ge 2$, $x_1 = N^{1/2}G_k(a)$, and thus

$$a_1 = \frac{x_1 - \sqrt{p}(x_2 + \cdots + x_k - x_{k+1} - \cdots - x_{\nu})}{\Gamma_1\sqrt{N}}, \tag{2.22}$$

(2.14) is, up to a small error, equal to

$$\sum_{k=1}^{\nu}\int_{D_k''}\frac{\mathrm{d}x\, e^{-\frac{1}{2}\sum_{i=2}^{\nu}x_i^2}}{\Gamma_1 N^{1/2}(2\pi)^{\nu/2}}\exp\left(-\frac{\gamma}{\beta}x_1-\frac{x_1^2}{2\Gamma_1^2 N}\right)\exp\left(-\frac{a_1^2}{2}+\frac{x_1^2}{2\Gamma_1^2 N}\right) \times\left\{1-\exp\left(-\sum_{i=1}^{\nu}g\left(e^{(1+\theta)\beta x_1}e^{-\left(2\beta\sqrt{p}\sum_{j=i+1}^{k}x_j+2p\gamma|k-i|\right)(1+\theta)}\right)\right)\right\}. \tag{2.23}$$

The last exponential term on the first line can be omitted. Indeed,

$$-\frac{a_1^2}{2}+\frac{x_1^2}{2\Gamma_1^2 N}=\frac{1}{\Gamma_1^2 N}\left[x_1\sqrt{p}(x_2+\cdots-x_\nu)-2p(x_2+\cdots-x_\nu)^2\right]\xrightarrow{N\to\infty}0 \tag{2.24}$$

uniformly for all $|x_1|\le N^{(1+\delta)/2}$ and $|x_2+\cdots-x_\nu|\le \nu^{(1+\delta)/2}$ if $\delta>0$ is sufficiently small. The integral over the remaining x is again at most $e^{-N^{\delta'}}$, again by a large-deviation argument.

Now we estimate the integral over $x_2,\ldots,x_\nu$ in (2.23). Namely,

$$\int_{\bar{D}_k''}\frac{\mathrm{d}x e^{-\frac{1}{2}\sum_{i=2}^{\nu}x_i^2}}{(2\pi)^{(\nu-1)/2}}\left[1-\exp\left(-\sum_{i=1}^{\nu}g\left(e^{(1+\theta)\beta x_1-\left(2\beta\sqrt{p}\sum_{j=i+1}^{k}x_j+2p\gamma|k-i|\right)(1+\theta)}\right)\right)\right], \tag{2.25}$$

where $\bar{D}_k''$ is the restriction of D_k'' to the last $\nu-1$ coordinates (which does not depend on the value of the first one). Let $V=(V_2,\ldots,V_\nu)$ be a sequence of i.i.d. standard normal random variables and let $R_i^k=\sum_{j=i+1}^{k}V_j$. Using this notation we rewrite (2.25) as

$$\mathbb{E}\left[1-\exp\left(-\sum_{i=1}^{\nu}g\left(e^{(1+\theta)\beta x_1}e^{-\left(2\beta\sqrt{p}R_i^k+2p\gamma|k-i|\right)(1+\theta)}\right)\right);\, V\in\bar{D}_k''\right]. \tag{2.26}$$

Restricting the summation to $i=k$, we get a lower bound

$$\mathbb{E}\left[1-\exp\left(-g\left(e^{(1+\theta)\beta x_1}\right)\right);\, V\in\bar{D}_k''\right]\sim e^{(1+\theta)\beta x_1}\mathbb{P}[V\in\bar{D}_k''] \quad \text{as } x_1\to-\infty. \tag{2.27}$$

The probability $\mathbb{P}[V\in\bar{D}_k'']$ is bounded from below by the probability that a two-sided random walk $(R(i), i\in\mathbb{Z})$ with standard normal increments and $R(0)=0$ satisfies $R(i)\ge-\gamma\sqrt{p}|i|/\beta$ for all $i\in\mathbb{Z}$. This probability is positive and does not depend on N. This implies that there exists $c>0$, independent of k, N, and u, such that, for all $x_1<0$,

$$(2.25)\ge ce^{(1+\theta)\beta x_1}. \tag{2.28}$$

Using $g(x)\le x$ and $1-e^{-x}\le x$ we get an upper bound for (2.25), namely

$$(2.25)\le e^{(1+\theta)\beta x_1}\sum_{i=1}^{\nu}\mathbb{E}\left[e^{-\left(2\beta\sqrt{p}R_i^k+2p\gamma|k-i|\right)(1+\theta)};\, V\in\bar{D}_k''\right]. \tag{2.29}$$

Relaxing the condition $V \in \bar{D}''_k$ to $R_i^k \geq -\frac{\gamma\sqrt{p}}{\beta}|k-i|$ in the i^{th} term of the summation and using the fact that R_i^k is a centered normal random variable with variance $|k-i|$, we get, by a straightforward Gaussian computation, that

$$(2.25) \leq e^{(1+\theta)\beta x_1} \sum_{i=1}^{\nu} \frac{C}{\sqrt{|k-i|}} e^{-\gamma^2 p|k-i|/(2\beta^2)} \leq C e^{(1+\theta)\beta x_1}, \tag{2.30}$$

where C depends only on β, γ, and p.

The bounds (2.28) and (2.30) imply that (2.23) is bounded from above and from below (with different constants) by

$$C'\nu N^{-1/2} \int_{\mathbb{R}} \mathrm{d}x_1 \exp\left(-\frac{\gamma}{\beta}x_1 - \frac{x_1^2}{2\Gamma_1^2 N}\right)(1 \wedge ce^{(1+\theta)\beta x_1}) = C''\nu N^{-1/2}. \tag{2.31}$$

This proves that (2.23), and thus (2.14), are of the right order. Moreover, these bounds imply that we can restrict the domain of integration over x_1 in (2.23) to a large compact interval $[-\bar{C}, \bar{C}]$ without losing precision.

Finally, observe that (2.25) is an increasing function of $\min(k, \nu - k)$. Therefore, there exists a bounded function, $\bar{c} : \mathbb{R} \to [0, \infty)$, such that, as this minimum increases, (2.25) converges to $\bar{c}(x_1)(e^{(1+\theta)\beta x_1} \wedge 1)$. Using this fact and the bounds (2.28) and (2.30), it is easy to see that there exists $\mathcal{C} > 0$ such that (2.23) behaves as $\mathcal{C}\nu N^{-1/2}(1 + o(1))$, as $N \to \infty$. Finally, observe that if $x_1 \in [-\bar{C}, \bar{C}]$ then the derivative of the integrand in (2.25) with respect to θ is bounded uniformly in N and k. This can be used to show that the constant $\mathcal{C}$ is independent of u. This completes the proof of Proposition 2.1. □

We close this section with a short description of the shape of the valleys mentioned in the introduction. First, it follows from (2.10) and the following computations that the most important contribution to the Laplace transform comes from realizations for which $\max\{U_i : 1 \leq i \leq \nu\} \sim \gamma\sqrt{N}/\beta$ with an error of order $N^{-1/2}$. It is the "geometrical" sequence in (2.30) which shows that only finitely many neighbors of the maximum actually contribute to the Laplace transform. The same can be seen, at least heuristically, from a simple calculation

$$\mathbb{E}\left[U_{k+i} \middle| U_k = \frac{\gamma}{\beta}\sqrt{N}\right] = \frac{\gamma\sqrt{N}}{\beta} - C_{\beta,\gamma,p}\frac{|i|}{\sqrt{N}}, \tag{2.32}$$

which means that, disregarding the fluctuations, the energy decreases linearly with the distance from the local maximum and thus the mean waiting times decrease exponentially.

3. Comparison Between the Real and the Clock Process

We now come to the main task, the comparison of the clock-process sums with those in which the real Gaussian process is replaced by a simplified process. For a given realization, Y_N, of the SRW, we set $X_N^0(i) = H_N(Y_N(i))$ (the dependence on Y_N will be suppressed in the notation). Then $X_N^0(i)$ is a centered Gaussian process indexed by $\mathbb{N}$ with covariance matrix

$$\Lambda_{ij}^0 = \mathbb{E}[X_N^0(i)X_N^0(j)] = R_N(Y_N(i), Y_N(j))^p. \tag{3.1}$$

We further define the comparison process, $X_N^1(i)$, as a centered Gaussian process with covariance matrix

$$\Lambda_{ij}^1 = \mathbb{E}[X_N^1(i)X_N^1(j)] = \begin{cases} 1-2pN^{-1}|i-j|, & \text{if } \lfloor i/\nu \rfloor = \lfloor j/\nu \rfloor, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

For $h \in (0,1)$ we define the interpolating process $X_N^h(i) \equiv \sqrt{1-h}X_N^0(i) + \sqrt{h}X_N^1(i)$.

Let $\ell \in \mathbb{N}$, $0 = t_0 < \cdots < t_\ell = T$, and $u_1, \ldots, u_\ell \in \mathbb{R}_+$ be fixed. For any Gaussian process $X = (X(i), i \in \mathbb{N})$ we define a function, $F_N(X) = F_N(X; \{t_i\}, \{u_i\})$, as

$$\begin{aligned} F_N(X; \{t_i\}, \{u_i\}) &\equiv \mathbb{E}\left[\exp\left(-\sum_{k=1}^{\ell} \frac{u_k}{e^{\gamma N}} \sum_{i=\lfloor t_{k-1} r(N) \rfloor}^{\lfloor t_k r(N) \rfloor - 1} e_i e^{\beta\sqrt{N}X(i)} \right) \Big| X \right](X) \\ &= \exp\left(-\sum_{k=1}^{\ell} \sum_{i=\lfloor t_{k-1} r(N) \rfloor}^{\lfloor t_k r(N) \rfloor - 1} g\left(\frac{u_k}{e^{\gamma N}} e^{\beta\sqrt{N}X(i)} \right) \right), \end{aligned} \tag{3.3}$$

where

$$r(N) = \left\lceil N^{1/2} e^{N\gamma^2/2\beta^2} \right\rceil. \tag{3.4}$$

Observe that $\mathbb{E}[F(X^0; \{t_i\}, \{u_i\})|\mathcal{Y}]$ is a joint Laplace transform of the distribution of the properly rescaled clock process at times t_i. The following approximation is the crucial step of the proof.

Proposition 3.1. *If the assumptions of Theorem 1.1 are satisfied, then, for all sequences* $\{t_i\}$ *and* $\{u_i\}$,

$$\lim_{N\to\infty} \mathbb{E}\left[F_N\left(X_N^0; \{t_i\}, \{u_i\} \right) |\mathcal{Y}\right] - \mathbb{E}\left[F_N\left(X_N^1; \{t_i\}, \{u_i\} \right) \right] = 0, \qquad \mathcal{Y}\text{-a.s.} \tag{3.5}$$

Proof. The well-known interpolation formula for functionals of two Gaussian processes due (probably) to Slepian and Kahane (see e.g. [LT91]) reads in our context

$$\mathbb{E}[F_N(X_N^0) - F_N(X_N^1)|\mathcal{Y}] = \frac{1}{2}\int_0^1 \mathrm{d}h \sum_{\substack{i,j=1 \\ i\neq j}}^{r(N)T} (\Lambda_{ij}^0 - \Lambda_{ij}^1)\mathbb{E}\left[\frac{\partial^2 F_N(X_N^h)}{\partial X(i) \partial X(j)} \Big| \mathcal{Y} \right]. \tag{3.6}$$

We will show that the integral in (3.6) converges to 0.

Let $k(i)$ be defined by $\lfloor t_{k(i)-1} r(N) \rfloor \le i < \lfloor t_{k(i)} r(N) \rfloor$. The second derivative in (3.6) is equal to

$$\begin{aligned} &\frac{u_{k(i)} u_{k(j)} \beta^2 N}{e^{2\gamma N}} e^{\beta\sqrt{N}(X_N^h(i) + X_N^h(j))} \\ &\quad \times g'\left(\frac{u_{k(i)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(i)} \right) g'\left(\frac{u_{k(j)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(j)} \right) F_N(X_N^h) \\ &\le \frac{u_{k(i)} u_{k(j)} \beta^2 N}{e^{2\gamma N}} e^{\beta\sqrt{N}(X_N^h(i) + X_N^h(j))} \\ &\quad \times \exp\left[-2g\left(\frac{u_{k(i)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(i)} \right) - 2g\left(\frac{u_{k(j)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(j)} \right) \right], \end{aligned} \tag{3.7}$$

where we used that $g'(x) = (1+x)^{-1} = \exp(-g(x))$ (recall (2.5)), and we omitted in the summation of $F_N(X_N^h)$ all terms different from i and j. To estimate the expected value of this expression we need the following technical lemma.

Lemma 3.2. *Let $c \in [-1,1]$ and let U_1, U_2 be two standard normal variables with covariance $\mathbb{E}[U_1U_2] = c$, λ a small constant, $0 < \lambda < \min\{1-\gamma\beta^{-2}, \gamma\beta^{-2}\}$ (which will stay fixed), and $u, v > 0$. Define $\Xi_N(c) = \Xi_N(c,\beta,\gamma,u,v)$ and $\bar{\Xi}_N(c) = \bar{\Xi}_N(c,\beta,\gamma,u,v,\lambda)$ by*

$$\Xi_N(c) = \frac{uv\beta^2 N}{e^{2\gamma N}} \mathbb{E}\left[\exp\left(\beta\sqrt{N}(U_1+U_2) - 2g\left(ue^{\beta\sqrt{N}U_1-\gamma N}\right) - 2g\left(ve^{\beta\sqrt{N}U_2-\gamma N}\right)\right)\right] \tag{3.8}$$

and

$$\bar{\Xi}_N(c) = \begin{cases} C\left\{(1-c)^{-1/2} \wedge \sqrt{N}\right\} \exp\left\{-\frac{\gamma^2 N}{\beta^2(1+c)}\right\}, & \text{if } 1 \ge c > \gamma\beta^{-2}+\lambda-1, \\ CN\exp\left\{N(\beta^2(1+c)-2\gamma)\right\}, & \text{if } c \le \gamma\beta^{-2}+\lambda-1, \end{cases} \tag{3.9}$$

where $C \equiv C(\gamma,\beta,u,v,\lambda)$ is a suitably chosen constant independent of N and c. Then

$$\Xi_N(c) \le \bar{\Xi}_N(c). \tag{3.10}$$

Proof. Set $\kappa_\pm = \sqrt{2(1\pm c)}$. Let $\bar{U}_1, \bar{U}_2$ be two independent standard normal variables. Then U_1 and U_2 can be written as

$$U_1 = \frac{1}{2}(\kappa_+\bar{U}_1 + \kappa_-\bar{U}_2), \qquad U_2 = \frac{1}{2}(\kappa_+\bar{U}_1 - \kappa_-\bar{U}_2). \tag{3.11}$$

Hence, $U_1 + U_2 = \kappa_+\bar{U}_1$. For $x \ge 0$ and $y \ge 0$ the function g satisfies $g(x)+g(y) = g(x+y+xy) \ge g(x+y)$. Moreover, $ue^x + ve^{-x} \ge \min(u,v)e^{|x|}$. Hence,

$$\begin{aligned} g&\left(ue^{\beta\sqrt{N}U_1-\gamma N}\right) + g\left(ve^{\beta\sqrt{N}U_2-\gamma N}\right) \\ &\ge g\left(\min(u,v)\exp\left(\frac{\kappa_+\beta\sqrt{N}\bar{U}_1}{2} + \left|\frac{\kappa_-\beta\sqrt{N}\bar{U}_2}{2}\right| - \gamma N\right)\right). \end{aligned} \tag{3.12}$$

Setting $\min(u,v) = \bar{u}$, we can bound $\Xi_N(c)$ from above by

$$\frac{uv\beta^2 N}{e^{2\gamma N}} \int_{\mathbb{R}^2} \frac{\mathrm{d}y}{2\pi} \exp\left[-\frac{y_1^2+y_2^2}{2} + \beta\sqrt{N}\kappa_+ y_1 - 2g\left(\bar{u}e^{\frac{1}{2}\kappa_+\beta\sqrt{N}y_1 + \frac{1}{2}\kappa_-\beta\sqrt{N}|y_2| - \gamma N}\right)\right]. \tag{3.13}$$

Substituting $z_1 = y_1 - \beta\sqrt{N}\kappa_+$ and $z_2 = y_2$ we get

$$\begin{aligned} &\frac{uv\beta^2 N}{e^{2\gamma N}} e^{\beta^2\kappa_+^2 N/2} \int_{\mathbb{R}^2} \frac{\mathrm{d}z}{2\pi} \exp\left(-\frac{z_1^2+z_2^2}{2}\right) \\ &\times \exp\left(-2g\left(\bar{u}\exp\left\{\sqrt{N}\left[\left(\frac{\beta^2\kappa_+^2}{2} - \gamma\right)\sqrt{N} + \frac{\beta\kappa_+}{2}z_1 + \frac{\beta\kappa_-}{2}|z_2|\right]\right\}\right)\right). \end{aligned} \tag{3.14}$$

The second line of the last expression is always smaller than one. Therefore,

$$\Xi_N(c) \le \frac{uv\beta^2 N}{e^{2\gamma N}} e^{\beta^2\kappa_+^2 N/2} = C(\gamma,\beta,u,v) N \exp\left\{N(\beta^2(1+c)-2\gamma)\right\}, \quad (3.15)$$

which is the same expression as appears on the second line of definition (3.9) of $\bar{\Xi}$. This estimate is however not always optimal. Indeed, note that the function $\exp(-2g(\bar{u}e^{\sqrt{N}x}))$ converges to the indicator function $\mathbf{1}_{x<0}$ as $N \to \infty$. The role of x will be played by the square bracket in the expression (3.14). If $\beta^2\kappa_+^2/2 - \gamma > 0$, this bracket is positive for z close to 0 and the integrand in (3.14) is typically very small.

We thus fix λ satisfying the assumptions of Lemma 3.2, and consider c such that $c > \gamma\beta^{-2} + \lambda - 1$. This is equivalent to $\beta^2\kappa_+^2/2 - \gamma > \lambda'$ for some $\lambda' = \lambda'(\lambda) > 0$. In this case we need another substitution, namely

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{N}}\left[v_1 - \frac{\kappa_-}{\kappa_+}|v_2| - N\left(\beta\kappa_+ - \frac{2\gamma}{\beta\kappa_+}\right)\right], \\ z_2 &= \frac{v_2}{\sqrt{N}}. \end{aligned} \quad (3.16)$$

This substitution transforms the domain where the square bracket of (3.14) is negative into the half plane $v_1 < 0$: The expression inside of the braces in (3.14) equals $\beta\kappa_+ v_1/2$. Substituting (3.16) into $(z_1^2 + z_2^2)/2$ produces an additional exponential prefactor $\exp\left(-\frac{(\beta^2\kappa_+^2 - 2\gamma)^2 N}{2\beta^2\kappa_+^2}\right)$. Another prefactor N^{-1} comes from the Jacobian. The remaining terms can be bounded from above by

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}v}{2\pi} \exp\left\{-\frac{v_2^2}{2N} + \left(\beta\kappa_+ - \frac{2\gamma}{\beta\kappa_+}\right)\left(v_1 - \frac{\kappa_-}{\kappa_+}|v_2|\right) - 2g(\bar{u}e^{\beta\kappa_+ v_1/2})\right\}, \quad (3.17)$$

which can be separated into a product of two integrals. The integral over v_2 contains two terms: one with v_2^2 and the second with $|v_2|$. If we ignore the quadratic one (which can be done only if $\kappa_- \neq 0$, that is $c \neq 1$), then the integral over v_2 can be bounded from above by

$$\left(\left(\beta\kappa_+ - \frac{2\gamma}{\beta\kappa_+}\right)\frac{\kappa_-}{\kappa_+}\right)^{-1} \le C(\lambda)\kappa_-^{-1} \le C(\lambda)(1-c)^{-1/2}, \quad (3.18)$$

where $C(\lambda)$ is a constant depending only on λ. If $\kappa_- = 0$, then the term with $|v_2|$ disappears and the integration over v_2 gives a factor $C\sqrt{N}$.

To bound the integral over v_1 in (3.17) observe that the integrand behaves as $\exp\{-2v_1\gamma/\beta\kappa_+\}$ as $v_1 \to \infty$, and as $\exp\{(\beta\kappa_+ - (2\gamma/\beta\kappa_+))v_1\}$ as $v_1 \to -\infty$. Therefore, the integral over v_1 is bounded uniformly by some λ-dependent constant for all values of $c \ge -1 + (\gamma/\beta^2) + \lambda$. Putting everything together we get

$$\begin{aligned} \Xi_N(c) &\le C\left\{(1-c)^{-1/2} \wedge \sqrt{N}\right\} \frac{uv\beta^2 N}{e^{2\gamma N}} e^{\beta^2\kappa_+^2 N/2} \frac{1}{N} \exp\left(-\frac{(\beta^2\kappa_+^2 - 2\gamma)^2 N}{2\beta^2\kappa_+^2}\right) \\ &= C(\gamma,\beta,u,v,\lambda)\left\{(1-c)^{-1/2} \wedge \sqrt{N}\right\} \exp\left\{-\frac{\gamma^2 N}{\beta^2(1+c)}\right\} = \bar{\Xi}_N(c). \end{aligned} \quad (3.19)$$

This finishes the proof of Lemma 3.2. □

We can now return to the proof of Proposition 3.1, that is to formula (3.6). Let $D_{ij} = \mathrm{dist}(Y_N(i), Y_N(j))$. Observe that D_{ij} is always smaller than $|i-j|$. Hence, for $\lfloor i/\nu \rfloor = \lfloor j/\nu \rfloor$,

$$\Lambda^0_{ij} = \left(1 - 2N^{-1}D_{ij}\right)^p \geq \left(1 - 2N^{-1}|i-j|\right)^p \geq \Lambda^1_{ij}. \tag{3.20}$$

Since $\Lambda^1_{ij} = 0$ for (i,j) with $\lfloor i/\nu \rfloor \neq \lfloor j/\nu \rfloor$, $\Lambda^0_{ij} - \Lambda^1_{ij} < 0$ if and only if $\Lambda^0_{ij} < 0$. The absolute value of (3.6) is thus bounded from above by

$$\frac{1}{2}\int_0^1 \mathrm{d}h \sum_{\substack{i,j=1\\ i\neq j}}^{r(N)T} (\Lambda^0_{ij} - \Lambda^1_{ij})_+ \mathbb{E}\left[\frac{\partial^2 F_N(X^h_N)}{\partial X(i)\partial X(j)}\,|\mathcal{Y}] + (\Lambda^0_{ij})_- \mathbb{E}\left[\frac{\partial^2 F_N(X^h_N)}{\partial X(i)\partial X(j)}\right|\mathcal{Y}\right]. \tag{3.21}$$

Given the sequence $\{u_i\}$ that was fixed at the beginning of this section, we define

$$\tilde{\Xi}_N(c) = \max\{\bar{\Xi}_N(c,\beta,\gamma,u_i,u_j,\lambda) : 1 \leq i,j \leq \ell\}. \tag{3.22}$$

Clearly, $\tilde{\Xi}_N(c)$ can be written as in (3.9) with a new large constant C. Observe that $\tilde{\Xi}_N(c)$ is an increasing function of $c \in [-1,1]$. Lemma 3.2 and the computation just preceding it then imply that (3.21) is bounded from above by

$$\frac{1}{2}\int_0^1 \mathrm{d}h \sum_{\substack{i,j=1\\ i\neq j}}^{r(N)T} (\Lambda^0_{ij} - \Lambda^1_{ij})_+ \tilde{\Xi}_N\left((1-h)\Lambda^0_{ij} + h\Lambda^1_{ij}\right) + (\Lambda^0_{ij})_- \tilde{\Xi}_N\left((1-h)\Lambda^0_{ij}\right). \tag{3.23}$$

We define, with a slight abuse of notation, $\Lambda^0_d = (1-2dN^{-1})^p$. That is to say, Λ^0_d is the covariance of $X^0_N(i)$ and $X^0_N(j)$ if $D_{ij} = d$. Using this notation, (3.23) is smaller than

$$\sum_{d=0}^{N}\left\{\sum_{\substack{i,j=1\\ \lfloor i/\nu\rfloor\neq\lfloor j/\nu\rfloor}}^{r(N)T} \mathbf{1}\{D_{ij}=d\}(\Lambda^0_d)_+ \int_0^1 \tilde{\Xi}_N\left((1-h)\Lambda^0_d\right)\mathrm{d}h \right.$$
$$+ \sum_{\substack{i,j=1,i\neq j\\ \lfloor i/\nu\rfloor=\lfloor j/\nu\rfloor}}^{r(N)T} \mathbf{1}\{D_{ij}=d\}(\Lambda^0_d - \Lambda^1_{ij})\tilde{\Xi}_N\left(\Lambda^0_d\right) \tag{3.24}$$
$$\left.+ \sum_{i,j:|i-j|\geq N/2}^{r(N)T} \mathbf{1}\{D_{ij}=d\}(\Lambda^0_d)_- \tilde{\Xi}_N(0)\right\}.$$

In the last two lines we used the fact that the integral over h is bounded by the supremum of its integrand, and the fact that $\tilde{\Xi}_N$ is increasing. Finally, the definition of $\tilde{\Xi}_N$ implies that, for $c \geq 0$,

$$\int_0^1 \tilde{\Xi}_N((1-h)c)\mathrm{d}h \leq C e^{-\frac{\gamma^2 N}{\beta^2(1+c)}} \int_0^1 \left((1-(1-h)c)^{-1/2} \wedge \sqrt{N}\right)\mathrm{d}h \leq 2. \tag{3.25}$$

To control (3.24) we need to count the pairs (i, j) with $D_{ij} = d$. The following proposition, that is proved in the next section, provides sufficiently good estimates for our purposes.

Proposition 3.3. *Let γ and β satisfy the hypothesis of Theorem 1.1, let $T > 0$, and let ν be as in (2.1). Then, for any $\eta > 0$, there exists a constant, $C = C(\beta, \gamma, \nu, \eta)$, such that, $\mathcal{Y}$-a.s., for all but finitely many values of N, for all $d \in \{0, \dots, N\}$,*

$$\sum_{\substack{i,j=1\\ \lfloor i/\nu\rfloor \neq \lfloor j/\nu\rfloor}}^{r(N)T} \mathbf{1}\{D_{ij} = d\} \leq C\left[T^2 r(N)^2 2^{-N}\binom{N}{d} + r(N)T\nu^{-1}e^{\eta\|d\|}\right], \tag{3.26}$$

where $\|d\| = \min(d, N-d)$. Moreover, we can choose $\varepsilon < 1$ such that, $\mathcal{Y}$-a.s., for all but finitely many values of N,

$$\sum_{\substack{i,j=1, i\neq j\\ \lfloor i/\nu\rfloor = \lfloor j/\nu\rfloor}}^{r(N)T} \mathbf{1}\{D_{ij} = d\}(\Lambda_d^0 - \Lambda_{ij}^1) \leq \begin{cases} C\nu r(N)T\{d \vee 1\}N^{-2}, & \text{if } d \leq \varepsilon N\nu^{-1},\\ Cr(N)T\nu^2 N^{-1}, & \text{if } \nu \geq d \geq \varepsilon N\nu^{-1}. \end{cases} \tag{3.27}$$

This proposition and the estimate (3.25) imply that the first line of (3.24) is smaller than the sum of the following two terms:

$$C\sum_{d=0}^{N} T^2 r(N)^2 2^{-N}\binom{N}{d}(\Lambda_d^0)_+ \exp\left\{-\frac{\gamma^2 N}{\beta^2(1+\Lambda_d^0)}\right\} \tag{3.28}$$

and

$$C\sum_{d=0}^{N} \frac{r(N)Te^{\eta\|d\|}}{\nu}(\Lambda_d^0)_+ \exp\left\{-\frac{\gamma^2 N}{\beta^2(1+\Lambda_d^0)}\right\}. \tag{3.29}$$

By (3.27), the second line of (3.24) is bounded by

$$C\sum_{d=0}^{\varepsilon N/\nu} \frac{r(N)T\nu\{d\vee 1\}}{N^2}\tilde{\Xi}_N(\Lambda_d^0) + C\sum_{d=\varepsilon N/\nu}^{\nu} \frac{r(N)T\nu^2}{N}\tilde{\Xi}_N(\Lambda_d^0). \tag{3.30}$$

The third line is non-zero only if p is odd. By (3.26) it is smaller than

$$\sum_{d=N/2}^{N} C\left[T^2 r(N)^2 2^{-N}\binom{N}{d} + r(N)T\nu^{-1}e^{\eta\|d\|}\right]\left(1-\frac{2d}{N}\right)_-^p \tilde{\Xi}_N(0). \tag{3.31}$$

We estimate (3.28) first. Let $I(u)$ be defined by

$$I(u) = u\ln u + (1-u)\ln(1-u) + \ln 2, \tag{3.32}$$

and let

$$J_N(u) = 2^{-N}\binom{N}{\lfloor Nu\rfloor}\sqrt{\frac{\pi N}{2}}e^{NI(u)}. \tag{3.33}$$

By Stirling's formula we have that $J_N(u) \xrightarrow{N\to\infty} (4u(1-u))^{-1}$, uniformly on compact subsets of (0, 1). Moreover, $J_N(u) \le CN^{1/2}$ for all $u \in [0, 1]$. From the definitions of $r(N)$ and $\tilde{\Xi}_N$, we find that

$$(3.28) \le C \sum_{d=0}^{N} T^2 N^{1/2} \left(1 - \frac{2d}{N}\right)_+^p \exp\left\{N \Upsilon_{p,\beta,\gamma}\left(\frac{d}{N}\right)\right\} J_N\left(\frac{d}{N}\right), \tag{3.34}$$

where

$$\Upsilon_{p,\beta,\gamma}(u) = \frac{\gamma^2}{\beta^2} - I(u) - \frac{\gamma^2}{\beta^2(1+|1-2u|^p)}. \tag{3.35}$$

Lemma 3.4. *There exists a function, $\zeta(p)$, such that, for all $p \ge 2$ and γ, β satisfying $\gamma < \min\{\zeta(p)\beta, \beta^2\}$, there exist positive constants δ, δ', and c such that*

$$\Upsilon_{p,\beta,\gamma}(u) \le -\delta, \qquad \textit{for all } u \in [0,1] \setminus (1/2-\delta', 1/2+\delta'), \tag{3.36}$$

and

$$\Upsilon_{p,\beta,\gamma}(u) \le -c(u-1/2)^2, \qquad \textit{for all } u \in (1/2-\delta', 1/2+\delta'). \tag{3.37}$$

Moreover, $\zeta(p)$ is increasing and satisfies (1.10)*, that is,*

$$\zeta(2) = 2^{-1/2}, \quad \zeta(3) \simeq 1.0291, \qquad \textit{and} \qquad \lim_{p\to\infty} \zeta(p) = \sqrt{2\ln 2}. \tag{3.38}$$

Proof. The function $\Upsilon_{p,\beta,\gamma}$ and its derivatives satisfy $\Upsilon_{p,\beta,\gamma}(1/2) = \Upsilon'_{p,\beta,\gamma}(1/2) = 0$ and

$$\Upsilon''_{p,\beta,\gamma}(1/2) = \begin{cases} 4\left(\frac{2\gamma^2}{\beta^2} - 1\right), & \text{if } p = 2, \\ -4 & \text{otherwise.} \end{cases} \tag{3.39}$$

The second derivative is always negative for β, γ, p satisfying the assumptions of the lemma. Therefore (3.37) holds.

For any $\delta' > 0$ and $|u - 1/2| \ge \delta'$, the function $I(u)$ is strictly positive and the function $\Phi(u) \equiv 1 - 1/(1+|1-2u|^p)$ is bounded. Therefore, if γ/β is sufficiently small (how small defines the function $\zeta(p)$), then $\Upsilon_{p,\beta,\gamma}(u) < -\delta$. This proves (3.36). The monotonicity of $\zeta(p)$ follows from the monotonicity of Φ in p.

The function $\Upsilon_{2,\beta,\gamma}(u)$ is increasing in γ^2/β^2 and $I(u) \ge (1-2u)^2/2$. Thus, for all $\gamma < 2^{-1/2}\beta$,

$$\Upsilon_{2,\beta,\gamma}(u) < \frac{1}{2}\left(1 - \frac{1}{1+(1-2u)^2}\right) - \frac{1}{2}(1-2u)^2. \tag{3.40}$$

The right-hand side of the last inequality is equal to 0 for $u = 1/2$ and its derivative,

$$2(1-2u)\left(1 - \frac{1}{(1+(1-2u)^2)^2}\right) \gtrless 0, \quad \text{for all } u \lessgtr 1/2, \tag{3.41}$$

which implies that (3.36) is true for all $\gamma < 2^{-1/2}\beta$, and so the first part of (3.38) holds.

Obviously, $\Phi(0) = 1/2$, $\Phi'(0) = -2p$, $I(0) = \ln 2$ and $I'(0) = -\infty$. Hence, for $\gamma/\beta = \sqrt{2\ln 2}$, there exists $u > 0$ small enough such that $\Upsilon_{p,\beta,\gamma}(u)$ is positive. This

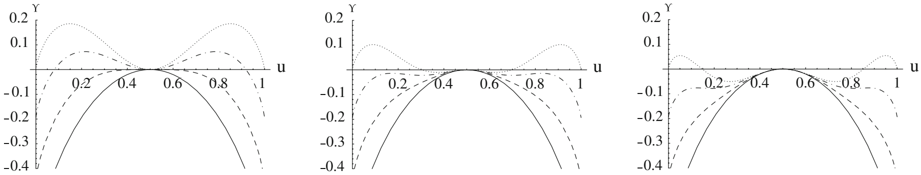

Fig. 1. Function $\Upsilon_{p,\gamma,\beta}$ for $p = 2, 3, 4$ and various values of γ/β

implies that $\zeta(p) < \sqrt{2\ln 2}$. If $u \in (0, 1/2)$ then $\lim_{p\to\infty} \Phi(u) = 0$. This yields the third claim of (3.38). The value of $\zeta(3)$ was obtained numerically.

For illustration the graphs of the function $\Upsilon_{p,\beta,\gamma}$ are plotted in Fig. 1 for $p = 2, 3, 4$, $\beta = 1$, and $\gamma = 0$ (solid lines), $\gamma = \sqrt{1/2}$ (dashed lines), $\gamma = 1$ (dash-dotted lines) and $\gamma = \sqrt{2\ln 2}$ (dotted lines). □

We can now finish the bound of (3.28), resp. of (3.34). Lemma 3.4 and the bounds on the function J_N yield that, for $d/N \notin (1/2 - \delta', 1/2 + \delta')$, the summands decrease exponentially in N. Therefore they can be neglected. The remaining part can be bounded by

$$
\begin{aligned}
C \sum_{d=(1/2-\delta')N}^{(1/2+\delta')N} & T^2 N^{1/2} \left(1 - \frac{2d}{N}\right)_+^p \exp\left\{-cN(d/N - 1/2)^2\right\} \\
& \le CT^2 N^{3/2} \int_{-\delta'}^{\delta'} |x|^p e^{-c'Nx^2} \mathrm{d}x \\
& \le CT^2 N^{3/2} N^{-(p+1)/2} \int_{-\infty}^{\infty} |u|^p e^{-c'u^2} \mathrm{d}u \xrightarrow{N\to\infty} 0,
\end{aligned}
\tag{3.42}
$$

however only if $p \ge 3$.

Similarly, for (3.29) we have

$$
(3.29) \le C \sum_{d=0}^{N} T N^{1/2} \nu^{-1} \left(1 - \frac{2d}{N}\right)_+^p \exp(N\tilde{\Upsilon}(d/N)), \tag{3.43}
$$

where, setting $\|u\| = \min(u, 1-u)$,

$$
\tilde{\Upsilon}_{p,\beta,\gamma}(u) = \frac{\gamma^2}{2\beta^2} - \frac{\gamma^2}{\beta^2(1 + |1-2u|^p)} + \eta\|u\|. \tag{3.44}
$$

It is easy to check that there are positive values of δ, δ', and η, such that $\tilde{\Upsilon}_{p,\beta,\gamma}(u) < -\delta$, for all $\|u\| \ge \delta'$. Therefore all such d can be neglected. Around $d = 0$ the function $\tilde{\Upsilon}_{p,\beta,\gamma}(x)$ can be approximated by a linear function $-cx$, $c > 0$, and the summation by an integration. As an upper bound we get, using (2.1),

$$
CTN^{3/2}\nu^{-1} \int_0^{\delta'} e^{-cNx} \mathrm{d}x \le CTN^{1/2}\nu^{-1} \xrightarrow{N\to\infty} 0. \tag{3.45}
$$

An analogous bound works for d close to N and p even.

For (3.30) we have

$$(3.30) \leq \frac{CT\nu}{N} e^{N\tilde{\Upsilon}_{p,\beta,\gamma}(0)} + C \sum_{d=1}^{\varepsilon N/\nu} \frac{T\nu d}{N^{3/2}} \left[1 - \left(1 - \frac{2d}{N}\right)^p\right]^{-1/2} e^{N\tilde{\Upsilon}_{p,\beta,\gamma}(d/N)} + C \sum_{d=\varepsilon N/\nu}^{\nu} \frac{T\nu^2}{N^{1/2}} \left[1 - \left(1 - \frac{2d}{N}\right)^p\right]^{-1/2} e^{N\tilde{\Upsilon}_{p,\beta,\gamma}(d/N)}. \tag{3.46}$$

The first term converges to zero. The linear approximation of $\tilde{\Upsilon}_{p,\beta,\gamma}$ and of the bracket in the second term yields an upper bound

$$CTN^{1/2}\nu \int_0^{\varepsilon} x^{1/2} e^{-c'Nx} \mathrm{d}x \leq CTN^{-1}\nu \xrightarrow{N\to\infty} 0. \tag{3.47}$$

The third term is smaller than $\nu^3 e^{-c'N/\nu}$, which is also negligible.

Finally, since $\tilde{\Xi}_N(0) = Ce^{-N\gamma^2/\beta^2}$, it is easy to see that the contribution of the second term in the bracket of (3.31) tends to 0. The contribution of the first term is equal (up to a constant) to

$$\sum_{d=N/2}^{N} \left(\frac{2d}{N} - 1\right)^p T^2 N 2^{-N} \binom{N}{d} \leq CT^2 \left\{ \sum_{d \geq N/2 + N^{3/5}} N 2^{-N} \binom{N}{d} + \sum_{i=1}^{2N^{3/5}} \left(\frac{N+i}{N} - 1\right)^p N^{1/2} e^{-i^2/2N} \right\}, \tag{3.48}$$

where we used that $\binom{N}{d} \leq CN^{-1/2} 2^N e^{-i^2/2N}$ for $d = (N+i)/2$ and $i \ll N^{2/3}$. The first term in (3.48) tends to 0 by a standard moderate deviation argument. The second one can be approximated by

$$CT^2 N^{1-(p/2)} \int_0^{\infty} x^p e^{-x^2/2} \mathrm{d}x \xrightarrow{N\to\infty} 0 \tag{3.49}$$

for $p \geq 3$. This completes the proof of Proposition 3.1. □

4. Random Walk Properties

In this section we prove Proposition 3.3. We write $\mathbb{P}_x$ for the law of the simple random walk Y_N conditioned to start in x. Let $Q = \{Q_i, i \in \mathbb{N}\}$ be the Ehrenfest Urn Markov chain, that is a birth-death process on $\{0, \dots, N\}$ with transition probabilities $p_{i,i-1} = 1 - p_{i,i+1} = i/N$. We use P_k and E_k to denote the law of (the expectation with respect to) Q conditioned on $Q_0 = k$. Under P_0, Q_i has the same law as $\mathrm{dist}(Y_N(0), Y_N(i))$. We define T_k as the hitting time of k by Q, $T_k = \min\{i \geq 1 : Q_i = k\}$. It is a well-known fact that, for $k < l < m$,

$$P_l[T_m < T_k] = \frac{\sum_{i=k}^{l-1} \binom{N-1}{i}^{-1}}{\sum_{i=k}^{m-1} \binom{N-1}{i}^{-1}}. \tag{4.1}$$

Finally, let $p_k(d) = P_0(Q_k = d)$. We need the following lemma to estimate $p_k(d)$ for large k.

Lemma 4.1. *There exists K large enough such that, for all $k \geq KN^2 \ln N \equiv \mathcal{K}(N)$ and $x, y \in \mathcal{S}_N$,*

$$\left| \frac{\mathbb{P}_y[Y_N(k) = x \cup Y_N(k+1) = x]}{2} - 2^{-N} \right| \leq 2^{-8N}, \tag{4.2}$$

and thus

$$\left| \frac{p_k(d) + p_{k+1}(d)}{2} - 2^{-N} \binom{N}{d} \right| \leq 2^{-4N}. \tag{4.3}$$

Proof. The beginning of the argument is the same as in [Mat87]. We construct a coupling between Y_N (which by definition starts at site $\mathbf{1} = (1, \dots, 1) \in \mathcal{S}_N$) and another process $Y_N^\star$. This process is a simple random walk on $\mathcal{S}_N$, with initial distribution $\mu_N^\star$ being uniform on those $x \in \mathcal{S}_N$ with $\mathrm{dist}(x, \mathbf{1})$ even. The coupling is the same as in [Mat87]. This coupling provides a certain random time, $\mathcal{T}_N$, which can be used to bound the variational distance d_∞ between $\mu^\star$ and the distribution μ_N^k of $Y_N(k)$: for k even

$$d_\infty(\mu_N^\star, \mu_N^k) \equiv \max_{A \subset \mathcal{S}_N} |\mu_N^\star(A) - \mu_N^k(A)| \leq \mathbb{P}[\mathcal{T}_N > k]. \tag{4.4}$$

The law of $\mathcal{T}_N$ is as follows. Let $U = \mathrm{dist}(Y_N^\star(0), \mathbf{1})$. That is U is a binomial random variable with parameters N and $1/2$ conditioned on being even. Consider another simple random walk, $\tilde{Y}_U$, on $\mathcal{S}_U$, started from $\mathbf{1}$. The distribution of $\mathcal{T}_N$ is then the same as the distribution of the hitting time of the set $\{x \in \mathcal{S}_U : \mathrm{dist}(\mathbf{1}, x) = U/2\}$ by $\tilde{Y}_U$. It is proved in [Mat87] that $\mathbb{P}(\mathcal{T}_N > N \ln N) \to c < 1$. It is then easy to see that,

$$\mathbb{P}[\mathcal{T}_N \geq \mathcal{K}(N)] \leq c^{KN} \leq 2^{-8N}, \tag{4.5}$$

if K is large enough. Thus, for even $k \geq \mathcal{K}(N)$, $d_\infty(\mu_N^\star, \mu_N^k) \leq 2^{-8N}$, and thus $|\mu_N^\star(x) - \mu_N^k(x)| \leq 2^{-8N}$, for all $x \in \mathcal{S}_N$. A similar claim for k odd is then not difficult to prove. The second part of the lemma is a direct consequence of the first part. □

Lemma 4.2. *Let γ, β, ν satisfy the hypothesis of Proposition 3.3. Then, there exists a constant, $C = C(\beta, \gamma, \nu)$, such that, $\mathcal{Y}$-a.s., for all but finitely many values of N, for all $d \in \{0, \dots, N\}$,*

$$\sum_{\substack{i,j=1, i \neq j \\ \lfloor i/\nu \rfloor = \lfloor j/\nu \rfloor}}^{r(N)T} \mathbf{1}\{D_{ij} = d\} \leq C r(N) T \mathbf{1}\{d \leq \nu\}. \tag{4.6}$$

Proof. The lemma is trivially true for $d > \nu$. For $d \leq \nu$, observe first that (4.6) is bounded from above by an i.i.d. sum of $m = \lceil r(N)T/\nu \rceil$ "block" random variables which have the same distribution as $\sum_{i,j=1, i \neq j}^{\nu} \mathbf{1}\{D_{ij} = d\}$. To control these block variables we first compute

$$\rho(d) = E_0 \sum_{i=1}^{\nu} \mathbf{1}\{Q_i = d\}. \tag{4.7}$$

We have $\rho(0) \geq N^{-1}$ and $\rho(d) \geq P_0[T_d \leq \nu]$. This probability is decreasing in d and

$$P_0[T_\nu \leq \nu] = \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-\nu+1}{N} \geq e^{-\nu^2/N}. \tag{4.8}$$

Thus $\rho(d) \geq e^{-\nu^2/N}$ for all $d \leq \nu$. Using $T_\nu \leq \nu$, the decomposition on the first visit of d, and the standard relation between the Green's function and the escape probability, we get

$$\rho(d) \leq E_0\left[\sum_{i=1}^{T_\nu} \mathbf{1}\{Q_i = d\}\right] = 1 + E_d\left[\sum_{i=1}^{T_\nu} \mathbf{1}\{Q_i = d\}\right] = 1 + \frac{1}{P_d[T_\nu < T_d]}. \tag{4.9}$$

However, using (4.1),

$$P_d[T_\nu < T_d] = \frac{N-d}{N} P_{d+1}[T_\nu < T_d] = \frac{N-d}{N} \frac{\binom{N-1}{d}^{-1}}{\sum_{i=d}^{\nu-1} \binom{N-1}{i}^{-1}} = 1 - O(\nu N^{-1}). \tag{4.10}$$

Since $\nu \ll N$, $\rho(d) \leq 2$.

Consider now the one-block contribution to (4.6),

$$\sum_{i,j=1}^{\nu} \mathbf{1}\{D_{ij} = d\} \equiv \nu^2 \tilde{Z}. \tag{4.11}$$

Of course, $\tilde{Z} \in [0, 1]$ and, using the results of the previous paragraph,

$$e^{-\nu^2/N}(2\nu)^{-2} \leq \mathbb{E}[\tilde{Z}] \leq 4\nu^{-1}. \tag{4.12}$$

Denoting by $\tilde{Z}_k$ a sequence of i.i.d. copies of $\tilde{Z}$, we obtain from Hoeffding's inequality [Hoe63]:

$$\mathbb{P}\left[\sum_{i=1}^{m} \tilde{Z}_k \geq 2m\mathbb{E}[\tilde{Z}_k]\right] \leq \exp\{-2m\mathbb{E}[\tilde{Z}_k]^2\} \leq \exp\{-me^{-2\nu^2/N}(2\nu)^{-4}\}, \tag{4.13}$$

where we used the lower bound from (4.12). Since $\nu^2/N \ll N$, by the Borel-Cantelli lemma, the left-hand side of (4.6) is a.s. bounded by

$$\nu^2 2m\mathbb{E}[\tilde{Z}] \leq Cr(N)T, \tag{4.14}$$

for all N large enough and all $d \leq \nu$. This completes the proof of Lemma 4.2. □

Proof of Proposition 3.3. We prove (3.27) first. Since it is trivially verified for $d \in \{\nu - 1, \nu\}$, we will assume that $d \leq \nu - 2$. Observe that, for i, j in the same block,

$$\Lambda_d^0 - \Lambda_{ij}^1 = \left(1 - \frac{2d}{N}\right)^p - \left(1 - \frac{2p|i-j|}{N}\right) = \frac{2p(|i-j| - d)}{N} + O\left(\frac{d^2}{N^2}\right). \tag{4.15}$$

The contribution of the error term is smaller than the right-hand side of (3.27), as follows from Lemma 4.2.

To compute the contribution of the main term, let

$$\tilde{\rho}(d) = E_0\left[\sum_{i=1}^{\nu}(i-d)\mathbf{1}\{Q_i = d\}\right] = \sum_{i=1}^{\nu}(i-d)p_i(d). \tag{4.16}$$

We need upper and lower bounds on $\tilde{\rho}(d)$ to proceed with a Hoeffding-type argument. The lower bound is easy: by considering the path with $Q_2 = 0$ and $Q_{d+2} = d$, we get using (4.8) that $\tilde{\rho}(d) \geq N^{-1}e^{-\nu^2/N}$. The upper bound is slightly more complicated,

$$\begin{aligned}\tilde{\rho}(d) &= \sum_{k=1}^{(\nu-d)/2} 2kp_{d+2k}(d) \leq \sum_{k=1}^{(\nu-d)/2} 2k\binom{d+2k}{k}\left(\frac{d+k}{N}\right)^k \\ &\leq C\sum_{k=1}^{(\nu-d)/2} k\frac{(d+2k)^k}{k^k e^{-k}\sqrt{k}}\left(\frac{d+k}{N}\right)^k \\ &\leq C\sum_{k=1}^{(\nu-d)/2} (2e)^k(dk^{-1}+2)^k(\nu N^{-1})^k \leq C\sum_{k=1}^{(\nu-d)/2}\left(c(d+2)\nu N^{-1}\right)^k \\ &\leq C(d\vee 1)\nu N^{-1}\end{aligned} \tag{4.17}$$

if $d \leq \varepsilon N/\nu$ for some small ε. Otherwise, trivially, $\tilde{\rho}(d) \leq \nu^2$. The one-block contribution of the first term of (4.15) to (3.27) is then given by

$$\frac{2p}{N}\sum_{i,j=1}^{\nu}(|i-j|-d)\mathbf{1}\{D_{ij} = d\} \equiv \frac{2p}{N}\nu^3\tilde{Z}, \tag{4.18}$$

with $\tilde{Z} \in [0,1]$, $\mathbb{E}[\tilde{Z}] \geq cN^{-1}e^{-\nu^2/N}\nu^{-3}$, and

$$\mathbb{E}[\tilde{Z}] \leq \begin{cases} C\{d\vee 1\}N^{-1}\nu^{-1}, & \text{if } d \leq \varepsilon N/\nu, \\ 1, & \text{if } \nu \geq d \geq \varepsilon N/\nu. \end{cases} \tag{4.19}$$

Then, as in the proof of Lemma 4.2, Hoeffding's inequality and (4.19) imply that the contribution of the first term of (4.15) to (3.27) is smaller than $Cr(N)T\{d\vee 1\}\nu N^{-2}$ or $Cr(N)T\nu^2N^{-1}$, respectively, which was to be shown.

Finally, we prove (3.26). We restrict the summation to $i < j$, since the terms with $j > i$ give the same contribution. We first consider the contribution of pairs (i, j) such that $j - i \geq \mathcal{K}(N)$, so that in particular, $\lfloor i/\nu\rfloor \neq \lfloor j/\nu\rfloor$. With $R = r(N)T$, Lemma 4.1 yields

$$\mathbb{E}\left[\sum_{j-i\geq\mathcal{K}(N)}^{R}\mathbf{1}\{D_{ij} = d\}\right] = \sum_{j-i\geq\mathcal{K}(N)}^{R} p_{j-i}(d) \leq CR^2 2^{-N}\binom{N}{d}. \tag{4.20}$$

Moreover,

$$\begin{aligned}&\mathrm{Var}\left[\sum_{j-i\geq\mathcal{K}(N)}^{R}\mathbf{1}\{D_{ij} = d\}\right] \\ &= \sum_{j_1-i_1\geq\mathcal{K}(N)}^{R}\ \sum_{j_2-i_2\geq\mathcal{K}(N)}^{R} \mathbb{P}\left[D_{i_1,j_1} = D_{i_2,j_2} = d\right] - \mathbb{P}\left[D_{i_1,j_1} = d\right]\mathbb{P}\left[D_{i_2,j_2} = d\right].\end{aligned} \tag{4.21}$$

We can again suppose that $i_1 \le i_2$. The right-hand side of (4.21) is non-zero only if $i_1 \le i_2 \le j_1 < j_2$ or $i_1 \le i_2 < j_2 \le j_1$. We will consider only the first case. The second one can be treated analogously. In is not difficult to see, using Lemma 4.1, that if $i_2 - i_1 \ge \mathcal{K}(N)$ or $j_2 - j_1 \ge \mathcal{K}(N)$ then the difference of probabilities in the above summation is at most 2^{-4N}. Therefore, the contribution of such (i_1, i_2, j_1, j_2) to the variance is at most $R^4 2^{-4N}$.

If $i_2 - i_1 < \mathcal{K}(N)$ and $j_2 - j_1 < \mathcal{K}(N)$ then, using again Lemma 4.1,

$$\mathbb{P}\left[D_{i_1,j_1} = D_{i_2,j_2} = d\right] \le \mathbb{P}\left[D_{i_1,j_1} = d\right] \le C 2^{-N}\binom{N}{d}. \tag{4.22}$$

We choose $\varepsilon > 0$. For $\|d\| \le (1-\varepsilon)N/2$ we have

$$\begin{aligned} &\sum_{\substack{j_1-i_1\ge\mathcal{K}(N)\\ i_2-i_1<\mathcal{K}(N)}} \sum_{\substack{j_2-i_2\ge\mathcal{K}(N)\\ j_2-j_1<\mathcal{K}(N)}} \mathbb{P}\left[D_{i_1,j_1} = D_{i_2,j_2} = d\right] \\ &\quad \le C\mathcal{K}(N)^2 R^2 2^{-N}\binom{N}{d} \le C\mathcal{K}(N)^2 R^2 e^{-NI((1-\varepsilon/2)/2)} \ll N^{-3}R^2\nu^{-2}, \end{aligned} \tag{4.23}$$

say. For $\|d\| \ge (1-\varepsilon)N/2$, that is $|d - N/2| \le \varepsilon N/2$, we have, for ε small enough (how small depend on γ and β), that $2^{-N}\binom{N}{d} \gg N^7 R^{-2}$. Then,

$$\begin{aligned} &\sum_{\substack{j_1-i_1\ge\mathcal{K}(N)\\ i_2-i_1<\mathcal{K}(N)}} \sum_{\substack{j_2-i_2\ge\mathcal{K}(N)\\ j_2-j_1<\mathcal{K}(N)}} \mathbb{P}\left[D_{i_1,j_1} = D_{i_2,j_2} = d\right] \\ &\quad \le CN^4 R^2 2^{-N}\binom{N}{d} \ll N^{-3}R^4 2^{-2N}\binom{N}{d}^2. \end{aligned} \tag{4.24}$$

We have thus found that the expectation of the summation over $j - i > \mathcal{K}(N)$ is smaller than the right-hand side of (3.26) and the variance of the same summation is much smaller than N^{-3} times the right-hand side of (3.26) squared. A straightforward application of the Chebyshev inequality and the Borel-Cantelli Lemma then gives the desired a.s. bound for pairs $j - i \ge \mathcal{K}(N)$ and all $d \in \{0, \dots, N\}$.

Choose again $\varepsilon > 0$. For $j - i < \mathcal{K}(N)$, observe first that if $\|d\| \ge (\ln N)^{1+\varepsilon} \gg \ln N$ then the summation over such pairs (i, j) in (3.26) is always smaller than $\mathcal{K}(N)R \ll R\nu^{-1}e^{\eta\|d\|}$, for all $\eta > 0$. For the remaining d's, that is $\|d\| < (\ln N)^{1+\varepsilon}$, let $K_N \ge K$ be the smallest constant such that $K_N N^2 \ln N$ is a multiple of ν. Since $\nu \ll N^2$, $K_N - K \ll 1$. As the difference between K and K_N is negligible, we will use the same notation $\mathcal{K}(N)$ for $K_N N^2 \ln N$ and we will simply suppose that $\mathcal{K}(N)$ is a multiple of ν. The summation in (3.26) for $j - i \le \mathcal{K}(N)$ can be bounded from above by

$$\sum_{\substack{0<j-i<\mathcal{K}(N)\\ \lfloor i/\nu\rfloor \ne \lfloor j/\nu\rfloor}}^{r(N)T} \mathbf{1}\{D_{ij} = d\} \le \sum_{k=0}^{\mathcal{K}(N)-1} \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} \sum_{m=j_k}^{\mathcal{K}(N)} \mathbf{1}\{D_{\mathcal{K}(N)\ell+k,\mathcal{K}(N)\ell+k+m} = d\}, \tag{4.25}$$

where j_k is the smallest integer such that $\lfloor(\mathcal{K}(N)\ell + k)/\nu\rfloor \ne \lfloor(\mathcal{K}(N)\ell + k + j_k)/\nu\rfloor$, which does not depend on ℓ. We define random variables, $Z_\ell(j, d)$, by

$$Z_\ell(j, d) = \frac{1}{\mathcal{K}(N)} \sum_{m=j}^{\mathcal{K}(N)} \mathbf{1}\{D_{\mathcal{K}(N)\ell+k,\mathcal{K}(N)\ell+k+m} = d\}. \tag{4.26}$$

The sequence $\{Z_\ell(j,d) : \ell \geq 0\}$, for fixed j and d, is a sequence of i.i.d. variables with values in [0, 1].

Let $E_N = \{d : \|d\| < (\ln N)^{1+\varepsilon}, d \geq N/2\}$. For $d \in E_N$, we have

$$\begin{aligned}\mathbb{P}[Z_\ell(k,d) > 0] \leq P_0[T_d \leq \mathcal{K}(N)] &\leq \binom{N}{d}\mathbb{P}_{z_d}[T_{\mathbf{1}} < \mathcal{K}(N)] \\ &\leq \binom{N}{d} e^{\lambda K}\mathbb{E}_{z_d}\left[e^{-\lambda T_{\mathbf{1}}/N^2\ln N}\right],\end{aligned} \tag{4.27}$$

where z_d is any point on the hypercube with $\mathrm{dist}(\mathbf{1}, z_d) = d$ and, with a slight abuse of notation, $T_{\mathbf{1}}$ is the hitting time of $\mathbf{1}$ by the simple random walk Y_N. According to Lemma 3.4 of [ČG08],

$$\mathbb{E}_{z_d}\left[\exp(-\lambda T_{\mathbf{1}} m(N)^{-1})\right] \leq (2^{-N} m(N)\lambda^{-1} + \xi_N(d))(1 + o(1)), \tag{4.28}$$

for $N \ln N \ll m(N) \ll 2^N$, with $\xi_n(k) = 2^{-n}\frac{n}{2}\binom{n}{k}^{-1}\sum_{j=1}^{n-k}\binom{n}{k+j}\frac{1}{j}$. Taking $m(N) = N^2 \ln N$ and $d \in E_N$ it is not difficult to check that, for ε small enough,

$$\mathbb{E}_{z_d}\left[e^{-\lambda T_{\mathbf{1}}/N^2\ln N}\right] \leq 2^{-N(1-\varepsilon)}. \tag{4.29}$$

Hence,

$$\begin{aligned}&\mathbb{P}\left[\bigcup_{d\in E_N}\left\{\sum_{k=0}^{\mathcal{K}(N)-1}\sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(j_k, d) > 0\right\}\right] \\ &\leq C\binom{N}{\lceil(\ln N)^{1+\varepsilon}\rceil} R(\ln N)^{1+\varepsilon}2^{-N(1-\varepsilon)} \leq C2^{-\varepsilon' N},\end{aligned} \tag{4.30}$$

for some ε' small. Hence, $d \in E_N$ do not pose any problem, by the Borel-Cantelli lemma again.

To treat $d \leq (\ln N)^{1+\varepsilon} \ll \nu$ we will distinguish two cases: $j_k \leq d+6$ and $j_k > d+6$. For the first case, observe that, for any $d < \nu$, there are at most $(d+6)\mathcal{K}(N)/\nu$ values of $k \in \{0, \ldots, \mathcal{K}(N) - 1\}$ such that $j_k \leq d + 6$. Clearly, $Z_\ell(j_k, d) \leq Z_\ell(0, d)$. Moreover, by similar arguments as in Lemma 4.2, $\mathbb{E}[Z_\ell(0,d)] \geq 1/(N\mathcal{K}(N))$, and $\mathbb{E}[Z_\ell(0,d)] \leq C/\mathcal{K}(N)$. Hence, by Hoeffding's inequality, the probability

$$\mathbb{P}\left[\mathcal{K}(N)\sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(0,d) \geq \frac{CR}{\mathcal{K}(N)}\right] \tag{4.31}$$

decreases, for C large, at least exponentially with N. Hence, for $j_k \leq d + 6$, $\mathcal{Y}$-a.s,

$$\mathcal{K}(N)\sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(j_k,d) \leq \mathcal{K}(N)\sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(0,d) \leq \frac{CR}{\mathcal{K}(N)}. \tag{4.32}$$

For $j > d + 6$ and N large enough, $Z_\ell(j,d) \leq Z_\ell(d+6,d)$. We claim that

$$cN^{-6} \leq \mathcal{K}(N)\mathbb{E}[Z_\ell(d+6,d)] \ll \nu^{-1}. \tag{4.33}$$

Indeed, the lower bound is trivially obtained by considering a path that returns 6 times to its starting point in the first 12 steps and then continues without backtracking to a distance d. To get the upper bound in (4.33), we first bound the probability that the chain Q started at 0 hits d between times $d+6$ and $\mathcal{K}(N)$. This probability is bounded by

$$P_0[T_{d+6} \neq d+6] + P_0[T_{d+6} = d+6]P_{d+6}[T_d < \mathcal{K}(N)]. \tag{4.34}$$

The first term is smaller than $c(d+6)^2N^{-1} \ll \nu^{-1}$. For the second term, $P_{d+6}[T_d < T_{d+6}] \le CN^{-5}(\ln(N))^{5(1+\varepsilon)} \ll N^{-4}$. Moreover, before time $\mathcal{K}(N)$, there are at most $\mathcal{K}(N)$ trials to reach d, so $P_{d+6}[T_d < \mathcal{K}(N)] \le \mathcal{K}(N)N^{-4} \ll \nu^{-1}$. So (4.34) $\ll \nu^{-1}$. If Q hits d after $d+6$ it spends there on average a time less than 2. This gives the upper bound in (4.33).

From (4.33), it follows by another Hoeffding's type argument that, for $j > d+6$,

$$\mathbb{P}\left[\mathcal{K}(N) \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(j,d) \ge \frac{R}{\nu\mathcal{K}(N)}\right] \tag{4.35}$$

decreases at least exponentially in N and thus the inequality question fails $\mathcal{Y}$-a.s. for all but finitely many values of N.

Putting together all arguments of the last three paragraphs and summing over k we get, $\mathcal{Y}$-a.s., for all but finitely many N,

$$\sum_{k=0}^{\mathcal{K}(N)-1} \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} \mathcal{K}(N) Z_\ell(j_k,d) \le d\mathcal{K}(N)\nu^{-1}\frac{CR}{\mathcal{K}(N)} + \mathcal{K}(N)\frac{R}{\nu\mathcal{K}(N)} \le CR\nu^{-1}e^{\eta d}. \tag{4.36}$$

This completes the proof. □

5. Convergence of Clock Process

We will prove the convergence of the rescaled clock process to the stable subordinator on the space $D([0,T],\mathbb{R})$ equipped with the Skorokhod M_1-topology. This topology is not commonly used in the literature, therefore we shortly recall some of its properties and compare it with the more standard Skorokhod J_1-topology, which we will need later, too. The reader is referred to [Whi02] for more details on both topologies, and to [Bil68] for a thorough account on the J_1-topology.

5.1. Topologies on the Skorokhod space. Consider the space $D = D([0,T],\mathbb{R})$ of càdlàg functions. The J_1-topology is the topology given by the J_1-metric d_{J_1},

$$d_{J_1}(f,g) = \inf_{\lambda\in\Lambda}\{\|f\circ\lambda - g\|_\infty \vee \|\lambda - e\|_\infty\}, \quad f,g \in D, \tag{5.1}$$

where Λ is the set of strictly increasing functions mapping $[0,T]$ onto itself such that both λ and its inverse are continuous, and e is the identity map on $[0,T]$.

Also the M_1-topology is given by a metric. For $f \in D$ let Γ_f be its completed graph, namely

$$\Gamma_f = \{(t,z) \in [0,T]\times\mathbb{R} : z = \alpha f(t-) + (1-\alpha)f(t), \alpha \in [0,1]\}. \tag{5.2}$$

A parametric representation of the completed graph Γ_f (or of f) is a continuous bijective mapping $\phi(s) = (\phi_1(s), \phi_2(s))$, $[0, 1] \mapsto \Gamma_f$, whose first coordinate ϕ_1 is increasing. If $\Pi(f)$ is the set of all parametric representation of f, then the M_1-metric, d_{M_1}, is defined by

$$d_{M_1}(f, g) = \inf\{\|\phi_1 - \psi_1\|_\infty \vee \|\phi_2 - \psi_2\|_\infty : \phi \in \Pi(f), \psi \in \Pi(g)\}. \quad (5.3)$$

The space D equipped with both M_1- and J_1-topologies is Polish. The M_1-topology is weaker than the J_1-topology: As an example, consider the sequence

$$f_n = \mathbf{1}\{[1 - 1/n, 1)\} + 2 \cdot \mathbf{1}\{[1, T]\}, \quad (5.4)$$

which converges to $f = 2 \cdot \mathbf{1}\{[1, T]\}$ in the M_1-topology but not in the J_1-topology. One often says that the M_1-topology allows "intermediate jumps".

We will need a criterion for tightness of probability measures on D. To this end we define several moduli of continuity,

$$\begin{aligned}
w_f(\delta) &= \sup\left\{\min\left(|f(t) - f(t')|, |f(t'') - f(t)|\right) : t' \le t \le t'' \le T, t'' - t' \le \delta\right\},\\
w'_f(\delta) &= \sup\left\{\inf_{\alpha\in[0,1]} |f(t) - \alpha f(t') - (1-\alpha) f(t'')| : t' \le t \le t'' \le T, t'' - t' \le \delta\right\},\\
v_f(t, \delta) &= \sup\left\{|f(t') - f(t'')| : t', t'' \in [0, T] \cap (t - \delta, t + \delta)\right\}.
\end{aligned} \quad (5.5)$$

The following result is a restatement of Theorem 12.12.3 of [Whi02] and Theorem 15.3 of [Bil68].

Lemma 5.1. *The sequence of probability measures $\{P_n\}$ on $D([0, T], \mathbb{R})$ is tight in the J_1-topology if*

(i) *For each positive ε there exists c such that*

$$P_n[f : \|f\|_\infty > c] \le \varepsilon, \qquad n \ge 1. \quad (5.6)$$

(ii) *For each $\varepsilon > 0$ and $\eta > 0$, there exist a δ, $0 < \delta < T$, and an integer n_0 such that*

$$P_n[f : w_f(\delta) \ge \eta] \le \varepsilon, \qquad n \ge n_0, \quad (5.7)$$

and

$$P_n[f : v_f(0, \delta) \ge \eta] \le \varepsilon \text{ and } P_n[f : v_f(T, \delta) \ge \eta] \le \varepsilon, \qquad n \ge n_0. \quad (5.8)$$

The same claim holds for the M_1-topology with $w_f(\delta)$ in (5.7) replaced by $w'_f(\delta)$.

5.2. Proof of Theorem 1.1. To prove the convergence of the rescaled clock process $\bar{S}_N(\cdot) = e^{-\gamma N} S_N(\cdot r(N))$ to the stable subordinator V_{γ/β^2}, we check first the convergence of finite-dimensional marginals. Let $\ell \in \mathbb{N}$, $0 = t_0 < \cdots < t_\ell = T$ and $u_1, \dots, u_\ell \in \mathbb{R}_+$ be fixed. Then,

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{-\sum_{i=1}^{\ell} u_i \left(\bar{S}_N(t_k) - \bar{S}_N(t_{k-1})\right)\right\} |\mathcal{Y}\right] \\ &\quad = \mathbb{E}\left[F_N(X_N^0; \{t_i\}, \{u_i\})\,|\mathcal{Y}\right] = \mathbb{E}\left[F_N(X_N^1; \{t_i\}, \{u_i\})\right] + o(1),\end{aligned} \tag{5.9}$$

as follows from Proposition 3.1.

The value of $\mathbb{E}\left[F_N(X_N^1; \{t_i\}, \{u_i\})\right]$ is not difficult to calculate. Define $j_N(i) = \lfloor t_i r(N)/\nu \rfloor$. Then

$$\begin{aligned}\mathbb{E}\left[F_N(X_N^1; \{t_i\}, \{u_i\})\right] &= \mathbb{E}\left[\exp\left(-\sum_{k=1}^{\ell} \frac{u_k}{e^{\gamma N}} \sum_{i=t_{k-1}r(N)}^{t_k r(N)-1} e_i e^{\beta\sqrt{N} X_N^1(i)}\right)\right] \\ &\geq \mathbb{E}\left[\prod_{k=1}^{\ell} \prod_{j=j_N(k-1)+1}^{j_N(k)} \exp\left(-\frac{u_k}{e^{\gamma N}} \sum_{i=0}^{\nu-1} e_{j\nu+i} e^{\beta\sqrt{N} X_N^1(j\nu+i)}\right)\right].\end{aligned} \tag{5.10}$$

Since the process X_N^1 is a piecewise independent process, the product in (5.10) is a product of independent random variables. The expectations of all of them can be then bounded using Proposition 2.1. We get, for $\delta > 0$ fixed and N large enough,

$$\begin{aligned}\mathbb{E}\left[F_N(X_N^1; \{t_i\}, \{u_i\})\right] &\geq \prod_{k=1}^{\ell} \prod_{j=j_N(k-1)+1}^{j_N(k)} \mathcal{F}_N(u_k) \\ &\geq \prod_{k=1}^{\ell} \left(1 - (1+\delta)\nu N^{-1/2} e^{-N\gamma^2/2\beta^2} K u_k^{\gamma/\beta^2}\right)^{j_N(k)-j_N(k-1)-1} \\ &\geq \prod_{k=1}^{\ell} \exp\left\{-(1+2\delta)(t_k - t_{k-1}) K u_k^{\gamma/\beta^2}\right\},\end{aligned} \tag{5.11}$$

which is (up to $1+2\delta$ term) the Laplace transform of $V_{\gamma/\beta^2}(K\cdot)$. A corresponding upper bound can be constructed analogously.

To check the tightness for $\bar{S}_N$ in $D([0,T],\mathbb{R})$ equipped with the Skorokhod M_1-topology we use Lemma 5.1. Since the processes $\bar{S}_N$ are increasing, it is easy to see that condition (i) is equivalent to the tightness of the distribution of $\bar{S}_N(T)$, which can be checked easily from the convergence of the Laplace transform of the marginal at time T (the limiting Laplace transform tends to 1 as $u \to 0$).

Since $\bar{S}_N$ are increasing, the oscillation function $w'_{\bar{S}_N}(\delta)$ is always equal to zero. So checking (ii) boils down to controlling the boundary oscillations $v_{\bar{S}_N}(0,\delta)$ and $v_{\bar{S}_N}(T,\delta)$. For the first quantity (using again the monotonicity of $\bar{S}_N$) this amounts to check that $\mathbb{P}[\bar{S}_N(\delta) \geq \eta] < \varepsilon$ if δ is small enough and N large enough. Using the convergence of the marginal at time δ, it is sufficient to take δ such that $\mathbb{P}[V_{\gamma/\beta^2}(K\delta) \geq \eta] \leq \varepsilon/2$, and to take N_0 such that, for all $N \geq N_0$,

$$\left|\mathbb{P}[\bar{S}_N(\delta) \geq \eta] - \mathbb{P}[V_{\gamma/\beta^2}(K\delta) \geq \eta]\right| \leq \varepsilon/2. \tag{5.12}$$

The reasoning for $v_{\bar{S}_N}(T,\delta)$ is analogous. □

5.3. Coarse-grained clock process. To prove our aging result, that is Theorem 1.2, we need to modify the result of Theorem 1.1 slightly. Let $\tilde{S}_N$ be the "coarse-grained" clock processes,

$$\tilde{S}_N(t) = \frac{1}{e^{\gamma N}} S_N(\nu \lfloor tr(N)\nu^{-1} \rfloor). \tag{5.13}$$

For these processes we can strengthen the topology used in Theorem 1.1, that is we can replace the M_1- by the J_1-topology.

Theorem 5.2. *If the hypothesis of Theorem 1.1 is satisfied, then*

$$\tilde{S}_N(t) \xrightarrow{N\to\infty} V_{\gamma/\beta^2}(Kt), \qquad \mathcal{Y} - a.s., \tag{5.14}$$

weakly in the J_1-topology on the space of càdlàg functions $D([0,T],\mathbb{R})$.

Unfortunately, we cannot prove the theorem with the estimates we have already at our disposition. We should return and improve some of them. First we show that traps with energies "much smaller" than $\gamma\sqrt{N}/\beta$ almost do not contribute to the clock process. Let $B_m = \gamma\sqrt{N}/\beta - m/(\beta\sqrt{N})$ and let

$$\bar{S}_N^m(t) = e^{-\gamma N} \sum_{i=0}^{\lfloor tr(N)\rfloor} e_i \exp\left\{\beta\sqrt{N} X_N^0(i)\right\} \mathbf{1}\{X_N^0(i) \le B_m\}. \tag{5.15}$$

Lemma 5.3. *For every T and η, $\varepsilon > 0$ there exists m large enough such that*

$$\mathbb{P}[\bar{S}_N^m(T) \ge \eta | \mathcal{Y}] \le \varepsilon, \qquad \mathcal{Y}\text{-}a.s. \tag{5.16}$$

Proof. To prove this lemma we should improve/modify slightly the calculations of Sects. 2 and 3. With the notation of Sect. 2 define

$$\mathcal{F}_N^m = \mathbb{E}\left[\exp\left\{-e^{-\gamma N} \sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N} U_i} \mathbf{1}\{U_i \le B_m\}\right\}\right] \tag{5.17}$$

(comparing with (2.2) observe that we set $u = 1$). We will show that

$$\lim_{N\to\infty} N^{1/2} \nu^{-1} e^{N\gamma^2/2\beta^2} [1 - \mathcal{F}_N^m] = K_m, \tag{5.18}$$

with $K_m \to 0$ as $m \to \infty$. The proof of this claim is completely analogous to the proof of Proposition 2.1. One should only modify the domains of integrations. More precisely, the definition of D_k which appears after (2.9) should be replaced by $D_k^m = D_k \cap \{z : G_k(z) \le B_m\}$. Hence, D_k' becomes $D_k'^m = D_k' \cap \{b : G_k(b) \le -m/(\beta/\sqrt{N})\}$, which then restricts the domain of integration in (2.31) to $(-\infty, -m/\beta]$. Hence, the constant K_m can be made arbitrarily small by choosing m large.

We define, similarly as in Sect. 3,

$$F_N^m(X) = \exp\left(-\sum_{i=0}^{Tr(N)-1} g\left(e^{-\gamma N} e^{\beta\sqrt{N} X(i)} \mathbf{1}\{X(i) \le B_m\}\right)\right). \tag{5.19}$$

As in Proposition 3.1 we show

$$\lim_{N\to\infty} \mathbb{E}\left[F_N^m(X_N^0)\,|\mathcal{Y}\right] - \mathbb{E}\left[F_N^m(X_N^1)\right] = 0, \qquad \mathcal{Y}\text{-a.s.} \tag{5.20}$$

We use again (3.6) to show this claim. Although the indicator function is not differentiable, we will proceed as if it was, setting $(\mathbf{1}\{x \le B\})' = -\delta(x-M)$, where δ denotes the Dirac delta function. As usual, this can be justified e.g. by using smooth approximations of the indicator function. The second derivative of $F_N^m(X)$ equals

$$\begin{aligned}
&\frac{u^2\beta^2 N}{e^{2\gamma N}} e^{\beta\sqrt{N}(X(i)+X(j))} g'\left(ue^{\beta\sqrt{N}X(i)-\gamma N}\right) g'\left(ue^{\beta\sqrt{N}X(j)-\gamma N}\right) F_N^m(X)\\
&\quad\times\left(\mathbf{1}\{X(i)\le B_m\} - \frac{\delta_{B_m}(X(i))}{\beta\sqrt{N}}\right)\left(\mathbf{1}\{X(j)\le B_m\} - \frac{\delta_{B_m}(X(j))}{\beta\sqrt{N}}\right)\\
&\le \frac{u^2\beta^2 N}{e^{2\gamma N}} e^{\beta\sqrt{N}(X(i)+X(j))} \exp\left(-2g\left(ue^{\beta\sqrt{N}X(i)-\gamma N}\right) - 2g\left(ue^{\beta\sqrt{N}X(j)-\gamma N}\right)\right)\\
&\quad\times\left(\mathbf{1}\{X(i)\le B_m\} - \frac{\delta_{B_m}(X(i))}{\beta\sqrt{N}}\right)\left(\mathbf{1}\{X(j)\le B_m\} - \frac{\delta_{B_m}(X(j))}{\beta\sqrt{N}}\right).
\end{aligned} \tag{5.21}$$

We should now bound the contributions of four terms. The one with the product of two indicator functions is easy, because we can use directly the result of Lemma 3.2. For the remaining three terms, those with the product of one indicator and one delta function, and the one with two delta functions, the calculation should be repeated. However, in the end we find that (5.21) is bounded by $\bar{\Xi}(\mathrm{Cov}(X(i), X(j)))$ as before. The presence of the delta functions actually simplifies the calculations slightly. The proof then proceeds as in Sect. 3.

We can now finish the proof of Lemma 5.3. By (5.17) and (5.20),

$$\begin{aligned}
\mathbb{E}\left[\exp(-\bar{S}_N^m(T))|\mathcal{Y}\right] &= \mathbb{E}\left[F_N^m(X_N^0)|\mathcal{Y}\right] = \mathbb{E}\left[F_N^m(X_N^1)|\mathcal{Y}\right] + o(1)\\
&= (1 - K_m f(N)^{-1} e^{-N\gamma^2/2\beta^2})^{Tr(N)/\nu} + o(1) = e^{-K_m T} + o(1).
\end{aligned} \tag{5.22}$$

Since $K_m \to 0$ as $m \to \infty$,

$$\mathbb{P}[\bar{S}_N^m(T) \ge \eta|\mathcal{Y}] \le \frac{1 - \mathbb{E}\left[\exp(-\bar{S}_N^m(T))\,|\mathcal{Y}\right]}{1 - e^{-\eta}} \tag{5.23}$$

can be made arbitrarily small by taking m large enough. □

We study now how the blocks where the process visits sites with energies larger than B_m are distributed along the trajectory. To this end we set for any Gaussian process X,

$$s_N^m(i; X) = \mathbf{1}\{\exists j : i\nu < j \le (i+1)\nu, X(j) > B_m\}. \tag{5.24}$$

We define the point process $H_N^m(X)$ on $[0, T]$ by

$$H_N^m(X; \mathrm{d}x) = \sum_{i=0}^{Tr(N)/\nu} s_N^m(i; X)\delta_{i\nu/r(N)}(\mathrm{d}x). \tag{5.25}$$

Lemma 5.4. *For every $m \in \mathbb{R}$ the point processes $H_N^m(X_N^0)$ converge to a homogeneous Poisson point process on $[0, T]$ with intensity $\rho_m \in (0, \infty)$, $\mathcal{Y}$-a.s.*

Proof. To show this lemma we use Proposition 16.17 of Kallenberg [Kal02]. According to it, to prove the convergence of $H_N^m(X_N^0)$ to a Poisson point process with intensity ρ_m it is sufficient to check that, for any interval $I \subset [0, T]$,

$$\lim_{N\to\infty} \mathbb{P}[H_N^m(X_N^0; I) = 0|\mathcal{Y}] = e^{-\rho_m|I|} \tag{5.26}$$

and

$$\limsup_{N\to\infty} \mathbb{E}[H_N^m(X_N^0; I)|\mathcal{Y}] \le \rho_m|I|, \tag{5.27}$$

where $|I|$ denotes the Lebesgue measure of I.

The proof of the first claim is completely similar to the previous ones. We start with a one-block estimate for (5.26):

$$\lim_{N\to\infty} N^{1/2}\nu^{-1}e^{N\gamma^2/2\beta^2}\,\mathbb{E}[s_N^m(0, U)] = \rho_m. \tag{5.28}$$

Using the notation of Sect. 2, we get

$$\mathbb{E}[s_N^m(0, U)] = \int_{A_m} \frac{\mathrm{d}z}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} z_i^2}, \tag{5.29}$$

where $A_m = \{z : \exists k \in \{1, \dots, \nu\} G_k(z) > B_m\}$. Dividing the domain of integration according to the maximal $G_k(z)$, this is equal to

$$\sum_{k=1}^{\nu} \int_{D_k} \frac{\mathrm{d}z}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} z_i^2}, \tag{5.30}$$

where $D_k = \{z : G_k(z) > B_m, G_i(z) \le G_k(z)\forall i \ne k\}$. Using the substitution $z_i = b_i \pm \Gamma_i B_m$ on D_k (where + sign is used for $i \le k$ and − sign for $i > k$) we get

$$e^{-N\gamma^2/2\beta^2} e^{m\gamma/\beta^2} \sum_{k=1}^{\nu} \int_{D_k'} \frac{\mathrm{d}b}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} b_i^2} e^{-B_m G_k(b)}, \tag{5.31}$$

where $D_k' = \{b : G_k(b) > 0, \sum_{j=i+1}^{k} b_j + |k-i|\Gamma_\nu B_m \ge 0 \forall i \ne k\}$. The same reasoning as before then allows to show that the last expression behaves like $\rho_m \nu N^{-1/2} e^{-\gamma^2 N/2\beta^2}$, as $N \to \infty$.

To compare the real process with the block-independent process, let

$$F_N(I; X) = \mathbf{1}\{\max\{X(i) : i\nu/r(N) \in I\} \le B_m\}. \tag{5.32}$$

The difference between $\mathbb{E}[F_N(I; X_N^0)|\mathcal{Y}]$ and $\mathbb{E}[F_N(I; X_N^1)]$ is again given by the Gaussian comparison formula (3.6). This time the second derivative equals

$$\delta(X(i) - B_m)\delta(X(j) - B_m) \prod_{k\ne i,j} \mathbf{1}\{X(k) \le B_m\} \le \delta(X(i) - B_m)\delta(X(j) - B_m). \tag{5.33}$$

If the covariance of $X(i)$ and $X(j)$ is equal to c, then the expectation of the last expression is given by the value of the joint density of $X(i)$, $X(j)$ at the point (B_m, B_m), which is

$$(2\pi(1 - c^2))^{-1} e^{-B_m^2/(1+c)} \le C(1 - c^2)^{-1} \exp\left\{-\frac{\gamma^2 N}{\beta^2(1 + c)}\right\}. \tag{5.34}$$

The exponential term is the same as in $\bar{\Xi}(c)$. The polynomial prefactor is however different, it diverges faster as $c \to 1$. We should thus return to (3.24) with $\tilde{\Xi}$ replaced by the right-hand side of (5.34). First

$$\int_0^1 \left(1-(1-h)^2c^2\right)^{-1} \mathrm{d}h = c^{-1} \arg\tanh(c) \approx -\frac{1}{2}\ln(1-c) \tag{5.35}$$

as $c \to 1$, which is not bounded for all c as before. The estimates (3.28) and (3.29) are influenced by this change. For (3.28) we can actually neglect this change, because the main contribution to this term came from the neighborhood of $d = N/2$ (or $c = 0$) and was exponentially small in the neighborhood of $d = 1$ (or $c \sim 1/N$). In the treatment of (3.29), the change has a stronger effect and after some computations (3.45) turns into

$$CtN^{3/2}\nu^{-1}\int_0^{\delta'} \ln(c/x)e^{-cNx}\mathrm{d}x \le CtN^{1/2}\nu^{-1}\ln N \xrightarrow{N\to\infty} 0. \tag{5.36}$$

Finally, the change of the polynomial prefactor of $\bar{\Xi}$ implies a change in the control of (3.30). Equation (3.46) becomes

$$(3.30) \le C\sum_{d=0}^{\nu} tN^{-3/2}d^2[1-(1-2dN^{-1})^2p]^{-1}\exp(N\tilde{\Upsilon}_{p,\beta,\gamma}(d/N)), \tag{5.37}$$

and the linearization of $\tilde{\Upsilon}_{p,\beta,\gamma}$ gives a new form of (3.47), namely

$$CtN^{3/2}\int_0^{\varepsilon} xe^{-c'Nx}\mathrm{d}x \le CtN^{-1/2} \xrightarrow{N\to\infty} 0. \tag{5.38}$$

Therefore, using (5.28)

$$\begin{aligned}\mathbb{P}[H_N^m(X_N^0; I) = 0|\mathcal{Y}] &= \mathbb{E}[F_N(I; X_N^0)|\mathcal{Y}] = \mathbb{E}[F_N(I; X_N^1)] + o(1)\\ &= (1-\mathbb{E}[s_N^m(0,U)])^{|I|r(N)/\nu} \to e^{-\rho_m|I|}.\end{aligned} \tag{5.39}$$

This completes the proof of (5.26).

It is easy to check Eq. (5.27). By definition,

$$\mathbb{E}[H_N^m(X_N^0; I)|\mathcal{Y}] = \sum_{i:i\nu/r(N)\in I} \mathbb{E}[s_N^m(i, X_N^0)|\mathcal{Y}]. \tag{5.40}$$

Since $\Lambda_{ij}^0 \ge \Lambda_{ij}^1$ for i, j in the same block, $\mathbb{E}[s_N^m(i, X_N^0)|\mathcal{Y}] \le \mathbb{E}[s_N^m(i, X_N^1)]$. Therefore,

$$(5.40) \le |I|r(N)/\nu\mathbb{E}[s_N^m(0,U)] = \rho_m|I|. \tag{5.41}$$

This completes the proof of Lemma 5.4. □

Proof of Theorem 5.2. Checking the convergence of finite-dimensional marginals as well as condition (i) and the second part of (ii) of Lemma 5.1 is analogous to the case of the original clock process $\bar{S}_N$. We should thus only prove the first part of condition (ii). Namely, for any η and ε there exist δ such that

$$\mathbb{P}[w_{\bar{S}_N}(\delta) \ge \eta] \le \varepsilon, \tag{5.42}$$

for all N large enough.

Let

$$w_f([\tau,\tau+\delta]) = \sup\{\min(|f(t_2)-f(t)|, |f(t)-f(t_1)|) : \tau \le t_1 \le t \le t_2 \le \tau+\delta\}. \tag{5.43}$$

Fix m such that $\mathbb{P}[\bar{S}_N^m(T) \ge \eta/2] \le \varepsilon/2$, which is possible according to Lemma 5.3. If $H_N^m(X_n^0; [\tau,\tau+\delta]) \le 1$ then

$$w_{\bar{S}_N}([\tau,\tau+\delta]) \le \bar{S}_N^m(\tau+\delta) - \bar{S}_N^m(\tau) \le \bar{S}_N^m(T). \tag{5.44}$$

Hence,

$$\mathbb{P}[w_{\bar{S}_N}([\tau,\tau+\delta]) \ge \eta | i\bar{S}_N^m(T) \le \eta/2] \le \mathbb{P}[H_N^m(X_N^0; [\tau,\tau+\delta]) \ge 2] \le C\rho_m\delta^2. \tag{5.45}$$

We can now show (5.42). The estimate

$$w_{\tilde{S}_N}(\delta) \le \max\{w_{\tilde{S}_N}([\tau,\tau+2\delta]) : 0 \le \tau \le T, \tau = k\delta, k \in \mathbb{N}\} \tag{5.46}$$

yields

$$\begin{aligned}\mathbb{P}[w_{\tilde{S}_N}(\delta) \ge \eta|\mathcal{Y}] &\le \sum_{k=0}^{T\delta^{-1}} \mathbb{P}[w_{\tilde{S}_N}([k\delta,(k+2)\delta]) \ge \varepsilon|\mathcal{Y}] \\ &\le \mathbb{P}[\bar{S}_N^m(T) \ge \eta/2] + \sum_{k=0}^{T\delta^{-1}} \mathbb{P}[H_N^m(X_N^0; [k\delta,(k+2)\delta]) \ge 2] \\ &\le \varepsilon/2 + CT\delta^{-1}\rho_m\delta^2 \le \varepsilon\end{aligned} \tag{5.47}$$

if δ is chosen small enough. This completes the proof. □

Proof of Theorem 1.2. Let $\mathcal{R}_N$ be the range of the coarse grained process $\tilde{S}_N$. Obviously, for any $1 > \varepsilon > 0$,

$$A_N^\varepsilon(t,s) \supset \{\mathcal{R}_N \cap (t,s) = \emptyset\}, \tag{5.48}$$

because if the above intersection is empty, then σ_N makes less than ν steps in the time interval $[te^{\gamma N}, se^{\gamma N}]$, and thus the overlap of $\sigma_N(te^{\gamma N})$ and $\sigma_N(se^{\gamma N})$ is $O(\nu/N)$.

If $\mathcal{R}_N \cap (t,s) \neq \emptyset$, then there exist u such that $\tilde{S}_N(u) \in (t,s)$. Moreover, it follows from Theorem 5.2 that, for any δ, there exist η such than

$$\mathbb{P}[\tilde{S}_N(u+\eta) \in (s,t)] \ge 1-\delta. \tag{5.49}$$

This means that the process σ_N makes at least $\eta r(N)$ steps between times t and s and thus the overlap between $\sigma_N(te^{\gamma N})$ and $\sigma_N(se^{\gamma N})$ is with high probability close to 0.

Hence $\mathbb{P}[A_N^\varepsilon(t,s)|\mathcal{Y}]$ is very well approximated by $\mathbb{P}[\mathcal{R}_N \cap (t,s) = \emptyset|\mathcal{Y}]$. Since the stable subordinators do not hit points, that is $\mathbb{P}[\exists u : V_{\gamma/\beta^2}(u) = t] = 0$, and $\tilde{S}_N$ converge in the J_1-topology,

$$\mathbb{P}[\mathcal{R}_N \cap (t,s) = \emptyset|\mathcal{Y}] \xrightarrow{N\to\infty} \mathbb{P}[\{V_{\gamma/\beta^2}(u) : u \ge 0\} \cap (s,t) = \emptyset]. \tag{5.50}$$

The right-hand side of this equation is given by the formula (1.13), as follows from the arc-sine law for stable subordinators. □

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