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Review Article

Unquenched Flavor in the Gauge/Gravity Correspondence

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Within the AdS/CFT correspondence, we review the studies of field theories with a large number of adjoint and fundamental fields, in the Veneziano limit. We concentrate in set-ups where the fundamentals are introduced by a smeared set of D-branes. We make emphasis on the general ideas and then in subsequent chapters that can be read independently and describe particular considerations in various different models. Some new material is presented along the various sections.

1. Introduction, General Idea, and Outline

1.1. Introduction and Outline

The AdS/CFT conjecture originally proposed by Maldacena [1, 2], refined in [3, 4] and reviewed in [5], has been one of the most interesting developments in theoretical physics of the last decades. It has become one of the most powerful analytic tools to deal with strong coupling effects of some particular gauge theories in the planar limit $N_c \to \infty$. The most studied and best understood case corresponds to $\mathrm{SU}(N_c)\,\mathcal{N} = 4\,\mathrm{SYM}$ which is a highly supersymmetric conformal theory and which only contains matter in the adjoint representation of the gauge group. Certainly, there are many interesting field theories which do not share these properties and this fact has lead to an enormous amount of effort devoted to extending the duality along different paths. Consequently, people have constructed gravity duals of nonsupersymmetric, non-conformal gauge theories, in different vacua and with diverse matter contents. One can mention the attempt of building a dual as close as possible to QCD as an aim for these generalizations. However, one should keep in mind that this is just one among many desirable motivations, since understanding gauge theories at strong coupling (or using gauge theories to understand gravity) is a very relevant problem *per se*, with both theoretical and phenomenological possible implications.

An important development of AdS/CFT has been to generalise the matter content of the gauge theories under consideration and, in particular, to include fields which transform in the fundamental representation of the gauge group, as the QCD quarks do. (With an abuse of language, we will use throughout this review the words *quark* or *flavor* to refer to any field, fermionic or bosonic, transforming in the fundamental representation of the gauge group. Accordingly, by *mesons* we will mean bound states of *quarks*.) A first possibility is to add the flavors in the *quenched* approximation. The word *quenched* comes from the lattice literature and, in that context, it amounts to setting the quark fermion determinant to one. In more physical terms, *quenching* corresponds to discarding the quark dynamical effects. This means that quantum effects produced by the fundamentals are neglected; the quarks are considered as external nondynamical objects in the sense that they do not run in the loops. (In the lattice, usually, quenching is thought to be a good approximation for heavy quarks whereas for the gauge-gravity examples the relevance of the quenched approximation comes from having parametrically less fundamental than adjoint fields $N_f \ll N_c$.) From the string side, adding quenched quarks to a given gauge theory corresponds to incorporating a set of brane probes in the dual background, which is not modified with respect to the quark-less case. By analysing the worldvolume physics of these flavor branes (typically using the Dirac-Born-Infeld + Wess-Zumino action) a lot of physically interesting questions can be understood. For instance and just to mention a few, chiral symmetry breaking can be neatly described, phase diagrams can be constructed, and meson spectra can be exactly computed. It is hard to do justice to the huge literature in the subject; so let us just mention the seminal papers [6, 7] and a recent review [8].

Thus, it is fair to say that the study of quenched flavor within the gauge-gravity correspondence has been a very fruitful program. Nonetheless, there are physical features which are intimately related to the quantum effects of the quarks. Examples are the consequences of the presence of fundamentals on the running couplings, which may ultimately lead to conformal points, conformal windows [9], or Seiberg-like dualities [10]. More phenomenologically, multihadron production, the screening of the color charge, or the large mass of the η' meson are spin-offs of these quantum effects. Let us also mention that the most successful application of string duals towards phenomenology has been the construction of solutions that can be used as toy models for the experimental quark-gluon plasma. Thus, a very interesting program is to build black hole solutions with unquenched flavor which are really dual to *quark*-gluon plasmas, that is, such that the effect of the dynamical quarks affects the plasma physics, as is expected to be the case in the real world.

These observations largely motivate the study of theories with unquenched quarks from the string theory dual point of view. Unquenching the flavors of the gauge theory has a very precise implication for the dual theory: the gravity background has to be modified by the inclusion of the quarks; namely, one needs to take into account the back-reaction on the geometry produced by the flavor branes. The main goal in the following will be to present methods to compute such back-reacted solutions. This will be done by presenting different examples that, hopefully, will help the reader to gain insight in both the physical questions and the technical tools used to address them.

In this review, we will focus on a specific family of unquenched constructions. Namely, we will discuss at length just solutions of type IIA or type IIB string theory in which the fundamentals come from a *smeared* set of flavor branes. In Section 1.5, we will try to provide a general understanding of this notion of smearing the flavor and argue why we find it a case of particular interest. As we will see, by considering the case of smeared D-branes we can build a systematic approach applicable to different situations and which typically results in large simplifications as compared to other kinds of flavor D-brane distributions. This smearing procedure referring to flavors was first introduced in [11] in a noncritical string framework and in [12] in a well-controlled ten-dimensional context.

It is important to remark that this smearing is by no means the only possibility to introduce unquenched fundamentals in gauge-gravity duals. Many important works have followed alternative paths to construct different models. We are not able to review them here, but we provide a survey of the literature in Section 1.6.

Outline

We will devote the rest of Section 1 to further clarifying the kind of physical problems we want to address and to give the general methods and notions which are common to all the constructions we will present later.

Then, Sections 2–6 will analyse different models that are ordered in increasing order of complexity. Each section can be read mostly independently from the rest. The discussion of each model can always be regarded as a two-step process. First, one has to solve the equations for finding the back-reacted solutions of type II supergravity coupled to a set of D-brane sources. Second, one can use these solutions to extract the physics of the conjectured gauge theory duals with unquenched flavors. Readers interested in different aspects of the problem can consult the different parts independently. We would like to stress that, even without making any reference to the gauge-gravity correspondence, the string theory solutions and methods developed to find them are interesting by themselves.

Section 2 deals with the backreaction of D7-branes on $\mathrm{AdS}_5 \times X^5$ spaces, where X^5 stands for a Sasaki-Einstein space. As a matter of fact, a large part of the discussion can be carried out without specifying the X^5. Notwithstanding, the two most interesting cases correspond to $X^5 = S^5$ and $X^5 = T^{1,1}$. At different points during Section 2, we will refer to these particular examples in order to explain concrete features. We will present supersymmetric solutions for massless and massive quarks and also non-supersymmetric black hole solutions which are dual to theories at finite temperature, in a deconfined plasma phase. All these solutions share the property of having a singularity, associated with a UV Landau pole in the field theory (when quarks are massless and the temperature is zero, there is also a naked IR singularity). We will show how to make well-defined IR predictions from the geometry, even in the presence of the UV singularity (in much the same spirit as in field theory renormalization).

In Section 3, we will discuss a model in which both the color and flavor branes are D5's. It is dual to a (3+1)-dimensional $\mathcal{N} = 1$ theory with a UV completion. Among several nice features of the model that will be presented, we would like to remark here that it incorporates a geometrical description of a Seiberg-like duality. Section 4 is also built from a D5-D5 intersection and in fact shares several similarities with the previous model. The construction corresponds to color D5's wrapping a compact 3-cycle and therefore the dual field theory is (2+1)-dimensional.

In Section 5 we examine the addition of D7-branes to the conspicuous Klebanov-Strassler model [13]. From the physical point of view, how the unquenched flavors affect a duality cascade is particularly interesting. From a technical point of view, the system is slightly more involved than the previous ones because different RR and NSNS forms are turned on. However, despite this complication, it is remarkable that almost all functions of the ansatz can be integrated in a closed form.

Section 6 reviews a different class of models. The dual gauge theories are built on wrapped supersymmetric D-branes with the peculiarity that some of the adjoint scalars remain massless. As we will explain, it is not sensible to smear the branes in all the transverse directions. The associated main technical difficulty will be the fact that one has to solve partial differential equations to find the background.

Profiting from the experience gained by discussing these examples one by one, in Section 7, we will give a more mathematical viewpoint of the constructions. In particular, we will take some tools of differential geometry to describe in a concise and compact way the distributions of mass and charge due to the presence of the flavor branes.

Finally, in Section 8 we will conclude by summarizing the whole topic trying to give a general perspective of the results obtained and by also providing an outlook of the subject.

1.2. Presentation of the Problem

As anticipated in the introduction, we will discuss the addition of flavors to field theories (mostly focusing on SUSY examples, but this is not mandatory) using AdS/CFT or, more generally, gauge-strings duality.

We hope it is clear to the reader that the addition of flavors (fields transforming in the fundamental representation) is a very interesting exercise from a dynamical point of view. Indeed, in a theory with adjoint fields (let us, for the sake of this discussion, consider the case of a confining field theory) the presence of fields transforming in the fundamental will produce the breaking of the "QCD-string" or screening. Of course, the fundamentals will add a new symmetry, that can be $\mathrm{SU}(N_f)$ or, like in massless QCD, $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$; a baryonic $U(1)_B$ symmetry should also appear. Obviously the presence of global symmetries (and their possible spontaneous or explicit breaking) will directly reflect in the spectrum. Apart from this, it will happen that the states before the addition of fundamentals, that is the glueballs, will interact and mix with the mesons, giving place to new diagonal combinations that will be the observed states. Moreover, anomalies will be modified, as fermions that transform in the fundamental will run in the triangles. Also gauge couplings will run differently and finally new dualities (Seiberg-like [10]) may appear. In the rest of this article, we will discuss how all of the above mentioned features are encoded in string backgrounds.

It is clear that we need to add new objects to our string background. These new objects are D-branes, on which a gauge field propagates, encoding the presence of a $U(N_f)$ gauge symmetry (in the bulk), dual to the global $U(N_f)$ in the dual QFT. Also, it is on these D-branes that the meson fields, represented by excitations of the branes, propagate and interact. Following a nomenclature that by now became standard, we will call these D-branes "flavor branes".

It is then clear that to add flavors to a field theory whose dual we know, we should consider the original (unflavored) string background and add flavor-branes. Now, the point is how to proceed technically to add these new branes. It may be useful to consider two developments of the 1970s, that will turn to illuminate on the answer to this question.

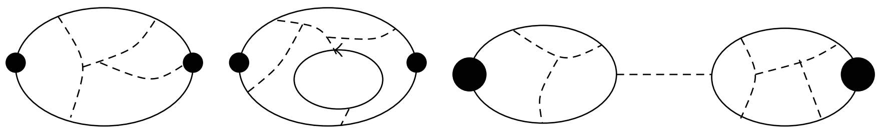

Figure 1: Diagrams for a meson propagator, with two insertions of the meson operator ($n = 2$) shown as thick points on the boundaries. The dashed lines are gluons that fill the diagram in the large N_c limit and the thick lines are quarks. (a) Planar diagram with no internal quark loops ($h = 0$, $w = 0$), the scaling is ~1. (b) Planar diagram with an internal quark loop $h = 0$, $w = 1$, the scaling is $\sim N_f/N_c$. (c) Nonplanar diagram with no internal quark loops $h = 0$, $w = 0$, $b = 2$, the scaling is $\sim 1/N_c$.

In [14, 15] 't Hooft and Veneziano, respectively, considered the influence of fundamentals when the following scaling is taken:

$$N_c \longrightarrow \infty, \qquad \lambda = g_{\mathrm{YM}}^2 N_c = \text{fixed}, \tag{1.1}$$

and considered the two possible cases ('t Hooft and Veneziano, resp.)

$$\begin{aligned} N_f &= \text{fixed}, \qquad x = \frac{N_f}{N_c} \longrightarrow 0, \\ N_f &\longrightarrow \infty, \qquad x = \frac{N_f}{N_c} = \text{fixed}. \end{aligned} \tag{1.2}$$

It is very illuminating to see how different diagrams contributing to the same physical process (to fix ideas, an n-point correlator of mesonic currents) scale in these two cases. In this respect, a formula for the kinematical factor of the scattering of n mesons was produced in [16], considering diagrams with w being internal fermion loops (windows), h nonplanar handles, and b boundaries:

$$\langle B_1 \cdots B_n \rangle \sim \left(\frac{N_f}{N_c}\right)^w N_c^{(2-(n/2)-2h-b)}. \tag{1.3}$$

Consider the case of scattering of two mesons $n = 2$. We see that diagrams like the first one in the figure ($w = h = 0, b = 1, n = 2$) scale like a constant $N_c^0 \sim 1$, the second diagram (with $w = 1, h = 0, b = 1, n = 2$) scales like N_f/N_c, while the third one (with $w = 0, h = 0, b = 2, n = 2$, that is nonplanar) goes like N_c^{-1}.

So, we see that from this view point, the Veneziano scaling captures more physics, represented here by diagrams like (b) in Figure 1. Nevertheless, there may be some particular problems for which studying things in the 't Hooft scaling may be enough.

From the view point of a lattice theorist, working in the 't Hooft scaling, hence neglecting the effects of fundamentals running inside loops, is the same as working in what they would call the "quenched approximation". We can think of the field theory as being quenched when the fundamental fields do not propagate inside the loops. One natural way of doing this is to consider the case of very massive quarks. Indeed, when quenching, one considers an expansion of the fermionic determinant (for massive quarks) of the form:

$$\log\,\det\left[\gamma^{\mu}D_{\mu}+m\right]=\mathrm{Tr}\,\log\left(\gamma^{\mu}D_{\mu}+m\right)=\mathrm{Tr}\,\log\left(m\left[1+\frac{\gamma^{\mu}D_{\mu}}{m}\right]\right)\sim\mathrm{Tr}\,\log(m)+O\left(\frac{1}{m}\right). \tag{1.4}$$

Keeping only the constant term (or considering a very large mass) is equivalent to saying that fundamentals are very difficult to pair-produce; hence their presence inside loops will be very suppressed. Another way to quench in the field theory is to consider the case in which the quotient $x = N_f/N_c$ is very small. Notice that the quenched theory is not equivalent to a theory with only adjoints, as fundamentals can occur in external lines, like in a correlator of two mesonic currents as exemplified in the diagram (a) of Figure 1. Needless to say, lattice theorists developed techniques to quench fundamental fields with arbitrary mass. Also, while at first sight the quenching as described above is not a good operation as it breaks unitarity (not including all possible diagrams), this kind of troubles will be avoided when working in the 't Hooft scaling, where unitarity problems will be suppressed in $1/N_c$ (but of course will be present in a lattice version of theories with finite N_c).

The interesting point to be taken from this, by a physicist working in gauge-string duality, is that both scalings ('t Hooft's and Veneziano's) can be realized with D-branes. Indeed, in both cases we must add D-branes (to realize symmetries and new states as discussed above), but we can add these flavor branes in two ways.

(i) We can add N_f flavor branes in such a way that we will only *probe* the geometry produced by the N_c color branes. In this case the dynamics of the probe-flavor branes (the mesons) will be influenced by the dynamics of the color branes (the glueballs) but not viceversa. This is a good approximation if $x = N_f/N_c \to 0$, which immediately sets us in the 't Hooft scaling limit. Notice, however, that when the N_c contribution to some particular quantity vanishes, the flavor effects may be the leading ones even when $N_f \ll N_c$.

(ii) We can add N_f flavor branes, in such a way that they will deform the already existing geometry, in other words *backreacting* on the original "color" geometry. In field theory language, we would say that the dynamics of the glueballs and that of the mesons influence each other, leading to new states that will be a mixture of mesons and glueballs. This is surely what we need to do if $x = N_f/N_c$ = fixed and doing this will set us in the Veneziano scaling limit.

More technically, in the 't Hooft scaling limit we are studying the Born-Infeld-Wess-Zumino dynamics for a Dk flavor brane in a background created by N_c color Dp-branes (we will always work in Einstein frame in the following):

$$S_{\mathrm{BIWZ}}=-T_k\int d^{k+1}x e^{((k-3)/4)\phi}\sqrt{\det\left[\hat{g}_{ab}+2\pi\alpha' e^{-\phi/2}\mathcal{F}_{ab}\right]}+T_k\int C\wedge e^{\mathcal{F}}, \tag{1.5}$$

where $\hat{g}_{ab}$, $\mathcal{F}_{ab}$ are fields *induced* by the color branes background on the (few) flavor branes. The "shape" of the flavor branes (induced metric) will then influence the mass spectrum and interactions of the fluctuations of the flavor branes (the mesons), explicitly realizing the picture advocated above. This line of research was initiated by Karch and Katz [6] and substantially clarified in subsequent papers [7, 17–21]. This line kept on growing in the last few years, finding numerous applications. See [8] for a comprehensive review.

On the other hand, working in the Veneziano scaling limit implies that we will need to study the action

$$S = S_{\mathrm{IIA/IIB}} + S_{\mathrm{BIWZ}}. \tag{1.6}$$

There will be new equations of motion, encoding explicitly the numbers N_f, N_c. As discussed above, it is now very clear that proceeding like this will be the only possibility when the number of flavors is comparable with the number of colors. Also, it makes manifest the fact that the dynamics of glueballs (represented on the string side by $S_{\mathrm{IIA/IIB}}$ which is of order $g_s^{-2} \sim N_c^2$) is influenced and influences back on the dynamics of fundamentals (represented by S_{BIWZ}, of order $g_s^{-1} N_f \sim N_c N_f$). The rest of this review will focus on this second scaling (Veneziano).

Notice that (in both scalings) we are making an explicit difference between the color $\mathrm{SU}(N_c)$ *gauged* symmetry and the flavor $\mathrm{SU}(N_f)$ *global* symmetry on the field theory side. From the string theory construction, this qualitative difference is connected to the fact that the volume of the flavor branes is infinite, as compared to the volume of the color branes. Indeed, in the bulk, we only need to realize the field theory global symmetry, and we do it with the gauge field present in the Born-Infeld-Wess-Zumino action. Searching for solutions of D_p color branes in interaction with D_k flavor branes in pure IIA/IIB supergravity is an interesting problem but will not represent the physical system we are after, as only flavor singlet states would be included in the dynamics.

Before we proceed studying the formalism and examples to clarify the details, some comments are in order.

1.3. The String Action and the Scaling Limit in N_c and N_f

Let us study a bit more the expression of (1.6), being careful about coefficients. We will consider the case of a set of N_c "color" Dp-branes and N_f "flavor" Dk-branes. The action for this system will be, in Einstein frame,

$$\begin{aligned} S &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{g_{10}} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{e^{-\phi}}{12} H_3^2 - \sum_l \frac{e^{((5-l)/2)\phi}}{2 \times l!} F_l^2 \right] + \int \text{CS-terms} \\ &\quad - N_f T_k \int d^{k+1} x e^{((k-3)/4)\phi} \sqrt{\det\left[\hat{g}_{ab} + 2\pi\alpha' e^{-\phi/2} \mathcal{F}_{ab}\right]} + N_f T_k \int_{k+1} C \wedge e^{\mathcal{F}}, \\ S &= \frac{1}{2\kappa_{10}^2} \left[\int L\left(\frac{\mathrm{IIA}}{\mathrm{IIB}}\right) - 2\kappa_{10}^2 N_f T_k \int L_{\mathrm{BIWZ}} \right], \end{aligned} \tag{1.7}$$

where by F_l we have denoted the various RR fields and with CS-terms the possible Chern-Simons terms. We have taken the simplification of writing the action for the set of flavor branes as N_f times that of a single D-brane, which is enough for the large N counting we want to undertake here. The gravitational constant and D-brane tension are

$$2\kappa_{10}^2 = (2\pi)^7 g_s^2 \alpha'^4, \qquad T_k = \frac{1}{(2\pi)^k g_s (\alpha')^{(k+1)/2}}. \tag{1.8}$$

The typical quantization condition for the color branes reads

$$\frac{1}{2\kappa_{10}^2}\int F_{8-p} = N_c T_p. \tag{1.9}$$

As a consequence of (1.7), we will have equations of motion, that generically will read for the metric and dilaton:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= T_{\mu\nu}\left[\frac{\text{IIA}}{B}\right] - 2\kappa_{10}^2 N_f T_k T_{\mu\nu}[\text{brane}], \\ \nabla^2 \phi &= \frac{\partial}{\partial\phi}\left[L\left[\frac{\text{IIA}}{B}\right] - 2\kappa_{10}^2 N_f T_k L[\text{BIWZ}]\right], \end{aligned} \tag{1.10}$$

together with the modified (by the CS-terms) Maxwell equations and, importantly, the Bianchi identity for the (magnetic) Ramond-Ramond field F_{8-k} that couples to the flavor Dk-branes:

$$dF_{8-k} = 2\kappa_{10}^2 N_f T_k \delta^{9-k}(\vec{r}), \tag{1.11}$$

indicating that the flavor branes are localized (all together) at the position $\vec{r} = 0$. Similarly the $T_{\mu\nu}$[brane] contains delta functions with support on the position of the flavor branes. In principle, one will need to solve second-order, nonlinear, partial differential equations.

Instead of directly dealing with the above equations, we want to present here an argument to understand which parameter controls the size of the flavor effects on the action and, therefore, on the solution. We remark that the reasoning below is qualitative and in particular we will just write a background for flat Dp-branes as considered, for instance, in [22]. This will be enough for understanding the scaling with the parameters, at least in the cases studied in this review. In the following, we just focus on the behaviour with respect to N_f, N_c, g_{YM}^2 and do not care about numerical prefactors. We will use notation similar to [22]. Consider the background associated to a stack of Dp color branes (in Einstein frame):

$$\begin{aligned} ds^2 &= e^{-\phi/2}\alpha'\left[\frac{\left(\sqrt{\alpha'}U\right)^{(7-p)/2}}{\alpha' c_p\sqrt{g_s N_c}} dx_{1,p}^2 + \frac{\alpha' c_p\sqrt{g_s N_c}}{\left(\sqrt{\alpha'}U\right)^{(7-p)/2}} dU^2 + c_p\sqrt{g_s N_c}\left(\sqrt{\alpha'}U\right)^{(p-3)/2} d\Omega_{8-p}^2\right], \\ e^{\phi} &\sim \left(\frac{g_s N_c}{\left(\sqrt{\alpha'}U\right)^{7-p}}\right)^{(3-p)/4}, \end{aligned} \tag{1.12}$$

where c_p is a known numerical constant and U is an energy scale. On a background of this kind, we want to introduce N_f Dk-flavor branes and to know which is the relative importance of the associated terms in the action (1.7) and equations of motion (1.10). With that aim, let us start by computing the coefficient in front of the term coming from the RR-form sourced by the color branes in (1.7), namely, $(2\kappa_{10}^2)^{-1}\sqrt{g_{10}}e^{((p-3)/2)\phi}F_{8-p}^2$. Using (1.9) and (1.12), we find that the Lagrangian density associated to the color branes goes as:

$$\mathcal{L}_{\text{color}} \sim \left(2\kappa_{10}^2\right)^{-1}\sqrt{g_{10}}e^{5\phi/2}\left(\alpha'^{-1}\right)(g_s N_c)^{(p-4)/2}\left(\sqrt{\alpha'}U\right)^{(p-3)(p-8)/2}. \tag{1.13}$$

Let us now look at the DBI term. We assume that the flavor Dk-branes are extended along the Minkowski directions, the radial direction U, and $k-p-1$ directions within the sphere. We find

$$\mathcal{L}_{\text{flavor,DBI}} \sim N_f T_k e^{((k-3)/4)\phi}\sqrt{\hat{g}_{k+1}} \sim \frac{N_f}{N_c} g_{\text{eff}}^{(k-p)/2}\mathcal{L}_{\text{color}}, \tag{1.14}$$

where in order to get the last expression we have used (1.8), (1.12), and (1.13) and defined a dimensionless effective coupling as in [22]:

$$g_{\text{eff}}^2 \sim g_{\text{YM}}^2 N_c U^{p-3} \sim g_s N_c\left(\sqrt{\alpha'}U\right)^{p-3}. \tag{1.15}$$

Thus, parametrically, the action from the flavor branes as compared with that from the color background is weighed by $(N_f/N_c)g_{\text{eff}}^{(k-p)/2}$. We now want to take a low-energy decoupling limit as in [22] (see also [23]); namely, the dimensionless effective coupling g_{eff} and U are fixed as $\alpha' \to 0$. Thus, the Veneziano scaling limit in this framework amounts to

$$N_c, N_f \longrightarrow \infty, \qquad g_{\text{eff}} \text{ fixed}, \qquad \frac{N_f}{N_c} g_{\text{eff}}^{(k-p)/2} \text{ fixed}, \tag{1.16}$$

where the last relation comes from demanding that the flavor effects are also fixed. Staying in the supergravity limit requires

$$1 \ll g_{\text{eff}}^2 \ll N_c^{4/(7-p)}, \tag{1.17}$$

a constraint that limits the range of energy scales U for which the supergravity description is valid [22]. Notice that if we further require that the flavor terms do not parametrically dominate over the color ones, this can further restrict U, depending on p and k.

The probe limit, in which the flavor action is negligible as compared to the gravity action, comes from making the last quantity in (1.16) vanishingly small. (In the literature, it is usually written that the probe limit is good when $N_f \ll N_c$. That is not strictly correct. For instance, in the D3–D7 case, the probe approximation is valid when, parametrically, $\lambda N_f \ll N_c$.) As expected, that term is strictly zero in the ’t Hooft limit. We now comment on the values of p, k that will appear in the following sections.

For the D3–D7 case of Section 2, the parameter weighing the flavor effects is $(g_{\text{YM}}^2 N_c)(N_f/N_c) \sim \lambda(N_f/N_c)$. For the cascading case of Section 5, the result is similar but one has to replace N_c by the number of fractional branes M. Getting ahead of the discussion of upcoming sections we notice that, in these cases, it is not enough to take this parameter fixed, but it should also be small. This will be due to positive beta functions, as will be thoroughly discussed.

From (1.16), we see that $k = p$ (Sections 3, 4 and 6) is particularly interesting since it is really N_f/N_c what has to be taken fixed. (Even if in all these sections we will deal with wrapped branes and therefore the backgrounds are not that similar to (1.12), the argument above still yields the correct result.) For this reason, only in these cases one can hope to describe—within gravity—phenomena as Seiberg-like dualities (see Section 3.6.2). Loosely speaking, the Klebanov-Strassler duality cascade [13] lies in this class of $k = p$ theories, since it can be understood as the interplay between two sets of D5-branes wrapping vanishing two-cycles.

We close this section by mentioning other brane intersections that will not be discussed further in later sections. In a D2–D6 system, the effective coupling (1.15) decreases at large U and therefore the flavor backreaction on the glue fades away in the UV—see (1.16)—as expected in a superrenormalizable theory. This was observed in [24] when studying a solution with localized D6-branes. In a D4–D8 intersection, the opposite happens. The probe brane approach can be valid in an intermediate regime $1 \ll g_{\rm eff}^2 \ll N_c/N_f$ but at a given U the fundamentals eventually take over and dominate. Notice that the value of U for which the D4-D8 theory loses its validity is parametrically smaller than that for which the unflavored D4-brane theory becomes pathological, which is set by (1.17).

1.4. The Method

Looking back at (1.7), one can appreciate that in general finding the solution describing the backreaction between the type II closed strings and the open strings described by the Born-Infeld action is quite a challenging problem. Indeed, the fact that the flavor branes (BIWZ) are localized in the ten-dimensional space implies that we will have to solve second-order, nonlinear, coupled, and partial differential equations with localized sources. Basically what makes things so difficult are the presence of delta function sources and the fact that the differential equations describing the dynamics are "partial" (in principle depending on all the variables describing the space transverse to the flavor branes). In order to get some intuition of the answer, we may consider the case in which we will "erase" the dependence on these transverse coordinates (this is like considering the "s-wave" of the putative multipole decomposition of the full solution in this transverse space) and delocalize the sources. To achieve this, we will propose to *smear* the flavor branes over their perpendicular space.

On the field theory side, this will amount to considering systems where the addition of the degrees of freedom transforming in the fundamental does not break any of the global symmetries of the unflavored QFT. Also, it may happen that the original $U(N_f)$ is explicitly broken to $U(1)^{N_f}$ as we are separating the flavor branes—see the discussion in [25, Section 7] and in [26, Section 2]. An intuitive understanding of the smearing procedure will be discussed in Section 1.5, while a more formal approach will be treated in Section 7. For technical reasons, this procedure is cleaner in examples preserving some amount of SUSY, since the force between flavor branes is cancelled and the smearing is at no cost of energy.

In the examples described in the following sections, we will proceed like this.

(i) Consider an unflavored string background and find the embedding of flavor branes that will preserve some SUSY, or (in non-SUSY examples) that will be stable and solve the equations of motion for the brane. In the SUSY cases, this can be achieved by considering kappa-symmetric embeddings, that we review generically below.

(ii) Consider now N_f flavor branes in that particular embedding and smear them, getting an action in ten dimensions, as will be explained with all generality in Section 7.

(iii) Solve the equations derived from (1.7), that will now contain smeared branes and will be ordinary differential equations. In SUSY cases, there will be a set of BPS equations to be solved. In non-SUSY examples one might manage to get a fake superpotential and fake-BPS equations [27].

Moreover, one has to check that the flavor embeddings considered are still a solution in the backreacted geometry.

Let us review briefly the main technical points collected above.

1.4.1. BPS Equations, Kappa Symmetry (SUSY Probes), and Smearing

Let us consider the case of a supersymmetric background, namely, a solution of type II sugra for which the supersymmetry variations of the gravitino and the dilatino vanish $\delta\psi_\mu = \delta\lambda = 0$. We will not give here details on the form of these expressions, which can be found elsewhere. For instance, the string frame SUSY transformations of both type IIA and type IIB are written down in [28, Appendix A].

Given a background that preserves some amount of SUSY, the idea is to find the hypersurfaces in which to extend the flavor branes (in other words, finding the embeddings for flavor branes) so that these will preserve all (or a fraction) of the SUSY of the background.

One then writes an eigenvalue problem, imposing that the preserved spinors of the background are eigenspinors of the kappa-symmetry matrix:

$$\Gamma_\kappa \epsilon = \epsilon. \tag{1.18}$$

See [29, 30] for the definition of Γ_κ.

Once we have the kappa-symmetric embeddings as described above, we now proceed to write an action describing the dynamics of the closed and open strings, as in (1.7). We then realize that the problem will lead (unless we are adding D9-branes) to a system of partial differential equations. As discussed above, we then proceed to smear these flavor branes. For this we propose an ansatz for the metric, where the embedding of the flavor branes is clear and distribute them along the directions of their transverse space. This distribution of the flavor branes can be done in a uniform way. In some sense, we are "deconstructing" the transverse space to the flavor branes by adding at each point one of the many N_f flavor branes.

The key point is that once the BPS equations and kappa-symmetry conditions are simultaneously satisfied, the problem is solved. In fact, it is a general result [31] that the SUSY equations $\delta\psi_\mu = \delta\lambda = 0$, together with the Bianchi identities—and equations of motion—for the different forms modified by calibrated (namely, kappa-symmetric) sources imply the full set of equations of motion.

In the following, we will discuss first an intuitive way of understanding this smearing. Then we will apply this to different examples in Sections 2–6. Finally, in Section 7, we will present a formal way of implementing the backreaction from smeared sources.

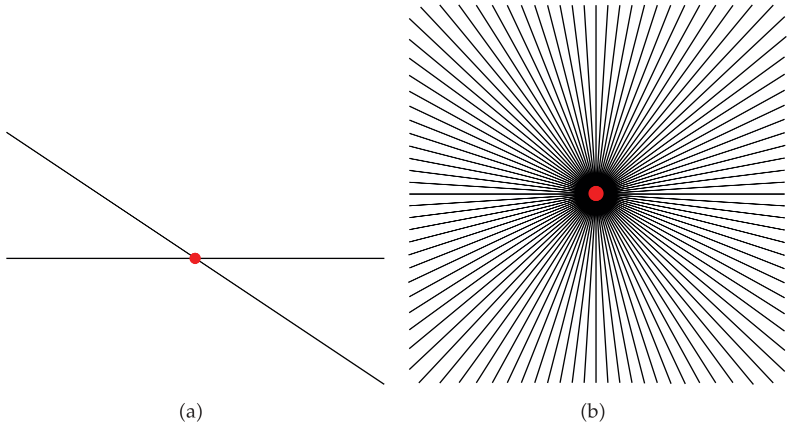

Figure 2: (a) A point-like charge (in red) and two lines of charge at different angles. (b) A configuration with many lines of charge. In the asymptotic limit of an infinite number of lines, they just correspond to a radial charge density. This picture depicts an analogous situation to the case of smeared flavored branes, when the fundamental fields are massless.

1.5. A Heuristic Viewpoint

In the following sections, we will introduce the necessary mathematical machinery to consistently compute solutions of string theory in which smeared backreacting flavor branes are present. Before that, it is worth to make a digression in order to explain the general set-up in simple, heuristic terms.

Let us make an analogy with electrostatics. Suppose that we want to compute the electric field generated by a point-like charge and a couple of lines of charge, as depicted on Figure 2(a). In order to depict the situation, we show dimension 1 lines of charge in a total space of dimension 2, but clearly the situation can be generalized by changing such dimensions. Since in the left plot there is no particular symmetry in the configuration, the resulting electric field will have a not so simple expression. But let us imagine that we consider a huge number of lines of charge as in the plot of the right and homogeneously distribute them in the angle they form with the horizontal axis. In the limit of many lines, radial symmetry is recovered, and the charge density is “smeared” and will be just given by a single (monotonically decreasing) function $\rho(r)$. The electric field, accordingly, will also be radially symmetric. Notice that this process of describing a large number of discrete objects by a continuous distribution is ubiquitous in physics: for instance, a “homogeneous” gas is a collection of atoms or the “homogeneous” Universe considered in cosmological models contains a collection of galaxy clusters. Also, solutions with different kinds of smeared sources have been considered many times in string theory contexts not necessarily related to gauge-gravity duality; see, for instance, [32, 33].

When comparing to the string theory set-up, the point-like charge in the center corresponds to the color branes and the lines to the flavor branes (which have to extend to infinity). The limiting radially symmetric configuration corresponds to the kind of smeared situations that we will discuss in this review. (More precisely, it corresponds to the situations analysed in Sections 2–5. For the cohomogeneity 2 cases analysed in Section 6, the different functions depend on two radial variables. A heuristic picture for such situations is presented in Section 6.) All functions of the ansatz can then be considered to depend on a single radial

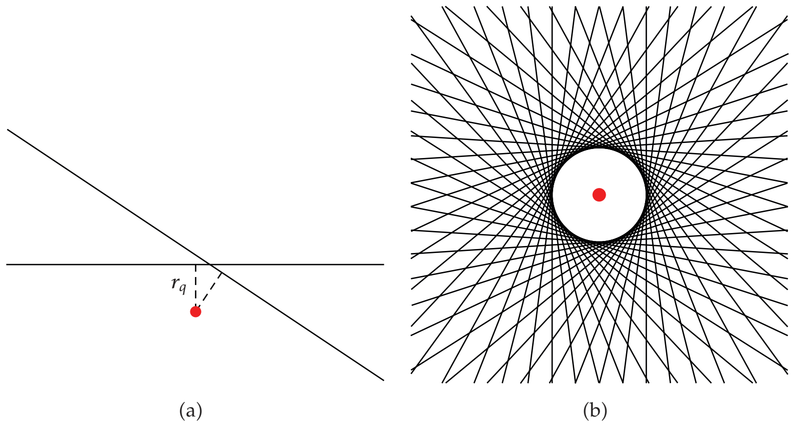

Figure 3: These pictures depict analogous situations to the case of flavored branes, when the fundamental fields are massive. (b) Again, we add a large number of lines, such that in the limit radial symmetry is recovered.

variable. For flavor branes, the different "angles" correspond to adding fundamental matter which couples differently to the rest of the fields. In some of the cases discussed in the following, we will see how this is reflected in the field theory superpotential (Section 2.3.1).

We can still get further intuition from this simple analogy. In Figure 2(b), we see that all lines intersect at the center. From the string point of view, that means that the flavor branes are stretched down to the bottom of the geometry and the quarks are massless. In this situation, the charge density $\rho(r)$ is highly peaked at $r = 0$. Essentially, that is the reason why for the solutions with massless quarks described in the following sections there is a curvature singularity at the origin, where all flavor branes meet.

Then, a simple way of getting rid of such a singularity is to displace the lines of charge from the origin, while still keeping the radial symmetry. This is depicted in Figure 3. If we dub the distance from any of the lines to the center as r_q, the density of charge $\rho(r)$ will vanish for $0 < r < r_q$, while it will asymptote to the "massless" $r_q = 0$ one as $r \gg r_q$. From the brane construction, this displacement typically corresponds to giving a mass to the fundamentals (or, in particular cases, it could correspond to a nontrivial vacuum expectation value). The solution of Section 2.3.2, which indeed is regular in the IR, is a neat example of this notion. Another possibility is to add temperature and to hide the singularity behind a horizon; see Section 2.5.

Going back to electrostatics for Figure 3(b), we know from Gauss' law that the charge density outside does not affect the central region. The corresponding statement in the field theory is that the massive fields decouple from the IR physics below the scale given by their mass. We find it interesting that, through this heuristic reasoning, Gauss' law is connected to the decoupling of heavy particles (or holomorphic decoupling in the SUSY cases).

Even if the example of electrostatics is useful to qualitatively picture what we will do in the following, the analogy is by no means perfect. We note two differences: first, we will be working with gravity, which is nonlinear and, thus, one cannot find the final solution by superposing the fields generated by different sources (which in the case of electrostatics would make it rather trivial to find the electric field for the configurations depicted on Figures 2(a) and 3(a)). Second, our "lines of charge" (the flavor branes) are dynamical. This means that is it not enough to compute the background fields generated from the sources but one also has to check that the sources are stably embedded in the geometry.

We end this section by summarizing the pros and cons of looking for duals of unquenched theories for which the string solutions include smeared flavor branes, many of which can be inferred from the heuristic discussion above. On the positive side, one has the following.

(i) The smearing simplifies the situation allowing us to write ansätze depending on a single radial coordinate, and therefore the problem is eventually reduced to a set of ODEs. (For the cases of Section 6, they depend on two radial coordinates and thus one finds PDEs in terms of two independent variables, but again, without delta-function localized sources.)

(ii) Possible issues related to singularities and strong coupling are ameliorated in the same sense as they are washed out in electrostatics when considering a smooth charge density rather than a sum of delta-functions over a large number of electrons.

(iii) It allows a simple application of the powerful mathematical tools of calibrated geometry [34]; see Section 7.

On the negative side, one has the following.

(i) Obviously, if we require the flavor branes to be smeared, we are limiting ourselves to considering a very particular subset of all the possible flavored theories. In particular, we require the superpotentials to effectively recover (some of) the global symmetries of the theory without flavors.

(ii) Related to the previous point, one cannot realize, in general, theories with $U(N_f)$ flavor groups. Since the flavor branes are required to sit at different points in the internal space, the typical string connecting different flavor branes is heavy and the flavor symmetry is typically broken to $U(1)^{N_f}$ (one may also interpret the solutions as having flavor symmetry $U(k)^{N_f/k}$ for some $k \ll N_f$). From the point of view of the field theory, this amounts to having a Veneziano expansion with "one window graphs", as pointed out in [26]. In principle, this fact can hinder the realization of some interesting physical features in the dual set-ups considered.

1.6. Localized Sources and Other Approaches

As already remarked, this review focuses on solutions of string theory for which there are D-brane sources homogeneously smeared over a given family of possible embeddings and that can be interpreted as duals of strongly coupled gauge theories in the Veneziano limit. As stressed above, this is a very particular subset of all the possible duals of theories with flavor. In a generic case, one should consider the sources to be localized at certain positions, such that the density of charge is given by a sum over Dirac delta functions. Such generic case is technically more challenging. However, remarkable works along these lines have appeared, pursuing solutions with the flavor branes localized at a single point of space (notice this is not the most general case either). We will not review them in any detail here, but the goal of this section is to provide a brief guide to the literature on the subject.

The main ingredient of this approach consists of finding solutions of supergravity which can be interpreted as intersections of branes of different dimension, with each stack of branes localized at a fixed position of space-time. In the context of gauge-gravity duality, the search for such solutions was initiated in [35, 36]. These papers discussed D3-D7 intersections, which have been the most studied in the literature (see below for different set-ups). A lot of progress was reported in subsequent years [37–39]. Among other aspects, these papers presented a better understanding of the D3-D7 solutions, the inclusion of fractional branes, and clear matching with field theory issues such as the running of couplings and anomalies. Further work on the D3-D7 localized system was performed in [40] (where the conifold was also addressed) [41] (where D7 brane backreaction on bubbling geometries was considered), and [42], where the solution corresponding to D3-D7 in flat space was completed by providing an expression for the warp factor in closed form. It is also worth mentioning [43] where a flavor D7-brane in a cascading theory was considered and its backreaction introduced as a perturbation. The finite temperature generalization of the background of [43] was discussed in [44, 45].

Let us now outline the literature on D2-D6 localized intersections, which can be interpreted as duals of 2+1 supersymmetric gauge theories coupled to fundamentals introduced by the D6-branes. The construction of the type IIA solutions (and their relation to M-theory) was carried out in [24, 46, 47]. In [48], meson excitations of this background were discussed and, in particular, the holographic dictionary relating meson-like operators to certain (closed string) supergravity modes was presented. On the other hand, the authors of [49] found a finite temperature version of the solution, which was used to discuss the thermodynamics of the system. Very recent progress in the D2-D6 systems, their M-theory uplifts, and the detailed relation to Chern-Simons theories with flavor has been reported in [50, 51].

Regarding D4-D8 intersections, localized solutions in that set-up were constructed in [52] in an early attempt to build a QCD dual. In the context of the Sakai-Sugimoto model [20], backreaction from localized D8-$\overline{\text{D8}}$ branes was analysed in [53].

It is also worth mentioning recent solutions in heterotic string theory which were argued to be related to flavored theories [54].

Interestingly, there are a few papers in which similar situations were considered in subcritical string theory and therefore defined in dimensions lower than ten. In many of these cases, each flavor brane fills the whole space-time (therefore they are not localized, neither smeared). Some physics can then be extracted by using exact string theory methods but what these models have in common is that it is not possible to handle them within a well-controlled gravity description: gravity-like actions with just two derivatives suffer curvature corrections which cannot be neglected, nor consistently computed. However, there is the hope that the two-derivative actions can nevertheless provide additional nontrivial insights in the physics of the system. This idea was put forward by Klebanov and Maldacena in [55], who considered a D3-D5 system in a six-dimensional background (the cigar). Such a system is dual to 4D $\mathcal{N} = 1$ SQCD as was shown using exact worldsheet methods in [56, 57]. For a recent discussion on the dual to the flavor singlet sector of $\mathcal{N} = 2$ superconformal QCD in a subcritical string framework, see [58]. The set-up of [55] was generalized to different situations in [11, 59, 60]. The finite temperature physics of a model in [11] was analysed in [61]. Bottom-up approaches (in which a high-dimensional gravity theory is proposed to describe some specific features of QCD) with space-time filling flavor branes have been discussed in [62, 63]. Recently, a bottom-up approach to the conformal window along these lines has appeared [64].

Finally, let us mention a recent contribution by Armoni [65], in which a way of departing from the quenched approximation was proposed. The fermion determinant is expanded in terms of Wilson loops. It then turns out that a sum of correlators of an observable with the Wilson loops boils down to an expansion in N_f/N_c, which can in principle be computed. It would be nice to further develop possible implications of this observation in holographic set-ups.

2. Flavor Deformations of AdS$_5 \times X^5$

Our first concrete application of the procedure described will be the flavor deformation of AdS$_5 \times S^5$. This is the simplest possible case and, hopefully, it will neatly illustrate the comments of Section 1.2. In fact, for most of the discussion, the formalism applies to any AdS$_5 \times X^5$ geometry, X^5 being a five-dimensional compact Sasaki-Einstein (SE) space so we will refer to this more general case during this whole section. At some points, we will use the two notable examples $X^5 = S^5$ or $T_{1,1}$ to clarify particular issues.

Let us start with a general comment. Since the AdS$_5 \times S^5$ theories without flavor are conformal, we expect that once we include extra matter, a positive beta function is generated. This is in fact the case and leads to the appearance of a Landau pole. Nevertheless, as in QED, the theory renders meaningful IR physics even if the UV is ill-defined as long as the IR and UV are well-separated scales. However, this separation does not allow to have N_f and N_c of the same order. As we will see, one can define a parameter $\epsilon \sim \lambda N_f/N_c$ which weighs the internal flavor loops and that has to be kept small. The effect of the unquenched quarks can then be computed as an expansion in ϵ.

After introducing the framework in Sections 2.1 and 2.2, we present the unquenched supersymmetric ($\mathcal{N} = 1$ in 4d) solutions in Section 2.3. In Section 2.4, we present an instance of the effects of the unquenched flavors on a physical quantity, namely, on the mass of a particular meson tower. Then, in Section 2.5, we break supersymmetry by turning on temperature and analyse the physics of the dual quark-gluon plasma. We end in Section 2.6 by discussing the range of validity for the solutions and approximations used.

2.1. The Geometries and Field Theories without Flavors

The models we discuss here are obtained by placing a stack of N_c D3-branes at the origin of the six-dimensional cone over X^5. The corresponding type IIB background reads

$$\begin{aligned} ds^2 &= [h(r)]^{-1/2} dx_{1,3}^2 + [h(r)]^{1/2}\left[dr^2 + r^2 ds_{X^5}^2\right], \\ F_5 &= dh^{-1} dx^0 \wedge \cdots \wedge dx^3 + \text{Hodge dual}, \\ h(r) &= \frac{Q_c}{4r^4}, \qquad Q_c \equiv \frac{(2\pi)^4 g_s \alpha'^2 N_c}{\mathrm{Vol}(X^5)}, \end{aligned} \tag{2.1}$$

where we have taken the near horizon limit. The dilaton is constant and all the other fields of type IIB supergravity vanish. In general the metric of the SE space X^5 can be written as a Hopf fibration over a four-dimensional Kähler-Einstein (KE) manifold:

$$ds_{X^5}^2 = ds_{\mathrm{KE}}^2 + (d\tau + A_{\mathrm{KE}})^2, \tag{2.2}$$

where τ is the fiber and A_{KE} is the connection one-form whose exterior derivative gives the Kähler form J_{KE} of the KE base:

$$dA_{\mathrm{KE}} = 2J_{\mathrm{KE}}. \tag{2.3}$$

Let us first consider the particular case in which X^5 is the five-sphere S^5. In this case the KE base is the manifold CP^2 (with the Fubini-Study metric) and the space transverse to the color branes, with metric $dr^2 + r^2 ds^2_{S^5}$, is just $\mathbb{R}^6$. When $X^5 = S^5$, the coefficient Q_c appearing in (2.1) is just $Q_c = 16\pi g_s \alpha'^2 N_c$. Moreover, as is well known, the field theory dual to the $\mathrm{AdS}_5 \times S^5$ background is $\mathcal{N} = 4$ SYM in 4d, which, in $\mathcal{N} = 1$ language, can be written in terms of a vector multiplet and of three chiral superfields Φ_i $(i = 1,2,3)$ transforming in the adjoint representation of the gauge group and interacting by means of the cubic superpotential:

$$W_{\mathcal{N}=4} = \mathrm{Tr}[\Phi_1[\Phi_2, \Phi_3]]. \tag{2.4}$$

If we represent the transverse $\mathbb{R}^6$ of the $\mathrm{AdS}_5 \times S^5$ solution in terms of three complex variables Z_i $(i = 1,2,3)$, one can regard the Z_i's as the geometric realization of the adjoint superfields Φ_i.

The second prominent example which we will analyze in detail is the one in which X^5 is the $T^{1,1}$ space with metric:

$$ds^2_{T^{1,1}} = \frac{1}{6}\sum_{i=1}^{2}\left[d\theta_i^2 + \sin^2\theta_i d\varphi_i^2\right] + \frac{1}{9}\left[d\psi + \sum_{i=1}^{2}\cos\theta_i d\varphi_i\right]^2, \tag{2.5}$$

where the range of the angles is $\psi \in [0,4\pi)$, $\varphi_i \in [0,2\pi)$, $\theta_i \in [0,\pi]$. Since $\mathrm{Vol}(T^{1,1}) = 16\pi^3/27$, the coefficient Q_c for this solution is $Q_c = 27\pi g_s \alpha'^2 N_c$. In this case the space transverse to the color branes is the conifold, which is a 6d Calabi-Yau cone which can also be described as the locus of the solutions of the algebraic equation:

$$z_1 z_2 = z_3 z_4, \tag{2.6}$$

where the z_i are four complex coordinates. The relation between these variables and the coordinates used in (2.5) is the following:

$$\begin{aligned}
z_1 &= r^{3/2} e^{(i/2)(\psi-\varphi_1-\varphi_2)} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \qquad & z_2 &= r^{3/2} e^{(i/2)(\psi+\varphi_1+\varphi_2)} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}, \\
z_3 &= r^{3/2} e^{(i/2)(\psi+\varphi_1-\varphi_2)} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \qquad & z_4 &= r^{3/2} e^{(i/2)(\psi-\varphi_1+\varphi_2)} \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}.
\end{aligned} \tag{2.7}$$

Notice also that the metric written in (2.5) is of the form (2.2) where the KE base is just the $S^2 \times S^2$ space parameterized by the angles (θ_i, φ_i) and one should make the following identifications:

$$\begin{aligned} \tau &= \frac{\psi}{3}, \qquad A_{T^{1,1}} = \frac{1}{3}\left(\cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2\right), \\ J_{T^{1,1}} &= \frac{dA_{T^{1,1}}}{2} = -\frac{1}{6}\left(\sin\theta_1 d\theta_1 \wedge d\varphi_1 + \sin\theta_2 d\theta_2 \wedge d\varphi_2\right). \end{aligned} \tag{2.8}$$

The field theory dual to the $\mathrm{AdS}_5 \times T^{1,1}$ background is the $\mathcal{N} = 1$ superconformal quiver gauge theory with gauge group $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c)$ and bifundamental matter fields A_1, A_2 and B_1, B_2 transforming, respectively, in the $(N_c, \overline{N}_c)$ and in the $(\overline{N}_c, N_c)$ representations of the gauge group [66], that is, the so-called Klebanov-Witten (KW) model. The matter fields form two SU(2) doublets and interact through a quartic superpotential:

$$W_{\mathrm{KW}} = \widehat{h}\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\left[A_i B_k A_j B_l\right]. \tag{2.9}$$

For a single brane the fields A_i and B_i can be related to the coordinates z_i by means of the following relations:

$$z_1 = A_1 B_1, \qquad z_2 = A_2 B_2, \qquad z_3 = A_1 B_2, \qquad z_4 = A_2 B_1, \tag{2.10}$$

which automatically solve the defining conifold equation (2.6).

2.2. Flavor Branes and Smeared Charge Distribution

The flavor branes for the $\mathrm{AdS}_5 \times X^5$ backgrounds just described are D7-branes extended along the four Minkowski directions as well as along a noncompact submanifold of the cone over X^5. The type of flavor that the D7-branes add depends both on the space X^5 and on the submanifold they wrap in the transverse space. We first illustrate the situation with the two examples of $X^5 = S^5$ and $T_{1,1}$ and at the end display the general expressions.

The first instance is the case in which $X^5 = S^5$. In this case a simple kappa symmetry analysis shows that, in order to preserve eight supersymmetries, the D7-branes must be extended along a codimension two hyperplane in $\mathbb{R}^6$ which, in terms of the complex coordinates Z^i, can be written as

$$a_1 Z^1 + a_2 Z^2 + a_3 Z^3 = \mu, \tag{2.11}$$

with the a_i and μ being complex constants satisfying $\sum_1^3 |a_i|^2 = 1$. On the field theory side these flavor branes introduce $\mathcal{N} = 2$ fundamental hypermultiplets $(Q^r, \tilde{Q}_r)$ $(r = 1, \ldots, N_f)$—nonetheless, a generic collection of branes within the family (2.11) retains just $\mathcal{N} = 1$ susy.

The corresponding superpotential for an embedding such as the one in (2.11) can be written as

$$W = W_{\mathcal{N}=4} + \tilde{Q}_r \left[\sum_j a_j \Phi_j + m \right] Q^r, \tag{2.12}$$

where the mass m is related to the constant μ in (2.11). Notice that since the embeddings are holomorphic, it is not possible to smear them in a way in which the full SO(6) isometry is realised. After smearing over the embeddings (2.11), one can recover, at most, $\mathrm{SU}(3) \times U(1)$, as will be seen directly from the dual solution.

In the case of the $\mathrm{AdS}_5 \times T^{1,1}$ background there are two classes of holomorphic embeddings which correspond to different types of flavors in the KW theory. In terms of the z_i coordinates of (2.7) the representative embedding of the first class is given by the equation $z_1 = \mu$. This is the so-called Ouyang embedding [43], which has two branches in the massless limit $\mu = 0$. In each of these branches the D7-brane adds fundamental matter to one of the two nodes of the KW quiver and antifundamental matter to the second. The corresponding superpotential contains cubic couplings between the quark fields q_i and $\tilde{q}_i$ $(i = 1, 2)$ and the bifundamental fields A_i and B_i. For example, for the massless embedding $z_1 = 0$ the superpotential (2.9) is modified as

$$W_{z_1=0} = W_{\mathrm{KW}} + h_1 \tilde{q}_1 A_1 q_2 + h_2 \tilde{q}_2 B_1 q_1, \tag{2.13}$$

where, here and in the following, traces over color indices and sums over the N_f flavor indices are implied. The second class of D7-brane embeddings is the one giving rise to nonchiral flavors, whose representative element is given by the equation $z_1 - z_2 = \mu$. In this case every D7-brane adds fundamental and antifundamental flavor to one node of the KW quiver and the flavor mass terms do not break the classical symmetry of the massless theory. The corresponding superpotential contains only mass terms and quartic couplings, namely,

$$W = W_{\mathrm{KW}} + \hat{h}_1 \tilde{q}_1 [A_1 B_1 - A_2 B_2] q_1 + \hat{h}_2 \tilde{q}_2 [B_1 A_1 - B_2 A_2] q_2 + k_i (\tilde{q}_i q_i)^2 + m(\tilde{q}_i q_i). \tag{2.14}$$

In order to develop our program and construct backreacted gravity solutions for smeared distributions of flavor branes following [67], we should be able to find a family of equivalent embeddings for each type of configuration described above. In the case of the $\mathrm{AdS}_5 \times S^5$ background (2.11) provides such a family. Notice that, even if each individual embedding of the form (2.11) preserves $\mathcal{N} = 2$, which supersymmetries are preserved depends on the a_i's. Nevertheless, one can check that all the holomorphic embeddings of the type (2.11) are mutually supersymmetric and, due to the holomorphic nature of the linear equation (2.11), they preserve the same common four supersymmetries ($\mathcal{N} = 1$) for all values of the constants a_i. Thus, we can use these constants to parameterize the family of different planes that constitute our continuous distribution of flavor branes.

In the case of the $AdS_5 \times T^{1,1}$ background one can generalize the chiral embedding $z_1 = \mu$ by acting with the $SU(2) \times SU(2)$ symmetry of the conifold. The corresponding family of embeddings takes the following form:

$$\sum_{i=1}^{4} \alpha_i z_i = \mu, \tag{2.15}$$

with the complex constants α_i spanning a conifold (up to overall complex rescalings):

$$\alpha_1 \alpha_2 - \alpha_3 \alpha_4 = 0. \tag{2.16}$$

Notice that embeddings like $z_1 - z_2 = \mu$ are not in this family. Indeed, the nonchiral embeddings $z_1 - z_2 = \mu$ can be generalized as

$$\overline{p} z_1 - p z_2 + \overline{q} z_3 + q z_4 = \mu, \tag{2.17}$$

where p, q span a unit 3-sphere; that is, they satisfy $|p|^2 + |q|^2 = 1$.

In spite of the differences among the cases presented above, the charge distribution generated by these families of embeddings can be written in a common form. The reason for this universality is the underlying Sasaki-Einstein structure. In order to illustrate this fact, let us consider the chiral embeddings (2.15) in the case in which the mass parameter μ is zero. Without loss of generality we can rescale the α_i coefficients and fix $\alpha_1 = 1$. Then, (2.16) fixes $\alpha_2 = \alpha_3 \alpha_4$ and, after using (2.7), the massless embedding equation

$$z_1 + \alpha_3 \alpha_4 z_2 + \alpha_3 z_3 + \alpha_4 z_4 = 0 \tag{2.18}$$

nicely factorizes as

$$\left(\sin \frac{\theta_1}{2} + \alpha_3 e^{i\varphi_1} \cos \frac{\theta_1}{2} \right) \left(\sin \frac{\theta_2}{2} + \alpha_4 e^{i\varphi_2} \cos \frac{\theta_2}{2} \right) = 0. \tag{2.19}$$

Notice that the vanishing of each of the factors in (2.19) determines a branch in which the branes sit at a fixed point of one of the two two-spheres parameterized by the angles (θ_i, φ_i). The constants α_3 and α_4 determine the particular point at which each brane is sitting in each S^2. Indeed, if ξ_1^α and ξ_2^α are systems of worldvolume coordinates for the D7-branes, these two branches can be written as

$$\begin{aligned} \xi_1^\alpha &= \left\{ x^0, x^1, x^2, x^3, r, \theta_2, \varphi_2, \psi \right\}, \qquad \theta_1 = \theta_1^* = \text{const.}, \qquad \varphi_1 = \varphi_1^* = \text{const.}, \\ \xi_2^\alpha &= \left\{ x^0, x^1, x^2, x^3, r, \theta_1, \varphi_1, \psi \right\}, \qquad \theta_2 = \theta_2^* = \text{const.}, \qquad \varphi_2 = \varphi_2^* = \text{const.} \end{aligned} \tag{2.20}$$

In Figure 4 we have represented the two branches for the embedding (2.18). From the field theory side, which particular embedding we choose determines the coupling between the associated quarks and the bifundamentals. Roughly speaking, the contribution to

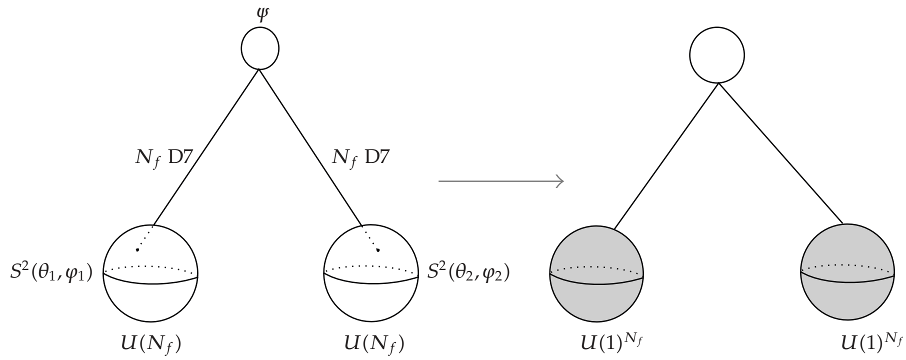

Figure 4: We see on the left side the two stacks of N_f flavor-branes localized on each of their respective S^2's (they wrap the other S^2). The flavor group is clearly $U(N_f) \times U(N_f)$. After the smearing on the right side of the figure, this global symmetry is broken to $U(1)^{N_f-1} \times U(1)^{N_f-1} \times U(1)_B \times U(1)_A$.

the superpotential of an embedding determined by some α_3, α_4 is $h_1\tilde{q}_1(A_1+\alpha_4 A_2)q_2+h_2\tilde{q}_2(B_1+\alpha_3 B_2)q_1$. Thus, when we *smear* and sum over all the possible α_3 and α_4, both SU(2)'s (the one rotating the A_i's and the one rotating the B_i's) are effectively recovered. Figure 4 is the geometric interpretation of this effect.

It is straightforward to compute the charge density produced by this localized D7-brane configuration. Indeed, taking into account the contribution of the two branches, one gets

$$\Omega^{\text{loc}} = \delta^{(2)}(\theta_1 - \theta_1^*, \varphi_1 - \varphi_1^*) d\theta_1 \wedge d\varphi_1 + \delta^{(2)}(\theta_2 - \theta_2^*, \varphi_2 - \varphi_2^*) d\theta_2 \wedge d\varphi_2. \tag{2.21}$$

To produce a homogeneous configuration of N_f D7-branes we should distribute in every branch the branes homogeneously along their transverse two-sphere. In the continuum limit $N_f \to \infty$ this procedure amounts to performing an integration over each S^2 with the corresponding volume element, namely,

$$\begin{aligned}\Omega &= \left[\int\!\!\int \frac{N_f}{4\pi} \sin\theta_1^* \delta^{(2)}(\theta_1 - \theta_1^*, \varphi_1 - \varphi_1^*) d\theta_1^* d\varphi_1^*\right] d\theta_1 \wedge d\varphi_1 \\ &\quad + \left[\int\!\!\int \frac{N_f}{4\pi} \sin\theta_2^* \delta^{(2)}(\theta_2 - \theta_2^*, \varphi_2 - \varphi_2^*) d\theta_2^* d\varphi_2^*\right] d\theta_2 \wedge d\varphi_2.\end{aligned} \tag{2.22}$$

The integrations over θ_i^* and φ_i^* in (2.22) can be immediately performed, yielding the following expression for the smeared charge distribution of D7-branes:

$$\Omega = \frac{N_f}{4\pi}(\sin\theta_1 d\theta_1 \wedge d\varphi_1 + \sin\theta_2 d\theta_2 \wedge d\varphi_2). \tag{2.23}$$

Notice that in (2.22) we have included the normalization factor $N_f/4\pi$ in such a way that the resulting distribution densities $\sin\theta_i^* \; N_f/4\pi$ are normalized to N_f when they are integrated over S^2. Notice that, as already pointed out above, the flavor symmetry of the smeared configuration is $U(1)^{N_f}$ rather than $U(N_f)$, since the branes are not placed on top of each other. Interestingly, a similar calculation for the embeddings (2.17) in the massless case $\mu = 0$ gives rise to the same charge density for the smeared configuration as in (2.23) [68, 69]. This is because the form of Ω in (2.23) is determined by the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times Z_2$ global symmetry which we want to recover after smearing.

Actually, one can rewrite (2.23) in a form which can be easily generalized to any continuous family of equivalent D7-brane massless embeddings in an arbitrary Sasaki-Einstein manifold. Indeed, by using (2.8) one can rewrite the right-hand side of (2.23) in terms of the Kähler form of $T^{1,1}$ as

$$\Omega = -\frac{3N_f}{2\pi} J_{T^{1,1}}. \tag{2.24}$$

For an arbitrary Sasaki-Einstein space X^5, the expression (2.24) generalizes to

$$g_s \Omega = -2Q_f J_{\mathrm{KE}}, \tag{2.25}$$

where Q_f is the following constant coefficient

$$Q_f = \frac{\mathrm{Vol}(X^3) g_s N_f}{4\mathrm{Vol}(X^5)}. \tag{2.26}$$

In (2.26) X^3 is the compact submanifold of X^5 wrapped by the D7-brane in a massless embedding. Notice that in this case the D7-brane worldvolume along the space transverse to the color branes is always of the form $I \times X^3$, where I is a noncompact interval along the holographic radial direction. It is worth noticing that the factor $\mathrm{Vol}(X^5)/\mathrm{Vol}(X^3)$ appearing on the right-hand side of (2.26) is just the volume transverse to any individual flavor brane, over which we are distributing the D7-branes. For the massless chiral embeddings in the conifold one can readily check, after taking into account the contribution of both branches in (2.20), that $\mathrm{Vol}(X^3) = 16\pi^2/9$. Since $\mathrm{Vol}(T^{1,1}) = 16\pi^3/27$, one can easily prove that (2.25) reduces to (2.24). In the case $X^5 = S^5$ the three-manifold X^3 is just a unit S^3 and $\mathrm{Vol}(X^3) = 2\pi^2$. Therefore, we obtain the following values of Q_f for $X^5 = S^5, T^{1,1}$:

$$Q_f = \begin{cases} \dfrac{g_s N_f}{2\pi} & \text{for } X^5 = S^5, \\[2ex] \dfrac{3 g_s N_f}{4\pi} & \text{for } X^5 = T^{1,1}. \end{cases} \tag{2.27}$$

The charge density Ω determines the ansatz of F_1 in the backreacted geometry. Indeed, the WZ part of the D7-brane action contains a term in which the RR eight-form potential C_8 is coupled to the D7-brane worldvolume. The continuous limit for this term amounts to performing the following substitution:

$$S_{\mathrm{WZ}} = T_7 \sum_{N_f} \int_{\mathcal{M}_8} \hat{C}_8 \longrightarrow T_7 \int_{\mathcal{M}_{10}} \Omega \wedge C_8, \tag{2.28}$$

which leads to the following violation of the Bianchi identity for F_1:

$$dF_1 = -2\kappa_{(10)}^2 T_7 \Omega = -g_s \Omega. \tag{2.29}$$

Taking into account the general expression of Ω for a massless embedding written in (2.25) as well as the relation (2.3) between the one-form A_{KE} and the Sasaki-Einstein Kähler form J_{KE}, one is led to adopt [67] the following ansatz for F_1:

$$F_1 = Q_f (d\tau + A_{\text{KE}}). \tag{2.30}$$

A simple modification of (2.30) for F_1 allows us to extend the ansatz to the case in which the quarks are massive [67]. This modification corresponds to introducing a function $p(\rho)$ of the holographic coordinate ρ and performing the substitution $Q_f \to Q_f p(\rho)$ in (2.30). The radial coordinate ρ will be conveniently chosen and, in general, will be different from the one we used so far. The function $p(\rho)$ encodes the effects of the nontrivial profile of the D7-branes. Indeed, when the quarks are massive, the brane does not extend along the full range of the radial coordinate ρ and, accordingly, $p(\rho)$ must vanish for $\rho < \rho_q$, where $\rho = \rho_q$ is the radial location of the tip of the D7-brane. Moreover, the function $p(\rho)$ should approach the value $p = 1$ when $\rho \gg \rho_q$ since in this region the quarks are effectively massless. The form of the function $p(\rho)$ is not universal and depends on the particular embedding of the D7-brane. For the three embeddings in the cases $X^5 = S^5$ and $T^{1,1}$ discussed above, the expressions for $p(\rho)$ are given below. At this point let us simply notice that the charge density Ω is modified with respect to the massless case as

$$g_s \Omega = -2p(\rho) Q_f J_{\text{KE}} - Q_f \dot{p}(\rho) d\rho \wedge (d\tau + A_{\text{KE}}), \tag{2.31}$$

where the dot denotes derivative with respect to the radial variable ρ.

2.3. Backreacted Ansatz and Solution

Let us now write an ansatz for the backreacted D3-D7 background for a generic Sasaki-Einstein space X^5 [67]. It is clear from the discussion of the previous subsection that, after performing the smearing, the resulting RR one-form F_1 introduces a distinction between the directions of the $U(1)$ fiber and of the KE base of X^5. Therefore, it seems clear that the effect of the smeared flavor branes on the metric should be an internal deformation of the X^5 in the form of a relative squashing between the KE space and the Hopf fiber. (Just in the case when X^5 is the sphere S^5, this squashing breaks part of the isometry $\text{SO}(6) \to \text{SU}(3) \times U(1)$, where SU(3) is the isometry of the Kähler-Einstein base CP^2.) Accordingly, let us adopt the following ansatz for the metric in Einstein frame:

$$ds^2 = [h(\rho)]^{-1/2} dx_{1,3}^2 + [h(\rho)]^{1/2} \left[e^{2f(\rho)} d\rho^2 + e^{2g(\rho)} ds_{\text{KE}}^2 + e^{2f(\rho)} (d\tau + A_{\text{KE}})^2 \right], \tag{2.32}$$

where $g(\rho)$ and $f(\rho)$ are the functions that implement the squashing mentioned above and the function multiplying $d\rho^2$ amounts to choosing a particular radial variable ρ which is convenient for our purposes. Moreover, the dilaton will depend on ρ and the RR forms F_5 and F_1 have the following form:

$$\phi = \phi(\rho), \qquad F_5 = Q_c(1+*)\varepsilon\left(X^5\right), \qquad F_1 = Q_f p(\rho)(d\tau + A_{\mathrm{KE}}), \tag{2.33}$$

where $\varepsilon(X^5)$ is the volume element of X^5 and Q_c and Q_f are written in (2.1) and (2.26), respectively. The function $p(\rho)$, whose form depends on the D7-brane embedding, takes into account the effects of massive quarks, as explained above.

Given the ansatz (2.32)-(2.33) one can easily study the supersymmetric variations of the dilatino and gravitino in type IIB supergravity and find the corresponding first-order BPS equations, which ensure the preservation of four supersymmetries. The resulting equations are [67]

$$\begin{gathered}\partial_\rho g = e^{2f-2g}, \qquad \partial_\rho f = 3 - 2e^{2f-2g} - \frac{Q_f}{2}p(\rho)e^{\phi}, \\ \partial_\rho \phi = Q_f p(\rho) e^{\phi}, \qquad \partial_\rho h = -Q_c e^{-4g}.\end{gathered} \tag{2.34}$$

Remarkably, the system (2.34) can be integrated analytically for any function $p(\rho)$. In order to present this solution, let us define the function $\eta(\rho)$ as follows:

$$\eta(\rho) = Q_f e^{\phi} \int_{\rho_q}^{\rho} e^{6\xi} p(\xi) d\xi, \tag{2.35}$$

where ρ_q is the value of the radial coordinate at the tip of the flavor brane ($p(\rho < \rho_q) = 0$). Then, we can write down quite simple expressions for f, g, ϕ, namely,

$$\begin{aligned} e^{-\phi} &= e^{-\phi_*} - Q_f \int_{\rho_*}^{\rho} p(\xi) d\xi, \\ e^{g} &= c_2 e^{\rho} e^{-\phi/6}\left(1 + e^{-6\rho}\left(c_1 e^{\phi} + \eta\right)\right)^{1/6}, \\ e^{f} &= c_2 e^{\rho} e^{-\phi/6}\left(1 + e^{-6\rho}\left(c_1 e^{\phi} + \eta\right)\right)^{-1/3}, \end{aligned} \tag{2.36}$$

where we have introduced a reference scale ρ_* and we have defined $\phi_* = \phi(\rho = \rho_*)$. Notice that the warp factor h can be obtained as the integral of e^{-4g} as follows from the last equation in the BPS system (2.34). In (2.36) c_1 and c_2 are integration constants that we now fix. First, if we demand IR regularity of the solution, we need $g = f$ when $\rho \leq \rho_q$. Since η vanishes at $\rho = \rho_q$, we need $c_1 = 0$. Moreover, the constant c_2 is just some overall scale and has no physical meaning. It is natural to fix it to $\alpha'^{1/2}e^{\phi_*/6}$ in order to give appropriate dimensions and to recover the usual expression for the metric when $Q_f = 0$. Therefore, we find

$$\begin{aligned} e^{\phi-\phi_*} &= \frac{1}{1 - e^{\phi_*}Q_f \int_{\rho_*}^{\rho} p(\xi)d\xi}, \\ e^{g} &= \sqrt{\alpha'}e^{\rho}e^{-(\phi-\phi_*)/6}\left(1 + e^{-6\rho}\eta\right)^{1/6}, \\ e^{f} &= \sqrt{\alpha'}e^{\rho}e^{-(\phi-\phi_*)/6}\left(1 + e^{-6\rho}\eta\right)^{-1/3}. \end{aligned} \tag{2.37}$$

Notice that when $Q_f = 0$, we recover the unflavored $\mathrm{AdS}_5 \times X^5$ background. Indeed, in this case $\phi = \phi_*$ and $\eta = 0$ and, after performing the change of the radial variable $r = \sqrt{\alpha'}e^{\rho}$, we get that $e^{g} = e^{f} = r$ and the background (2.32)-(2.33) coincides with the one written in (2.1).

Let us now introduce the following parameter:

$$\epsilon_* \equiv Q_f e^{\phi_*}, \tag{2.38}$$

which, as we will see in a while, controls the effects of quark loops in the backreacted supergravity solution. Indeed, the gauge/gravity dictionary for the type of theories we are studying relates the exponential of the dilaton to the Yang-Mills coupling constant. For example, for the (flavored) $\mathcal{N} = 4$ $\mathrm{SU}(N_c)$ theory, dual to the deformed $\mathrm{AdS}_5 \times S^5$ background, the gauge coupling is $g_{\mathrm{YM}}^2 = 4\pi g_s e^{\phi}$ and, thus, the 't Hooft coupling at the scale ρ_* is given by

$$\lambda_* = 4\pi g_s N_c e^{\phi_*}. \tag{2.39}$$

For the quiver theories that correspond to different X^5 geometries, the gauge groups are of the form $\mathrm{SU}(N_c)^n$. Let us generalize a relation from the orbifold constructions $\sum_i^n 4\pi g_{\mathrm{YM},i}^{-2} = (g_s e^{\phi})^{-1}$ [66, 70, 71] and consider all the gauge couplings $g_{\mathrm{YM},i}$ to be equal. Then $4\pi\ g_s N_c e^{\phi}$, strictly speaking, gives the 't Hooft coupling at each node of the quiver, divided by n. However, with an abuse of language we will simply refer to it as the 't Hooft coupling. Therefore, by using (2.39) and the definition of Q_f in (2.26) in (2.38), we get

$$\epsilon_* = \frac{\mathrm{Vol}(X^3)}{16\pi \mathrm{Vol}(X^5)}\lambda_* \frac{N_f}{N_c}. \tag{2.40}$$

In particular, when $X^5 = S^5$, this relation becomes

$$\epsilon_{*\ (X^5=S^5)} = \frac{1}{8\pi^2}\lambda_* \frac{N_f}{N_c}. \tag{2.41}$$

Notice that the fact that ϕ is not constant in the backreacted solution is simply a reflection, in the gauge theory dual, of the running of the Yang-Mills coupling constant when matter is added to a conformal theory.

In terms of ϵ_* the dilaton and the function η of (2.35) take the following form:

$$e^{\phi-\phi_*} = \frac{1}{1-\epsilon_* \int_{\rho_*}^{\rho} p(\xi)d\xi}, \qquad \eta = \epsilon_* \, e^{\phi-\phi_*} \int_{\rho_q}^{\rho} e^{6\xi} p(\xi)d\xi. \tag{2.42}$$

One of the prominent features of our solution is the fact that, for $N_f \neq 0$, the dilaton blows up at some UV scale $\rho = \rho_{\rm LP}$, determined by the following condition:

$$\int_{\rho_*}^{\rho_{\rm LP}} p(\xi)d\xi = \epsilon_*^{-1}. \tag{2.43}$$

Clearly, in order to have a well-defined solution, we should restrict the value of the radial coordinate ρ to the range $\rho < \rho_{\rm LP}$. In view of the relation between the Yang-Mills coupling $g_{\rm YM}$ and the dilaton ($g_{\rm YM}^2 \sim e^{\phi}$), the divergence of ϕ implies that $g_{\rm YM}$ blows up at some UV scale, that is, that the gauge theory develops a Landau pole. This UV pathology of our solution was expected on physical grounds since the flavored gauge theory has positive beta function. Indeed, we will check below in some particular case that our solution reproduces the running of the coupling constant of the dual field theory.

2.3.1. The Supersymmetric Solution with Massless Quarks

We now consider the particular case of massless quarks, which corresponds to taking the charge distribution given by (2.25) or simply $p(\rho) = 1$ and $\rho_q \to -\infty$. In this case (2.42) simply gives

$$e^{\phi-\phi_*} = \frac{1}{1+\epsilon_*(\rho_*-\rho)}, \qquad e^{-6\rho}\eta = \frac{\epsilon_*}{6} e^{\phi-\phi_*}, \tag{2.44}$$

and the solution written in (2.37) reduces to

$$\begin{aligned}
e^{g} &= \sqrt{\alpha'} e^{\rho} \left(1+\epsilon_*\left(\frac{1}{6}+\rho_*-\rho\right)\right)^{1/6}, \\
e^{f} &= \sqrt{\alpha'} e^{\rho} \left(1+\epsilon_*(\rho_*-\rho)\right)^{1/2} \left(1+\epsilon_*\left(\frac{1}{6}+\rho_*-\rho\right)\right)^{-1/3}, \\
\frac{dh}{d\rho} &= -\frac{Q_c}{\alpha'^2} e^{-4\rho} \left(1+\epsilon_*\left(\frac{1}{6}+\rho_*-\rho\right)\right)^{-2/3}.
\end{aligned} \tag{2.45}$$

Notice that the location of the Landau pole in this case is just $\rho_{\rm LP} = \rho_* + \epsilon_*^{-1}$ and that the range of ρ for which the solution (2.45) makes sense is $\rho \in (-\infty, \rho_{\rm LP})$. Moreover, by using the definition of ϵ_* in (2.38) one can immediately show that the dilaton can be written as

$$e^{\phi(\rho)} = \frac{1}{Q_f(\rho_{\rm LP}-\rho)}. \tag{2.46}$$

Let us now verify that the dependence on ρ of ϕ in (2.46) matches the expectations from field theory. For concreteness we will consider the case of $\mathcal{N} = 4$ SYM with matter. Similar checks can be done in other cases (see [67] for the case of the Klebanov-Witten theory). By using the relation between the Yang-Mills coupling and the dilaton discussed above as well as the value of Q_f for $X^5 = S^5$ written in (2.27), one gets

$$\frac{8\pi^2}{g_{\mathrm{YM}}^2} = N_f(\rho_{\mathrm{LP}} - \rho). \tag{2.47}$$

In order to read the running of the coupling constant from (2.47) we must convert the dependence on the coordinate ρ in (2.47) into a dependence on the energy scale of the corresponding dual field theory. At an energy scale μ much lower than the Landau pole scale Λ_{UV} (i.e., for $\rho \ll \rho_{\mathrm{LP}}$) the scaling dimensions of the adjoints and fundamentals take their canonical values and the natural radius/energy relation is

$$\rho_{\mathrm{LP}} - \rho = \log \frac{\Lambda_{\mathrm{UV}}}{\mu}. \tag{2.48}$$

Plugging this relation in (2.47) we get

$$\frac{8\pi^2}{g_{\mathrm{YM}}^2} = N_f \log \frac{\Lambda_{\mathrm{UV}}}{\mu}. \tag{2.49}$$

Therefore, we get a logarithmic scaling of the coupling of the type $8\pi^2/g_{\mathrm{YM}}^2 = b \log E$, with $b = -N_f$, which matches the one-loop field theory result in which one has that $b = 3N_c - 3N_c - N_f$ (see, e.g., [72]). (In principle, one could object that, being strongly coupled, the matter fields could get large anomalous dimensions making this result suspicious. However, since we are performing a small perturbative (in ϵ_*) deformation of the unflavored backgrounds, the anomalous dimensions for the fundamental multiplets cannot differ much from their quenched values. For the $X^5 = S^5$ case, those anomalous dimensions vanish. We thank F. Bigazzi for stressing this point to us.)

In order to have a clearer understanding of the deformation of the $\mathrm{AdS}_5 \times S^5$ metric introduced by the flavor, it is very convenient to change to a new radial variable r, which is defined by requiring that the warp factor takes the same form as in the unflavored case (see (2.1)):

$$h = \frac{R^4}{r^4}, \quad R^4 = \frac{Q_c}{4}. \tag{2.50}$$

By integrating the last equation in (2.45) we can get $h(\rho)$ and thus $r(\rho)$. We will perform this integration order by order in a series expansion in powers of ϵ_*. The additive integration constant will be fixed by requiring that $r(\rho_*) \equiv r_* = \sqrt{\alpha'}e^{\rho_*}$. One gets

$$r = \alpha'^{1/2} e^{\rho}\left[1 + \frac{\epsilon_*}{72}\left(e^{4\rho-4\rho_*} - 1 + 12(\rho_* - \rho)\right) + \frac{5\epsilon_*^2}{10368}\right.$$
$$\left.\times\left(e^{8\rho-8\rho_*} + 6e^{4\rho-4\rho_*}(3 + 4(\rho_* - \rho)) - \left(19 - 24(\rho_* - \rho) + 144(\rho_* - \rho)^2\right)\right) + O\left(\epsilon_*^3\right)\right]. \tag{2.51}$$

It is now straightforward to obtain the functions $f(r)$, $g(r)$ and the dilaton $\phi(r)$ as expansions in powers of ϵ_*. Up to second order we have

$$\begin{aligned}
e^f &= r\left[1 - \frac{\epsilon_*}{24}\left(1 + \frac{1}{3}\frac{r^4}{r_*^4}\right) + \frac{\epsilon_*^2}{1152}\left(17 - \frac{94}{9}\frac{r^4}{r_*^4} + \frac{5}{9}\frac{r^8}{r_*^8} - 48\log\left(\frac{r}{r_*}\right)\right) + O\left(\epsilon_*^3\right)\right],\\
e^g &= r\left[1 + \frac{\epsilon_*}{24}\left(1 - \frac{1}{3}\frac{r^4}{r_*^4}\right) + \frac{\epsilon_*^2}{1152}\left(9 - \frac{106}{9}\frac{r^4}{r_*^4} + \frac{5}{9}\frac{r^8}{r_*^8} + 48\log\left(\frac{r}{r_*}\right)\right) + O\left(\epsilon_*^3\right)\right],\\
\phi &= \phi_* + \epsilon_* \log\frac{r}{r_*} + \frac{\epsilon_*^2}{72}\left(1 - \frac{r^4}{r_*^4} + 12\log\frac{r}{r_*} + 36\log^2\frac{r}{r_*}\right) + O\left(\epsilon_*^3\right).
\end{aligned} \tag{2.52}$$

Equation (2.52) neatly displays the effects of quark loops in the deformation of the geometry and in the running of the dilaton (the latter is related to the running of the gauge coupling, as argued above). It is important to point out that the deformed geometry has a curvature singularity at the origin $r = 0$ (or $\rho = -\infty$) (this singularity is similar to the one that appears at $r = 0$ in a 2-dimensional manifold with metric $ds^2 = dr^2 + r^2(1 + r)d\varphi^2$). In the same IR limit, e^{ϕ} runs to 0. As argued in Section 1.5, the appearance of this singularity can be intuitively understood as due to the fact that, in this massless case, all branes of our smeared distribution pass through the origin and the charge density is highly peaked at that point. From the field theory side, one can think of the singularity as appearing because the theory becomes IR free, as first pointed out in [36]. Consistently with these interpretations and with the heuristic picture of Section 1.5, the IR singularity can be easily cured by giving a mass to the quarks (it is a "good" singularity according to the criteria of [73, 74]). We will explicitly verify this fact in the next subsection.

2.3.2. The Supersymmetric Solution with Massive Quarks

Let us now find the backreacted supergravity solution for massive quarks. As mentioned above, the function $p(\rho)$ entering the ansatz for F_1 in this case is not universal and depends on the particular Sasaki-Einstein space X^5 and on the family of D7-brane embeddings chosen.

For concreteness we first concentrate in discussing the case in which $X^5 = S^5$. The calculation of the function $p(\rho)$ in this case was performed in [75, Appendix C]. If $|\mu| = e^{\rho_q}$, one has

$$p(\rho) = \left[1 - e^{2(\rho_q-\rho)}\right]^2 \Theta(\rho - \rho_q). \tag{2.53}$$

When $\rho \geq \rho_q$ the function $p(\rho)$ is nonvanishing and one has to perform the integrals appearing in (2.42). These integrals can be straightforwardly done in analytic form and yield the following result:

$$\begin{aligned}
e^g &= \sqrt{\alpha'} e^\rho \left(1 + \epsilon_* \left(\frac{1}{6} + \rho_* - \rho - \frac{1}{6} e^{6\rho_q - 6\rho} - \frac{3}{2} e^{2\rho_q - 2\rho} + \frac{3}{4} e^{4\rho_q - 4\rho} - \frac{1}{4} e^{4\rho_q - 4\rho_*} + e^{2\rho_q - 2\rho_*}\right)\right)^{1/6}, \\
e^f &= \sqrt{\alpha'} e^\rho \\
&\quad \times \frac{\left(1 + \epsilon_* \left(\rho_* - \rho - e^{2\rho_q - 2\rho} + (1/4) e^{4\rho_q - 4\rho} + e^{2\rho_q - 2\rho_*} - (1/4) e^{4\rho_q - 4\rho_*}\right)\right)^{1/2}}{\left(1 + \epsilon_* \left(1/6 + \rho_* - \rho - (1/6) e^{6\rho_q - 6\rho} - (3/2) e^{2\rho_q - 2\rho} + (3/4) e^{4\rho_q - 4\rho} - (1/4) e^{4\rho_q - 4\rho_*} + e^{2\rho_q - 2\rho_*}\right)\right)^{1/3}}, \\
\phi &= \phi_* - \log\left[1 + \epsilon_* \left(\rho_* - \rho - e^{2\rho_q - 2\rho} + \frac{1}{4} e^{4\rho_q - 4\rho} + e^{2\rho_q - 2\rho_*} - \frac{1}{4} e^{4\rho_q - 4\rho_*}\right)\right].
\end{aligned} \tag{2.54}$$

As a check, notice that setting $\rho_q \to -\infty$ one recovers the massless solution of (2.44) and (2.45). We still have to write the solution for $\rho < \rho_q$. In this case $p(\rho)$ vanishes and the dilaton is constant and, by continuity, it has the value that can be read from (2.54) inserting $\rho = \rho_q$:

$$\phi_{\mathrm{IR}} = \phi_q = \phi_* - \log\left(1 + \epsilon_* \left(\rho_* - \rho_q - \frac{3}{4} + e^{2\rho_q - 2\rho_*} - \frac{1}{4} e^{4\rho_q - 4\rho_*}\right)\right). \tag{2.55}$$

The functions f and g are equal and given by

$$e^f = e^g = \alpha'^{(1/2)} \, e^\rho e^{-(1/6)(\Phi_{\mathrm{IR}} - \Phi_*)} \quad (\rho < \rho_q). \tag{2.56}$$

It follows straightforwardly from these results that the IR singularity at $\rho = -\infty$ of the massless case disappears when $\mu \neq 0$ since the background reduces to $\mathrm{AdS}_5 \times S^5$ for $\rho < \rho_q$. Moreover, one can verify that the metric is also regular at $\rho = \rho_q$. Thus, as stressed in Section 1.5, the smearing of massive flavors allows one to smooth out IR singularities.

Similar calculations can be done for the conifold theories. In this case we redefine the parameter μ of the embedding equations (2.15) and (2.17) as $|\mu| = e^{3\rho_q/2}$. The charge distribution for the family (2.15) of chiral embeddings was obtained in [26], with the following result:

$$p(\rho) = \left[1 - e^{3(\rho_q - \rho)}(1 + 3\rho - 3\rho_q)\right]\Theta(\rho - \rho_q). \tag{2.57}$$

Similarly, for the nonchiral embeddings (2.17) the function $p(\rho)$ is given by [76]

$$p(\rho) = \left[1 - e^{3(\rho_q - \rho)}\right]\Theta(\rho - \rho_q). \tag{2.58}$$

The corresponding supergravity solutions have been written down in [26, 76]. They are regular in the IR, much in the same way as in the $X^5 = S^5$ case detailed above.

2.4. Screening Effects on the Meson Spectrum

The holographic theories with flavors present mesonic excitations, meaning that there exists a spectrum of colorless physical states created by operators which are bilinears in the fundamental fields. They are associated to normalizable excitations of the flavor branes as was neatly explained in the seminal paper [7]. For a review of this broad subject, see [8]. Notice that the notion of "meson" we use here generalizes that used in QCD. For instance, the "mesons" of [7] are excitations of a nonconfining theory and in this case the dimensionful quantity that sets the meson masses is just the quark mass (divided by a power of the 't Hooft coupling), not a dynamically generated scale.

In the present section, we review how the presence of unquenched flavors can affect the discrete mesonic spectrum. Again, we will restrict ourselves to the smeared set-up and follow [77]. For discussions about screening effects on the spectrum in cases with localized rather than smeared backreacting flavor branes, we refer the reader to [8, 42]. The effect of the smeared flavors on the hydrodynamical transport coefficients (in a finite temperature setting) was studied in [75, 78]. It is also worth mentioning that, within the model we will introduce in Section 3, the screening effects on the glueball spectrum have been recently analyzed in [79].

For the sake of briefness, we will just focus on an example and discuss a particular mesonic excitation in the backreacted Klebanov-Witten model. The analysis and conclusions for different modes and/or different models should be similar; see [77] for some other examples. In particular, we will consider oscillations of a D7-brane which introduces massive nonchiral flavor [80] and just look at the oscillation of the gauge field that gives rise to a vector mode in the dual field theory. Thus, we discuss the physics of a meson whose "constituent quarks" are massive in the presence of many dynamical massless flavors.

We write the gauge field along the Minkowski directions as $A_\mu = a_v(\rho)\xi_\mu e^{ikx}$, where ξ_μ is a constant transverse vector. The equation describing this oscillation was found in [77], building on the formalism introduced in [80]. It reads

$$0 = \partial_\rho\left(e^{2g-3\rho}\left(e^{3\rho} - e^{3\rho_Q}\right)\partial_\rho a_v\right) + M_v^2 h e^{2g+2f}\left(1 + e^{3\rho_Q - 3\rho}\left(\frac{3}{4}e^{2g-2f} - 1\right)\right)a_v, \tag{2.59}$$

where $M_v^2 = -k^2$, the constant ρ_Q is the minimal value of ρ reached by the D7-brane (related to the quark mass), and f, g, h are given in (2.45).

Notice that for the meson excitation, we just use a D-brane probe; namely, we consider the oscillation of a single brane in a fixed background. At first sight, this could look contradictory, since our aim is always to take into account the effect of the flavor branes on the geometry. Then, one may think about considering coupled fluctuations of brane and background fields. Nevertheless, this is not necessary: there are $N_f \gg 1$ flavor branes which are affecting the background, but when we consider a meson, only one (or two) out of this N_f is fluctuating. Therefore, the effect of this oscillation on the background is suppressed by N_f^{-1} with respect to the contribution of the whole set of branes and therefore is consistently negligible. On the other hand, the existence of the rest of flavors and the associated quantum effects on the spectrum are taken into account through the deformation they have produced in the background geometry.

Following the standard procedure [7, 8], a discrete tower of values for M_v should be found when selecting solutions of (2.59) which are regular and normalizable. Since the background has a Landau pole, some prescription is needed for dealing with the UV limit (large ρ). Technically, we will just require that the fluctuation a_v vanishes at ρ_*. Physically, one can check that this is a consistent procedure if $\rho_Q \ll \rho_*$: we are interested in some IR physics which should be independent of the UV completion of the theory at $\rho > \rho_*$, up to corrections suppressed by powers of the UV scale. Namely, we neglect contributions of order $e^{\rho_Q-\rho_*} \sim \Lambda_{\rm IR}/\Lambda_{\rm UV}$ and check that the spectrum can be written in terms of IR quantities. The value ρ_* disappears from the final result, apart from the quoted negligible corrections. See [77] for further discussions on the issue. In Section 2.5, we will see similar examples of how to deal with the Landau pole. In that case, the IR scale, which has to be much smaller than the arbitrary UV scale at ρ_*, is set by the temperature rather than by the quark mass.

In order to estimate the spectrum from (2.59), we can use a WKB approximation. In [77], using a formalism developed in [81], an expression for the mass tower in terms of the principal quantum number n was found. Adapting notation to the one we are using here,

$$M_v^{(n)} \approx \frac{\pi}{\Sigma_v} n, \quad \Sigma_v \equiv \int_{\rho_Q}^{\rho_*} h^{1/2} e^f \sqrt{\frac{1+e^{3\rho_Q-3\rho}\left((3/4)e^{2g-2f}-1\right)}{1-e^{3\rho_Q-3\rho}}}\, d\rho. \tag{2.60}$$

Let us evaluate this integral at first order in ϵ_*, by inserting (2.45). We still have to fix the additive constant for h, which we can do by requiring $h(\rho_*) = 0$ (in [77] $h(\rho_{\rm LP}) = 0$ was used. It is crucial that both prescriptions give the same result, up to quantities in $e^{\rho_Q-\rho_*} \sim \Lambda_{\rm IR}/\Lambda_{\rm UV}$). We shift to a coordinate u such that $u \equiv e^{\rho-\rho_Q}$, $u_* \equiv e^{\rho_*-\rho_Q}$. Defining λ_Q as the 't Hooft coupling (2.39) at the quark mass scale, inserting the value of Q_c in (2.1), and defining $T_Q \equiv (e^{\phi/2}\sqrt{-g_{tt}g_{xx}})|_{\rho=\rho_Q}$ as the tension of a hypothetical fundamental string stretched at constant $\rho = \rho_Q$, we can write the estimate for the meson masses as

$$M_v^{(n)} \approx \frac{T_Q^{1/2}}{\lambda_Q^{1/4}} \left(\frac{\pi n}{3^{3/4}/4\sqrt{2\pi}} \int_1^{u_*} \left(\frac{\sqrt{4u^3-1}}{u^2\sqrt{u^3-1}} + \epsilon_Q \frac{7-4u^3+4\left(4u^3-1\right)\log u}{24u^2\sqrt{4u^3-1}\sqrt{u^3-1}} \right) du \right). \tag{2.61}$$

It is important to stress once again that this expression is written only in terms of IR quantities, once we discard terms of order $u_*^{-1} = e^{\rho_Q-\rho_*}$; namely, contributions like $\log u_*$ have cancelled out. Notice that the upper limit of the integrals can be taken to infinity if we again insist in discarding $O(u_*^{-1})$ contributions. The expression (2.61) is a neat example of how, even having a Landau pole, the holographic set-up is able to consistently obtain IR predictions, in exactly the same spirit as in field theory. We can perform numerically the integration in (2.61), and we get [77]

$$M_v^{(n)} \approx \frac{T_Q^{1/2}}{\lambda_Q^{1/4}} n \left(5.2 - 6 \times 10^{-3} \frac{N_f \lambda_Q}{N_c} + \cdots \right), \tag{2.62}$$

where in order to substitute ϵ_Q as in (2.40) we have used $\mathrm{Vol}(X^3) = (16/9)\pi^2$, $\mathrm{Vol}(X^5) = (16/27)\pi^3$. The expression (2.62) is the result quoted in [77], apart from a different factor of 2 in the definition of λ_Q.

The lesson we want to take from this section is that there is a well-defined method to obtain the shift produced by the flavor quantum effects on the meson spectrum (or, eventually, on any physical observable) as an expansion in the parameter $\epsilon \sim \lambda N_f / N_c$ which weighs the flavor loops. Heuristically, it may be useful to think of the computation leading to (2.62) as (partially) a strong coupling analogue of the Lamb shift corrections of QED.

2.5. Black Hole Solutions: D3-D7 Quark-Gluon Plasmas

In this subsection we will review the results in [75]. We start by showing how one can find a black hole solution which includes the backreaction effects due to massless quarks. To perform this analysis it is more convenient to work with a new radial variable σ such that the metric takes the following form:

$$ds^2 = h^{-1/2}\left[-b dt^2 + d\vec{x}_3^{\,2}\right] + h^{1/2}\left[b e^{8g+2f} d\sigma^2 + e^{2g} ds_{\mathrm{KE}}^2 + e^{2f}(d\tau + A_{\mathrm{KE}})^2\right]. \tag{2.63}$$

Notice that we have introduced a new function b which parameterizes the breaking of Lorentz invariance induced by the nonzero temperature T. All functions appearing in the metric (2.63), as well as the dilaton ϕ, depend on σ. Moreover, the RR field strengths F_5 and F_1 are given by the ansatz (2.33) with the function $p = 1$. We remind the reader that fixing $p = 1$ corresponds to taking massless quarks. (In [75], the more involved case of massive quarks $p \neq 1$ was also discussed. An extra complication is the necessity of finding the nontrivial D7-brane embeddings in the backreacted geometry.)

In this non-supersymmetric case we will not have the first-order BPS equations at our disposal and we will have to deal directly with the second-order equations of motion. Actually, since all the functions we need to compute depend only on the radial coordinate σ, it is possible to describe the system in terms of a one-dimensional effective action. One can find this effective action by directly substituting the ansatz in the gravity plus branes action (1.7). One gets

$$\begin{aligned} S_{\mathrm{eff}} = {} & \frac{\mathrm{Vol}(X^5) V_{1,3}}{2\kappa_{10}^2} \\ & \times \int d\sigma \Bigg(-\frac{1}{2}\frac{(\partial_\sigma h)^2}{h^2} + 12(\partial_\sigma g)^2 + 8\partial_\sigma g \partial_\sigma f - \frac{1}{2}(\partial_\sigma \phi)^2 + \frac{(\partial_\sigma b)}{2b}\left(\frac{\partial_\sigma h}{h} + 8\partial_\sigma g + 2\partial_\sigma f\right) \\ & \qquad + 24 b e^{2f+6g} - 4 b e^{4f+4g} - \frac{Q_c^2}{2}\frac{b}{h^2} - \frac{Q_f^2}{2} b e^{2\phi+8g} - 4 Q_f b e^{\phi+6g+2f} \Bigg). \end{aligned} \tag{2.64}$$

In (2.64) $V_{1,3}$ denotes the (infinite) integral over the Minkowski coordinates. The second derivatives coming from the Ricci scalar have been integrated by parts and, as is customary, only the angular part of F_5 is inserted in the F_5^2 term (otherwise the Q_c would not enter the effective action since, on-shell, $F_5^2 = 0$ due to the self-duality condition). The last term in (2.64), proportional to Q_f, comes from the DBI contribution in (1.5). Notice also that the WZ term does not enter (2.64) because it does not depend on the metric or the dilaton.

The equations of motion stemming from the effective action (2.64) are

$$\begin{aligned}
&\partial_\sigma^2(\log b) = 0,\\
&\partial_\sigma^2(\log h) = -Q_c^2\frac{b}{h^2},\\
&\partial_\sigma^2 g = -2be^{4g+4f} + 6be^{6g+2f} - Q_f be^{\phi+6g+2f},\\
&\partial_\sigma^2 f = 4be^{4g+4f} - \frac{Q_f^2}{2}be^{2\phi+8g},\\
&\partial_\sigma^2 \phi = Q_f^2 be^{2\phi+8g} + 4Q_f be^{\phi+6g+2f}.
\end{aligned} \tag{2.65}$$

It is straightforward to check that these equations solve the full set of Einstein equations provided that the following "zero-energy" constraint is also satisfied:

$$\begin{aligned}
0 = &-\frac{1}{2}\frac{(\partial_\sigma h)^2}{h^2} + 12(\partial_\sigma g)^2 + 8\partial_\sigma g\partial_\sigma f - \frac{1}{2}(\partial_\sigma\phi)^2 + \frac{(\partial_\sigma b)}{2b}\left(\frac{\partial_\sigma h}{h} + 8\partial_\sigma g + 2\partial_\sigma f\right) - 24be^{2f+6g}\\
&+ 4be^{4f+4g} + \frac{Q_c^2}{2}\frac{b}{h^2} + \frac{Q_f^2}{2}be^{2\phi+8g} + 4Q_f be^{\phi+6g+2f}.
\end{aligned} \tag{2.66}$$

This constraint can be thought of as the $\sigma\sigma$ component of the Einstein equations or, alternatively, as the Gauss law from the gauge fixing of $g_{\sigma\sigma}$ in the ansatz (2.63).

Let us now find a solution of the system of equations (2.65) and of the "zero-energy" constraint (2.66) that corresponds to a black hole for the backreacted D3-D7 system. We will require that such a solution is regular at the horizon and tends to the supersymmetric one at energy scales much higher than the black hole temperature T. Actually, the biggest advantage of the radial variable σ introduced above is that the equations of motion of b and h in (2.65) are decoupled from the ones corresponding to the other functions of the ansatz. These decoupled equations can be easily integrated in terms of an integration constant r_h as follows:

$$b = e^{4r_h^4\sigma}, \qquad h = \frac{Q_c}{4r_h^4}\left(1 - e^{4r_h^4\sigma}\right), \tag{2.67}$$

where $\sigma \in (-\infty, 0)$. We now define a new radial coordinate r by means of the following relation:

$$e^{4r_h^4\sigma} = 1 - \frac{r_h^4}{r^4}, \quad r \in (r_h, +\infty). \tag{2.68}$$

Then, b and h take the following form:

$$b = 1 - \frac{r_h^4}{r^4}, \quad h = \frac{R^4}{r^4}, \tag{2.69}$$

with $R^4 = Q_c/4$. Notice that h is given by the same expression as in (2.50). Moreover, it is clear from (2.69) that $r = r_h$ is the position of the horizon and, thus, the extremal limit is attained by sending r_h to zero. In terms of r the metric takes the following form:

$$ds^2 = \frac{r^2}{R^2}\left[\left(1 - \frac{r_h^4}{r^4}\right)dt^2 + d\vec{x}_3^2\right] + \frac{R^2}{r^2}\frac{e^{8\hat{g}+2\hat{f}}}{1 - r_h^4/r^4}(dr)^2 + R^2\left[e^{2\hat{g}}ds_{\rm KE}^2 + e^{2\hat{f}}(d\tau + A_{\rm KE})^2\right], \tag{2.70}$$

where we have defined the functions $\hat{f}$ and $\hat{g}$ as follows:

$$e^{\hat{f}} \equiv \frac{e^f}{r}, \qquad e^{\hat{g}} \equiv \frac{e^g}{r}. \tag{2.71}$$

In order to determine completely the background we still have to solve (2.65) and (2.66) for f, g, and the dilaton ϕ. We will find this solution by introducing a reference UV scale r_* and by expanding the functions in terms of the parameter ϵ_* defined in (2.38). We will impose that the functions f, g, and ϕ are equal to the SUSY ones of (2.52) when the extremality parameter r_h vanishes. Moreover, we will also require that these functions coincide with those in (2.52) at the UV scale r_*. These conditions fix uniquely a solution of (2.65) and (2.66). Up to second order in ϵ_* this solution is given by

$$\begin{aligned}
e^{\hat{f}} &= 1 - \frac{\epsilon_*}{24}\left(1 + \frac{2r^4 - r_h^4}{6r_*^4 - 3r_h^4}\right) \\
&\quad + \frac{\epsilon_*^2}{1152}\left(17 - \frac{94}{9}\frac{2r^4 - r_h^4}{2r_*^4 - r_h^4} + \frac{5}{9}\frac{\left(2r^4 - r_h^4\right)^2}{\left(2r_*^4 - r_h^4\right)^2} - \frac{8}{9}\frac{r_h^8\left(r_*^4 - r^4\right)}{\left(2r_*^4 - r_h^4\right)^3} - 48\log\left(\frac{r}{r_*}\right)\right) + O\left(\epsilon_*^3\right), \\
e^{\hat{g}} &= 1 + \frac{\epsilon_*}{24}\left(1 - \frac{2r^4 - r_h^4}{6r_*^4 - 3r_h^4}\right) \\
&\quad + \frac{\epsilon_*^2}{1152}\left(9 - \frac{106}{9}\frac{2r^4 - r_h^4}{2r_*^4 - r_h^4} + \frac{5}{9}\frac{\left(2r^4 - r_h^4\right)^2}{\left(2r_*^4 - r_h^4\right)^2} - \frac{8}{9}\frac{r_h^8\left(r_*^4 - r^4\right)}{\left(2r_*^4 - r_h^4\right)^3} + 48\log\left(\frac{r}{r_*}\right)\right) + O\left(\epsilon_*^3\right), \\
\phi &= \phi_* + \epsilon_*\log\frac{r}{r_*} \\
&\quad + \frac{\epsilon_*^2}{72}\left(1 - \frac{2r^4 - r_h^4}{2r_*^4 - r_h^4} + 12\log\frac{r}{r_*} + 36\log^2\frac{r}{r_*} + \frac{9}{2}\left(Li_2\left(1 - \frac{r_h^4}{r^4}\right) - Li_2\left(1 - \frac{r_h^4}{r_*^4}\right)\right)\right) + O\left(\epsilon_*^3\right),
\end{aligned} \tag{2.72}$$

where $Li_2(u) \equiv \sum_{n=1}^{\infty} u^n/n^2$ is a polylogarithmic function. The functions written in (2.72) determine a geometry that is regular at the horizon $r = r_h$. In the next subsection we will study its thermodynamics and we will extract some consequences for the dual field theory with dynamical quarks at nonzero temperature.

Let us conclude this section with some comments on the stability of our perturbative non-extremal solutions. A possible way to check for the latter is to consider worldvolume fluctuations of a D7-brane in the setup. If, as in our cases, the brane corresponds to massless flavors, the related quasinormal modes on the unflavored background all have frequencies with a negative imaginary part of the order of the temperature, signaling stability [82–84]. This result cannot be changed in the flavored case when a perturbative expansion in ϵ_* is done. Thus, in our regime of approximations, stability with respect to those fluctuations is guaranteed.

2.5.1. Thermodynamics of the Solution

In the previous subsections we have defined the backreacted background in terms of an arbitrary UV scale r_* as an expansion in powers of the parameter ϵ_* written in (2.40). This scale r_* should be well separated from the Landau pole scale in order to avoid having the pathologies of the latter. Moreover, we are interested in analyzing the physical consequences of this background at energies much lower than the UV scale r_*. In a black hole background dual to a quark gluon plasma the natural IR scale is the location r_h of the horizon, which should be related to temperature T of the plasma. Accordingly, we define ϵ_h as

$$\epsilon_h = \frac{\lambda_h \mathrm{Vol}(X^3)}{16\pi \mathrm{Vol}(X^5)} \frac{N_f}{N_c}, \tag{2.73}$$

where, in what follows, the subscript h means that the quantities are evaluated at the horizon $r = r_h$. Thus, λ_h is naturally identified with the 't Hooft coupling at the scale of the plasma temperature. We therefore have

$$\epsilon_h = \epsilon_* \frac{e^{\Phi_h}}{e^{\Phi_*}} = \epsilon_* + \epsilon_*^2 \log \frac{r_h}{r_*} + O\left(\epsilon_*^3\right). \tag{2.74}$$

We will use this relation to recast the expansions in powers of ϵ_* as series in ϵ_h. We will assume in what follows that r_h is well below the reference UV scale r_* to ensure that the IR physics does not depend on the UV completion of the theory. In a Wilsonian sense of the renormalization group flow, the UV details of the theory should not affect the IR physics. Moreover, since, as we will see below, r_h is proportional to the temperature (at leading order), we have

$$\frac{d\epsilon_h}{dT} = \frac{\epsilon_h^2}{T} + O\left(\epsilon_h^3\right), \tag{2.75}$$

and $T(d\lambda_h/dT) = \epsilon_h \lambda_h$ at leading order. These relations reflect the running of the gauge coupling induced by the dynamical flavors.

The thermodynamic properties of the black hole solution are determined by the metric functions at the horizon. After neglecting terms suppressed in powers of r_h^4/r_*^4, the values of the functions $\hat{f}$ and $\hat{g}$ at $r = r_h$ can be obtained from (2.72). One gets

$$e^{\hat{f}_h} = 1 - \frac{\epsilon_h}{24} + \frac{17}{1152}\epsilon_h^2 + \mathrm{O}\left(\epsilon_h^3\right), \qquad e^{\hat{g}_h} = 1 + \frac{\epsilon_h}{24} + \frac{1}{128}\epsilon_h^2 + O\left(\epsilon_h^3\right). \tag{2.76}$$

The black hole temperature can be obtained by requiring regularity of the euclideanized metric and by identifying the temperature with the inverse of the period of the euclideanized time. A simple computation yields

$$T = \frac{2r_h}{2\pi R^2 e^{4\hat{g}_h+\hat{f}_h}} = \frac{r_h}{\pi R^2}\left[1 - \frac{1}{8}\epsilon_h - \frac{13}{384}\epsilon_h^2 + O\left(\epsilon_h^3\right)\right], \tag{2.77}$$

where in the last step we have used the values of $\hat{f}_h$ and $\hat{g}_h$ written in (2.76).

The entropy density s is proportional to A_8, the volume at the horizon of the eight-dimensional part of the space orthogonal to the $\hat{t}, r$ plane (where $\hat{t}$ is the Euclidean time), divided by the infinite constant volume of the 3D space directions V_3. From the general form of the metric we get that

$$s = \frac{2\pi A_8}{\kappa_{10}^2 V_3} = \frac{r_h^3 R^2 e^{4\hat{g}_h+\hat{f}_h}\mathrm{Vol}(X^5)}{2^5\pi^6 g_s^2 \alpha'^4} = \frac{\pi^5}{2\mathrm{Vol}(X^5)} N_c^2 \frac{r_h^3}{\pi^3 R^6}\left[1 + \frac{1}{8}\epsilon_h + \frac{19}{384}\epsilon_h^2 + O\left(\epsilon_h^3\right)\right], \tag{2.78}$$

which in terms of the temperature reads

$$s = \frac{\pi^5}{2\mathrm{Vol}(X^5)} N_c^2 T^3 \left[1 + \frac{1}{2}\epsilon_h + \frac{7}{24}\epsilon_h^2 + O\left(\epsilon_h^3\right)\right]. \tag{2.79}$$

As for the other thermodynamic quantities which will follow, the leading term of this formula is the well-known unflavored result. The $O(\epsilon_h)$ term was already calculated in [84] with the probe brane technique, in the $X^5 = S^5$ case. Here we have reobtained this result in a quite standard way, by computing the increase of the horizon area produced by the flavor branes. This can be considered as a crosscheck of the validity of the whole construction. Finally, the order ϵ_h^2 was first obtained in [75].

The ADM energy of the solution can be computed as an integral of the extrinsic curvature of the eight-dimensional hypersurface of constant time and radius. This calculation is straightforward and has been done in the [75, Appendix B], with the following result:

$$\varepsilon = \frac{E_{\mathrm{ADM}}}{V_3} = \frac{3}{8}\frac{\pi^5}{\mathrm{Vol}(X^5)} N_c^2 T^4 \left[1 + \frac{1}{2}\epsilon_h(T) + \frac{1}{3}\epsilon_h(T)^2 + O\left(\epsilon_h(T)^3\right)\right]. \tag{2.80}$$

Again, terms suppressed as powers of r_h/r_* have been neglected. Moreover, since in the following derivatives with respect to T are going to be taken, we find it convenient to make explicit that ϵ_h depends on T (see (2.75)). Equation (2.80) yields the energy density of the plasma and, thus, it allows us to study the full thermodynamics. Indeed, from (2.75) and (2.80) we get immediately the heat capacity (density):

$$c_V = \partial_T \varepsilon = \frac{3}{2}\frac{\pi^5}{\mathrm{Vol}(X^5)} N_c^2 T^3 \left[1 + \frac{1}{2}\epsilon_h(T) + \frac{11}{24}\epsilon_h(T)^2 + \mathrm{O}\left(\epsilon_h(T)^3\right)\right]. \tag{2.81}$$

The free energy density, and so (minus) the pressure, reads

$$\frac{F}{V_3} = -p = \varepsilon - Ts = -\frac{1}{8}\frac{\pi^5}{\mathrm{Vol}(X^5)} N_c^2 T^4 \left[1 + \frac{1}{2}\epsilon_h(T) + \frac{1}{6}\epsilon_h(T)^2 + O\left(\epsilon_h(T)^3\right)\right]. \tag{2.82}$$

Notice that, consistently, this satisfies the relation $s = \partial_T p$ (where it is crucial to take (2.75) into account). This result is confirmed by the direct computation of F from the renormalized Euclidean action (see, again, [75, Appendix B]) and also by the calculation in [78] of the correlator of the tensorial mode in the hydrodynamical approximation.

The speed of sound v_s is obtained by combining (2.79) and (2.81), namely,

$$v_s^2 = \frac{s}{c_V} = \frac{1}{3}\left[1 - \frac{1}{6}\epsilon_h(T)^2 + O\left(\epsilon_h(T)^3\right)\right]. \tag{2.83}$$

Note that the correction to the speed of sound, which measures the deviation from conformality, only appears at second order and that the sign of the correction is consistent with the bound $v_s^2 \leq 1/3$ conjectured in [85]. It is also interesting to point out that the solution provides a direct measure of the breaking of conformality at second order, namely, the so-called interaction measure, given by

$$\frac{\varepsilon - 3p}{T^4} = \frac{\pi^5 N_c^2}{16\,\mathrm{Vol}(X^5)}\epsilon_h(T)^2. \tag{2.84}$$

Let us now analyze the viscosity of the plasma predicted by the flavored black hole. Since we are not introducing higher derivatives of the metric in the action, the usual theorems apply and the shear viscosity η saturates the Kovtun-Son-Starinets bound [86], that is, $\eta/s = 1/4\pi$. Therefore, the shear viscosity η can be obtained by dividing by 4π the entropy density written in (2.79). Again, the first-order term coincides with the one calculated in the probe approach in [87] while the second-order result was first computed in [75]. On the other hand, one can also compute the bulk viscosity ζ for this model, with the following result [78]:

$$\zeta = \frac{\pi^4}{72\,\mathrm{Vol}(X^5)} N_c^2 T^3 \left[\epsilon_h(T)^2 + O\left(\epsilon_h(T)^3\right)\right]. \tag{2.85}$$

Interestingly, the value of ζ written in (2.85) saturates the bound proposed in [88]:

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - v_s^2\right). \tag{2.86}$$

For the computation of other transport coefficients, we refer the reader to [78].

2.5.2. Energy Loss of Partons

One of the main phenomenological applications of holography is the analysis of the energy loss of a parton that moves through a quark-gluon plasma. One of the measures of this energy degradation is the so-called jet quenching parameter $\hat{q}$, which is a transport coefficient that measures the bremsstrahlung experienced by a parton probe due to its interactions with the quarks and gluons of the plasma [89]. At very high energy, and using the eikonal approximation, the authors of [90] found a nonperturbative prescription for calculating $\hat{q}$ as the coefficient of L^2 in an almost light-like Wilson loop with dimensions $L^- \gg L$. By using the generic formula in [91] (and cutting the integral at r_*), we can write

$$\hat{q}^{-1} = \pi\alpha' \int_{r_h}^{r_*} e^{-\phi/2} \frac{\sqrt{g_{rr}}}{g_{xx}\sqrt{g_{xx}+g_{tt}}} dr = \frac{\pi\alpha' R^4}{r_h^2} e^{-\phi_h/2} \int_{r_h}^{r_*} e^{-(\phi-\phi_h)/2} \frac{e^{4\hat{g}+\hat{f}}}{\sqrt{r^4 - r_h^4}} dr. \tag{2.87}$$

The dilaton enters the formula because we are considering the Einstein frame metric. By plugging in (2.87) the expressions of $\hat{f}$, $\hat{g}$, and ϕ written in (2.72), and by performing the corresponding integrals in r, one gets $\hat{q}$ as a power series expansion in ϵ_*. In the course of this calculation we will neglect terms that are suppressed by powers of r_h/r_* and we will write the result in a series in ϵ_h rather than in ϵ_*. In terms of gauge theory quantities one gets [75]

$$\hat{q} = \frac{\pi^3 \sqrt{\lambda_h}\,\Gamma(3/4)}{\sqrt{\mathrm{Vol}(X^5)}\;\Gamma(5/4)} T^3 \left[1 + \frac{1}{8}(2+\pi)\epsilon_h + \gamma\epsilon_h^2 + O\left(\epsilon_h^3\right)\right], \tag{2.88}$$

where we have introduced a constant γ:

$$\gamma = \frac{11}{96} + \frac{\pi}{48} + \frac{3\pi^2}{128} + \frac{1}{8}\mathcal{C} + \frac{1}{48}\,{}_4F_3\left(1,1,1,\frac{3}{2};\frac{7}{4},2,2;1\right) \approx 0.5565, \tag{2.89}$$

with $\mathcal{C} \sim 0.91597$ being the Catalan constant. Notice that the flavor correction to $\hat{q}$ is positive; that is, fundamentals enhance the jet quenching. Actually, (2.88) can be used to estimate this enhancement in the extrapolation to the realistic RHIC regime. Let us take $X^5 = S^5$, $N_c = N_f = 3$, and $\alpha_s = g_{\mathrm{YM}}^2/4\pi \sim 1/2$. Then, $\lambda_h \sim 6\pi$ and $\epsilon_h \sim N_f/4\pi \sim 0.24$. Using this value in (2.88) we have that $\hat{q}$ is increased by 20%. For example, at $T = 300\,\mathrm{MeV}$ we obtain $\hat{q} \sim 5.3\,(\mathrm{Gev})^2/\mathrm{fm}$, to be compared with the value [90] $\hat{q} \sim 4.5\,(\mathrm{Gev})^2/\mathrm{fm}$ of the unflavored plasma (the RHIC values are $\hat{q} \sim 5$–$15\,(\mathrm{Gev})^2/\mathrm{fm}$). It is also interesting to rewrite (2.88) in terms of the entropy density s. One gets

$$\hat{q} = c\,\sqrt{\lambda_h}\sqrt{\frac{s}{N_c^2}}\,T^{3/2}\left[1 + \frac{\pi}{8}\epsilon_h + \left(\gamma - \frac{11}{96} - \frac{\pi}{32}\right)\epsilon_h^2 + O\left(\epsilon_h^3\right)\right], \quad c = \sqrt{2\pi}\frac{\Gamma(3/4)}{\Gamma(5/4)}, \tag{2.90}$$

which shows a deviation (driven by ϵ_h) from the general expression put forward in [92]. In this setting, the presence of fundamentals and the breaking of conformality are inevitably mingled. It would be interesting to have the dual of a conformal theory with fundamentals to check whether the conjecture [92] holds in such situation. That was analyzed in [61] in a noncritical string framework and, interestingly, the result differs from [92]. The caveat is that the model studied in [61] suffers from the usual problem of gravity-like approaches to noncritical strings; namely there are uncontrolled approximations.

Another way of characterizing the energy loss of a parton probe in the plasma is by modeling it as a macroscopic string attached to a probe flavor brane. The string is dragged by a constant force f which keeps its velocity v fixed and transfers to the parton energy and momentum, which is then lost in the plasma at a constant rate. This energy loss is measured by the drag coefficient μ, which relates the force f and the parton momentum p: $f = \mu p$. To compute this drag force one can apply the general procedure of [93–95]. By using the Nambu-Goto action for a string in the black hole background one gets that the rate of momentum transferred to the medium is given by [95]

$$\frac{dp}{dt} = -\frac{1}{2\pi\alpha'}C = -\frac{r_h^2}{2\pi\alpha' R^2}e^{\Phi(r_c)/2}\frac{v}{\sqrt{1-v^2}} = -\mu M_{\mathrm{kin}}\frac{v}{\sqrt{1-v^2}}, \tag{2.91}$$

where C is the constant determined from the equation $g_{xx}(r_c)g_{tt}(r_c) + C^2 = 0$ with the point r_c given by $g_{tt}(r_c) + g_{xx}(r_c)v^2 = 0$, namely, $r_c = r_h(1-v^2)^{-1/4}$. In (2.91) we have introduced, following [93], the kinematical mass M_{kin} such that $p = M_{\mathrm{kin}}(v/\sqrt{1-v^2})$. From (2.91), using (2.72), (2.74), and (2.77), we find

$$\begin{aligned}\mu M_{\mathrm{kin}} = {}& \frac{\pi^{5/2}}{2}\frac{\sqrt{\lambda_h}}{\sqrt{\mathrm{Vol}(X^5)}}\\ &\times T^2\Bigg[1 + \frac{1}{8}\Big(2-\log\Big(1-v^2\Big)\Big)\epsilon_h\\ &\qquad + \frac{1}{384}\Big[44 - 20\log\Big(1-v^2\Big) + 9\log^2\Big(1-v^2\Big) + 12Li_2\Big(v^2\Big)\Big]\epsilon_h^2 + O\Big(\epsilon_h^3\Big)\Bigg].\end{aligned} \tag{2.92}$$

As happens with the jet quenching, the energy loss (at fixed v) is enhanced by the presence of fundamental matter. The quantity $\mu\ M_{\mathrm{kin}}$ grows when increasing the velocity. From (2.92), formally, it would diverge as $v \to 1$. However, (2.92) is not applicable in that limit since we have to require $\epsilon_h \log(1-v^2) \ll 1$ for the expansions to be valid.

2.6. A Discussion on the Range of Validity

We now discuss, following [75], the restriction on the physical parameters needed for the deconfined flavored plasma solution to be physically meaningful. Before we go on, two comments are in order: first, notice that, even if we will use here the plasma temperature as the IR scale at which the relevant physics takes place, this can be substituted by any other IR scale, depending on what one wants to study. Thus, for instance, when computing meson masses at zero temperature as in Section 2.4, the discussion below holds; just taking into account the IR scale there is set by the quark mass. Second, notice that the restriction of small ϵ_* (which leads to $N_f \ll N_c$) of the D3-D7 case at hand comes from the existence of a Landau pole. In holographic theories in which there is no Landau pole in the geometry (Sections 3, and 4), there is in principle no restriction to N_f. In particular, it is possible to consider in those theories N_f to be of the same order as N_c.

As we have already remarked, having a pathological UV means that there must exist a separation of scales between IR and UV. Concretely, there must exist a hierarchy, which in terms of the r radial coordinate reads

$$r_h \ll r_* \ll r_a < r_{\rm LP}. \tag{2.93}$$

The quantity r_h sets the scale of the plasma temperature $r_h/R^2 \sim \Lambda_{\rm IR} \sim T$, which is the scale at which we want to analyze the physics. The point $r_{\rm LP}$ is where the dilaton diverges, signaling a Landau pole in the dual theory. At a scale r_a the string solution starts presenting subtler pathologies, whose discussion we delay until the end of this section. Finally, r_* sets an (arbitrary) UV cutoff scale $r_*/R^2 \sim \Lambda_{\rm UV}$. The solution (2.72) will only be used for $r < r_*$. In a Wilsonian sense of a renormalization group flow, the UV details should not affect the IR physical predictions. This feature is reflected in the fact that physical quantities do not depend (up to suppressed contributions) on r_* or functions evaluated at that point, but only on IR parameters. Even if the precise value of r_* is arbitrary, we have to make sure that it is possible to choose it such that it is well above the IR scale (so that the UV completion only has negligible effects on the IR physics) and well below the pathological $r_a, r_{\rm LP}$ scales (so that the solution we use is meaningful and the expansions do not break down). To this we turn now.

Let us start by computing the hierarchy between r_* and $r_{\rm LP}$. Since at r_* we can approximate the solution by the supersymmetric one, we can read the position of the Landau pole from (2.45). If we insert the approximate relation between radial coordinates $r \approx \sqrt{\alpha'}e^{\rho}$, we find

$$\frac{r_*}{r_{\rm LP}} \approx e^{-1/\epsilon_*} \ll 1 \tag{2.94}$$

as long as $\epsilon_* \ll 1$.

Moreover, one has to make sure that the Taylor expansions (2.72) are valid in the region $r_h < r < r_*$. This of course requires $\epsilon_* \ll 1$, but also that $\epsilon_* |\log(r_h/r_*)| \ll 1$ (notice that the absolute value of the logarithm can be big because $r_h \ll r_*$). This means that $r_h/r_* \gg e^{-1/\epsilon_*}$. On the other hand, when computing physical quantities in the previous sections, we always neglect quantities suppressed as powers of $r_h/r_* \sim T/\Lambda_{\rm UV}$. This is the order of magnitude of the corrections due to the eventual UV completion of the theory at r_*. One has to make sure that the corrections in ϵ_* we are keeping are much larger than the neglected ones, namely, $\epsilon_* \gg r_h/r_*$. In summary, we have the following hierarchy of parameters (in the following, in order to avoid overly messy expressions, we insert the value of ϵ_* for the $X^5 = S^5$ case, remembering that for a generic X^5, its value is given by (2.40)):

$$e^{-1/\epsilon_*} \sim e^{-8\pi^2 N_c/\lambda_* N_f} \ll \frac{r_h}{r_*} \sim \frac{T}{\Lambda_{\rm UV}} \ll \epsilon_* \sim \frac{\lambda_*\ N_f}{8\pi^2\ N_c} \ll 1. \tag{2.95}$$

As long as $\epsilon_* \sim \lambda_*\ N_f/8\pi^2 N_c \ll 1$, there always exists a range of r_* such that this inequality is satisfied. Since we focus on the IR physics of the plasmas, at the scale set by their temperature, the actual physical constraint on the parameters will be $(\lambda_h/8\pi^2)(N_f/N_c) \ll 1$, which we have written in terms of the coupling at the scale of the horizon, $\lambda_h = \lambda_*(1 + O(\epsilon_*))$.

On top of this, we have to make sure that the SUGRA + DBI + WZ action we are using is valid. As usual, the suppression of closed string loops requires $N_c \gg 1$ whereas the suppression of α'-corrections is guaranteed by $\lambda_h \gg 1$. We have written the D7-brane worldvolume contribution to the action as a sum of N_f single brane contributions. This is justified if the typical energy of a string connecting two different branes is large (in α' units). Since the branes are distributed on a space whose size is controlled by $R \sim \lambda_h^{1/4}\sqrt{\alpha'}$, we again need $\lambda_h \gg 1$. The smearing approximation will be good if the distribution of D7-branes on the transverse space is dense, that is, $N_f \gg 1$. The discussion up to now is summarized in the following validity regime:

$$N_c \gg 1, \qquad \lambda_h \gg 1, \qquad N_f \gg 1, \qquad \epsilon_h = \frac{\lambda_h}{8\pi^2}\frac{N_f}{N_c} \ll 1. \tag{2.96}$$

Finally, we want to find the regime of parameters in which the flavor corrections are not only *valid* but are also the *leading* ones. With this aim, we ought to demand that the leading α'-corrections to the supergravity action (which typically scale as $\lambda_h^{-3/2}$ due to terms of the type $\alpha'^3\mathcal{R}^4$) are smaller than the flavor ones, controlled by ϵ_h, namely,

$$\lambda_h^{-3/2} \ll \epsilon_h. \tag{2.97}$$

Demanding that corrections to the D7-branes contributions (e.g., curvature corrections to the worldvolume action itself or corrections produced by possible modifications of the brane embeddings due to curvature corrections to the background metric) are subleading does not impose any further restriction. The reason is that their contribution is typically of order $\epsilon_h\lambda_h^{-c}$ for some $c > 0$ which is always subleading with respect to ϵ_h as long as (2.96) is satisfied.

The Holographic a-Function

As discussed in [77] and mentioned above, the string solution starts presenting pathologies at a scale $r_a < r_{\mathrm{LP}}$, where the holographic a-function is singular. The utility of the solution for $r > r_a$ is doubtful, but since we have only used the solutions up to $r_* \ll r_a$ in order to derive the IR physics, this subtlety does not affect the physical results. We now briefly review the argument in [77], which used the backreacted Klebanov-Witten solution at zero temperature. The qualitative picture holds for the rest of the cases addressed in the present section and for the case of Section 5 too.

Let us start by considering the metric of a generic dimensional reduction to five dimensions, giving a 5d Einstein frame metric of the form (the u here is, obviously, a redefined holographic coordinate, namely, $u = u(r)$):

$$ds_5^2 = H(u)^{1/3}\left[dx_{1,3}^2 + \beta(u)du^2\right]. \tag{2.98}$$

In standard set-ups, the function $H(u)^{1/6}$, which can be roughly identified with the dual field theory energy scale, monotonically varies with the radial coordinate. This is also required in order for the “holographic a-function” [96, 97]

$$a(u) \sim \beta(u)^{3/2}H(u)^{7/2}\left[H'(u)\right]^{-3}, \tag{2.99}$$

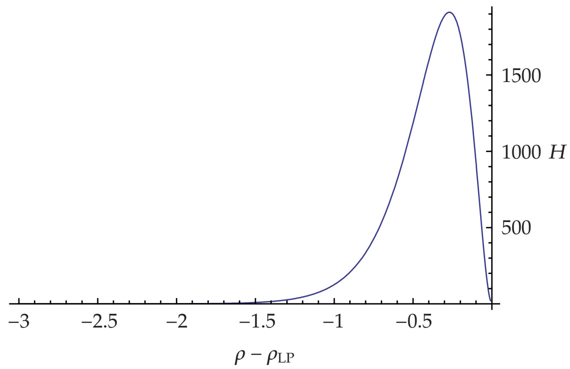

Figure 5: The function H in the massless-flavored KW model at zero temperature.

to be finite. (The monotonicity of $H(u)$ also plays a crucial role in holographic computations of the entanglement entropy; see [98]. The notations of that paper are used in the equations above.) Instead, the function $H(u)$ is not monotonic here: it increases with u from zero up to a maximum at a point u_a and then it decreases back to zero where h vanishes. (For the present discussion and in particular for Figure 5 we will choose the additive integration constant of h such that h is zero at the Landau pole. The specific point u_a (namely r_a) at which this UV pathology sets in depends on this choice. Again, we stress that the important point is the IR results do not depend on this choice (modulo suppressed contributions) as long as $r_a \gg r_*$. What we show here is that the integration constant can be naturally chosen such that this condition is satisfied.) In the flavored supersymmetric KW case, the H and a functions read:

$$H(\rho) \sim he^{2f+8g}, \qquad a(\rho) \sim h^{3/2} e^{3f} H^{7/2} [\partial_\rho H]^{-3}, \tag{2.100}$$

where we have not written unimportant overall factors. A representative plot is given in Figure 5. The nonmonotonic behavior of H implies that the holographic a-function is singular and discontinuous at the “a-scale”. From the plot, we see that $r_a \sim e^{\rho_a}$ is below, but not parametrically separated from r_{LP}.

3. A Dual to $\mathcal{N} = 1$ SQCD-Like Theories

In the following section, we will study a system that in some sense is qualitatively different from those of the previous sections, though the procedure to deal with the addition of flavors is identical. The main qualitative difference will be that there need not be a hierarchical difference between the number of flavors and the number of colors. The case treated here will represent the addition of fundamental matter to a field theory that is originally confining and four dimensional at low energies, but that gets some higher-dimensional completion in the UV (in principle this allows one to extend the range of the radial coordinate to arbitrarily large values). Some of the qualitative changes that observables of a confining theory undergo when fundamentals are added will be discussed. The developments described in the present section were applied to model possible aspects that could appear in physics beyond the standard model, as we will briefly mention below.

More concretely, in this section, we will study a dual to a version of $\mathcal{N} = 1$ SQCD. The model is based on D5-branes wrapped on two-cycles inside the resolved conifold—leading to a geometry related to the deformed conifold. We will first briefly present the model without flavors, then study the addition of flavors following the ideas described in the first sections of this paper: kappa symmetric embeddings, smearing, backreaction, system of BPS equations, and particular solutions to this system, and finally present a set of checks that the correspondence we are proposing is valid and robust; we also explain some predictions about the field theory obtained with the string background.

3.1. The Model without Flavors

The proposal is to construct a dual to a field theory with minimal SUSY in four dimensions using wrapped branes. Ideas of this kind were first explored by Witten in the early days of AdS/CFT. In [99], Witten presented a model dual to a version of Yang-Mills theory (with an extra massive scalar that gets mass due to loop corrections and UV-completed by an infinite tower of massive vectors, scalars, and fermions), by wrapping a set of N_c D4 branes on a circle with SUSY-breaking boundary conditions.

The idea here is very similar, only that we will work with D5-branes and we will preserve some amount of SUSY. We will compactify the five branes in a very subtle way (involving a twisting of the 6-d theory) so that only four supercharges will be preserved in the compactified theory for all energies [100] in other words, the partial SUSY breaking is *not* due to the presence of relevant operators, like mass terms. (The fact that a twisting procedure (see [101] for a very nice presentation of this idea) is at work implies that even in the far UV, the theory is still preserving only four supercharges.) This kind of compactification of the six-dimensional theory living on a stack of D5-branes (when the D5's wrap a two-cycle inside the resolved conifold) was well studied in various papers; see [102–104] for various reviews. We will follow mostly the detailed study of [105, 106].

One can show that a very generic string background describing a stack of N_c D5-branes wrapping a two cycle and preserving four supercharges includes a metric, RR-three form $F_3 = dC_2$, and a dilaton $\phi(\rho)$ and is given by

$$\begin{aligned} ds^2 &= \alpha' g_s e^{(\phi(\rho))/2}\Bigg[dx_{1,3}^2 + e^{2k(\rho)}d\rho^2 + e^{2h(\rho)}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \frac{e^{2g(\rho)}}{4} \\ &\qquad \times\left((\tilde{\omega}_1 + a(\rho)d\theta)^2 + (\tilde{\omega}_2 - a(\rho)\sin\theta d\varphi)^2\right) + \frac{e^{2k(\rho)}}{4}(\tilde{\omega}_3 + \cos\theta d\varphi)^2\Bigg], \\ F_{(3)} &= \frac{g_s\alpha' N_c}{4}\Big[-(\tilde{\omega}_1 + b(\rho)d\theta)\wedge(\tilde{\omega}_2 - b(\rho)\sin\theta d\varphi)\wedge(\tilde{\omega}_3 + \cos\theta d\varphi) \\ &\qquad + b' d\rho\wedge(-d\theta\wedge\tilde{\omega}_1 + \sin\theta d\varphi\wedge\tilde{\omega}_2) + \left(1 - b(\rho)^2\right)\sin\theta d\theta\wedge d\varphi\wedge\tilde{\omega}_3\Big], \end{aligned} \tag{3.1}$$

where $\tilde{\omega}_i$ are the left-invariant forms of SU(2):

$$\begin{aligned} \tilde{\omega}_1 &= \cos\psi d\tilde{\theta} + \sin\psi\sin\tilde{\theta}d\tilde{\varphi}, \\ \tilde{\omega}_2 &= -\sin\psi d\tilde{\theta} + \cos\psi\sin\tilde{\theta}d\tilde{\varphi}, \\ \tilde{\omega}_3 &= d\psi + \cos\tilde{\theta}d\tilde{\varphi}. \end{aligned} \tag{3.2}$$

For convenience, below we will set the parameters $g_s = \alpha' = 1$. The presence of the N_c color D5-branes is indicated in F_3 that satisfies the quantization condition:

$$\frac{1}{2\kappa_{(10)}^2}\int_{S^3} F_{(3)} = N_c T_5. \tag{3.3}$$

The S^3 on which we integrate is parameterized by $\widetilde{\theta}, \widetilde{\varphi}$, and ψ.

We then impose that a fraction of SUSY is preserved; hence we need to impose some projections on the Type IIB spinors and a set of BPS equations reflecting this arise (see [12, Appendix B] for generous details). The BPS equations are nonlinear, first order, and coupled for the functions of the background in (3.1)-for details see [12, Appendix B]. Certainly, solving first-order equations is simpler than solving the second order Einstein equations; nevertheless the BPS equations for the functions (ϕ, h, g, k, a, b) are nonlinear and coupled, rendering the problem complicated.

It is technically convenient to make a "change of basis" to another set of functions, so that the BPS equations become first order and nonlinear (of course) but can be decoupled, and then solved independently. A change of variables that does the job partially was obtained in [25]. The change of basis is from the set of functions $[\phi, h, g, k, a, b]$ into the functions $[P, Q, \tau, \Phi, Y, \sigma]$. The map reads [25]

$$e^{2h} = \frac{1}{4}\left(\frac{P^2 - Q^2}{P\cosh\tau - Q}\right), \quad e^{2g} = P\cosh\tau - Q, \quad e^{2k} = 4Y, \quad a = \frac{P\sinh\tau}{P\cosh\tau - Q}, \quad b = \frac{\sigma}{N_c}, \tag{3.4}$$

$$\Phi = \left(P^2 - Q^2\right)\sqrt{Y}e^{2\phi}. \tag{3.5}$$

As explained in detail in [25] (see Section 3 of that paper), the BPS equations can be solved one by one for these new functions, obtaining

$$\begin{gathered}
Q(\rho) = (Q_o + N_c)\cosh\tau + N_c(2\rho\cosh\tau - 1), \\
\sinh\tau(\rho) = \frac{1}{\sinh(2\rho - 2\rho_o)}, \quad \cosh\tau(\rho) = \coth(2\rho - 2\rho_o), \\
Y(\rho) = \frac{P'}{8}, \\
e^{4\phi} = \frac{e^{4\phi_o}\cosh(2\rho_o)^2}{(P^2 - Q^2)Y\sinh^2\tau}, \\
\sigma = \tanh\tau(Q + N_c) = \frac{(2N_c\rho + Q_o + N_c)}{\sinh(2\rho - 2\rho_o)}.
\end{gathered} \tag{3.6}$$

Note that both Q and the dilaton are given *algebraically* in terms of the rest of the functions parametrizing the backgrounds. Here Q_o and ϕ_o are constants of integration and we have chosen the integration constant in the dilaton field equation ϕ_o such that it admits a smooth limit as $\rho_o \to -\infty$ (this limit gives $\tau = \sigma = 0$ and so corresponds to what in [107] were called type **A** backgrounds).

The function P satisfies the following second-order equation:

$$P'' + P'\left(\frac{P' + Q'}{P - Q} + \frac{P' - Q'}{P + Q} - 4\coth(2\rho - 2\rho_o)\right) = 0. \tag{3.7}$$

We will refer to this equation as the "master" equation, since once we have a solution of (3.7) all other functions are determined via (3.6).

3.2. Some Solutions

There are many solutions to the master equation (3.7). A very simple one is given by

$$P = 2N_c\rho, \qquad Q_o = -N_c. \tag{3.8}$$

Once processed back, one computes the functions in the original background of (3.1) and one recovers an old solution [108]. To avoid nasty singular behaviors, in the following, we will choose the value of the integration constant $Q_o = -N_c$, so that the first term in the expression for $Q(\rho)$, namely, $(Q_o + N_c)$, vanishes. (If we do not make this choice, the space ends before $\rho = \rho_o$, since $Q > P$ possibly giving place to geodesically incomplete spaces and a divergent dilaton. Hence, we will choose the term proportional to $\cosh\tau(\rho)$ in $Q(\rho)$ to vanish.)

Aside from the simple solution presented above, there are a variety of very interesting solutions. For example, the function $P(\rho)$ near $\rho = 0$ has the following Taylor series:

$$P = h_1\rho + \frac{4h_1}{15}\left(1 - \frac{4N_c^2}{h_1^2}\right)\rho^3 + \frac{16h_1}{525}\left(1 - \frac{4N_c^2}{3h_1^2} - \frac{32N_c^4}{3h_1^4}\right)\rho^5 + \mathcal{O}\left(\rho^7\right), \tag{3.9}$$

where h_1 is again an arbitrary constant (notice that for $h_1 = 2N_c$ we get back to the solution in (3.8); we will also assume that $h_1 > 2N_c$). It is interesting that this solution can be numerically connected in a smooth way with a solution for large values of the radial coordinate ($\rho \to \infty$) that differs greatly from the linear behavior of the solution in (3.8). In this case, it is given by

$$P \sim e^{(4/3)\rho}\left[c\left(1 - \frac{8}{3}\rho e^{-4\rho}\right) + \frac{1}{64c}\left(256\rho^2 + 256Q_o\rho + 144N_c^2 + 64Q_o^2\right)e^{-(8/3)\rho} + O\left(e^{-4\rho}\right)\right]. \tag{3.10}$$

These solutions were studied explicitly in [12, Section 8] and have a variety of interesting applications that we will briefly mention in the following sections.

3.2.1. An Exact Recursive Solution

There is one recursive way of obtaining solutions, described in [109], that basically uses the fact that the master equation (3.7) can be written as (we choose here and in the following $\rho_o = 0$)

$$\partial_\rho\left(s\left(P^2 - Q^2\right)P'\right) + 4sP'QQ' = 0, \qquad s(\rho) = \sinh^2\tau = \frac{1}{\sinh^2(2\rho)}. \tag{3.11}$$

Integrating (3.11) twice we obtain

$$P^3 - 3Q^2P + 6\int_{\rho_2}^{\rho} d\rho' QQ'P + 12\int_{\rho_2}^{\rho} d\rho' s^{-1}\int_{\rho_1}^{\rho'} d\rho'' sP'QQ' = c^3R(\rho)^3, \tag{3.12}$$

where

$$R(\rho) \equiv \left(\cos^3\alpha + \sin^3\alpha(\sinh(4\rho) - 4\rho)\right)^{1/3}, \tag{3.13}$$

being (c, α) the two integration constants of the master equation.

Following [25] we write P in a formal expansion in inverse powers of c—the integration constant encountered above—as

$$P = \sum_{n=0}^{\infty} c^{1-n}P_{1-n}. \tag{3.14}$$

Inserting this expansion in (3.12) we obtain recursively

$$\begin{aligned}
P_1 &= R,\\
P_0 &= 0,\\
P_{-1} &= -\frac{1}{3}P_1^{-2}\left(-3Q^2P_1 + 6\int_{\rho_2}^{\rho} d\rho' QQ'P_1 + 12\int_{\rho_2}^{\rho} d\rho' s^{-1}\int_{\rho_1}^{\rho'} d\rho'' sQQ'P_1'\right),\\
P_{-2} &= 0,\\
P_{-n-2} &= -\frac{1}{3}P_1^{-2}\left\{\sum_{m=1}^{n+2}\left(2P_1P_{1-m}P_{m-n-2} + \sum_{k=1}^{n-m+3}P_{1-m}P_{1-k}P_{m+k-n-2}\right) - 3Q^2P_{-n}\right.\\
&\qquad \left. +6\int_{\rho_2}^{\rho} d\rho' QQ'P_{-n} + 12\int_{\rho_2}^{\rho} d\rho' s^{-1}\int_{\rho_1}^{\rho'} d\rho'' sQQ'P_{-n}'\right\}, \quad n \geq 1.
\end{aligned} \tag{3.15}$$

It follows by induction that $P_k = 0$ for all even k. The large ρ expansion of these solutions coincides with that described in (3.10) Once again, solutions written in this form have interesting applications to the physics of cascading quivers on the baryonic branch [110, 111].

We will not study the physics encoded in the solutions described above, suggesting the interested reader to consult the papers [12, 102–104, 109].

There is another set of solutions, proposed in [109] and whose physics content was developed in [112–114] that correspond to what are called "walking solutions". The idea here is to construct string backgrounds such that the dual QFT has a gauge coupling with very slow running (or "walking" coupling). See [109, 112–114] for detailed explanations on the physical implications of these solutions.

3.2.2. A Comment about the Dual Field Theory

The unflavored system of wrapped D5-branes has a field theory realized on its worldvolume, whose dual background and various solutions were described above. The field theory is a version of minimally SUSY Yang-Mills. Again, some UV completion takes over at high energies. (We are not saying that Super-Yang-Mills needs a UV completion, just that the system of D5-branes realizes a theory with these characteristics.) The field theory is minimally SUSY ($\mathcal{N} = 1$) and its *perturbative* spectrum, aside from a massless vector multiplet, contains a tower of massive vector and chiral multiplets. A careful study of the perturbative dual field theory obtained by compactification and twisting of the six-dimensional theory living on (unwrapped) D5-branes was done in [105, 106]. In that paper, the degeneracies and masses of the (perturbative) states in the tower mentioned above are given. More interestingly, the authors of [105, 106] showed that the theory is equivalent to $\mathcal{N} = 1^*$ Yang-Mills in a particular Higgs vacuum, where the extra dimensions appear by deconstruction. In this sense, we will think of the theory without flavors either as a six-dimensional theory compactified or as a four-dimensional theory with an infinite set of fields.

For our purposes, it will be enough to use the fact that the Lagrangian of the field theory reads

$$L = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}^2 - i\bar{\lambda}\gamma^\mu D_\mu\lambda + L(\Phi_k, W_k, W)\right], \tag{3.16}$$

where Φ_k and W_k represent the infinite number of massive chiral and vector multiplets and W denotes the massless vector multiplet. The term $L(\Phi_k, W_k, W)$, represents all the kinetic terms and interactions that can be deduced from [105, 106]. More comments about this field theory can be found in [25, Appendix A].

In what follows, we will summarize the procedure of adding flavors to this field theory. The flavor branes in this particular case are D5-branes.

3.3. Addition of Flavors

The study of supersymmetric embeddings in backgrounds of the form of (3.1), more precisely for the solution given in (3.8), was initiated in [19]. There the eigenspinors of the kappa symmetry matrix were found to be the spinors preserved by the background for a variety of D5-brane embeddings. For the purposes of this review, we will focus on the "cylinder embeddings" described in [19, Section 6.3] and in more detail in Section 6.5.3 of the third paper in [102–104]. In this case the flavor D5-branes are extended along the $R^{1,3}$ Minkowski directions, on the radial direction ρ, and also wrap the R-symmetry direction ψ. Intuitively, the flavor branes are localized in the directions $(\theta, \tilde{\theta}, \varphi, \tilde{\varphi})$, but interestingly enough, any constant value of these coordinates ensures that we have a kappa symmetric configuration. This is a very important fact, as we can put one flavor brane "at each point" of the four manifold $\Sigma[\theta, \tilde{\theta}, \varphi, \tilde{\varphi}]$ and still have a SUSY configuration.

This is precisely what we will take advantage of when smearing. Let us see this in more detail: if, as discussed in the first section, we write the action describing the closed strings (IIB) and the open strings (BIWZ), we will have

$$\begin{aligned} S &= \frac{1}{2\kappa_{(10)}^2} \int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{12} e^{\phi} F_{(3)}^2\right] \\ &\quad - T_5 \sum^{N_f} \int_{\mathcal{M}_6} d^6 x e^{\phi/2} \sqrt{-\hat{g}_{(6)}} + T_5 \sum^{N_f} \int_{\mathcal{M}_6} P[C_6], \end{aligned} \tag{3.17}$$

where the integrals are taken over the six-dimensional worldvolume of the flavor branes $\mathcal{M}_6$, and $\hat{g}_{(6)}$ stands for the determinant of the pull-back of the metric in such a worldvolume.

As discussed in previous sections, we then think of the $N_f \rightarrow \infty$ branes as being homogeneously smeared along the four transverse directions parameterized by the coordinates θ, φ and $\tilde{\theta}, \tilde{\varphi}$. The smearing erases the dependence on the angular coordinates and makes it possible to consider an ansatz with functions only depending on r, enormously simplifying computations. One has

$$\begin{aligned} -T_5 \sum^{N_f} \int_{\mathcal{M}_6} d^6 x e^{\phi/2} \sqrt{-\hat{g}_{(6)}} &\longrightarrow -\frac{T_5 N_f}{(4\pi)^2} \int d^{10}x \sin\theta \sin\tilde{\theta} e^{\phi/2} \sqrt{-\hat{g}_{(6)}}, \\ T_5 \sum^{N_f} \int_{\mathcal{M}_6} P[C_6] &\longrightarrow \frac{T_5 N_f}{(4\pi)^2} \int \mathrm{Vol}(\mathcal{Y}_4) \wedge C_{(6)}, \end{aligned} \tag{3.18}$$

where we have defined $\mathrm{Vol}(\mathcal{Y}_4) = \sin\theta \sin\tilde{\theta} d\theta \wedge d\varphi \wedge d\tilde{\theta} \wedge d\tilde{\varphi}$ and the new integrals span the full space-time. We will need the following expressions (with the choice explained above $\alpha' = g_s = 1$):

$$T_5 = \frac{1}{(2\pi)^5}, \qquad 2\kappa_{(10)}^2 = (2\pi)^7. \tag{3.19}$$

From here, we will have a set of BPS equations describing the dynamics of this open-closed string system. The same change of basis with the purposes described around (3.4) can be performed—see [25] for details. The solution in this case is dependent on the number of flavor branes N_f and reads (reinstating momentarily the integration constant ρ_o)

$$\sinh\tau = \frac{1}{\sinh(2(\rho - \rho_o))}, \tag{3.20}$$

for the function τ, while for Q, Φ we have

$$Q = \left(Q_o + \frac{2N_c - N_f}{2}\right)\cosh\tau + \frac{2N_c - N_f}{2}(2\rho\cosh\tau - 1), \tag{3.21}$$

$$e^{4(\phi-\phi_o)} = \frac{\cosh^2(2\rho_o)}{(P^2 - Q^2)Y\sinh^2\tau}. \tag{3.22}$$

In the case with flavors, like in the unflavored case previously discussed, both Q and the dilaton are given *algebraically* in terms of the rest of the functions parametrizing the backgrounds. As before, ρ_o, Q_o, and ϕ_o are constants of integration and we have chosen the integration constant in (3.22) such that it admits a smooth limit as $\rho_o \to -\infty$ (this limit gives $\tau = \sigma = 0$ and so corresponds to the type **A** backgrounds). The function Y is determined in terms of P as

$$Y = \frac{1}{8}(P' + N_f), \tag{3.23}$$

while the only remaining unknown, the function P, then satisfies the *new* decoupled second-order master equation:

$$P'' + (P' + N_f)\left(\frac{P' + Q' + 2N_f}{P - Q} + \frac{P' - Q' + 2N_f}{P + Q} - 4\coth(2\rho - 2\rho_o)\right) = 0. \tag{3.24}$$

One can redefine $P(\rho) = N_c p(\rho)$ and factor out N_c from the master equation. We will mention some solutions to (3.24), that explicitly include the quotient $x = N_f/N_c$; hence the solutions will capture the nontrivial physics of the fields transforming in the fundamental representation of the gauge group.

3.4. Study of Solutions

We now describe various solutions to the "flavored" master equation (3.24). Some solutions were found exactly, for the particular relation $N_f = 2N_c$ while some other are known as asymptotic expansions, near the UV (large ρ) and the IR (small ρ). In these latter cases, a smooth numerical interpolation can be found.

3.4.1. Exact Solutions for $N_f = 2N_c$

One can find some *exact* solutions for the case $N_f = 2N_c$ or $x = 2$. They were first discussed in the papers [12, 25].

For $N_f = 2N_c$ an exact type **A** ($\rho_o \to -\infty$) solution of (3.24) is

$$P = N_c + \sqrt{N_c^2 + Q_o^2}, \qquad Q = Q_o \equiv 4N_c\frac{(2-\xi)}{\xi(4-\xi)}, \quad 0 < \xi < 4. \tag{3.25}$$

Another solution with a qualitatively different UV behavior is

$$P = \frac{9N_c}{4} + ce^{4\rho/3}, \qquad c > 0, \; Q = \pm\frac{3N_c}{4}. \tag{3.26}$$

One can check that in these solutions the radial coordinate moves all over the real axis and that for $\rho \to -\infty$ the solutions take the same form but, as anticipated above, differ substantially in the far UV, for $\rho \to \infty$. Also, for the case $N_f = 2N_c$, [115, 116] discuss some extra solutions apart from the ones mentioned, including, interestingly, the generalization to near-extremal solutions. (The metric for the simplest nonextremal solution can be written in terms of a constant ξ and a function $\mathcal{F} = 1 - (z_h/z)^4$ as:

$$\begin{aligned} ds^2 = e^{\phi_o/2}\, z\bigg[&-\mathcal{F}dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + N_c\bigg(\frac{4}{z^2}\mathcal{F}^{-1}dz^2 + \frac{1}{\xi}\Big(d\theta^2 + \sin^2\theta d\varphi^2\Big) \\ &+\frac{1}{4-\xi}\Big(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\varphi}^2\Big) + \frac{1}{4}\Big(d\psi + \cos\theta d\varphi + \cos\tilde{\theta}d\tilde{\varphi}\Big)^2\bigg)\bigg]. \end{aligned} \tag{3.27}$$

The solution also contains nontrivial RR $F_{(3)}$ and dilaton; see [12] for details. Different features of this black hole solution have been analysed in [61, 117, 118]. An important remark is that the theory is in a Hagedorn phase and, indeed, the temperature coincides with the Hagedorn temperature of Little String Theory. For this reason, this solution is a bit problematic for studying the effect of quarks in a field theory plasma, unlike the finite temperature solution of Section 2.5.)

3.4.2. Asymptotic Expansions of Generic Solutions

Other solutions of interest have been discussed in [12, 25, 107]. We will summarize the results but suggest to the interested reader to go over those papers for details of all the metric functions.

In the UV (for $\rho \to \infty$), two possible asymptotics were found, that were called Class I and Class II in [25]. Table 1 summarizes the situation.

In the IR ($\rho \to 0$), three types of solutions were found, called Types I, II, and III (there exist other, qualitatively different solutions reported in [111]). The function $P(\rho)$ in these cases is

$$P = -N_f\rho + P_o + \frac{4}{3}c_+^3P_o^2\rho^3 - 2c_+^3N_fP_o\rho^4 + \frac{4}{5}c_+^3\left(\frac{4}{3}P_o^2 + N_f^2\right)\rho^5 + \mathcal{O}\left(\rho^6\right), \tag{3.28}$$

for Type I. For the Type II asymptotics, we assume that this behavior occurs when the IR is located at $\rho_{\mathrm{IR}} > \rho_o$. Without loss of generality we can choose $\rho_{\mathrm{IR}} = 0$. With this choice we then necessarily have $\rho_o < 0$. Expanding Q in (3.21) around $\rho = 0$ we obtain

$$Q = b_0 + b_1\rho + \mathcal{O}\left(\rho^2\right), \tag{3.29}$$

Table 1: The two classes of leading UV behaviors.

N_f	I	II
$< 2N_c$	$P \sim Q \sim \lvert 2N_c - N_f \rvert \rho$	
	$e^{2h} \sim \frac{1}{2}(2N_c - N_f)\rho$	
	$e^{2g} \sim N_c$	
	$Y \sim \frac{N_c}{4}$	
	$e^{4(\phi-\phi_o)} \sim \frac{e^{4(\rho-\rho_o)}\sinh^2(2\rho_o)}{2N_c^2(2N_c - N_f)\rho}$	
	$a \sim \frac{2}{N_c}(2N_c - N_f)e^{-2(\rho-\rho_o)}\rho$	
$> 2N_c$	$P \sim -Q \sim \lvert 2N_c - N_f \rvert \rho$	$P \sim c_+ e^{4\rho/3}$
	$e^{2h} \sim \frac{1}{4}(N_f - N_c)$	$e^{2h} \sim \frac{1}{4}c_+ e^{4\rho/3}$
	$e^{2g} \sim \frac{1}{2}(N_f - 2N_c)\rho$	$e^{2g} \sim c_+ e^{4\rho/3}$
	$Y \sim \frac{1}{4}(N_f - N_c)$	$Y \sim \frac{1}{6}c_+ e^{4\rho/3}$
	$e^{4(\phi-\phi_o)} \sim \frac{e^{4(\rho-\rho_o)}\sinh^2(2\rho_o)}{2(N_c - N_f)^2(N_f - 2N_c)\rho}$	$e^{4(\phi-\phi_o)} \sim 1$
	$a \sim e^{-2(\rho-\rho_o)}\rho$	$a \sim 2e^{-2(\rho-\rho_o)}$
$= 2N_c$	$P \sim N_c + \sqrt{N_c^2 + Q_o^2} \sim \frac{8N_c}{(4-\xi)\xi}$	
	$e^{2h} \sim \frac{N_c}{\xi}$	
	$e^{2g} \sim \frac{4N_c}{4-\xi}$	
	$Y \sim \frac{N_c}{4}$	
	$e^{4(\phi-\phi_o)} \sim e^{4(\rho-\rho_o)}\sinh^2(2\rho_o)\frac{(4-\xi)\xi}{16N_c^3}$	
	$a \sim \frac{4}{\xi}e^{-2(\rho-\rho_o)}$	

where

$$b_0 = -\coth(2\rho_o)\left(Q_o + \frac{2N_c - N_f}{2}\right) - \frac{2N_c - N_f}{2},$$
$$b_1 = -\frac{2}{\sinh^2(2\rho_o)}\left(Q_o + \frac{2N_c - N_f}{2}\right) - (2N_c - N_f)\coth(2\rho_o). \tag{3.30}$$

Looking for IR solutions of (3.24) we find that we must require that $b_0 > 0$. The corresponding asymptotic solution then takes the following form:

$$P = Q + h_1\rho^{1/2} - \frac{1}{6b_0}\left(h_1^2 + 12b_0(b_1 + N_f)\right)\rho$$
$$+ \frac{h_1}{72b_0^2}\left(5h_1^2 + 6(5b_1 + 2N_f)b_0 - 72b_0^2\coth(2\rho_o)\right)\rho^{3/2} + \mathcal{O}\left(\rho^2\right), \tag{3.31}$$

where h_1 is an arbitrary constant. Note that this expansion for P admits a smooth limit when $\rho_o \to -\infty$ and so it is valid for both solutions of type $\mathbf{A}$ $(\rho_o \to -\infty)$.

Finally, for Type III asymptotics we consider $\rho_{\mathrm{IR}} > \rho_o$ and we take $\rho_{\mathrm{IR}} = 0$. In terms of the expansion (3.29) this requires that $b_0 = 0$. We then find

$$P = h_1\rho^{1/3} - \frac{9N_f}{5}\rho - \frac{2h_1}{3}\coth(2\rho_o)\rho^{4/3} - \frac{1}{175h_1}\left(50b_1^2 - 18N_f^2\right)\rho^{5/3} + \mathcal{O}\left(\rho^2\right), \tag{3.32}$$

where $h_1 \neq 0$ is an arbitrary constant.

To leading order the solutions for large ρ—UV solutions—are quoted in Table 1. It is the presence of subleading terms that allow the smooth numerical interpolation with three possible IR behaviors discussed.

The physics of the dual field theory encoded in these solutions was discussed in detail in [12, 25, 107] by computing various observables using the string solution of (3.1) evaluated on the solutions above. (Finding a numerical interpolation between the IR solutions and the solutions of Class I in the far UV is (numerically) delicate. One can see some plots in [112, Section 5].) We move now to discuss general features of the dual field theory.

3.5. *The Dual Field Theory*

The proposal here is the following: without the addition of the flavor branes, the field theory is known to be a twisted version of six-dimensional Yang-Mills, or as we discussed above, a four-dimensional QFT with an infinite number of massive fields. See Section 3.2.2. To get an *intuitive* understanding of the modifications of the dynamics produced by the "quark" fields (that feature below), we will consider that all the infinite massive fields are chiral multiplets and then argue that the dynamics is ruled by a lagrangian of the following form:

$$\begin{aligned} L = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}^2 - i\bar{\lambda}\gamma^\mu D_\mu\lambda + L(\Phi_k, W)\right] \sim \int d^2\theta W_\alpha W^\alpha \\ + \sum_k \int d^4\theta \Phi_k^\dagger e^V \Phi_k + \int d^2\theta \mu_k |\Phi_k|^2 + \cdots . \end{aligned} \tag{3.33}$$

When the flavor branes come into play, we are adding "quark superfields" that are realized as the open strings going from the noncompact flavor branes to the compact (or wrapped) color branes (as usual, the open strings that begin and end on a flavor brane decouple and do not contribute to the four dimensional dynamics). More concretely, we add the quark and antiquark superfields $(Q, \widetilde{Q})$ and propose that we have a Lagrangian for the massive fields interacting with the quark-antiquark superfields $Q, \widetilde{Q}$ schematically of the following form (for more details, see [25, Appendix A]):

$$L\left(\Phi_k, \mathrm{W}, Q, \widetilde{Q}\right) = \sum_k \int d^4\theta \Phi_k^\dagger e^V \Phi_k + \kappa_k \int d^2\theta \widetilde{Q}\Phi_k Q + \mu_k |\Phi_k|^2 + \cdots , \tag{3.34}$$

and canonical kinetic terms for $(Q, \widetilde{Q})$. In this system, the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ symmetry is explicitly broken to the diagonal $\mathrm{SU}(N_f)_D$ by the presence of the coupling $\widetilde{Q}\Phi_k Q$. In this respect, the theory is qualitatively different from $\mathcal{N} = 1$ SQCD.

One may be interested in the theory at low energies and hence integrate out the massive fields (either massive vectors or massive chirals) and after some algebra end with a theory of the following form (again schematically):

$$L = \int d^4\theta \left(Q^\dagger e^V Q + \widetilde{Q}^\dagger e^{-V} \widetilde{Q}\right) + \int d^2\theta W_\alpha W^\alpha + \kappa \int d^2\theta \left(\widetilde{Q}Q\right)^2, \tag{3.35}$$

where we have a (naively irrelevant) deformation of $\mathcal{N} = 1$ SQCD.

We emphasize that this is an *intuitive* way of understanding the field theory dual to the flavored system described above. As we will summarize below there are various observables that can be computed that match the predicted (or expected) result. So, the *precise* dual QFT should be something similar to what we described above, or at least with the same qualitative physics.

3.6. Checks and Predictions

This subsection summarizes results developed in [12, 25, 107]. There is a point that should be emphasized here. All the solutions to the BPS equations or the master equation (3.24) that have been found up to the time of writing this review present a singularity in the IR (typically at $\rho = 0$). In spite of this being a "good singularity" according to some criteria developed in the literature [73], the presence of the singularity makes the interpretation of IR observables a bit unclear. In other words, though one gets the "correct or expected" result, one should perhaps handle those particular computations with care.

Let us then concentrate on various quantities computed in the UV and then we will specify some that are mostly influenced by the IR of the geometry.

3.6.1. Beta Function and Anomalies

The gauge coupling and the theta angle of the dual QFT can be defined as explained in various places; see, for example, [107, Section 4.1] or [25, Section 5]. One gets, after some algebra, that the gauge coupling is related to the functions of the background as

$$\frac{8\pi^2}{g^2} = e^{-\tau} P. \tag{3.36}$$

Choosing a particular radius-energy relation, that was discussed in [119–121], one can compute the variation of the coupling with respect to energy. Using the solutions where the dilaton asymptotes to a linear function $(e^{4\phi} \sim e^{4\rho}/\rho)$ and working to leading order in an expansion in inverse powers of the radial coordinate, we get

$$\beta_{8\pi^2/g^2} = \frac{3}{2}(2N_c - N_f), \tag{3.37}$$

that coincides with the result predicted by the NSVZ result, once we assign anomalous dimensions to the quark superfields $\gamma_Q = \gamma_{\tilde{Q}} = -1/2$.

Similarly, one can define a geometrical quantity that can be associated with the quartic coupling. See [107, Section 4.2]. The beta function can be computed using the anomalous dimensions discussed above and again get matching with the interesting fact that for $N_f > 2N_c$ the quartic coupling is irrelevant, for $N_f < 2N_c$ the coupling is relevant, while for $N_f = 2N_c$ the coupling is not running. See [72] for a nice explanation of this fact.

One can also assign a value of the R-charge to the quark superfields to get the correct R-symmetry transformation properties of the quartic superpotential of (3.35), that is, $R[Q] = R[\tilde{Q}] = 1/2$. This predicts that the R-symmetry anomaly, the triangle with one R-current and two gauge currents, is proportional to the quantity $(2N_c - N_f)$ times the phase by which we are rotating the fermions. This is the precise result that the string background gives. Indeed, if we compute the Θ-angle as explained in [107, Section 4.1] or in [25, Section 5], we will get

$$\Theta = \frac{\psi - \psi_0}{2}(2N_c - N_f), \tag{3.38}$$

where we associated $(\psi - \psi_0)/2$ with the change in phase of the fermions in the quark multiplet and the gauge multiplet to get perfect matching. In the same vein, it is possible to attempt a ’t Hooft matching of anomalies, that is of triangles involving three global currents. The reader will find it quite instructive to go over [107, Section 4.7]. There, a detailed study of the matching of the correlator of three global currents—some of them corresponding to discrete symmetries, some of them being continuous symmetries—is presented. The treatment is performed in the case of Type **A** backgrounds, that are characterized by the fact that the functions are $a = b = 0$ in (3.1). This translates to the fact that the R-symmetry is broken to $\mathbb{Z}_{2N_c - N_f}$ without the further (spontaneous) breaking to $\mathbb{Z}_2$.

3.6.2. Seiberg Duality

It is known that Seiberg duality manifests beautifully in a QFT like the one of (3.35). This is explained in [72, Section 1.10]. The backgrounds discussed here show this in a very nice way. Indeed, as discussed, for example, in [25], we can see that the master equation (and the whole system) is invariant under the following change:

$$P \longrightarrow P, \quad Q \longrightarrow -Q, \quad \sigma \longrightarrow -\sigma, \qquad N_c \longrightarrow N_f - N_c, \quad N_f \longrightarrow N_f, \tag{3.39}$$

while all other functions are invariant. Geometrically, this change amounts to swapping the two S^2 in the background, namely, those parameterized by θ, φ and $\widetilde{\theta}, \widetilde{\varphi}$ in (3.1). This should be interpreted as follows: suppose that we are presented with a background, representing the dynamics of a field theory with N_c colors and N_f flavors. This implies that we have a particular solution to the master equation (3.24) for the function P. With this solution, and applying the changes of (3.39), we can construct another solution, that will be related to the first one by a differomorphism and that will describe the physics of a field theory with $N_f - N_c$ colors and N_f flavors. Various aspects of this interesting duality have been discussed in [25, 79, 107] and probably elsewhere.

The implementation of Seiberg duality in $\mathcal{N} = 1$ subcritical string models was discussed in [57, 122]. One can think of the sphere exchange mentioned above as being the geometrical version of the mechanism described in these papers. Interestingly, similar methods were used to propose a non-supersymmetric Seiberg duality in [123].

3.7. IR Physics: Domain Walls and Some Comments on Wilson/'t Hooft Loops

One observable that can be computed and that strongly depends on the $\rho \rightarrow 0$ region of the solution (the IR) is the tension of domain walls. Indeed, domain walls can be thought of as D5-branes that wrap a three cycle inside the internal six-dimensional manifold and that extend on two of the Minkowski directions (and time, of course). We can compute the tension of a wall by considering a probe D5-brane that sits on the manifold $\Sigma_6 = [t, x_1, x_2, \widetilde{\theta}, \widetilde{\varphi}, \psi]$, at constant $\rho = 0$, constant θ, φ. The Born-Infeld action for this probe can be computed and one reads that the effective tension is given by (see, e.g., [12, Section 5.6]):

$$T_{\mathrm{DW}} = 4\pi^2 e^{2\phi+2g+k} T_{\mathrm{D5}}, \tag{3.40}$$

which, when evaluated at $\rho = 0$, gives a constant proportional to $2N_c - N_f$. (This would indicate that the walls are tensionless for $N_f = 2N_c$, a particular point of the QFT previously discussed, where it was argued that conformal symmetry is developed.) This is a good example for an observable, since even when a singularity is present (this typically reflects in some of the functions of the background being divergent), the combination above is finite. This is typical of "good singularities". There are other observables that can be computed using D-branes or fundamental strings; examples of these are Wilson or 't Hooft loops. A similar conspiracy of functions that avoids an infinite result occurs here. Nevertheless, one should be quite careful with these quantities as noted above. Indeed, it was found in [112, Section 5] that for the particular case of the backgrounds studied in this section, the Nambu-Goto action for the fundamental string might cease to be a good approximation as the string develops a cusp when approaching the singularity.

In other words, we believe that the solutions presented in this section surely capture correctly many UV aspects of the field theory, together with some IR observables. Probably, we could think of the presence of the singularity in the same way as we think about the singularity in the Klebanov-Tseytlin background that captures some of the physics, but some is lost and the singularity must be resolved. (In contrast to what happens for the solutions discussed here, the Klebanov-Tseytlin background presents a *bad* singularity and the IR physics computed with that solution is not trustable.) There are different ways of attempting a resolution of the singularity, for example, considering massive quarks. This is under present study.

3.7.1. Wilson Loops and First-Order Phase Transitions

One can consider the situation (of course, this is an idealized situation) in which all the flavors are massive and with a fixed sharp mass M_0 (corresponding to a given value of the radial coordinate, that we call $\rho = m_0$). The way to model this in a first approximation is to consider $N_f(\rho) = N_f\Theta(\rho - m_0)$, where $\Theta(\rho)$ is a Heaviside step function. Once again, there is some dynamics that is being lost in doing this, for example, the matching of the derivatives of the solutions is not smooth at the point $\rho = m_0$ and the curvature of the background is not well defined at that point. Nevertheless, it is possible (within this approximately correct way of proceeding) to find a solution that for energies below the scale set by m_0 corresponds to the theory *without* flavors, say those discussed around (3.9), and far in the UV corresponds to the flavored theory, as represented by solutions of the Class I in Table 1. One can then compute the Wilson loop following the well-known prescription [124, 125]. This was done explicitly in [126]. The qualitative result is the following: for a range of ratios between the mass of the quarks M_0 and the value of the gaugino condensate set by the function $a(\rho)$ one observes that the relation between the quark-antiquark potential V_{QQ} and their separation L_{QQ} presents a first-order transition. (Notice that we should talk of a "Quantum" phase transition, as the system is at zero temperature.) In other words, a point where dV_{QQ}/dL_{QQ} is discontinuous. The same kind of behavior was observed in systems where a more careful study is possible. Indeed, in the backreacted Klebanov-Witten (see Section 2) and Klebanov-Strassler (see Section 5) models, it was possible to find the precise form for the function $p(\rho)$—that shows that the "Heaviside approximation" described above is not a bad one. The same qualitative behavior for the first-order phase transition was found in [26, 127, 128].

This kind of first-order transitions for the Wilson loop string configurations is by no means particular of systems with dynamical fundamental fields. In fact, they were first found in a different context in [129], where a nice connection between these Wilson loops computations and the Van der Waals gas (paradigm of the first order transition) was put forward (further discussions can be found, e.g., in [112, 127]). Different examples of such phase transitions in systems without flavors have been worked out in [130, 131].

The "morale" seems to be the following: when we have a physical system that has two *independent* scales (i.e., two scales that can be tuned independently, in the present example, the mass of the quarks M_0 and the gaugino condensate Λ^3) the first-order phase transition for the quantity $V_{QQ}(L_{QQ})$ will be present. Of course, like in any other first-order transition, it will happen that the discontinuity in the derivative will disappear for some ratio between the scales mentioned above.

4. A Dual to a (2+1)-Dimensional $\mathcal{N} = 1$ SQCD-Like Model

In this section we will study gravity duals to minimal supersymmetric theories in 2+1 dimensions. These backgrounds can be obtained by wrapping D5-branes along three cycles of manifolds with G_2 holonomy [132–134]. The corresponding field theory dual is a (2+1)-dimensional $\mathcal{N} = 1$ supersymmetric $U(N_c)$ Yang-Mills theory with a level k Chern-Simons interaction. Such a theory coupled to an adjoint massive scalar field should arise on the domain walls separating the different vacua of pure $\mathcal{N} = 1$ super-Yang-Mills in 3+1 dimensions. The corresponding unflavored background was studied in [133], where it was argued to be dual to a $U(N_c)$ gauge theory with Chern-Simons level $k = N_c/2$. In what follows we will review a generalization of these results, following closely [135]. We will present the deformation of the background of [132–134] induced by a smeared distribution of massless flavors. In order to formulate these generalized backgrounds, let σ^i and ω^i $(i = 1, 2, 3)$ be two sets of SU(2) left-invariant one forms, obeying

$$d\sigma^i = -\frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k, \qquad d\omega^i = -\frac{1}{2}\epsilon_{ijk}\omega^j \wedge \omega^k. \tag{4.1}$$

The forms σ^i and ω^i parameterize two three-spheres. In the geometries we will be dealing with, these spheres are fibered by a one-form A^i. The corresponding ten-dimensional metric of the type IIB theory in the Einstein frame is given by

$$ds^2 = e^{\phi/2}\left[dx_{1,2}^2 + dr^2 + \frac{e^{2h}}{4}\left(\sigma^i\right)^2 + \frac{e^{2g}}{4}\left(\omega^i - A^i\right)^2\right], \tag{4.2}$$

where $\phi(r)$ is the dilaton of type IIB supergravity and g and h are functions of the radial variable r. In addition, the one-form A^i will be taken as

$$A^i = \frac{1 + w(r)}{2}\sigma^i, \tag{4.3}$$

with $w(r)$ being a new function of r. For convenience in this section we will take $g_s = \alpha' = 1$, as we did in Section 3. The backgrounds considered here are also endowed with an RR three-form F_3. We will represent F_3 as the sum of two contributions:

$$F_3 = \mathcal{F}_3 + f_3, \tag{4.4}$$

where $d\mathcal{F}_3 = 0$ and f_3 is the part of the RR three-form which is responsible for the violation of the Bianchi identity $(df_3 \neq 0)$ and which is sourced by the flavor D5-branes. Let us first parametrize the component $\mathcal{F}_3$ as

$$\frac{\mathcal{F}_3}{N_c} = -\frac{1}{4}\left(\omega^1 - B^1\right) \wedge \left(\omega^2 - B^2\right) \wedge \left(\omega^3 - B^3\right) + \frac{1}{4}F^i \wedge \left(\omega^i - B^i\right) + H, \tag{4.5}$$

where B^i is a new one-form and F^i are the components of its field strength, given by

$$F^i = dB^i + \frac{1}{2}\epsilon_{ijk}B^j \wedge B^k. \tag{4.6}$$

In (4.5) H is a three-form that is determined by imposing the Bianchi identity for $\mathcal{F}_3$, namely

$$d\mathcal{F}_3 = 0. \tag{4.7}$$

By using (4.1) one can easily check from the explicit expression written in (4.5) that, in order to fulfill (4.7), the three-form H must satisfy the following equation:

$$dH = \frac{1}{4}F^i \wedge F^i. \tag{4.8}$$

In what follows we shall adopt the following ansatz for B^i:

$$B^i = \frac{1+\gamma(r)}{2}\sigma^i, \tag{4.9}$$

where $\gamma(r)$ is a new function. After plugging the ansatz of B^i written in (4.9) into (4.6), one gets the expression for F^i in terms of $\gamma(r)$:

$$F^i = \frac{\gamma'}{2}dr \wedge \sigma^i + \frac{\gamma^2 - 1}{8}\epsilon_{ijk}\sigma^j \wedge \sigma^k, \tag{4.10}$$

where the prime denotes the derivative with respect to the radial variable r. Using this result for F^i in (4.8) one can easily determine the three-form H in terms of γ. Let us parameterize H as

$$H = \frac{1}{32}\frac{1}{3!}\mathcal{H}(r)\epsilon_{ijk}\sigma^i \wedge \sigma^j \wedge \sigma^k. \tag{4.11}$$

Then, by solving (4.8) for H, one can verify that $\mathcal{H}(r)$ is the following function of the radial variable:

$$\mathcal{H} = 2\gamma^3 - 6\gamma + 8\kappa, \tag{4.12}$$

with κ being an integration constant.

Let us now consider the contribution f_3 to the RR three-form F_3. As explained above, this contribution violates the Bianchi identity and is nonzero when flavor branes are present. Indeed, let us write the WZ term of the action of a system of flavor D5-branes as

$$S^{\text{WZ}}_{\text{flavor}} = T_5 \int_{\mathcal{M}_{10}} \Omega \wedge C_6, \tag{4.13}$$

with Ω being a four-form with components along the space transverse to the worldvolume of the branes. Then, the coupling to the RR potential C_6 written in (4.13) gives rise to the following modified Bianchi identity:

$$dF_3 = df_3 = 4\pi^2\Omega. \tag{4.14}$$

To write a specific ansatz for Ω and f_3 we have to select some family of supersymmetric embeddings for the flavor branes. As explained above, this can be done by using kappa symmetry. In the simplest case one looks for massless embeddings, which extend along the full range of the radial coordinate r. Those are the configurations considered in [135], in which the D5-brane is extended along the three Minkowski directions x^μ as well as along a three-dimensional cylinder spanned by r and two other angular directions. Actually, it was shown in [135] that these two angular directions could be the ones corresponding to σ^3 and ω^3. The corresponding transverse volume for this configuration is just

$$\mathrm{Vol}\left(\mathcal{Y}_4^{1,2}\right) = \sigma^1 \wedge \sigma^2 \wedge \omega^1 \wedge \omega^2. \tag{4.15}$$

However, there is nothing special in our background about these directions. Indeed, both in the metric and in the RR three-form $\mathcal{F}_3$, we are adopting a round ansatz which does not distinguish among the directions of the two three-spheres. Thus we could as well consider supersymmetric cylinder embeddings that span the σ^1, ω^1 or σ^2, ω^2 directions. The volume forms of the spaces transverse to these embeddings are clearly

$$\mathrm{Vol}\left(\mathcal{Y}_4^{2,3}\right) = \sigma^2 \wedge \sigma^3 \wedge \omega^2 \wedge \omega^3, \qquad \mathrm{Vol}\left(\mathcal{Y}_4^{1,3}\right) = \sigma^1 \wedge \sigma^3 \wedge \omega^1 \wedge \omega^3. \tag{4.16}$$

To construct a backreacted supergravity solution with the same type of ansatz as in (4.2) we should consider a brane configuration that combines these three possible types of embeddings in an isotropic way. The corresponding transverse volume form $\mathrm{Vol}(\mathcal{Y}_4)$ of this three-branch brane system would be just the sum of the three four-forms written in (4.15) and (4.16). The corresponding smearing form Ω is obtained by multiplying by the suitable normalization factor, namely,

$$\Omega = -\frac{N_f}{16\pi^2}\mathrm{Vol}(\mathcal{Y}_4) = -\frac{N_f}{64\pi^2}\epsilon_{ijk}\epsilon_{ilm}\sigma^j \wedge \sigma^k \wedge \omega^l \wedge \omega^m, \tag{4.17}$$

where the minus sign has its origin in the different orientation (required in the kappa symmetry analysis of [135]) of the D5-brane worldvolume with respect to the ten-dimensional space. It is now straightforward to use the Ω written in (4.17) and get an expression of f_3 whose modified Bianchi identity is the one of (4.14). One has

$$f_3 = \frac{N_f}{8}\epsilon_{ijk}\left(\omega^i - \frac{\sigma^i}{2}\right) \wedge \sigma^j \wedge \sigma^k. \tag{4.18}$$

Equation (4.18) completes our ansatz for the general flavored case. Using these expressions of the metric and RR three-form in the supersymmetry variations of the dilatino and gravitino of type IIB supergravity, after imposing that the background preserves two supersymmetries, we arrive at a system of first-order BPS equations. These equations, which are rather involved, have been derived and analyzed in detail in [135]. They admit several consistent truncations which lead to simpler solutions. One can, for example, first consider the unflavored case $N_f = 0$. If, in addition, we require that the function g is constant and that the fibering functions w and γ are equal, our ansatz reduces to the one considered in [132–134]. Actually, in this case the BPS equations fix the value of g to be $e^{2g} = N_c$ and, in order to have a regular solution, one should take the constant κ of (4.12) to be equal to $1/2$ (κ is related to the Chern-Simons level k of the dual field theory). Other unflavored solutions exist and have been studied in detail in [135]. Here we will concentrate on reviewing the case in which $N_f \neq 0$, starting from a particular truncation of the BPS system which is very interesting and serves to classify the different more involved solutions in the UV.

4.1. The Truncated System

In this section we will analyze the truncation of the general system of BPS equations that corresponds to taking $w = \gamma = \kappa = 0$. In this case the BPS equations of [135] for the remaining functions h and g of the metric and for the dilaton ϕ consistently reduce to the following simple system of differential equations:

$$\begin{aligned}
\phi' &= N_c e^{-3g} - \frac{3}{4}\left(N_c - 4N_f\right)e^{-g-2h}, \\
h' &= \frac{1}{2}e^{g-2h} + \frac{N_c - 4N_f}{2}e^{-g-2h}, \\
g' &= e^{-g} - \frac{1}{4}e^{g-2h} - N_c e^{-3g} + \frac{N_c - 4N_f}{4}e^{-g-2h}.
\end{aligned} \tag{4.19}$$

By inspecting the system (4.19) one readily realizes that there is a special solution for which the metric functions h and g are constant. Actually this solution only exists when $N_c < 2N_f$ and the corresponding expressions for g and h are the following:

$$e^{2g} = 4N_f - N_c, \qquad e^{2h} = \frac{1}{4}\frac{\left(4N_f - N_c\right)^2}{2N_f - N_c} \quad \left(N_c < 2N_f\right), \tag{4.20}$$

while the dilaton grows linearly with the holographic coordinate r, namely,

$$\phi = \frac{2\left(3N_f - N_c\right)}{\left[4N_f - N_c\right]^{3/2}} r + \phi_0. \tag{4.21}$$

Let us next consider solutions for which the function h is not constant. In this case we can use $\rho = e^{2h}$ as a radial variable and one can define a new function $F(\rho)$ as $F(\rho) = e^{2g}$. It follows from (4.19) that the BPS equation for $F(\rho)$ is now,

$$\frac{dF}{d\rho} = \frac{\left(F - N_c\right)\left(2 - F/2\rho\right) - \left(2N_f/\rho\right)F}{F + N_c - 4N_f}, \tag{4.22}$$

while the equation for the dilaton as a function of ρ can be written as,

$$\frac{d\phi}{d\rho} = \frac{N_c}{F(F+N_c-4N_f)}\left[1-\frac{3}{4\rho}\left(1-\frac{4N_f}{N_c}\right)F\right]. \tag{4.23}$$

Moreover, from the second equation in (4.19) we can obtain the relation between the two radial variables r and ρ, namely,

$$\frac{dr}{d\rho} = \frac{\sqrt{F(\rho)}}{F(\rho)+N_c-4N_f}. \tag{4.24}$$

Notice that the sign of the right-hand side of (4.24) could be negative when $N_f \neq 0$. This means that we have to be careful in identifying the UV and IR domains in terms of the new radial variable ρ. We can use the result of integrating (4.22)–(4.24) to obtain the metric in terms of the new variable ρ, which takes the following form:

$$ds^2 = e^{\phi/2}\left[dx_{1,2}^2 + \left(\frac{dr}{d\rho}\right)^2 (d\rho)^2 + \frac{\rho}{4}\left(\sigma^i\right)^2 + \frac{F}{4}\left(\omega^i - A^i\right)^2\right]. \tag{4.25}$$

Let us now study the different solutions of (4.22)-(4.23).

4.1.1. Linear Dilaton Backgrounds

When $N_f = 0$, (4.22) can be simply solved by taking $F = N_c$. However, it is clear from (4.22) that in the flavored case $F = N_c$ is no longer a solution of the equations. Nevertheless, there are solutions for which this constant value of F is reached asymptotically when $\rho \to \infty$. Indeed, one can check this fact by solving (4.22) as an expansion in powers of $1/\rho$. One gets:

$$F = N_c + N_c N_f \frac{1}{\rho} - \frac{3}{4} N_c N_f (N_c - 4N_f)\frac{1}{\rho^2} + \cdots \quad (\rho \longrightarrow \infty). \tag{4.26}$$

By plugging the expansion (4.26) into (4.23) one can prove that when $N_c \neq 2N_f$, these solutions have a dilaton that depends linearly on ρ in the UV and, actually, one can verify that

$$\frac{d\phi}{d\rho} = \frac{1}{2(N_c-2N_f)} - \frac{3N_c^2 - 12N_cN_f + 16N_f^2}{8(N_c-2N_f)^2}\frac{1}{\rho} + \cdots \quad (\rho \longrightarrow \infty). \tag{4.27}$$

Notice the different large ρ behavior of the dilaton in the two cases $N_c > 2N_f$ and $N_c < 2N_f$. Indeed, when $N_c > 2N_f$, the dilaton grows linearly with the holographic coordinate ρ (the behavior expected for a confining theory in the UV), while for $N_c < 2N_f$ the field ϕ decreases linearly with ρ. This seems to suggest that the sign of the beta function of the dual gauge theory depends on N_c and N_f through the combination $N_c - 2N_f$. Actually one can verify by means of a probe calculation in the complete system that the beta function is positive for $N_c > 2N_f$ and changes its sign when $N_c < 2N_f$ [135].

Equation (4.22) can be solved numerically by imposing the behavior (4.26) for large ρ. Once $F(\rho)$ is known, one can obtain the dilaton $\phi(\rho)$ by direct integration of the right-hand side of (4.23). The result of this numerical calculation was analyzed in detail in [135]. Let us only mention here that, in the most interesting case $N_c > 2N_f$, the function F diverges for $\rho \to 0$, while the dilaton ϕ remains finite for small ρ. This bad IR behavior of F is cured in the untruncated solution with the same leading UV form of F and ϕ but with $w, \gamma \neq 0$ (see below).

4.1.2. Flavored G_2 Cone

Let us now consider the solution of (4.22) and (4.23) that leads to a metric which is asymptotically a G_2-cone with constant dilaton in the UV. It can be checked that there exists a solution of (4.22) which can be expanded for large values of ρ as

$$F = \frac{4}{3}\rho + 4\left(N_f - N_c\right) + \frac{15N_c^2 - 39N_cN_f + 24N_f^2}{\rho} + \cdots . \tag{4.28}$$

The corresponding expansion for $\phi(\rho)$ is

$$\phi = \phi_* - \frac{9N_f}{4}\frac{1}{\rho} - \frac{27}{32}N_c\left(N_c + 2N_f\right)\frac{1}{\rho^2} + \cdots , \tag{4.29}$$

where ϕ_* is the constant limiting value of ϕ in the UV. In order to explore the asymptotic form of the metric for large ρ, it is convenient to perform a change in the radial variable, namely

$$\rho = \frac{1}{3}\tau^2, \tag{4.30}$$

in terms of which the metric asymptotically becomes the one corresponding to the direct product of (2+1)-dimensional Minkowski space and a seven-dimensional cone with G_2 holonomy, namely

$$ds^2 \approx e^{\phi_*/2}\left[dx_{1,2}^2 + (d\tau)^2 + \frac{\tau^2}{12}\left(\sigma^i\right)^2 + \frac{\tau^2}{9}\left(\omega^i - \frac{\sigma^i}{2}\right)^2\right]. \tag{4.31}$$

To find the solution in the whole range of the radial coordinate one can numerically integrate the system (4.22)-(4.23) by imposing the asymptotic behavior (4.28) to the function $F(\rho)$. For $N_c \geq 2N_f$ one can show that $F(\rho)$ is welldefined for $\rho > 0$ while it diverges for $\rho \to 0$ (see [135] for further details). Notice that, at least in the unflavored case $N_f = 0$, it is natural to regard these solutions with finite dilaton in the UV as corresponding to D5-branes wrapped on a three-cycle of a G_2 cone, in which the near horizon limit has not been taken and, thus, as we move towards the large ρ region the effect of the branes on the metric becomes asymptotically negligible and we recover the geometry of the G_2 cone where the branes are wrapped.

4.2. The Complete System

Let us now consider the solutions of the BPS equations for our general ansatz. These complete BPS equations have been derived in [135, Appendix A]. Here we will restrict ourselves from now on to the cases with $N_c > 2N_f$, which are the ones that lead to more sensible solutions. As in the truncated case of Section 4.1, we will use $\rho = e^{2h}$ as radial variable and $F = e^{2h}$ as a function of ρ. In order to solve the general BPS equations we must impose initial conditions to the functions $w(\rho)$ and $\gamma(\rho)$ introduced in (4.3) and (4.9), and we must fix the value of the constant κ of (4.12). These initial conditions are determined by imposing some regularity requirements at $\rho = 0$ that we now review (see [135] for additional details). First of all, we will demand that the function F approaches a constant finite value when $\rho \to 0$ (i.e., $F \sim F_0$ for $\rho \to 0$). In order to fix the value of the function $w(\rho)$ at $\rho = 0$ let us recall (see (4.3)) that w parameterizes the one-form A^i which, in turn, determines the mixing of the two three-spheres in the ten-dimensional fibered geometry. The curvature of the gauge connection A^i (defined as in (4.6) with $B^i \to A^i$) determines the nontriviality of this mixing. When this curvature vanishes, one can choose a new set of three one-forms in which the two three-spheres are disentangled in a manifest way and one can factorize the directions parallel and orthogonal to the color brane worldvolume in a well-defined way. From the wrapped brane origin of our solutions, one naturally expects such an unmixing of the two S^3's to occur in the IR limit $\rho = 0$ of the metric. Moreover, by a direct calculation using (4.1) it is easy to verify that for $w = 1$ the curvature of the one-form A^i vanishes. Thus, it follows that the natural initial condition for $w(\rho)$ is

$$w(\rho = 0) = 1. \tag{4.32}$$

Actually, the three-cycle that the color branes wrap can be identified with the one that shrinks when $\rho \to 0$, which is the one given by

$$\Sigma \equiv \left\{ \omega^i = \sigma^i \right\}. \tag{4.33}$$

In order to have a nonsingular flux at the origin, the RR three-form F_3 should vanish on Σ when $\rho \to 0$. It is easy to check that this occurs if the constant κ takes the following value:

$$\kappa = \frac{1}{2} - \frac{3N_f}{2N_c}. \tag{4.34}$$

Actually, (4.34) is also a necessary condition to have a finite dilaton at $\rho = 0$. Indeed, it was shown in [135] that, in addition to (4.34), the dilaton remains finite in the IR if the function $\gamma(\rho)$ takes the following value for $\rho = 0$:

$$\gamma(\rho = 0) = 1 - \frac{2N_f}{N_c}. \tag{4.35}$$

Equations (4.32) and (4.35) provide the initial conditions for the functions w and γ we were looking for.

4.2.1. Asymptotic Linear Dilaton

As explained above, we are interested in solutions of the BPS equations such that asymptotically F is constant. Actually, by solving the BPS system in powers of $1/\rho$, one can check that there are solutions in which F has the following asymptotic behavior:

$$F = N_c + \frac{a_1}{\rho} + \frac{a_2}{\rho^2} + \frac{a_3}{\rho^3} + \cdots, \tag{4.36}$$

where the coefficients a_1, a_2, and a_3 are given by

$$\begin{aligned} a_1 &= N_f N_c, \\ a_2 &= -\frac{3}{4} N_c N_f (N_c - 4N_f), \\ a_3 &= \frac{N_f N_c}{16}\left[21N_c^2 - 148 N_f N_c + 240 N_f^2\right]. \end{aligned} \tag{4.37}$$

Notice that the first two terms in (4.36) and (4.37) coincide with the one written in (4.26) for the truncated system. Similarly, the functions w and γ can be represented as

$$\begin{aligned} w &= \frac{b_1}{\rho} + \frac{b_2}{\rho^2} + \frac{b_3}{\rho^3} + \cdots, \\ \gamma &= \frac{c_1}{\rho} + \frac{c_2}{\rho^2} + \frac{c_3}{\rho^3} + \cdots, \end{aligned} \tag{4.38}$$

where the coefficients b_i and c_i are the following:

$$\begin{aligned} b_1 = c_1 &= \frac{1}{2}(N_c - 3N_f), \\ b_2 = c_2 &= \frac{5}{8}(N_c - 3N_f)(N_c - 2N_f), \\ b_3 &= \frac{1}{32}(N_c - 3N_f)\left[49N_c^2 - 184 N_c N_f + 204 N_f^2\right], \\ c_3 &= \frac{1}{32}(N_c - 3N_f)\left[49N_c^2 - 208 N_f N_c + 252 N_f^2\right]. \end{aligned} \tag{4.39}$$

Moreover, for $N_c > 2N_f$ the dilaton grows linearly with ρ as in (4.27), that is, $\phi \sim \rho/[2(N_c - 2N_f)]$ for large ρ.

The solution for the full range of the holographic coordinate can be found by numerical integration of the BPS system with the IR regularity conditions (4.32), (4.34), and (4.35) and with $F(\rho = 0) = F_0$ finite. One has to perform an interpolation between the $\rho \to 0$ and

$\rho \to \infty$ behaviors by means of a shooting technique in which the only free parameter F_0 is varied until a solution with $F(\rho) \approx N_c$ for large ρ is obtained (which only occurs when F_0 is fine tuned to a very precise value).

After obtaining this solution of the equations of motion of the gravity plus brane system, we can see if it incorporates some of the features that the supergravity dual of 2+1-dimensional gauge theory plus flavors should exhibit. In particular, we can study the evolution of the gauge coupling constant with the holographic coordinate. In order to do that, let us consider a D5-brane probe extended along the three Minkowski directions and wrapping the internal three-cycle Σ defined in (4.33) at a fixed value of the holographic coordinate ρ. By looking at the $\mathcal{F}^2$ terms in the DBI action of this probe, we get the value of the Yang-Mills coupling constant of the dual (2+1)-dimensional gauge theory, namely,

$$\frac{1}{g_{\mathrm{YM}}^2} \sim e^{-(3/4)\phi} \int_\Sigma \sqrt{-\det\left(\widehat{G}_3\right)} d^3\xi \sim \left[\rho + \frac{F}{4}(1-w)^2\right]^{3/2}, \tag{4.40}$$

where $\widehat{G}_3$ is the induced metric on the three-cycle Σ and we have neglected all constant numerical factors. Due to our initial condition (4.32), the right-hand side of (4.40) vanishes for $\rho = 0$, which corresponds to having $g_{\mathrm{YM}}^2 \to \infty$ in the IR, as expected in a confining theory. Moreover, $1/g_{\mathrm{YM}}^2$ grows as we move towards the UV region $\rho \to \infty$, in agreement with the expected property of asymptotic freedom. Other gauge theory observables for these backgrounds, such as the Wilson loops, can be also analyzed (see [135]). Notice that, despite the regularity conditions we have imposed, in the flavored case $N_f \neq 0$ the explicit calculation of the scalar curvature for the linear dilaton solutions shows that the metric is singular at the origin of the radial coordinate. Notice that, as argued for other backgrounds, it is physically reasonable to expect that massless flavors drastically alter the backreacted geometry in the deep IR. However, as our initial conditions are such that the dilaton is finite at the origin, the value of the g_{tt} component of the metric is also bounded and then, according to the criterium of [73], the singularity is "good" and the background can be used to extract nonperturbative information of the dual gauge theory.

4.2.2. Asymptotic G_2 Cones

When $F(\rho = 0)$ takes values in a certain range, the solutions of the BPS equations lead to the metric (4.31) at the UV, which is the direct product of 2+1-dimensional Minkowski space and a G_2 cone. The solutions in this case are very similar in the UV to the ones discussed in Section 4.1.2 (with better IR behavior) and we will not discuss them further here. Let us only mention that the asymptotic values of F, w, and γ for $\rho \to \infty$ can be determined analytically and are given by

$$\begin{aligned} F &\approx \frac{4}{3}\rho + 4\left(N_f - N_c\right) + \cdots, \\ w &\approx \frac{3\left(N_c - 3N_f\right)}{2\rho} + \cdots \quad (\rho \longrightarrow \infty), \\ \gamma &\approx \frac{1}{3} - \frac{N_f}{N_c} + \cdots. \end{aligned} \tag{4.41}$$

5. Flavors in the Klebanov-Strassler Model

The so-called Klebanov-Strassler (KS) solution [13] is dual to a cascading, confining theory, and has been a popular and successful laboratory in which to study numerous issues related to gauge-gravity duality and to cosmology. The gauge theory lives on a stack of regular and fractional D3-branes at the tip of the deformed conifold, as we now briefly review.

The deformed conifold is a regular, six-dimensional, noncompact manifold defined by the equation $z_1\, z_2 - z_3\, z_4 = \hat{\mu}^2$ in $\mathbb{C}^4$. When the complex deformation parameter $\hat{\mu}$ is turned off, it reduces to the singular conifold, which is invariant under complex rescaling of the z_i. The base of the conifold has $\mathrm{SU}(2) \times \mathrm{SU}(2) \times U(1)$ isometry and $S^2 \times S^3$ topology. The deformation parameter breaks the scale invariance, produces a blown-up S^3 at the apex of the conifold, and breaks the $U(1)$ isometry to $\mathbb{Z}_2$.

The low-energy dynamics of N regular and M fractional D3-branes on the deformed conifold is described by a cascading $\mathcal{N} = 1$ 4D gauge theory with gauge group $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ and bifundamental matter fields A, B transforming as $\mathrm{SU}(2) \times \mathrm{SU}(2)$ doublets and interacting with a quartic superpotential $W_{\mathrm{KW}} = \epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}[A_i B_k A_j B_l]$. The dual to this theory is the KS solution [13], that is relevant for the $N = nM$ case, where n is an integer. The related theory develops a Seiberg duality cascade which stops after $n-1$ steps when the gauge group is reduced to $\mathrm{SU}(2M) \times \mathrm{SU}(M)$. The regular KS solution precisely accounts for the physics of an $A \leftrightarrow B$-symmetric point in the baryonic branch of the latter theory, which exhibits confinement and $U(1)_R \rightarrow \mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$ where the second breaking is due to the formation of a gluino condensate $\langle \lambda\lambda \rangle \sim \Lambda_{\mathrm{IR}}^3$. The complex parameter $\hat{\mu}$ is the geometric counterpart of this condensate.

In this section, we will discuss how the solution is modified when a smeared distribution of D7-branes is introduced. In the dual theory, they correspond to fundamental fields, but the precise way in which they couple to the rest of fields depends on the D7-brane embeddings, as we will discuss below. In what follows, we only discuss cases in which the flavor D7-branes do not break any supersymmetry, such that the four-dimensional $\mathcal{N} = 1$ of the KS solution is preserved. The material we summarize in this section was developed in [68, 69, 127, 136].

5.1. Backreaction with Nonchiral Flavors

5.1.1. Brane Embeddings

Let us start by choosing an appropriate family of supersymmetric D7-brane embeddings. A particularly interesting example is given by D7-branes wrapping the holomorphic 4-cycle defined by an equation of the following form [80]:

$$z_1 - z_2 = \mu, \tag{5.1}$$

where μ is a constant. It was shown in [80] that this embedding is κ-symmetric and hence preserves the four supercharges of the deformed conifold theory.

A D7-brane wrapping the 4-cycle defined above is conjectured to add a massless (if $\mu = 0$) or massive (anti)fundamental flavor to a node of the KS model. The resulting gauge theory is said to be "nonchiral" because the flavor mass terms do not break the classical flavor symmetry of the massless theory. The related perturbative superpotential is just as in

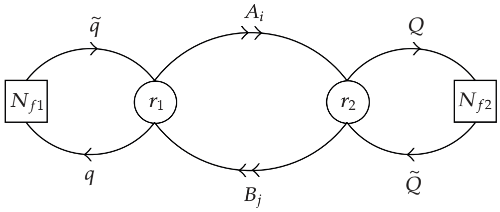

Figure 6: The quiver diagram of the gauge theory. Circles are gauge groups, squares are flavor groups, and arrows are bifundamental chiral superfields. N_{f1} and N_{f2} sum up to N_f.

the singular conifold case [43], which we wrote in (2.14). The complex mass parameter m in W is mapped to the geometrical parameter μ. The different fields are summarized in the quiver diagram of Figure 6.

By acting on this fiducial embedding (5.1) with the generators of the broken symmetries, we can build the family of embeddings over which we want to smear. This is the obvious generalization to the deformed conifold case of the discussion in Section 2 and, in fact, a generic nonchiral embedding is still given by (2.17).

5.1.2. The Ansatz

We now write the ansatz for the metric and forms. It is similar to the ansatz for the KS solution, but, due to the presence of D7-branes, the RR one-form $F_{(1)}$ is non-trivial and the dilaton runs. It is useful to introduce the g_i one-forms used in [13]:

$$g^1 = \frac{-\sin\theta_1 d\varphi_1 - \cos\psi \sin\theta_2 d\varphi_2 + \sin\psi d\theta_2}{\sqrt{2}}, \qquad g^2 = \frac{d\theta_1 - \sin\psi \sin\theta_2 d\varphi_2 - \cos\psi d\theta_2}{\sqrt{2}},$$
$$g^3 = \frac{-\sin\theta_1 d\varphi_1 + \cos\psi \sin\theta_2 d\varphi_2 - \sin\psi d\theta_2}{\sqrt{2}}, \qquad g^4 = \frac{d\theta_1 + \sin\psi \sin\theta_2 d\varphi_2 + \cos\psi d\theta_2}{\sqrt{2}},$$
$$g^5 = d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2. \tag{5.2}$$

The Einstein frame metric ansatz is (a more generic form of the ansatz was used in [68, 69, 127]; by requiring supersymmetry and performing some algebra, one ends up with (5.3); we will skip those intermediate steps here for the sake of briefness)

$$ds^2 = h^{-1/2}(\tau)dx_{1,3}^2 + h^{1/2}(\tau)\frac{1}{2}\hat{\mu}^{4/3}e^{-\phi(\tau)/3}\mathcal{K}(\tau) \times\left[\frac{1}{3\mathcal{K}^3(\tau)}\left(d\tau^2+\left(g^5\right)^2\right)+\cosh^2\left(\frac{\tau}{2}\right)\left(\left(g^3\right)^2+\left(g^4\right)^2\right)+\sinh^2\left(\frac{\tau}{2}\right)\left(\left(g^1\right)^2+\left(g^2\right)^2\right)\right], \tag{5.3}$$

where $\hat{\mu}$ is the complex deformation parameter of the conifold, $dx_{1,3}^2$ denotes the four-dimensional Minkowski metric, and $\mathcal{K}(\tau)$, $h(\tau)$, and the dilaton ϕ are unknown functions of the radial variable to be determined. (The relation of the z_i complex variables as used above to the $\tau, \theta_1, \varphi_1, \theta_2, \varphi_2, \psi$ coordinates can be found, for instance in [127]. The embedding equation (2.17) expressed in terms of the "deformed conifold τ variable" looks the same in terms of the "backreacted ansatz τ variable". See [127] for details.)

For the forms we will adopt the following ansatz:

$$\begin{aligned}
F_5 &= dh^{-1}(\tau) \wedge dx^0 \wedge \cdots \wedge dx^3 - \dot{h}\frac{\widehat{\mu}^{8/3}}{16} e^{-2\phi/3}\sinh^2\tau \mathcal{K}^2 g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5, \\
B_2 &= g_s\alpha' \frac{M}{2}\left[f g^1 \wedge g^2 + k g^3 \wedge g^4\right], \\
H_3 &= g_s\alpha' \frac{M}{2}\left[d\tau \wedge \left(\dot{f} g^1 \wedge g^2 + \dot{k} g^3 \wedge g^4\right) + \frac{1}{2}(k - f) g^5 \wedge \left(g^1 \wedge g^3 + g^2 \wedge g^4\right)\right], \\
F_1 &= g_s \frac{N_f p(\tau)}{4\pi} g^5, \\
F_3 &= g_s\alpha' \frac{M}{2}\left\{g^5 \wedge \left[\left(F + \frac{g_s N_f p(\tau)}{4\pi} f\right) g^1 \wedge g^2 + \left(1 - F + \frac{g_s N_f p(\tau)}{4\pi} k\right) g^3 \wedge g^4\right]\right. \\
&\qquad \left. + \dot{F} d\tau \wedge \left(g^1 \wedge g^3 + g^2 \wedge g^4\right)\right\},
\end{aligned} \tag{5.4}$$

where $f = f(\tau)$, $k = k(\tau)$, $F = F(\tau)$ are functions of the radial coordinate (and where the dot denotes derivatives with respect to τ). We have implemented the self-duality condition for F_5.

Notice that, consistently, $dF_1 = -g_s\Omega$, where Ω is the symmetry preserving D7-brane density distribution form analogous to (2.31):

$$\Omega = \frac{N_f}{4\pi}\left(p(\tau)\left(\sin\theta_1 d\theta_1 \wedge d\varphi_1 + \sin\theta_2 d\theta_2 \wedge d\varphi_2\right) - \dot{p}(\tau) d\tau \wedge g_5\right). \tag{5.5}$$

When quarks are massless [68, 69], one just has $p(\tau) = 1$, whereas $p(\tau)$ becomes nontrivial when quarks are massive. In what follows, we will keep $p(\tau)$ generic. We refer the reader to [127] for the computation of $p(\tau)$ from the massive nonchiral brane embeddings (2.17). The source contributions to the modified Bianchi identities for F_3 and F_5

$$\begin{aligned}
dF_3 &= H_3 \wedge F_1 - g_s\Omega \wedge B_2, \\
dF_5 &= H_3 \wedge F_3 - \frac{1}{2} g_s\Omega \wedge B_2 \wedge B_2
\end{aligned} \tag{5.6}$$

follow from the WZ term of the smeared D7-brane action [127]. Given (5.4) and (5.5), equations in (5.6) are satisfied provided that

$$\dot{h}\frac{\widehat{\mu}^{8/3}}{16} e^{-2\phi/3}\sinh^2\tau \mathcal{K}^2 = \text{const} - \frac{1}{4}\left(g_s\alpha' M\right)^2\left[f - (f - k)F + \frac{g_s N_f}{4\pi} p(\tau) f k\right]. \tag{5.7}$$

5.1.3. The BPS Equations

By requiring the vanishing of the bulk fermionic supersymmetry variations, one finds a set of first-order BPS equations. The computation is lengthy but straightforward and was carried out in [68, 69] (since in that paper $p(\tau) = 1$, the substitution $N_f \to N_f p(\tau)$ has to be implemented in the equations of [68, 69]). In the present notation, the differential equations are

$$\begin{aligned}
\dot{\phi} &= \frac{g_s N_f p(\tau)}{4\pi} e^{\phi}, \\
\dot{k} &= e^{\phi}\left(F + \frac{g_s N_f p(\tau)}{4\pi} f\right)\coth^2\frac{\tau}{2}, \\
\dot{f} &= e^{\phi}\left(1 - F + \frac{g_s N_f p(\tau)}{4\pi} k\right)\tanh^2\frac{\tau}{2}, \\
\dot{F} &= \frac{1}{2}e^{-\phi}(k - f), \\
\frac{\dot{\mathcal{K}}}{\mathcal{K}} &= \frac{2}{3\mathcal{K}^3 \sinh\tau} + \frac{\dot{\phi}}{3} - \coth\tau,
\end{aligned} \tag{5.8}$$

supplemented by the algebraic constraint:

$$e^{-\phi}(k - f) = \tanh\frac{\tau}{2} - 2F\coth\tau + \frac{g_s N_f p(\tau)}{4\pi}\left[k\tanh\frac{\tau}{2} - f\coth\frac{\tau}{2}\right]. \tag{5.9}$$

Quite remarkably, (5.8)-(5.9) can be (almost) explicitly integrated. In the following, we will use notations similar to those employed in Section 2. We introduce an arbitrary value of the radial coordinate τ_* at which the dilaton is ϕ_*. Then, we can write the dilaton as

$$e^{\phi-\phi_*} = \frac{1}{1 + \epsilon_* \int_\tau^{\tau_*} p(\xi)d\xi}, \tag{5.10}$$

where we have introduced the deformation parameter which weighs the flavor loops as

$$\epsilon_* = \frac{N_f}{16\pi^2 M}\lambda_* \quad \text{with } \lambda_* \equiv 4\pi g_s M e^{\phi_*}. \tag{5.11}$$

Let us also introduce a function

$$\eta(\tau) = \epsilon_* e^{\phi-\phi_*}\int_0^\tau (\sinh 2\xi - 2\xi)p(\xi)d\xi. \tag{5.12}$$

Then, we can integrate for the rest of the functions of the ansatz

$$\mathcal{K} = \frac{\left[\sinh 2\tau - 2\tau + \eta(\tau)\right]^{1/3}}{2^{1/3}\sinh\tau}, \qquad F = \frac{\sinh\tau - \tau}{2\sinh\tau},$$
$$f = e^{\phi}\frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1), \qquad k = e^{\phi}\frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1). \tag{5.13}$$

Finally, the function h can be obtained by integrating (5.7). The KS solution without flavors [13] is obtained by taking $\epsilon_* = 0$, such that the dilaton is constant and $\eta(\tau) = 0$. For $p(\tau) = 1$, we find the solution backreacted with massless flavors [68, 69]. In this case, the integrals for the dilaton and $\eta(\tau)$ can be explicitly performed:

$$\eta(\tau) = \epsilon_* e^{\phi-\phi_*}\left(\sinh^2\tau - \tau^2\right),$$
$$e^{\phi-\phi_*} = \frac{1}{1+\epsilon_*(\tau_* - \tau)} \quad \left(\text{for } p(\tau) = 1\right). \tag{5.14}$$

In this massless case, the solution has a curvature singularity in the IR $\tau = 0$. Some cases where $p(\tau)$ is nontrivial were discussed in [127].

5.1.4. Some Physical Features

The solution presented in the preceding sections has been used to extract some of the physics encoded in the unquenched background. In [68, 69], the running of the couplings and anomalies were discussed. As anticipated above, in [127], the solution with massive flavors was found. Quark masses erase the IR singularity in the same way as explained in Section 1.5 or in Section 2.3.2. Quark-antiquark potentials, screening lengths, and associated quantum phase transitions were discussed in the same paper. Finally, in [77], it was computed how the screening effects due to unquenched fundamental matter affect the mass spectra of the KS model, with results similar to Section 2.4. Due to space constraints, we cannot go explicitly through all of these features and we refer the interested reader to the original papers. Here, we will just briefly discuss how the solution captures the phenomenon of a duality wall [68, 69] and how gauge groups ranks change upon Seiberg duality.

We will make use of the following holographic formulae, which can be derived in the $\mathcal{N} = 2$ orbifold case by looking at the Lagrangian of the low-energy field theory living on probe (fractional) D3-branes:

$$\frac{4\pi^2}{g_{\mathrm{YM}}(l)^2} + \frac{4\pi^2}{g_{\mathrm{YM}}(s)^2} = \frac{\pi e^{-\phi}}{g_s}, \qquad \frac{4\pi^2}{g_{\mathrm{YM}}(l)^2} - \frac{4\pi^2}{g_{\mathrm{YM}}(s)^2} = \frac{2\pi e^{-\phi}}{g_s}\left[\frac{1}{4\pi^2\alpha'}\int_{S^2} B_2 - \frac{1}{2}\right]. \tag{5.15}$$

The labels $(l), (s)$ in (5.15) refer to the gauge group with the larger or smaller rank. Strictly speaking, these formulae need to be corrected for small values of the gauge couplings and are only valid in the large ’t Hooft coupling regime (see [67, 72, 137, 138]), which is the case under consideration. Moreover, they are also expected to be precise just in the UV region, where the cascade takes place and the region on which we will focus below. The expressions

(5.15) give positive squared couplings only if the expression inside the square bracket is in the range $[-1/2, 1/2]$. Define

$$b_0(\tau) \equiv \frac{1}{4\pi^2\alpha'}\int_{S^2} B_2 = \frac{g_s M}{\pi} f = \frac{\lambda_*}{8\pi^2}\frac{\tau - 1}{1 - \epsilon_*(\tau - \tau_*)}, \quad \tilde{b}_0 \equiv b_0 - [b_0] \in [0, 1], \tag{5.16}$$

where $[b_0]$ denotes the integer part of b_0. In order to get the explicit expression for b_0 we have integrated over the S^2 parameterized by $\theta_1 = \theta_2$, $\varphi_1 = 2\pi - \varphi_2$, $\psi = \text{const}$ [68, 69], considered the UV limit of (5.10), (5.13) such that $p(\tau) \approx 1$ and $f \approx k \approx e^{\phi}(\tau - 1)/2$, and inserted the definitions (5.11). Now we see that what we have to insert in (5.15) is indeed $\tilde{b}_0$. This is the physical content of the cascade: at a given energy scale we must perform a large gauge transformation on B_2 in supergravity to shift $\int B_2$ by a multiple of $4\pi^2\alpha'$ to get a field theory description with positive squared couplings.

Let us restrict our attention to an energy range, between two subsequent Seiberg dualities, where a field theory description in terms of specific ranks holds. When flowing towards the IR, $\tilde{b}_0$ decreases from 1 to 0. From (5.15) and inserting the solution, we can find an expression for each of the gauge couplings:

$$\frac{1}{\lambda_l} = \frac{1}{\lambda_*}(1 - \epsilon_*(\tau - \tau_*))\tilde{b}_0, \qquad \frac{1}{\lambda_s} = \frac{1}{\lambda_*}(1 - \epsilon_*(\tau - \tau_*))\left(1 - \tilde{b}_0\right). \tag{5.17}$$

In this energy range, the coupling λ_l starts different from zero and flows to ∞ at the end of this range, where a Seiberg duality on its gauge group is needed. The coupling λ_s of the gauge group with smaller rank is the one which starts very large (actually divergent) after the previous Seiberg duality on its gauge group and then flows toward weak coupling.

The qualitative picture of the RG flow in the UV can be extracted from our supergravity solution even without discussing the precise radius-energy relation, simply recalling that the radius must be a monotonic function of the energy scale. First, notice that at a finite τ (and therefore at a finite energy scale E_{UV}), the dilaton diverges making both gauge couplings diverge. This happens at

$$\tau_{dw} = \tau_* + \frac{1}{\epsilon_*}. \tag{5.18}$$

From (5.16), we see that the derivative $db_0/d\tau$ grows unbounded near τ_{dw}, meaning that the interval (in τ) between Seiberg dualities becomes shorter and shorter. The Seiberg dualities pile up the more we approach the UV cut-off E_{UV}. The picture which stems from the flavored Klebanov-Tseytlin/Strassler solution is that τ_{dw} is a so-called "Duality Wall", namely, an accumulation point of energy scales at which a Seiberg duality is required in order to have a weakly coupled description of the gauge theory [139]. Above the duality wall, Seiberg duality does not proceed and a weakly coupled dual description of the field theory is not known. See Figure 7. Nevertheless, in full analogy with the discussion of Section 2, the derivative of the holographic a-function changes sign at a finite distance in τ below τ_{dw} and so one should not trust the solution all the way up to the singular point τ_{dw}.

Duality walls were studied in the context of quiver gauge theories first by Fiol [140] and later in a series of papers by Hanany and collaborators [141, 142]. To our knowledge,

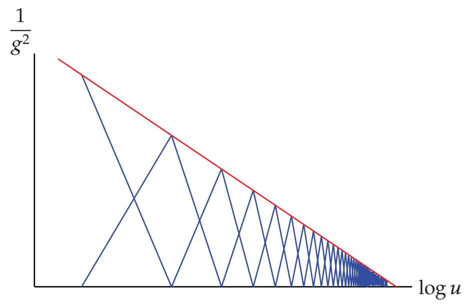

Figure 7: Qualitative plot of the running gauge couplings as functions of the logarithm of the energy scale in the cascading gauge theory. The blue lines are the inverse squared gauge couplings while the red line is their sum.

the solution above is the only explicit realization of this exotic ultraviolet phenomenon on the supergravity side of the gauge/gravity correspondence.

To end this section, we discuss how the effective number of regular and fractional D3-branes changes when undergoing a step of the cascade of Seiberg dualities. We will not compute the explicit shift in τ but rather the shift in the function f ($\approx k$). From (5.16), we have

$$b_0(\tau) \longrightarrow b_0(\tau') = b_0(\tau) - 1 \Longrightarrow f(\tau) \longrightarrow f(\tau') = f(\tau) - \frac{\pi}{g_s M}. \tag{5.19}$$

On the other hand, we compute the effective number of branes at a given energy scale by integrating the appropriate RR-forms:

$$\begin{aligned} N_{\text{eff}}(\tau) &\equiv \frac{1}{(2\pi)^4 g_s \alpha'^2} \int_{\mathcal{M}_5} F_5 = N_0 + \frac{g_s M^2}{\pi}\left[f + \frac{g_s N_f}{4\pi} f^2\right], \\ M_{\text{eff}}(\tau) &\equiv \frac{1}{4\pi^2 g_s \alpha'} \int_{S^3} F_3 = M\left[1 + \frac{g_s N_f}{2\pi} f\right]. \end{aligned} \tag{5.20}$$

In these expressions we have substituted (5.4), and (5.7) and already taken the UV limit $f = k$ and $p(\tau) = 1$. The S^3 for the second integral is the one parameterized by $\theta_2 =$ constant, $\varphi_2 =$ constant. Notice that N_{eff} and M_{eff} are not quantized. This is because they are *Maxwell* charges, as opposed to *Page* charges. See [68, 69] for thorough explanations.

We can compute how N_{eff} and M_{eff} vary in a Seiberg duality step (5.19). A bit of algebra shows that

$$\begin{aligned} M_{\text{eff}}(\tau) &\longrightarrow M_{\text{eff}}(\tau') = M_{\text{eff}}(\tau) - \frac{N_f}{2}, \\ N_{\text{eff}}(\tau) &\longrightarrow N_{\text{eff}}(\tau') = N_{\text{eff}}(\tau) - M_{\text{eff}}(\tau) + \frac{N_f}{4}, \end{aligned} \tag{5.21}$$

whereas N_f remains unchanged. A careful analysis in [68, 69] showed that this is in full agreement with field theory expectations.

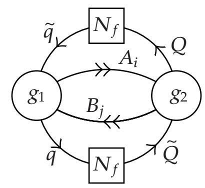

Figure 8: Quiver diagram of the KS theory with chiral flavors.

5.2. Backreaction with Chiral Flavors

In a remarkable paper [136], Benini discussed the solution dual to having smeared chiral flavors on the conifold. In the probe approximation, the D7-brane embeddings that correspond to chiral flavors were discussed in [43, 143]. How these flavors transform under the gauge groups is shown in the quiver diagram of Figure 8. In this case, the quiver theory is not self-similar under the duality cascade; in each step of the cascade a meson field is generated. Its couplings to the rest of the fields are, however, irrelevant [136].

The backreacted solution of [136] uses the singular conifold and therefore it can be considered as the deformation of the Klebanov-Tseytlin solution [144] due to smeared chiral flavors. From the gravity point of view, the extra complication with respect to Section 5.1 is that the worldvolume gauge field on the D7s has to be turned on. In fact, this is crucial when matching the shifts in the ranks of the gauge groups upon Seiberg dualities to the supergravity background (there are subtle differences with respect to the nonchiral case). We will not report further on this solution here but refer the reader to [136].

6. Models with Cohomogeneity 2

In this section we present some situations in which, even smearing the flavor branes, the system cannot be reduced to a one-dimensional problem. In fact, the different fields will depend on two different radial coordinates and, accordingly, one has to solve partial differential equations rather than ordinary differential equations.

In order to provide a heuristic picture, the situation is depicted in Figure 9. Concretely, we will refer here to the model of Section 6.1, but the situation is very similar for all the cases discussed in this section.

In Figure 9, the color branes are placed at the tip of a Calabi-Yau (σ is a radial coordinate along the CY; the rest of the directions of the CY are omitted from the plot). The $\rho - \phi_2$ plane is transverse both to the color branes and to the CY. Each flavor brane lies at a point in this plane and is extended along σ. Distributing the flavor branes along ϕ_2, it is possible to recover (in the smeared limit) the associated $U(1)$ isometry. On the contrary, as is apparent from Figure 9, there is no way in which one can place the flavor brane to recover the full radial symmetry. Hence, the solution associated to this brane configuration must be cohomogeneity two, meaning that all functions of the eventual ansatz will depend on ρ and σ. In the examples considered below the coordinate ρ represents the modulus of the quark mass and, therefore, we should not smear along this direction.

As a matter of fact, if one wishes to construct a deformation of $AdS_5 \times S^5$ with smeared flavor such that the supersymmetry preserved is $\mathcal{N} = 2$ (rather than $\mathcal{N} = 1$ as in Section 2),

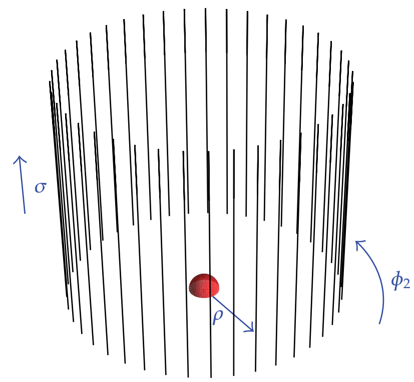

Figure 9: A qualitative plot of the situation with cohomogeneity 2 models. The red dot in the center represents the color branes and each vertical line is a flavor brane. Taking a lot of them smeared along the ϕ_2 angle, the rotational symmetry associated to this angle is effectively recovered. Any function of the ansatz depends on the radial coordinates ρ and σ.

Table 2: A scheme of the set-up: for the brane configuration, a line — means that the brane spans a noncompact dimension, a point · that it is point-like in that direction, a circle ○ that it wraps a compact cycle and ~ indicates smearing in the direction. Above, it is shown which directions spanned the Calabi-Yau and which the transverse plane before backreaction.

				CY_2			$\mathbb{R}_2$
	$x_{1,3}$	σ	ϕ_1	$\tilde{\theta}$	$\tilde{\varphi}$	ρ	ϕ_2
N_c D5	—	·	·	○	○	·	~
N_f D5	—	—	○	~	~	·	~

the solution would have cohomogeneity two and, presumably, would share some similarities with the examples presented in this section. This is an interesting open problem for the future.

6.1. A Dual to (3+1)-Dimensional $\mathcal{N} = 2$ SQCD-Like Theory

In this section, we study the dual solution to the brane intersection summarized in Table 2. The gauge theory lives on N_c D5-branes wrapping a two-sphere with the appropriate twisting to preserve eight supercharges, that is, $\mathcal{N} = 2$ in the effective four-dimensional low-energy theory. Geometrically, it corresponds to wrapping the branes along a compact SLag two-cycle inside a noncompact Calabi-Yau twofold. This leaves two flat transverse dimensions which are identified with the moduli space corresponding to giving vevs to the complex scalar inside the $\mathcal{N} = 2$ vector multiplet. The N_f flavor D5-branes do not further break supersymmetry and provide fundamental hypermultiplets in order to build $\mathcal{N} = 2$ SQCD. They are extended in the noncompact σ direction and, thus, their volume is infinite, making exactly zero the effective four-dimensional gauge coupling living on them. They would provide a global symmetry group $U(N_f)$ if they were placed on top of each other, but due to the smearing, only $U(1)^{N_f}$ is left. The dual solution without flavors was found in [145, 146], and the flavored case was discussed in [147].

We start by writing an ansatz for the metric consistent with the symmetries of the problem. In Einstein frame,

$$
\begin{aligned}
ds_{10}^2 = g_s N_c \alpha' e^{\Phi/2} \Bigg[& \frac{1}{g_s N_c \alpha'} dx_{1,3}^2 + z\left(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\varphi}^2\right) \\
& + e^{-2\Phi}\left(d\rho^2 + \rho^2 d\phi_2^2\right) + \frac{e^{-2\Phi}}{z}\left(d\sigma^2 + \sigma^2\left(d\phi_1 + \cos\tilde{\theta} d\tilde{\varphi}\right)^2\right)\Bigg],
\end{aligned}
\tag{6.1}
$$

where z and Φ depend on both radial coordinates ρ, σ. The Calabi-Yau twofold directions are $0 \leq \tilde{\theta} \leq \pi$, $0 \leq \tilde{\varphi} < 2\pi$, $0 \leq \sigma < \infty$, $0 \leq \phi_1 < 2\pi$ (of course, in this solution with fluxes there is not a Calabi-Yau anymore, but it can be thought of as a deformation of the Calabi-Yau that was present before backreaction). The coordinates $0 \leq \rho < \infty$, $0 \leq \phi_2 < 2\pi$ span the transverse two-dimensional plane, so they should be identified with the moduli space, and therefore rotations in ϕ_2 are related to the $U(1)_R$ symmetry of the field theory. Out of the $SU(2)_R$ symmetry, only its diagonal $U(1)_J$ is manifest in the geometry, as rotations in ϕ_1. The extra SO(3) isometry which acts on $\tilde{\theta}, \tilde{\varphi}, \phi_1$ does not play a role in the low-energy $\mathcal{N} = 2$ SQCD theory [145, 146].

As anticipated in Table 2, we want to consider a set of N_f D5-branes extended in $x_{1,3}$, σ and wrapped in ϕ_1. They lie at fixed $\rho = \rho_Q$, where ρ_Q is proportional to the modulus of the mass of the fundamental hypermultiplets. These D5-branes are homogeneously smeared over the S^2 parameterized by $\tilde{\theta}$, $\tilde{\varphi}$ and on the angle ϕ_2, which corresponds to the phase of the mass of the hypers. This distribution is described by the four-form:

$$
\Omega = \frac{N_f}{8\pi^2} \delta(\rho - \rho_Q) \sin\tilde{\theta} d\rho \wedge d\phi_2 \wedge d\tilde{\theta} \wedge d\tilde{\varphi},
\tag{6.2}
$$

such that the source-modified Bianchi identity for $F_{(3)}$ reads

$$
dF_{(3)} = 2\kappa_{(10)}^2 T_5 \Omega = g_s \alpha' \frac{N_f}{2} \delta(\rho - \rho_Q) \sin\tilde{\theta} d\rho \wedge d\phi_2 \wedge d\tilde{\theta} \wedge d\tilde{\varphi}.
\tag{6.3}
$$

We can write an ansatz for $F_{(3)}$ consistent with this expression:

$$
\begin{aligned}
F_{(3)} = N_c g_s \alpha' \Bigg[& - g' d\phi_2 \wedge d\rho \wedge \left(d\phi_1 + \cos\tilde{\theta} d\tilde{\varphi}\right) - \dot{g} d\phi_2 \wedge d\sigma \wedge \left(d\phi_1 + \cos\tilde{\theta} d\tilde{\varphi}\right) \\
& + \left(g + \frac{N_f}{2N_c}\Theta(\rho - \rho_Q)\right) \sin\tilde{\theta} d\phi_2 \wedge d\tilde{\theta} \wedge d\tilde{\varphi}\Bigg],
\end{aligned}
\tag{6.4}
$$

where Θ is the Heaviside step function (notice that, as opposed to Section 3.7.1 where a Heaviside function was introduced as an approximation to the effect of the massive flavors, the Θ here is exactly what comes from the family of D-brane embeddings considered, since they all lie at fixed $\rho = \rho_Q$), g a new function of ρ and σ that needs to be determined, and we have introduced the following notation for the partial derivatives:

$$
{}' \equiv \partial_\rho, \qquad \dot{} \equiv \partial_\sigma
\tag{6.5}
$$

The next step is to insert the ansatz (6.1) and (6.4) into the type IIB supersymmetry transformations $\delta\psi_\mu = \delta\lambda = 0$, as outlined in Section 1.4.1. This procedure was carefully performed in [147], whereas here we just quote the resulting system of first-order equations:

$$
\begin{gathered}
g + \frac{N_f}{2N_c}\Theta(\rho - \rho_Q) = -\rho z', \qquad g' = -2e^{-2\Phi}\rho\sigma\dot{\Phi}, \\
e^{2\Phi} = \frac{\sigma}{z\dot{z}}, \qquad \dot{g} = -z^{-2}e^{-2\Phi}\sigma\left(g + \frac{N_f}{2N_c}\Theta(\rho - \rho_Q)\right) + 2z^{-1}\rho\sigma e^{-2\Phi}\Phi'.
\end{gathered}
\tag{6.6}
$$

It is easy to check that the last equation is not independent of the previous ones and that the equations in (6.6) ensure the equation of motion for the 3-form $d(e^{\Phi}\ {}^*F_{(3)}) = 0$. This system of equations can be recast as a single, nonlinear, second-order PDE for $z(\rho, \sigma)$:

$$
\sigma\frac{N_f}{2N_c}\delta(\rho - \rho_Q) + \rho z(\dot{z} - \sigma\ddot{z}) = \sigma\left(\rho\dot{z}^2 + z' + \rho z''\right). \tag{6.7}
$$

Once $z(\rho, \sigma)$ is computed, g and Φ are read from (6.6). In general, (6.7) cannot be solved explicitly. In the unflavored case $N_f = 0$, there is in fact an exact solution [145, 146] (see [147] and the first paper of [120, 121] for the adaptation of the solution [120, 121] to the present coordinate system).

Equation (6.7), however, can be studied numerically [147]. We will not pursue that here, but we will verify using (6.6) that the expected beta-function for the gauge coupling stems from the differential equations. In order to read the effective four-dimensional gauge coupling from the geometry, we consider a "color" D5-brane probing the Coulomb branch of the theory, namely, a D5 wrapping the S^2 parameterized by $\widetilde{\theta}$, $\widetilde{\varphi}$, sitting at $\sigma = 0$ [120, 121]. After integrating the volume of the S^2, we find

$$
\frac{1}{g_{\mathrm{YM}}^2} = \frac{N_c}{4\pi^2}\left(z|_{\sigma=0}\right). \tag{6.8}
$$

Thus, in order to understand the running of the coupling it is not necessary to know the geometry everywhere, but just at $\sigma = 0$. From the second equation of (6.6), we see that g is a constant at $\sigma = 0$, which then results in the fact that the first equation of (6.6) can be trivially integrated. But before doing that, let us find out which is the value of $g|_{\sigma=0}$. With that purpose, let us consider the normalization condition:

$$
\frac{1}{2\kappa_{(10)}^2}\int F_{(3)} = N_c T_5, \tag{6.9}
$$

where we have to integrate along ϕ_1, ϕ_2 and an angle built in the "plane" of the two radial directions ρ, σ (heuristically, think of introducing some polar coordinates r, θ such that $\rho = r\sin\theta$ and $\sigma = r\cos\theta$. Then we want to integrate in θ from 0 to $\pi/2$ at fixed and large r). Inserting (6.4), we find

$$
g\big|_{(\sigma=\infty,\rho=0)} - g\big|_{(\sigma=0,\rho=\infty)} = 1. \tag{6.10}
$$

But from the first equation in (6.6) we read that $g|_{\rho=0} = 0$ and, thus, $g|_{\sigma=0} = -1$. We are now ready to integrate the first equation in (6.6) at $\sigma = 0$:

$$\frac{N_c}{4\pi^2}(z|_{\sigma=0}) = \frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{4\pi^2}\left[\left(N_c - \frac{N_f}{2}\Theta(\rho-\rho_Q)\right)\log\rho + \frac{N_f}{2}\Theta(\rho-\rho_Q)\log\rho_Q + \mathrm{const}\right], \tag{6.11}$$

where the next to last term comes from requiring continuity of the metric at $\rho = \rho_Q$. Making use of the radius-energy relation $\rho = \mu/\Lambda$ found in [120, 121], we get

$$\beta(g_{\mathrm{YM}})(\mu) = -\frac{g_{\mathrm{YM}}^3}{8\pi^2}\left(N_c - \frac{N_f(\mu)}{2}\right), \tag{6.12}$$

where $N_f(\mu)$ is defined as the number of flavors for which the modulus of their masses is smaller than the scale. Matter fields with bigger mass are holomorphically decoupled at lower scales, as expected. The expression (6.12) fits field theory expectations and is a nontrivial check of the described unquenched set-up. For further discussion of this model, see [147].

6.2. Flavors in Lower-Dimensional SQCD Models

The approach described in Section 6.1 can be also applied to construct supergravity duals of SQCD-like models in two and three dimensions by considering lower-dimensional branes wrapping different cycles of Calabi-Yau manifolds. In this subsection we will review two of such constructions. First of all, following [73, 148], we will consider the case of D3-branes wrapping a two-cycle of a Calabi-Yau twofold, which is dual to a two-dimensional gauge theory with $\mathcal{N} = (4,4)$ supersymmetry. Secondly, we will review the similar construction of [73, 128, 149] of the gravity dual of three-dimensional $\mathcal{N} = 4$ gauge theories from D4-branes wrapping two-cycles in a CY_2. Backgrounds dual to 2D and 3D flavored theories with reduced supersymmetry have been also constructed [34, 150, 151], and they will be also very briefly reviewed.

6.2.1. Two-Dimensional Theories

Let us consider the following setup for two sets of D3-branes in a Calabi-Yau cone of complex dimension two (see Table 3), where S^2 represents the directions of a compact two-cycle and N_2 are the directions of the corresponding normal bundle. Notice also that the symbols "—" and "·" represent, respectively, unwrapped worldvolume directions and transverse directions, while a circle denotes wrapped directions. Let us parameterize the cycle by means of two angular coordinates (θ, ϕ) and let σ be the radial coordinate of the CY cone. The ansatz for the string frame metric which we will adopt is the following:

$$\begin{aligned} ds_{st}^2 &= H^{-1/2}\left[dx_{1,1}^2 + \frac{z}{m^2}\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \\ &\quad + H^{1/2}\left[\frac{1}{z}\left(d\sigma^2 + \sigma^2(d\psi + \cos\theta d\phi)^2\right) + d\rho^2 + \rho^2 d\Omega_3^2\right], \end{aligned} \tag{6.13}$$

Table 3

			CY$_2$							
	$\mathbb{R}^{1,1}$		S^2		N_2		$\mathbb{R}^4$			
N_c D3 (color)	—	—	○	○	·	·	·	·	·	·
N_f D3 (flavor)	—	—	·	·	—	—	·	·	·	·

where m is a constant with units of mass which, for convenience, we will take as

$$\frac{1}{m^2} = \sqrt{4\pi g_s N_c}\alpha'. \tag{6.14}$$

Notice that in this set-up there is another radial coordinate ρ, which represents the distance along $\mathbb{R}^4$, the directions orthogonal to both the D3-brane worldvolume and the CY cone. Moreover, $d\Omega_3^2$ is the metric of a unit three-sphere. Furthermore, the function z (which controls the size of the cycle) and the warp factor H should be considered as functions of the two radial variables (ρ, σ): $H = H(\rho, \sigma)$, $z = z(\rho, \sigma)$.

As in any background created by D3-branes, our solution should be endowed with a self-dual RR five-form F_5, that we write as

$$F_5 = \mathcal{F}_5 + {}^*\mathcal{F}_5. \tag{6.15}$$

The presence of N_f flavor D3-branes induces a violation of the Bianchi identity of F_5. Indeed, the WZ term of the flavor brane action contains the term $\sum_{N_f} \int_{\mathcal{M}_4} \widehat{C}_4$ that acts as a source for this violation. Actually, the smearing procedure amounts to performing the following substitution in this term:

$$\sum_{N_f} \int_{\mathcal{M}_4} \widehat{C}_4 \longrightarrow \int_{\mathcal{M}_{10}} \Omega \wedge C_4, \tag{6.16}$$

where Ω is a six-form proportional to the volume form of the space transverse to the worldvolume of the flavor brane. The modified Bianchi identity takes the form $dF_5 = 2\kappa_{10}^2 T_3 \Omega$. As in the four-dimensional example discussed in Section 6.1, we shall locate the flavor branes at a particular value $\rho = \rho_Q$ of the ρ coordinate (the mass of the matter fields is just $m_Q = \rho_Q/(2\pi\alpha'))$. Moreover, we will smear the N_f D3-branes along the angular directions (θ, ϕ) of the cycle as well as along the external three-sphere. The corresponding smearing form is

$$\Omega = -\frac{N_f}{8\pi^3}\delta(\rho - \rho_Q) d\rho \wedge \omega_3 \wedge \omega_2, \tag{6.17}$$

with $\omega_2 = \sin\theta d\theta \wedge d\phi$ and ω_3 is the volume element of the external S^3 with line element $d\Omega_3^2$ (the minus sign in (6.17) is due to the orientation of the worldvolume required by supersymmetry). It is clear that the modified Bianchi identity in this case is

$$dF_5 = -2\pi g_s (\alpha')^2 N_f \delta(\rho - \rho_Q) d\rho \wedge \omega_3 \wedge \omega_2. \tag{6.18}$$

Accordingly, let us represent F_5 as in (6.15) with $\mathcal{F}_5$ being given by

$$\mathcal{F}_5 = f_5 - 2\pi g_s (\alpha')^2 N_f \Theta(\rho - \rho_Q) \omega_3 \wedge \omega_2, \tag{6.19}$$

with f_5 such that $df_5 = 0$. We shall represent f_5 in terms of a potential $\mathcal{C}_4$ as $f_5 = d\mathcal{C}_4$, where $\mathcal{C}_4$ is given by the ansatz:

$$\mathcal{C}_4 = g\omega_3 \wedge (d\psi + \cos\theta d\phi), \quad g = g(\rho, \sigma). \tag{6.20}$$

Proceeding as in Section 6.1, one gets in this case the following set of BPS equations:

$$\begin{gathered} m^2\left[g - 2\pi g_s (\alpha')^2 N_f \Theta(\rho - \rho_Q)\right] = \rho^3 z', \\ m^2 H = \frac{z\dot{z}}{\sigma}, \qquad g' = -\sigma\rho^3 \dot{H}, \\ \dot{g} = \frac{\sigma\rho^3}{z} H' - \frac{\sigma}{z^2} H m^2 \left[g - 2\pi g_s (\alpha')^2 N_f \Theta(\rho - \rho_Q)\right], \end{gathered} \tag{6.21}$$

where the prime and the dot have the same meaning as in (6.5). The fulfillment of (6.21) ensures the preservation of eight supersymmetries by the background, which corresponds to $\mathcal{N} = (4,4)$ SUSY of the dual gauge theory. Moreover, one can prove that $z(\rho, \sigma)$ satisfies the following PDE:

$$\rho z(\dot{z} - \sigma\ddot{z}) = \sigma\left(\rho\dot{z}^2 + \rho z'' + 3z'\right) + \frac{N_f}{2N_c} \frac{\sigma}{m^2\rho^2} \delta(\rho - \rho_Q). \tag{6.22}$$

In the unflavored case $N_f = 0$, the BPS system (6.21) (and the PDE equation (6.22)) can be solved analytically [148] by constructing the solution in five-dimensional gauged supergravity and by uplifting it to ten dimensions [73]. After a suitable change of variables one can show [148] that the metric and RR five-form of this gauged supergravity solution can be written as in our ansatz. In the general flavored case one has to apply numerical techniques. However, as in the four-dimensional case, one only needs to know the solution for $\sigma = 0$ in order to get the behavior of the gauge coupling. Indeed, by means of a probe calculation one can check [148] that the supersymmetric locus of a color D3-brane occurs precisely at $\sigma = 0$ and that the gauge coupling is related to $z(\rho, \sigma = 0)$ by means of the following relation:

$$\frac{1}{g_{\mathrm{YM}}^2(\rho)} = \frac{z(\rho, \sigma = 0)}{m^2 g_s}. \tag{6.23}$$

It follows from the system (6.21) that $g(\rho, \sigma = 0)$ is constant. Actually, by using a flux quantization condition similar to the one employed for the 4D case, one can verify that $g(\rho, \sigma = 0) = 1/m^4$, where m is the constant defined in (6.14). By using this result in the first

Table 4

				CY_2						
	$\mathbb{R}^{1,2}$			S^2		N_2		$\mathbb{R}^3$		
N_c D4 (color)	—	—	—	○	○	·	·	·	·	·
N_f D4 (flavor)	—	—	—	·	·	—	—	·	·	·

equation in (6.21) one readily integrates $z(\rho, \sigma = 0)$. By imposing continuity of the solution at $\rho = \rho_Q$, one gets

$$z(\rho, 0) = z_* - \frac{\pi m^2 g_s (\alpha')^2}{\rho_Q^2} N_f \Theta(\rho - \rho_Q) - \frac{2\pi m^2 g_s (\alpha')^2}{\rho^2} \left[N_c - \frac{N_f}{2} \Theta(\rho - \rho_Q) \right], \tag{6.24}$$

where z_* is a constant of integration. Plugging this result in (6.23), and assuming that the energy scale μ is related to the holographic coordinate ρ as $\rho = 2\pi\alpha'\mu$, one gets

$$\frac{1}{g_{\rm YM}^2(\mu)} = \frac{1}{g_{\rm YM}^2} \left(1 - \frac{g_{\rm YM}^2}{2\pi\mu^2} \left(N_c - \frac{N_f(\mu)}{2} \right) \right), \tag{6.25}$$

where $N_f(\mu)$ is again the number of flavors with mass smaller than the scale μ and $g_{\rm YM}$ is the bare UV Yang-Mills coupling. The dependence on the scale μ of the Yang-Mills coupling displayed in (6.25) matches precisely the one in field theory, which constitutes a nontrivial test of the gravity result.

Backgrounds dual to 2D theories with $\mathcal{N} = (2, 2)$ SUSY can be obtained by wrapping D5-branes along a four-cycle of a Calabi-Yau threefold [150]. An alternative construction, which improves the UV behavior of the solution, involves D3-branes wrapping a two-cycle of a CY_3 [73, 151]. One can further reduce the amount of supersymmetry by considering a D5-brane wrapping a four-cycle of a manifold of G_2 holonomy, which leads to a dual of an $\mathcal{N} = (1, 1)$ supersymmetric gauge theory. In all these cases the flavor branes are extended along some of the noncompact normal directions of the cycle wrapped by the color branes and the corresponding backreacted solutions can be obtained numerically and are similar to the one reviewed here.

6.2.2. Three-Dimensional Theories

A similar analysis can be carried out to obtain the gravity dual of $\mathcal{N} = 4$ three-dimensional gauge theories. In this case one must consider flavor and color D4-branes wrapping two-cycles according to the array (see Table 4).

The concrete ansatz for the ten-dimensional string frame metric we will adopt in this case is very similar to the 2D and 4D cases studied above, namely

$$\begin{aligned} ds^2 &= e^{2\Phi} \left[dx_{1,2}^2 + \frac{z}{m^2} \left(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2 \right) \right] \\ &\quad + e^{-2\Phi} \left[\frac{1}{z} \left(d\sigma^2 + \sigma^2 \left(d\psi + \cos\tilde{\theta} d\tilde{\phi} \right)^2 \right) + d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \end{aligned} \tag{6.26}$$

where $\Phi = \Phi(\rho, \sigma)$ is the dilaton and the constant m is now given by

$$\frac{1}{m^3} = 8\pi g_s N_c (\alpha')^{3/2}. \tag{6.27}$$

As before, $z = z(\rho, \sigma)$ and the background should include an RR form whose Bianchi identity is violated due to the presence of flavor branes. For D4-branes the appropriate RR form is a four-form F_4. If we locate the flavor branes at a fixed distance $\rho = \rho_Q$ in the transverse $\mathbb{R}^3$ and we smear them along their orthogonal angular directions, the modified Bianchi identity is

$$dF_4 = 2\kappa_{10}^2 T_4 \Omega = \frac{N_f}{2N_c}\frac{1}{8m^3}\delta(\rho - \rho_Q) d\rho \wedge \omega_2 \wedge \tilde{\omega}_2, \tag{6.28}$$

where ω_2 and $\tilde{\omega}_2$ are the volume forms of the unit (θ, ϕ) and $(\tilde{\theta}, \tilde{\phi})$ two-spheres. Let us solve (6.28) by means of the following ansatz:

$$F_4 = dC_3 + \frac{N_f}{2N_c}\frac{1}{8m^3}\Theta(\rho - \rho_Q)\tilde{\omega}_2 \wedge \omega_2, \tag{6.29}$$

where C_3 is the following potential depending on the function $g(\rho, \sigma)$:

$$C_3 = -g\omega_2 \wedge \left(d\psi + \cos\tilde{\theta} d\tilde{\phi}\right). \tag{6.30}$$

By imposing that the system preserves eight supersymmetries, we arrive at the following system of BPS equations:

$$\begin{gathered} g + \frac{N_f}{2N_c}\frac{1}{8m^3}\Theta(\rho - \rho_Q) = -\frac{\rho^2 z'}{m^2}, \qquad e^{-4\Phi}\sigma = \frac{1}{m^2} z\dot{z}, \\ g' = -4\sigma\rho^2 e^{-4\phi}\dot{\Phi}, \\ \dot{g} = -m^2 \sigma z^{-2} e^{-4\Phi}\left[g + \frac{N_f}{2N_c}\frac{1}{8m^3}\Theta(\rho - \rho_Q)\right] + 4\sigma\rho^2 z^{-1} e^{-4\Phi}\Phi'. \end{gathered} \tag{6.31}$$

Again, one can combine the different equations in (6.31) and get a single second-order PDE for $z(\rho, \sigma)$, namely,

$$\rho^2 z(\dot{z} - \sigma\ddot{z}) = \rho\sigma\left(\rho\dot{z}^2 + \rho z'' + 2z'\right) + \sigma\frac{N_f}{2N_c}\frac{1}{8m^3}\delta(\rho - \rho_Q). \tag{6.32}$$

As in the 4D and 2D cases, (6.31) and (6.32) can be solved analytically when $N_f = 0$ by using gauged supergravity [73, 149]. In the general flavored case one can get analytically the form of the solution for $\sigma = 0$ [128]. Indeed, one can verify from (6.31) and the corresponding flux quantization condition that $g(\rho, \sigma = 0) = -1/(8m^3)$ and that $z(\rho, 0)$ is

$$z(\rho, 0) = z_* - \frac{1}{8\rho_Q m}\frac{N_f}{2N_c}\Theta(\rho - \rho_Q) - \frac{1}{8m\rho}\left[1 - \frac{N_f}{2N_c}\Theta(\rho - \rho_Q)\right], \tag{6.33}$$

with z_* being a constant. Moreover, by means of a probe calculation one readily verifies that $\sigma = 0$ is the SUSY locus of the color D4-branes and that the relation between the YM coupling and $z(\rho, 0)$ is $g_{\rm YM}^{-2}(\rho) = z(\rho, 0)/(2\pi g_s\sqrt{\alpha'}m^2)$. Using this result and the radius-energy relation $\rho = 2\pi\alpha'\mu$, one can convert (6.33) into the following equation for the running of the YM coupling of the 3D theories:

$$\frac{1}{g_{\rm YM}^2(\mu)} = \frac{1}{g_{\rm YM}^2}\left[1 - \frac{g_{\rm YM}^2}{4\pi\mu}\left(N_c - \frac{N_f(\mu)}{2}\right)\right], \tag{6.34}$$

which again matches the corresponding field theory result.

A gravity dual of $\mathcal{N} = 2$ three-dimensional gauge theory based on D5-branes wrapping a three-cycle was found in [152, 153]. The addition of flavor to this background along the lines discussed here is carried out in [34, 151]. Alternatively, for this same amount of supersymmetry one can consider D4-branes wrapping a two-cycle of a Calabi-Yau threefold [73, 151].

7. A Mathematical Viewpoint

In the approach we have followed up to now in this review on how to add unquenched flavor; we considered a family of equivalent embeddings of the flavor branes. This family can be generated by acting with the isometries of the background on a fiducial representative embedding. When the number N_f of flavor branes is large, considering the set of branes as a continuous distribution is a good approximation. We then computed the RR charge density generated by the branes, that is, the smearing form Ω, by explicitly performing the average over the set of embeddings and, subsequently, we have studied the deformation induced on the metric and forms due to the backreaction.

The outcome of this microscopic approach is a system of supergravity plus delocalized sources preserving some amount of supersymmetry. It turns out that, in this process, very interesting mathematical structures emerge. The reason for this is the fact that the supersymmetric sources that we are using satisfy a calibration condition. As a consequence, one can use the methods of modern geometry to find backgrounds with smeared flavors in a systematic way [34]. In this procedure one does not deal with the set of embeddings and, for this reason, we will refer to it as the macroscopic approach, as opposed to the microscopic approach reviewed in previous sections. The goal is computing (or at least constraining) the smearing form Ω by using the same type of technology as the one employed in the analysis of flux compactifications of string theory (see [31, 154–157]).

The central object in this geometric approach is the so-called "calibration form" $\mathcal{K}$. For Dp-branes $\mathcal{K}$ is a $(p+1)$-form, which can be represented in a vielbein basis as

$$\mathcal{K} = \frac{1}{(p+1)!}\mathcal{K}_{a_1\cdots a_{p+1}} e^{a_1\cdots a_{p+1}}, \tag{7.1}$$

with $e^{a_1\cdots a_{p+1}} = e^{a_1} \wedge \cdots \wedge e^{a_{p+1}}$. The different components $\mathcal{K}_{a_1\cdots a_{p+1}}$ are given by fermionic bilinears of the following type:

$$\mathcal{K}_{a_1\cdots a_{p+1}} = \epsilon^{\dagger}\tau\Gamma_{a_1\cdots a_{p+1}}\epsilon, \tag{7.2}$$

where ϵ are Killing spinors of the background, conveniently normalized, and τ is a constant matrix which (in the type IIB theory) is $\tau = \tau_3^{(p-3)/2} i\tau_2$, where τ_2 and τ_3 are Pauli matrices and the spinor ϵ is represented as a two-dimensional vector of Majorana-Weyl spinors (τ is the same matrix that appears in the expression of the kappa symmetry matrix Γ_κ of a Dp-brane when all worldvolume fluxes are switched off). The form $\mathcal{K}$ can be used to characterize $(p+1)$-dimensional surfaces in the ten-dimensional geometry. A $(p+1)$-dimensional surface $\mathcal{M}_{p+1}$ is said to be calibrated by $\mathcal{K}$ if its pullback to $\mathcal{M}_{p+1}$ is equal to the induced volume form on $\mathcal{M}_{p+1}$, namely,

$$\widehat{\mathcal{K}} = \sqrt{-\det \widehat{g}}\, d^{p+1}\xi, \tag{7.3}$$

where the ξ's are a set of local coordinates of $\mathcal{M}_{p+1}$. When there are no NSNS fluxes or worldvolume gauge fields the calibration condition (7.3) characterizes the supersymmetric embeddings of Dp-branes (this can be easily established by using kappa symmetry). Actually, a Dp-brane whose worldvolume $\mathcal{M}_{p+1}$ is calibrated by $\mathcal{K}$ is electrically charged with respect to an RR$(p+2)$-form field strength F_{p+2} and, in the Einstein frame, F_{p+2} is related to $\mathcal{K}$ as

$$F_{p+2} = d\left(e^{((p-3)/4)\phi}\mathcal{K}\right). \tag{7.4}$$

Equation (7.4) is a consequence of supersymmetry [154] and, actually, in our backreacted backgrounds it follows from the system of BPS equations. Moreover, as a consequence of (7.3), the action of a localized embedding of a Dp-brane (without NSNS flux and with worldvolume gauge fields switched off) can be written as

$$S_{\mathrm{D}p}^{\mathrm{loc}} = -T_p \int_{\mathcal{M}_{p+1}} \left[e^{((p-3)/4)\phi}\widehat{\mathcal{K}} - \widehat{C}_{p+1}\right]. \tag{7.5}$$

Following our prescription, the smeared version of the brane action is obtained by performing the wedge product with Ω of the $(p+1)$-form inside the brackets in (7.5) and by integrating the result over the full ten-dimensional spacetime:

$$S_{\mathrm{D}p}^{\mathrm{smeared}} = -T_p \int_{\mathcal{M}_{10}} \left[e^{((3-p)/4)\phi}\mathcal{K} - C_{p+1}\right] \wedge \Omega. \tag{7.6}$$

Let us now define the $(8-p)$-form F_{8-p}, under which the Dp-brane is magnetically charged, as

$$F_{8-p} = \pm e^{((p-3)/2)\phi}\, {}^{*}F_{p+2}, \tag{7.7}$$

where the sign depends on the particular value of p and on the conventions used. As in the examples studied in previous sections, the Dp-brane modifies the Bianchi identity of F_{8-p}, namely,

$$dF_{8-p} = \pm 2\kappa_{10}^2 T_p \Omega. \tag{7.8}$$

Equation (7.8) establishes a crucial connection between the smearing form Ω and the calibration form $\mathcal{K}$. Indeed, by using (7.4) and (7.7), the right-hand side of (7.8) can be written in terms of $\mathcal{K}$ and its exterior derivative. Moreover, from the inspection of the smeared brane action (7.6), one concludes that Ω can be regarded as a kind of orthogonal complement (the Poincare dual) of $\mathcal{K}$ in $\mathcal{M}_{10}$. Interestingly, the possible calibration forms $\mathcal{K}$ in a manifold are known and are related to its supersymmetric cycles and G-structures. In the case of a manifold preserving minimal SUSY in 4D, $\mathcal{K}$ can be written in terms of powers of the Kähler form and of the holomorphic volume form. Thus, geometry and topology constrain the form of the charge density distribution of supersymmetric configurations and, actually, one could adopt the point of view in which the expression of Ω is partially determined from these constraints without explicitly performing the average over the family of embeddings, although, in order to fix Ω completely, an explicit microscopic calculation is needed. This macroscopic approach was followed in [34, 150, 151, 158] for some particular brane set-ups.

To finish this section let us detail the implementation of these mathematical concepts in the case discussed in Section 2, namely, the D3-D7 system. From now on we will assume that the metric, dilaton, and forms are given by the expressions written in (2.32) and (2.33). It is convenient to define the following two-form:

$$\mathcal{J} = h^{1/2}\left[e^{2g} J_{\mathrm{KE}} + e^{2f} d\rho \wedge (d\tau + A_{\mathrm{KE}})\right], \tag{7.9}$$

which is such that $h^{-1/2}\mathcal{J}$ is the Kähler form of the transverse 6d space. Actually, one can immediately verify that $d[h^{-1/2}\mathcal{J}] = 0$ as a consequence of the BPS equation for g in (2.34). By explicitly computing the fermion bilinear in (7.2) and by using the projections satisfied by the Killing spinor of the flavored $\mathrm{AdS}_5 \times S^5$ background, one gets that the calibration form $\mathcal{K}$ in this case is given by

$$\mathcal{K} = \frac{1}{2}\mathrm{Vol}(M_{1,3}) \wedge \mathcal{J} \wedge \mathcal{J}, \tag{7.10}$$

with $\mathrm{Vol}(M_{1,3}) = h^{-1} d^4x$ being the volume form of the Minkowski part of the space. Using the fact that $dA_{\mathrm{KE}} = 2J_{\mathrm{KE}}$, one gets

$$d\left(e^{\phi}\mathcal{K}\right) = \frac{1}{2}e^{2g+\phi}\left[(4g' + \phi')e^{2g} - 4e^{2f}\right] d^4x \wedge J_{\mathrm{KE}} \wedge J_{\mathrm{KE}} \wedge d\rho = F_9, \tag{7.11}$$

where, in the last step, we have used the condition (7.4) for $p = 7$. Let us now verify that the value of F_9 obtained in (7.11) is consistent with the expression for F_1 written in our ansatz (2.33) and, thus, with the Ω displayed in (2.31). Taking into account that the volume form of the KE space is $(1/2)J_{\mathrm{KE}} \wedge J_{\mathrm{KE}}$, one can easily compute the Hodge dual of F_1 and get the following result for F_9:

$$F_9 = -e^{2\phi}\, {}^*F_1 = \frac{Q_f}{2} p(\rho) e^{4g+2\phi} d^4x \wedge J_{\mathrm{KE}} \wedge J_{\mathrm{KE}} \wedge d\rho. \tag{7.12}$$

The expressions (7.11) and (7.12) for F_9 coincide if the following relation holds:

$$Q_f p(\rho) = e^{-\phi}\left[4g' + \phi' - 4e^{2f-2g}\right]. \tag{7.13}$$

One can easily check that (7.13) is a consequence of the BPS system (2.34).

8. Discussion

In hindsight, we can say that the program of finding solutions dual to theories with unquenched fundamentals with smeared flavor branes has been quite successful. As expected, it simplifies matters both when looking for the background solution and when discussing the physics they encode.

We have presented a series of example of solutions of ten-dimensional type IIA or type IIB supergravity coupled to D-brane sources. The philosophy and methods used in the different cases are quite similar. Finding a consistent solution requires solving at the same time the closed string degrees of freedom (namely, finding solutions of the generalised Einstein equations in the presence of sources) and the open string degrees of freedom (namely, checking that the D-brane embeddings which generate the mass and charge source density are indeed solutions of the background). Supersymmetric solutions are easier to deal with and indeed preserving SUSY simplifies enormously the technical work. It is rather remarkable that sometimes such complicated coupled systems can be (at least almost) completely integrated and the solutions given in a simple closed form (in particular in Sections 2 and 5; for the other sections, profuse numerical integration was necessary). However, supersymmetry is not mandatory for the construction and we have presented non-supersymmetric black hole solutions.

The solutions are conjectured to be dual to theories with unquenched quarks. Since we have always dealt with the particular case of smearing the flavor branes over the transverse directions, we have built duals of a very particular class of such unquenched theories. We have used the solutions to discuss many physical features of the different set-ups. Many crosschecks of field theory expectations have been discussed. Just to mention a few instances, the running of the gauge coupling in different theories, the behaviour of the cascade in Section 5, or the direct computation of the first flavor contribution to the entropy of the D3-D7 plasma (Section 2.5) which was previously known from an indirect method (namely, from first computing the free energy) [84]. All this asserts that the dualities discussed in this review are on firm ground. Due to obvious space constraints, we have not been able to include all the material that may deserve to be reviewed, but we hope that we have given enough references to the original literature.

It is worth recapitulating about the presence of singularities in the different solutions discussed. First, in all the cases presented there are IR curvature singularities when all of the flavor branes reach the bottom of the geometry; see the heuristic picture of Section 1.5. They pertain to the kind of singularities usually called *good*. In fact, we have shown explicitly in the examples of Sections 2 and 5 how adding (even an infinitesimal) quark mass leads to regular backgrounds (the analogous generalization for the set-ups of Sections 3 and 4 remains an interesting open question). Moreover, heating up the theories can result in the formation of a black hole horizon behind which the IR singularity is hidden; see Section 2.5 and (3.27) for examples.

On the other hand, the solutions of Sections 2 and 5 are singular in the UV (an effect connected to having flavor D7-branes) since the dilaton diverges at a finite position of the radial variable. This is expected, since it is the consequence of the Landau pole of the dual theory (more precisely, in the case of Section 5 it is a duality wall). Despite the singularity, we have shown that it is possible to consistently compute IR observables as long as the IR scale is well separated from the pathological UV. Clear examples are the meson spectrum (Section 2.4) and the black hole thermodynamical properties (Section 2.5.1). In the D5D5 set-ups of Sections 3 and 4, the dilaton diverges linearly in the UV, signalling a little string theory-like UV completion of the dual field theory. We want to stress here that this already happens in the unflavored solutions and thus is not problem associated to the backreaction.

Finally, all the solutions in Section 6 have a singularity in the IR. This singularity is not associated to the flavors as it is already present in the flavorless solutions and, at least in some cases, can be resolved by the worldsheet CFT [159]. On top of that, for the same models, typically, when N_f is sufficiently large, a Landau pole is generated and, jointly, a UV singularity appears in the geometry.

Notice that when choosing a particular radial coordinate, we still have reparametrization invariance; that is, we can still redefine $\rho = f(r)$. So, the fact that different energy-radius relations have appeared in different duals should not be a matter of concern as it is a physically motivated choice (inspired, e.g., in the gaugino condensate or some other operators whose scaling is known). What is certainly more important is the rate of change of different quantities with the radial coordinate. This should be thought as choosing a renormalization scheme.

We end this discussion with two clarifications.

(i) We have repeatedly stressed that our main goal is to build duals to theories in which N_c and N_f are of the same order. Nevertheless, for the set-ups discussed in Section 2, which include the specially interesting flavored $AdS_5 \times S^5$ case, $N_f \ll N_c$ is needed; see Section 2.6 (similar comments apply to Section 5). This is because, starting with a conformal theory, the introduction of extra matter generates an UV pathology, namely, a Landau pole. Then, roughly speaking, in order to have a meaningful IR, it has to be well separated from the pathological region, enforcing the number of flavors not to be too large. However, backreaction effects and, accordingly, the effect of unquenched quarks, can still be computed as an expansion in N_f/N_c. On the other hand, for the models in Sections 3, 4, and 6, this restriction is not present and, indeed, it makes sense to talk about solutions with $N_f \sim N_c$. In fact, this is imperative, for instance, when discussing Seiberg-like dualities as in Section 3.6.2.

(ii) Since the (DBI) action is used to model the D-brane sources, one could be wary for the following reason: the effective string coupling on a stack of N_f D-branes is

$g_s N_f \sim N_f/N_c$ and this should be small for the DBI to be a good approximation [160], whereas there is not a good effective description for strong string coupling. However, this caveat is circumvented because we do not deal with stacks of localized flavor branes: due to the smearing, the typical distance between any pair of flavor branes is of the order of the size of the transverse space, which is typically large. As a result, the flavor symmetry is usually broken to $U(1)^{N_f}$ and the effective open string coupling remains small. As already pointed out, this amounts to keeping "one window graphs" in the Veneziano expansion [26].

Outlook

There are still many open questions that deserve to be addressed within the framework presented in this review. We briefly mention a few examples of possible future projects. They comprise both making further progress in studying the models here presented and building new solutions that could be useful in exploring the consequences of the formalism for different physical points. Along the first of these lines, it would be nice to generalise the solutions of Section 3 to the massive quark case in order to remove the IR singularity. Also, we expect the black hole solution of Section 2.5 to encode interesting physical information. For instance, one could consider massive embeddings in the search of a first-order phase transition similar to those in [17, 84, 161]. The peculiarity of the back-reacted setting would be, conceivably, that the area of the horizon would undergo a finite jump at the transition. Along the second line, a back-reacted D4-D6 solution building on the model of [17] could be useful in discussing QCD-like properties. Another conceivable program is to look for a solution, which, as in [162], may correspond to a color-flavor locking phase. The study of fluctuations in these backgrounds, that will also contain fluctuations of the fields in the flavor branes, with a view on understanding holographic renormalization would be a highly interesting result.

Aside from this, it would be nice to find solutions (with backreacted flavor branes) that contain an AdS_5 factor. The study of conformal anomalies there may give interesting results.

As stressed in the introduction, finding the kind of solutions discussed here, including the D-brane backreaction, has an interest on their own, independently of AdS/CFT. It would be nice to understand whether they may turn out to be useful for different physical applications. For instance, for models of inflation built with D3-D7 systems on the conifold (see [163] for recent progress in this direction), the analysis of Section 5 could have some relevance.

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