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ORIGINAL RESEARCH

Unreliable bulk retrial queues with delayed repairs and modified vacation policy

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Abstract The present investigation deals with the bulk arrival M/G/1 retrial queue with impatient customers and modified vacation policy. The incoming customers join the virtual pool of customers called orbit if they find the server being busy, on vacation or in broken down state otherwise the service of the customer at the head of the batch is started by the server. The service is provided in k essential phases to all the customers by the single server which may breakdown while rendering service to the customers. The broken down server is sent to a repair facility wherein the repair is performed in d compulsory phases. As soon as the orbit becomes empty, the server goes for vacation and takes at most J vacations until at least one customer is noticed. The incoming customers are impatient and may renege on seeing a long queue of the customers for the service. The probability generating functions and queue length for the number of customers in the orbit and queue have been obtained using supplementary variable technique. Various system characteristics viz. average number of customers in the queue and the orbit, long run probabilities of the system states, etc. are obtained. Furthermore, numerical simulation has been carried out to study the sensitivity of various parameters on the system performance measures by taking an illustration.

Keywords Retrial queue · Modified vacation policy · Batch input · Supplementary variable · Reneging · Phase service · Phase repair

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Introduction

In many queueing scenarios, it may happen sometimes that a customer/job that did not receive service at the first attempt from the server tries again and again to avail the required service. These type of situations give rise to special type of queues known as retrial queues. Such queues are characterized by the phenomenon that a customer if deprived of service is forced to join the virtual pool of customers called orbit from where he tries again and again for the service. The customers retrying for the service are known as retrial customers. The retrial queues are visible in all day-to-day congestion situations from supermarkets to ATM, from hospitals to admission counters, etc. wherein the customer tries again from the retrial orbit for the service. The detailed account of the retrial queues along with their applications can be found in the books by Falin and Templeton (1997) and Artalejo and Corral (2008). The elaborate surveys on retrial queues can be found in the articles by Yang and Templeton (1987), Artalejo (1999a, b), Artalejo and Falin (2002) and Artalejo (2010) and many more.

The modeling and analysis of queueing models, especially retrial queueing models, have attracted the queue theorists since past many years. The retrial queueing models enriched with various concepts like vacation, discouragement, bulk, etc. had been studied in numerous ways by a number of researchers. The study of bulk queues with retrying customers is always in demand as it is directly related to the study of telecommunication, manufacturing and computer systems. In telecommunication system, the fraction of the signal/message at the head of the queue is transmitted to receive service and rest other customers/messages are kept in buffer/retrial orbit so as to wait for the server to get service. A bulk arrival retrial M/G/1 queue has been analyzed by Kahraman and Gosavi (2011). Choudhury and Ke (2012) studied a batch arrival retrial queue with delaying repair and Bernoulli vacation schedule.

A number of retrial queueing models have also been analyzed by researchers keeping in mind the vacation policy of the server. It is quite realistic that a server may go for vacation of any kind when no customers are present in the system and may come back when some customers are present in the system or as per his vacation policy. The various concepts related to queueing systems with server vacations can be found in articles by Doshi (1986) and Takagi (1991). Chang and Ke (2009) considered a batch retrial model where the server can take at most J vacations; the customer can go for a series of continuous J vacations if no customers/jobs are available in the orbit. Ke and Chang (2009) investigated modified vacation policy for M/G/1 retrial queue and obtained various performance measures. Ke et al. (2011) discussed performance measures and randomized optimization for an unreliable server vacation system. Recently, Dimitriou (2012) studied a mixed priority retrial queueing model with multiple vacations and negative customers.

The pattern of servicing also plays a significant role in the modeling of retrial queues. In past investigations, various kinds of services like single service, optional services, multioptional services and phase services have been studied. Choudhury et al. (2010) discussed steady-state behavior of M^x/G/1 retrial queueing system with two phases of service. Choudhury and Tadj (2011) studied the optimal control of bulk arrival M/G/1 unreliable server with two phases of service and Bernoulli vacation schedule. The servicing in phases clearly relates to many realistic day-to-day situations. The admission in any institute requires a number of formalities and filling of a number of forms. This process is completed in various compulsory phases from getting the form to the submission of completed application form. In between, it goes through the affidavit, medical fitness certificates, previous educational qualifications proof, completion of admit card, fees deposition, etc. All these are compulsory phases and admission cannot be completed if any of these compulsory steps is skipped. Jain and Agarwal (2010) analyzed a batch arrival queueing system with N-policy and Bernoulli vacation schedule wherein the customer undergoes l-essential stage service procedure to avail the service.

In real life situations, the repair of broken down server is also an important factor. In real practice, the unreliable server may breakdown or stops working during any phase of service and needs to be repaired. Similar to service, repair can also be completed either in single phase or in a series of compulsory or optional phases depending upon the severity of the breakdown. Atencia et al. (2006) studied M/G/1 retrial queue with active breakdowns and Bernoulli schedule. Choudhury and Ke (2012) investigated a batch arrival retrial queue under Bernoulli vacation schedule for unreliable server and delaying repair.

Discouragement of the customers while waiting for the service has also grabbed the attention of various researchers working in the field of queueing theory. It may happen sometimes that the customers waiting for the service in the queue/orbit get impatient and decide to quit the queue without availing the service. Using supplementary variable technique, Arrar et al. (2012) investigated asymptotic behaviour of M/G/1 retrial queues with batch arrivals and impatience phenomenon. Bhagat and Jain (2013) investigated unreliable M^x/G/1 retrial queue with multioptional services and impatience to obtain the queue length of the system and other performance indices.

The maximum entropy approach (MEP) was introduced by Shannon in (1948) to study the problems of information theory as the measurement of uncertainty. This principle is also applicable to select the appropriate probability distributions for the queueing situation. Wang et al. (2002) used the maximum entropy principle to examine the M/G/1 queueing system in different frameworks. Wang et al. (2007) carried out the maximum entropy analysis of M^x/M/1 queueing system with multiple vacations. A comparative study between the exact analytical results and approximate results obtained using maximum entropy method has been done by Wang and Huang (2009). Maximum entropy principle has also been used for discrete time unreliable server queue with working vacation by Jain et al. (2012).

In this investigation, a bulk arrival retrial vacation queue with unreliable server has been studied. Both service of the customers and repair of broken down server are done in a series of fixed compulsory phases in succession. In many cases, the server takes some time to start the repair so as to make some preliminary settings, known as setup time. Moreover, the concept of modified vacation has been incorporated along with the discouragement behaviour of the customers. The present analysis seems to be novel and fascinating as compared to earlier existing retrial models because of so many features incorporated at the same time. The present work has been organized in the following manner. Section "Model description" describes the requisite assumptions to formulate the model. The governing equations along with the boundary conditions and generating functions of the queue size distribution are obtained in Sect. "Queue size distribution". The performances measures are derived in Sect. "Performance indices". Section "Maximum entropy analysis" presents maximum entropy principle to faciliate approximate results for waiting time of the retrial model. Section "Numerical simulation" is devoted to the sensitivity analysis which is carried out by taking numerical illustration. The cost function has been formulated in Sect. "Cost analysis".

Finally, the conclusions have been drawn in Sect. "Concluding remarks".

Model description

Consider a bulk arrival M/G/1 retrial queue model with unreliable server and vacation policy. The basic assumptions underlying the model are as follows.

Arrival process

The customers arrive in batches in the system following the Poisson distribution with state-dependent arrival rate λ_I depending on server's status; 'I' takes value 1, 2, 3, 4 and 5 when the server is in retrial state, busy state, setup state, repair state and in vacation state, respectively. Let X be the random variable denoting the batch size defined by $\Pr\{X = m\} = c_m$; $m \geq 1$ such that $\sum_{m=1}^{\infty} c_m = 1$.

Retrial process

The incoming customers are served if they find the server idle; otherwise, they are forced to join the virtual pool of the customers called orbit from where they try again for the service. The customers waiting in the retrial orbit are known as retrial customers and they retry with retrial rate γ. The cdf, pdf and LST of pdf for the retrial process are denoted by $A(t)$, $a(t)$ and $\tilde{a}(s)$, respectively.

Service process

If an incoming batch of the customers finds the server in idle state, then a customer at the head of the batch joins the server to get served. All the customers are served in k essential phases with service rate μ_i $(1 \leq i \leq k)$ for a customer availing ith phase of service. The service time distribution is assumed to be general distributed. The cdf, pdf and LST of pdf for the service time are denoted by $B_i(t)$, $b_i(t)$ and $\tilde{b}_i(s)$, respectively.

Breakdown and repair process

The server under consideration is unreliable which can breakdown during any course of service. The unreliable server breakdowns exponentially with rate α_i $(1 \leq i \leq k)$. The broken down server is sent for repair immediately so as to become as good as before failure. The repair process is completed in d essential phases while the server is broken down during any ith $(1 \leq i \leq k)$ phase of service. The repair rate of jth phase of repair is β_{ij} $(1 \leq i \leq k)$, $(1 \leq j \leq \mathrm{d})$ for server broken down during any ith phase of service. The cdf, pdf and LST of pdf for the repair time are denoted by $G_{i,j}(t)$ and $g_{i,j}(t)$ and $\tilde{g}_{i,j}(s)$, respectively.

Set up before repair

Before starting the repair process of broken down server, some set up time is required to make some preliminary settings, i.e. there is delay-in-repair with setup rate ξ_i $(1 \leq i \leq k)$. The setup process is also general distributed with cdf, pdf and LST of pdf for the set up process denoted by $N_i(t)$, $\eta_i(t)$ and $\tilde{\eta}_i(s)$, respectively.

Vacation policy

If no more customers are present in the system, then the server takes at most J vacations repeatedly with rate θ_l $(1 \leq l \leq J)$ for lth vacation and returns back if at least one job is found in the orbit after returning from the vacation. This process repeats again if no more jobs are available in the system, i.e. the server may reactivate at the end of lth $(1 \leq l \leq J)$ vacation if any customer/job is available in the system. But the server remains dormant in the system if no job is present in the system at the end of Jth vacation. The vacation time is assumed to be general distributed with cdf, pdf and LST of pdf for the set up time denoted by $U_l(t)$, $u_l(t)$ and $\tilde{u}_l(s)$, respectively.

Reneging

If a primary customer arrives earlier as compared to retrial customer, then either retrial customer quits the system forever with probability $(1 - r)$ or it may cancel its attempt for service and returns back to its initial position with probability r.

Queue size distribution

To analyze the retrial queueing system, we need to construct the mathematical equations for the system state probabilities. The retrial process, service process, vacation policy and repair process are assumed to be general distributed; therefore, the model under consideration is non-markovian. In order to formulate the equations for the present non-markovian system, the supplementary variable technique has been employed by introducing the supplementary variables 'w' for elapsed retrial time, 'x' for elapsed service time as well as for the elapsed vacation time and 'y' for the elapsed repair time and elapsed setup time. Also at time t, $\xi(t)$ denotes the elapsed service and vacation time whereas $\sigma(t)$ denotes the elapsed repair and set up time of the customers.

Let $N(t)$ represent the number of customers in the system and $U_1(t)$, $U_2(t)$ and $U_3(t)$ denote the phase of the service, phase of repair and state of vacation, respectively at any time t.

The state of the server at any time t is given by

$$I(t) = \begin{cases} \text{1. When the server is in idle state.} \\ \text{2. When the server is busy in providing} \\ \quad \text{service to the customers.} \\ \text{3. When the server is broken down and under} \\ \quad \text{setup before repair.} \\ \text{4. When the server is broken down and under repair.} \\ \text{5. When the server is in vacation state.} \end{cases}$$

In the steady state, the joint distributions of the server state and queue size are defined as

$$D_n = \lim_{t\to\infty} \Pr\{I(t) = 1,\, N(t) = n\},\, n \geq 0$$

$$P_{i,n}(x) = \lim_{t\to\infty} \Pr\{I(t) = 2,\quad x \leq \xi(t) \leq x + \mathrm{d}x,\ N(t) = n,\quad U_1(t) = i\},\quad n \geq 0,\quad (1 \leq i \leq k)$$

$$S_{i,n}(x,y) = \lim_{t\to\infty} \Pr\{I(t) = 3,\quad \xi(t) = x, y \leq \sigma(t) \leq y + \mathrm{d}y,\ N(t) = n,\, U_1(t) = i, U_2(t) = j\},\ n \geq 0,\ (1 \leq i \leq k), (1 \leq j \leq d)$$

$$R_{i,j,n}(x,y) = \lim_{t\to\infty} \Pr\{I(t) = 4,\, \xi(t) = x,\, y \leq \sigma(t) \leq y + \mathrm{d}y,\ N(t) = n,\ U_1(t) = i,\ U_2(t) = j\},\ n \geq 0,\, (1 \leq i \leq k),\, (1 \leq j \leq d)$$

$$V_{l,n}(x) = \lim_{t\to\infty} \Pr\{I(t) = 5,\, x \leq \xi(t) \leq x + \mathrm{d}x,\ N(t) = n,\, U_1(t) = i,\, U_2(t) = j, U_3(t) = l\},\ n \geq 0,\, (1 \leq i \leq k),\, (1 \leq j \leq d),\, (1 \leq l \leq J)$$

Mathematical formulation

Before framing the governing equations for the model, we give the proposition stating the stability condition for the model as follows:

Proposition *The necessary and sufficient condition for the system to be stable is*

$$r(1 - \tilde{a}(\lambda_1)) + Y' < 1,$$

where

$$Y' = \prod_{q=1}^{k} b_q^{(1)} H_q'(1), H_i'(z) = -\lambda_2 C'(1) - \alpha_i M_i'(1) \& M_i'(z) = \eta_i^{(1)}(-\lambda_3 C'(1)) + \prod_{r=1}^{j} g_{i,r}^{(1)}(-\lambda_4 C'(1))$$

Proof In order to deal with the steady-state behaviour of the system, we need to establish the stability condition for the model. Wang et al. (2001) presented the proof for the establishment of stability condition for M/G/1 model. Following the same approach, we have derived the stability condition for our model.

Now, we formulate the set of equations along with the boundary conditions governing the model by introducing the supplementary variables technique as follows:

Governing equations

$$\lambda_1 D_0 = \int_0^\infty V_{J,0}(x)\, \theta_J(x) \mathrm{d}x \tag{1}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}w} + \lambda_1 + \gamma(w)\right] D_n(w) = 0;\ n \geq 1 \tag{2}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + \lambda_2 + \alpha_i + \mu_i(x)\right] P_{i,n}(x) = \lambda_2 \sum_{m=1}^{n} c_m P_{i,n-m}(x) + \int_0^\infty R_{i,j,n}(x,y)\beta_{i,j}(y)\mathrm{d}y;\ (1 \leq i \leq k),\, (1 \leq j \leq d) \tag{3}$$

$$\left[\frac{\partial}{\partial y} + \xi_i(y) + \lambda_3\right] S_{i,n}(x,y) = \lambda_3 \sum_{m=1}^{n} c_m S_{i,n-m}(x,y);\ (1 \leq i \leq k),\, n \geq 0 \tag{4}$$

$$\left[\frac{\partial}{\partial y} + \lambda_4 + \beta_{i,j}(y)\right] R_{i,j,n}(x,y) = \lambda_4 \sum_{m=1}^{n} c_m R_{i,j,n-m}(x,y);\ (1 \leq i \leq k),\, (1 \leq j \leq d),\ n \geq 0 \tag{5}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + \lambda_5 + \theta_l(x)\right] V_{l,0}(x) = 0;\quad (1 \leq l \leq J) \tag{6}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + \lambda_5 + \theta_l(x)\right] V_{l,n}(x) = \lambda_5 \sum_{m=1}^{n} c_m V_{l,n-m}(x);\ (1 \leq l \leq J),\ n \geq 1 \tag{7}$$

Boundary conditions

$$D_n(0) = \sum_{s=1}^{l} \int_0^\infty V_{l,n}(x)\theta_l(x)\mathrm{d}x + \int_0^\infty P_{k,n}(x)\mu_k(x)\mathrm{d}x \tag{8}$$

$$P_{i,n}(0) = \int_0^\infty P_{i-1,n}(x)\mu_{i-1}(x)\mathrm{d}x;\ (1 \leq i \leq k),\, n \geq 1 \tag{9}$$

$$S_{i,n}(x,0) = \alpha_i P_{i,n}(x),\ \ (1 \leq i \leq k);\ n \geq 1 \tag{10}$$

$$R_{i,1,n}(x,0) = \int_0^\infty S_{i,n}(x,y)\xi_i(y)\mathrm{d}y;\ (1 \leq i \leq k),\, n \geq 1 \tag{11}$$

$$R_{i,j,n}(x,0) = \int_0^\infty R_{i,j,n-1}(x,y)\beta_{i,j-1}\,\mathrm{d}y;\ (1 \leq j \leq d),\ (1 \leq i \leq k),\ n \geq 1 \tag{12}$$

$$P_{0,0}(0)=\int_0^\infty D_1(w)\gamma(w)\mathrm{d}w+(1-r)\lambda_1\int_0^\infty D_1(w)\mathrm{d}w+\lambda_1 D_0 \tag{13}$$

$$P_{1,n}(0)=\int_0^\infty D_{n+1}(w)\gamma(w)\mathrm{d}w+(1-r)\lambda_1\int_0^\infty D_{n+1}(w)\mathrm{d}w + r\lambda_1\int_0^\infty D_n(w)\mathrm{d}w;\quad n\geq 1 \tag{14}$$

$$V_{1,0}(0)=\begin{cases}\int_0^\infty P_{1,0}(x)\mu_1(x)\mathrm{d}x & n=0\\ 0 & n\geq 1\end{cases} \tag{15}$$

$$V_{l,0}(0)=\begin{cases}\int_0^\infty V_{l-1,0}(x)\theta_{l-1}(x)\mathrm{d}x & n=0 \quad l=2,3,..,J\\ 0 & n\geq 1 \quad l=2,3,..,J\end{cases} \tag{16}$$

Also, the normalizing condition is given as follows:

$$D_0+\sum_{n=1}^\infty\int_0^\infty D_n(w)\mathrm{d}w+\sum_{i=1}^k\sum_{n=0}^\infty\int_0^k P_{i,n}(x)\mathrm{d}x +\sum_{n=0}^\infty\sum_{i=0}^k\sum_{j=1}^d\int_0^\infty\int_0^\infty R_{i,j,n}(x,y)\mathrm{d}x\mathrm{d}y +\sum_{i=1}^k\sum_{n=0}^\infty\int_0^\infty\int_0^\infty S_{i,n}(x,y)\mathrm{d}x\mathrm{d}y+\sum_{l=1}^J\sum_{n=0}^\infty\int_0^\infty V_{l,n}(x)\mathrm{d}x=1 \tag{17}$$

Probability generating functions

We use probability generating functions (pgf) corresponding to different states of the server to solve the set of differential difference equations so as to obtain the steady-state solution of the retrial queueing model. We define the probability generating functions corresponding to the various states as

$$D(w,z)=\sum_{n=1}^\infty D_n(w)z^n;\ P_i(x,z)=\sum_{n=0}^\infty P_{i,n}(x)z^n;\ R_{i,j}(x,y,z)=\sum_{n=0}^\infty R_{i,j,n}(x,y)z^n\ (1\leq i\leq k)\text{ and }(1\leq j\leq d)$$

$$V_l(x,z)=\sum_{n=0}^\infty V_{l,n}(x)z^n;\ S_i(x,y,z)=\sum_{n=0}^\infty S_{i,n}(x,y)z^n;$$

$$C(z)=\sum_{n=1}^\infty C_n z^n;\quad |z|\leq 1,(1\leq l\leq J)$$

The corresponding hazard rates are, respectively, given by

$$\gamma(w)=\frac{a(w)}{1-A(w)},\quad \mu_i(x)=\frac{b_i(x)}{1-B_i(x)}$$
$$\beta_{i,j}(y)=\frac{g_{i,j}(x)}{1-G_{i,j}(x)}$$
$$\xi_i(y)=\frac{\eta_i(y)}{1-N_i(y)},\ \theta_l(x)=\frac{u_l(x)}{1-U_l(x)},$$
$$(1\leq i\leq k),\ (1\leq j\leq d),\ (1\leq l\leq J)$$

Now, we establish some theorems to present queue size distributions as follows:

Theorem 1 *The partial generating functions for the server being in idle state, in busy state, under repair state while broken down, during set up state, in lth vacation ($1\leq l\leq J$) at random epoch respectively, are*

$$D(w,z)=D(0,z)\exp\{-\lambda_1 w\}\bar{A}(w) \tag{18}$$

$$P_i(x,z)=P_1(0,z)\prod_{q=1}^{i-1}\tilde{b}_q(H_q(z))\exp\{-H_i(z)x\}\bar{B}_i(x) \tag{19}$$

$$R_{i,j}(x,y,z) =\alpha_i P_1(0,z)\prod_{q=1}^{i-1}\tilde{b}_q(H_q(z))\exp\{-H_i(z)x\}\bar{B}_i(x)\tilde{\eta}_i(-\lambda_3(\bar{C}(z))) \times\prod_{r=1}^{j-1}\tilde{g}_i(-\lambda_4(\bar{C}(z)))\exp\{-\lambda_4(\bar{C}(z))y\}\overline{B_i}(x)\bar{G}_{i,j}(y) \tag{20}$$

$$S_i(x,y,z)=\alpha_i P_1(0,z)\prod_{q=1}^{i-1}\tilde{b}_q(H_q(z))\exp\{-H_i(z)x\} \bar{B}_i(x)\exp\{-\lambda_3(\bar{C}(z))y\}\bar{N}_i(y) \tag{21}$$

$$V_l(x,z)=V_l(0,z)\exp\{-\lambda_5(\bar{C}(z))x\}\bar{U}_l(x), \tag{22}$$

where

$$V_l(0,z)=\frac{\lambda_1 D_0}{[\tilde{u}(\lambda_5)]^{J-l+1}}\ ,\ \text{ for }\ l=1,2,..,J \tag{23}$$

$$D(0,\mathrm{z})=\frac{\lambda_1 D_0\left[z\left(\left(\frac{[\tilde{u}_l(\lambda_5(\bar{C}(z))-1)][1-(\tilde{u}(\lambda_5))^J]}{\lambda_5[1-\tilde{u}(\lambda_5)](\tilde{u}(\lambda_5))^J}\right)+\left[\tilde{b}_k(H_k(z))\prod_{q=1}^{k-1}\tilde{b}_q(H_q(z))\right]-1\right)\right]}{z-\left[\tilde{b}_k(H_k(z))\prod_{q=1}^{k-1}\tilde{b}_q(H_q(z))\right][1+r(z-1)(1-\tilde{a}(\lambda_1))]} \tag{24}$$

$$P_1(0,z) = \frac{\lambda_1 D_0 \left[[1 + r(z-1)(1-\tilde{a}(\lambda))] \left(\left(\frac{[\tilde{u}_l(\lambda_5(\bar{C}(z)) - 1)][1 - (\tilde{u}(\lambda_5))^J]}{\lambda_5 [1 - \tilde{u}(\lambda_5)](\tilde{u}(\lambda_5))^J} \right) - 1 \right) + z \right]}{z - \left[\tilde{b}_k(H_k(z)) \prod_{q=1}^{k-1} \tilde{b}_q(H_q(z)) \right] [1 + r(z-1)(1-\tilde{a}(\lambda_1))]} \tag{25}$$

$$M_i(z) = \tilde{\eta}_i(\lambda_3(\bar{C}(z)))\tilde{g}_{i,j}(\lambda_4(\bar{C}(z))) \prod_{r=1}^{j-1} g_{i,r}(\lambda_4(\bar{C}(z))) \tag{26}$$

$$H_i(z) = [\lambda_2(\bar{C}(z)) + \alpha_i(1 - M_i(z))] \tag{27}$$

Proof Multiplying Eqs. (2)–(7) by z^n and summing over all values of n and then using generating functions, the above set of Eqs. (2)–(7) reduces to

$$\frac{d}{dw} D(w,z) + \sum_{n=0}^{\infty} D(w,z)[\lambda_1 + \gamma(w)] = 0 \tag{28}$$

$$\frac{d}{dx} P_i(x,z) + [\mu_i(x) + \lambda_2(\bar{C}(z)) + \alpha_i] P_i(x,z) = \int_0^{\infty} R_{i,j}(x,y,z)\beta_{i,j}(y)dy \tag{29}$$

$$\frac{\partial}{\partial y} S_i(x,y,z) + [\lambda_3(\bar{C}(z)) + \xi_i(y)] S_i(x,y,z) = 0 \tag{30}$$

$$\frac{\partial}{\partial y} R_{i,j}(x,y,z) + [\lambda_4(\bar{C}(z)) + \beta_{ij}(y)] R_{i,j}(x,y,z) = 0 \tag{31}$$

$$\frac{d}{dx} V_l(x,z) + [\lambda_5(\bar{C}(z)) + \theta_l(x)] V_l(x,z) = 0 \tag{32}$$

On solving Eq. (28) we get results given in Eq. (18).

On solving Eqs. (30), (31) and (32), we get

$$S_i(x,y,z) = S_i(x,0,z) \exp\{-\lambda_3(\bar{C}(z))y\} \bar{N}_i(y)\,; (1 \le i \le k) \tag{33}$$

$$R_{i,j}(x,y,z) = R_{i,j}(x,0,z) \exp\{-\lambda_4(\bar{C}(z))y\} \bar{G}_{i,j}(y); \quad (1 \le i \le k) \,\&\, (1 \le j \le d) \tag{34}$$

$$V_l(x,z) = V_l(0,z) \exp\{-\lambda_5(\bar{C}(z))x\} \bar{W}_l(x); (1 \le l \le J) \tag{35}$$

On solving (6), we get

$$V_{l,0}(x) = V_{l,0}(0) \exp\{(-\lambda_5)x\} \bar{U}_l(x) \tag{36}$$

Multiplying (7) by z^n and summing over all values of n ($\ge$1) and then adding in (1), and using further generating functions, we get

$$D(0,z) = \sum_{l=1}^{J} \int_0^{\infty} V_l(x,z)\theta_l(x)dx + \int_0^{\infty} P_k(x,z)\mu_k(x)dx - \sum_{s=1}^{l} V_{l,0}(0) - \lambda_1 D_0 \tag{37}$$

Similarly, multiplying Eqs. (9)–(12) by z^n and summing over all values of n ($\ge$1) and then using generating functions, we get

$$P_i(0,z) = \int_0^{\infty} P_{i-1}(x,z)\mu_{i-1}(x)dx, \quad (1 \le i \le k) \tag{38}$$

$$R_{i,j}(x,0,z) = \int_0^{\infty} R_{i,j-1}(x,y,z)\beta_{i,j-1}(y)dy \, (2 \le j \le d), (1 \le i \le k) \tag{39}$$

$$R_{i,1}(x,0,z) = \int_0^{\infty} S_i(x,y,z)\xi_i(y)dy; n \ge 1, (1 \le i \le k) \tag{40}$$

$$S_i(x,0,z) = \alpha_i P_i(x,z), \quad (1 \le i \le k) \tag{41}$$

Now, multiplying (14) by z^n and summing over all values of n ($\ge$1) and then adding them in (13) and after using generating functions, we get

$$P_0(0,z) = \frac{1}{z} \int_0^{\infty} D(w,z)\gamma(w)dw + \frac{\lambda(1 - r + rz)}{z} \int_0^{\infty} D(w,z)dw + \lambda D_0 \tag{42}$$

Multiplying (36) by $\eta_l(x)$ on both sides for $l = J$ and integrating w.r.t. 'x', we have

$$\int_0^{\infty} V_{J,0}(x)\theta_J(x)dx = \int_0^{\infty} V_{J,0}(0) \exp((-\lambda_5)x) \bar{U}_l(x)\theta_l(x)dx \tag{43}$$

Using (1) and (43), we get

$$V_{J,0}(0) = \frac{\lambda_1 D_0}{\tilde{u}(\lambda_5)} \tag{44}$$

Using (16) for $l = J$, we get

$$V_{l,0}(0) = \frac{\lambda_1 D_0}{[\tilde{u}(\lambda_5)]^{J-l+1}} \quad \text{for} \quad l = 1, 2, 3, \ldots, (J-1) \tag{45}$$

Also,

$$V_l(0,z) = \frac{\lambda_1 D_0}{[\tilde{u}(\lambda_5)]^{J-l+1}} \quad \text{for} \quad l = 1, 2, 3, \ldots, J \tag{46}$$

Integrating (36) from 0 to ∞ and using (45), we obtain

$$V_{l,0} = \frac{\lambda_1 D_0[1-\tilde{u}(\lambda_5)]}{\lambda_5[\tilde{u}(\lambda_5)]^{J-l+1}},\ (1 \le l \le J) \quad (47)$$

Also, the probability that no customer arrives in the system when the server is on vacation is obtained using (47)

$$V_0 = \frac{\lambda_1 D_0[1-\tilde{u}(\lambda_5)]}{\lambda_5[\tilde{u}(\lambda_5)]^{J}}$$

Using (40) and (41), we get

$$R_{i,1}(x,0,z) = \alpha_i P_i(x,z)\tilde{\eta}_i(\lambda_3(\bar{C}(z)));\ (1 \le i \le k) \quad (48)$$

Solving (39) for $j = 2, 3, 4, \ldots$, we obtain

$$R_{i,j}(x,0,z) = \alpha_i P_i(x,z)\tilde{\eta}_i(\lambda_3(\bar{C}(z)))\prod_{r=1}^{j-1}\tilde{g}_{i,r}(\lambda_4(\bar{C}(z)));$$
$$(1 \le i \le k),\ (2 \le j \le d) \quad (49)$$

Therefore, we have

$$R_{i,j}(x,y,z) = \alpha_i P_i(x,z)\tilde{\eta}_i(\lambda_3(\bar{C}(z)))\prod_{r=1}^{j-1}\tilde{g}_{i,r}(\lambda_4(\bar{C}(z))) \times \exp(-\lambda_4(\bar{C}(z))y)\bar{G}_{i,j}(y) \quad (50)$$

Using (50) in (29) and solving (38) recursively for $i = 2, 3$, we get (19).

Further, on solving (37) and (42) simultaneously as a pair of linear equations, we get (24) and (25).

D_0 can be determined by using normalizing condition (17) as

$$D_0 = \frac{(1-\chi_2'-\chi_1')}{(1-\chi_2'-\chi_1')\tilde{a}(\lambda_1) + \frac{\lambda_1(1+\chi_3'-r(1-\tilde{a}(\lambda_1)))}{\mu_i}\left(1+\alpha_i\left(\frac{1}{\beta_{ij}}+\frac{1}{\xi_i}\right)\right) + \frac{\lambda_1[1-\tilde{u}(\lambda_5)^J]}{\lambda_5[1-\tilde{u}(\lambda_5)][\tilde{u}(\lambda_5)]^J}\frac{1}{\theta_l}},$$

where

$$\chi_1' = \prod_{q=1}^{k} b_q^{(1)}H_q'(1),\ \chi_2' = r(1-\tilde{a}(\lambda_1)),$$

$$\chi_1' = \frac{[1-\tilde{u}(\lambda_5)^J]}{\lambda_5[1-\tilde{u}(\lambda_5)][\tilde{u}(\lambda_5)]^J}u_l^{(1)}(-\lambda_5 C'(1)).$$

Theorem 2 *The marginal probability generating functions at random epochs, when the server is in idle state, busy with ith (1 ≤ i ≤ k) phase service, under jth (1 ≤ j ≤ d) phase repair while breakdown, under set up before repair and under lth (1 ≤ l ≤ J) vacation are given, respectively, by*

$$D(z) = \frac{D(0,z)(1-\tilde{a}(\lambda_1))}{\lambda_1} \quad (51)$$

$$P_i(z) = \frac{P_1(0,z)\left(\prod_{q=1}^{i-1}\tilde{b}_q(H_q(z))\right)(1-\tilde{b}_i(H_i(z))}{H_i(z)};\ (1 \le i \le k) \quad (52)$$

$$R_{i,j}(z) = \frac{\alpha_i P_1(0,z)\left(\prod_{q=1}^{i-1}\tilde{b}_q(H_q(z))\right)\left(\prod_{r=1}^{j-1}\tilde{g}_{i,r}(\lambda_4(\bar{C}(z))\right)\tilde{\eta}_i(\lambda_3(\bar{C}(z))(1-\tilde{b}_i(H_i(z))(1-\tilde{g}_{i,j}(\lambda_4(\bar{C}(z)))}{H_i(z)(\lambda_4(\bar{C}(z)))};\ (1 \le i \le k),\ (1 \le j \le d) \quad (53)$$

$$S_i(z) = \frac{\alpha_i P_1(0,z)\left(\prod_{q=1}^{i-1}\tilde{b}_q(H_q(z))\right)(1-\tilde{b}_i(H_i(z)))(1-\tilde{\eta}_i(\lambda_3(\bar{C}(z)))}{H_i(z)(\lambda_3(\bar{C}(z))};\ (1 \le i \le k) \quad (54)$$

$$V_l(z) = \frac{\lambda_1 D_0[\tilde{u}_l(\lambda_5(1-C(z)))]}{[\tilde{u}_l(\lambda_5)]^{J-l+1}};\ (1 \le l \le J) \quad (55)$$

Proof The marginal generating functions for the different states of the server are obtained by using the following results:

$$D(z) = \int_0^\infty D(w,z)\mathrm{d}w,\ P_i(z) = \int_0^\infty P_i(x,z)\mathrm{d}x,\ R_{i,j}(z) = \int_0^\infty\int_0^\infty R_{i,j}(x,y,z)\mathrm{d}x\mathrm{d}y$$

$$S_i(z) = \int_0^\infty \int_0^\infty S_i(x,y,z)\mathrm{d}x\mathrm{d}y, \quad V_l(z) = \int_0^\infty V_l(x,z)\mathrm{d}x, \quad (1 \le \mathrm{i} \le \mathrm{k}), \ (1 \le \mathrm{j} \le d), \ (1 \le l \le J)$$

Theorem 3 The generating function for the number of customers in the retrial queue is

$$\begin{aligned} K(z) = {} & D_0 + \frac{D(0,z)(1-\tilde{a}(\lambda_1))}{\lambda_1} + \sum_{i=1}^{k} P_1(0,z) \frac{\left(\prod_{q=1}^{i-1} \tilde{b}_q(H_q(z)\right)(1-\tilde{b}_i(H_i(z))}{H_i(z)} \\ & + \sum_{i=1}^{k}\sum_{j=1}^{d} \frac{\alpha_i P_1(0,z)\left(\prod_{q=1}^{i-1} \tilde{b}_q(H_q(z))\right)\left(\prod_{r=1}^{j-1} \tilde{g}_{i,r}(\lambda_4(\bar{C}(z))\right)\tilde{\eta}_i(\lambda_3(\bar{C}(z))(1-\tilde{b}_i(H_i(z))(1-\tilde{g}_{i,j}(\lambda_4(\bar{C}(z)))}{H_i(z)(\lambda_4(\bar{C}(z)))} \\ & + \sum_{i=1}^{k} \frac{\alpha_i P_1(0,z)\left(\prod_{q=1}^{i-1} \tilde{b}_q(H_q(z))\right)(1-\tilde{b}_i(H_i(z)))(1-\tilde{\eta}_i(\lambda_3(\bar{C}(z)))}{H_i(z)(\lambda_3(\bar{C}(z))} + V_0[\tilde{u}_l(\lambda_5(\bar{C}(z)))] \end{aligned} \tag{56}$$

Proof We can obtain the probability generating function for the number of customers in the retrial queue using

$$K(z) = D_0 + D(z) + \sum_{i=1}^{k} P_i(z) + \sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j}(z) + \sum_{i=1}^{k} S_i(z) + \sum_{l=1}^{J} V_l(z) \tag{57}$$

Theorem 4 The generating function for the number of customers present in the system is

$$\begin{aligned} L(z) = {} & D_0 + \frac{D(0,z)(1-\tilde{a}(\lambda_1))}{\lambda_1} + z\sum_{i=1}^{k} P_1(0,z) \frac{\left(\prod_{q=1}^{i-1} \tilde{b}_q(H_q(z)\right)(1-\tilde{b}_i(H_i(z))}{H_i(z)} \\ & + z\sum_{i=1}^{k}\sum_{j=1}^{d} \frac{\alpha_i P_1(0,z)\left(\prod_{q=1}^{i-1} \tilde{b}_q(H_q(z))\right)\left(\prod_{r=1}^{j-1} \tilde{g}_{i,r}(\lambda_4(1-C(z))\right)\tilde{\eta}_i(\lambda_3(1-C(z))(1-\tilde{b}_i(H_i(z))(1-\tilde{g}_{i,j}(\lambda_4(1-C(z)))}{H_i(z)(\lambda_4(1-C(z)))} \\ & + z\sum_{i=1}^{k} \frac{\alpha_i P_1(0,z)\left(\prod_{q=1}^{i-1} \tilde{b}_q(H_q(z))\right)(1-\tilde{b}_i(H_i(z)))(1-\tilde{\eta}_i(\lambda_3(1-C(z)))}{H_i(z)(\lambda_3(1-C(z))} + V_0[\tilde{u}_l(\lambda_5(1-C(z)))] \end{aligned} \tag{58}$$

Proof The generating function of the number of customers in the system is obtained using the results of marginal generating functions given by

$$L(z) = D_0 + D(z) + z\sum_{i=1}^{k} P_i(z) + z\sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j}(z) + z\sum_{i=1}^{k} S_i(z) + \sum_{l=1}^{J} V_l(z) \tag{59}$$

Performance indices

The performance measures to quantify the system characteristics are of vital utility to improve the effectiveness of any system. Similarly, the applicability of any queueing model can also be best deciphered by means of its performance indices. Some of the important performance indices are derived using generating functions in various categories.

Long-run probabilities

We derive analytic expressions for the long-run probabilities of the server states. These results are significant in the sense that these probabilities completely describe the behaviour of a system when the system is analyzed after a long period of time. The long-run probabilities of the server being in idle (P_{I}), busy (P_{B}), repair (P_{R}), set up (P_{S}) and vacation (P_{V}) states, respectively, are given below in the form of theorem.

Theorem 5 (i) The long-run probability of the server being in idle state is

$$P_{\mathrm{I}} = (1-\tilde{a}(\lambda_1))D_0\left(\frac{(\chi_3' + \chi_1')}{(1-\chi_1'-\chi_2')}\right) \tag{60}$$

(ii) The long-run probability of the server being in busy state is

$$P_{\mathrm{B}} = \sum_{i=1}^{k} \lambda_1 D_0\left(\frac{\varsigma_9''}{\varsigma_8''}\right) \tag{61}$$

(iii) The long-run probability of the server being in repair state is

$$P_{\mathrm{R}} = \sum_{i=1}^{k} \alpha_i\lambda_1 D_0\left(\frac{\varsigma_{12}'''}{\varsigma_{11}'''}\right) \tag{62}$$

(iv) The long-run probability of the server being under set up state is

$$P_{\mathrm{S}} = \sum_{i=1}^{k} \alpha_i\lambda_1 D_0\frac{(\varsigma_{13}''')}{(\varsigma_{14}''')} \tag{63}$$

(v) The long-run probability of the server being under vacation state is

$$P_{\mathrm{V}} = \frac{\lambda_1 D_0}{\lambda_5[\tilde{u}(\lambda_5)]^{J-l+1}}\left(\frac{\varsigma_7'}{\chi_8'}\right) \tag{64}$$

Proof The expressions for the long-run probabilities are obtained using

$$P_{\mathrm{I}} = \lim_{z\to1} D(z),\; P_{\mathrm{B}} = \lim_{z\to1}\sum_{i=1}^{k} P_i(z), P_{\mathrm{R}} = \lim_{z\to1}\sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j}(z)$$

$$P_{\mathrm{S}} = \lim_{z\to1}\sum_{i=1}^{k} S_i(z), P_{\mathrm{V}} = \lim_{z\to1}\sum_{s=1}^{l} V_s(z)$$

Queueing measures

Queue length is the most important and effective performance measure for any queueing system as it directly corresponds to the effectiveness of the system and further guides the system engineers to design the system as desired. We proceed to find out the analytic expressions for the queue length of both system and retrial orbit in the following theorem:

Theorem 6 *The mean queue length of the retrial orbit (L_R) and that of the system (L_S) are*

$$\begin{aligned} L_{\mathrm{R}} = {} & \frac{D_0(1-\tilde{a}(\lambda_1))(1-\chi_1'-\chi_2')\left(\left(\chi_3'+\chi_1'\right)+\left(\chi_3''+\chi_1''\right)\right)+\left(\chi_4''+\chi_2''\right)\left(\chi_3'+\chi_1'\right)}{2(1-\chi_1'-\chi_2')^2} + \sum_{i=1}^{k}\frac{\lambda_1 D_0(a''b'''-a'''b'')}{12(1-\chi_1'-\chi_2')^2\chi_5'^2} \\ & + \sum_{i=1}^{k}\frac{\alpha_i\lambda_1 D_0(5e^{iv}d'''-5d^{iv}e''')}{624(1-\chi_1'-\chi_2')^2\chi_5'^2\chi_6'^2} + \sum_{i=1}^{k}\frac{\alpha_i\lambda_1 D_0(5f^{iv}h'''-5h^{iv}f''')}{624(1-\chi_1'-\chi_2')^2\chi_5'^2\chi_7'^2} + \sum_{l=1}^{J}\frac{\lambda_1 D_0}{\lambda_5[\tilde{u}(\lambda_5)]^{J-l+1}}\left(\frac{\chi_8'\chi_9''-\chi_8''\chi_9'}{2\chi_8'^2}\right) \end{aligned} \tag{65a}$$

$$L_{\mathrm{S}} = L_{\mathrm{R}} + \frac{\lambda_1 D_0\varsigma_9''}{\varsigma_8''} + \sum_{i=1}^{k}\frac{\alpha_i\lambda_1 D_0\varsigma_{12}'''}{\varsigma_{11}'''} + \sum_{i=1}^{k}\frac{\alpha_i\lambda_1 D_0\varsigma_{13}'''}{\varsigma_{14}'''}, \tag{65b}$$

where

$$\chi_1 = 1 + r(z-1)(1-\tilde{a}(\lambda_1));\;\; \chi_1' = r(1-\tilde{a}(\lambda_1))$$

$$M_i(z) = \tilde{\eta}_i(\lambda_3(1-C(z)))\tilde{g}_{i,j}(\lambda_4(1-C(z)))\prod_{r=1}^{j-1} g_{i,r}(\lambda_4(1-C(z)))$$

$$M_i'(z) = \eta_i^{(1)}(-\lambda_3 C'(1)) + \prod_{r=1}^{j} g_{i,r}^{(1)}(-\lambda_4 C'(1)),$$

$$\begin{aligned} M_i''(z) = {} & \eta_i^{(2)}(-\lambda_3 C'(1))^2 + \eta_i^{(1)}(-\lambda_3 C''(1)) + \eta_i^{(1)}(-\lambda_3 C'(1)) \\ & \times \prod_{r=1}^{j} g_{i,r}^{(1)}(-\lambda_4 C'(1)) \\ & + \prod_{r=1}^{j} g_{i,r}^{(2)}(-\lambda_4 C'(1))^2 + \prod_{r=1}^{j} g_{i,r}^{(1)}(-\lambda_4 C''(1)), \end{aligned}$$

$$\begin{aligned} \chi_5 = H_i(z) = \lambda_2(1-C(z)) + \alpha_i(1-M_i(z)),\;\; \chi_5' = H_i'(z) \\ = -\lambda_2 C'(1) - \alpha_i M_i'(1),\;\; \chi_5'' = H_i''(z) = -\lambda_2 C''(1) - \alpha_i M_i''(1) \end{aligned}$$

$$\chi_4 = \prod_{q=1}^{k-1} \tilde{b}_q(H_q(z))\tilde{b}_k(H_k(z)),\ \chi_4' = \prod_{q=1}^{k} b_q^{(1)} H_q'(1),$$

$$\chi_4'' = \prod_{q=1}^{k} b_q^{(2)} H_q'(1) + \prod_{q=1}^{k} b_q^{(1)} H_q''(1)$$

$$\varsigma_1 = \prod_{q=1}^{i-1} \tilde{b}_q(H_q(z)),\ \varsigma_1' = \prod_{q=1}^{i-1} b_q^{(1)}(H_q'(1)),\ \varsigma_1'' = \prod_{q=1}^{i-1} b_q^{(2)}(H_q'(1)) + \prod_{q=1}^{i-1} b_q^{(1)}(H_q''(1))$$

$$\varsigma_2 = 1 - \tilde{b}_i(H_i(z)),\ \varsigma_2' = -b_i^{(1)} H_i'(1),\ \varsigma_2'' = -b_i^{(2)} H_i''(1)^2 - b_i^{(1)} H_i''(1)$$

$$\varsigma_3 = 1 - \tilde{g}_{i,j}(\lambda_4(1 - C(z))),\ \varsigma_3' = -g_{i,j}^{(1)}(-\lambda_4 C'(1)),$$
$$\varsigma_3'' = -g_{i,j}^{(2)}(\lambda_4 C'(1))^2 - g_{i,j}^{(1)}(-\lambda_4 C''(1))$$

$$\chi_6 = \lambda_4(1 - C(z)),\ \chi_6' = -\lambda_4 C'(1),\ \chi_6'' = -\lambda_4 C''(1)$$

$$\chi_3 = \frac{[1 - \tilde{u}(\lambda_5)^J]}{\lambda_5[1 - \tilde{u}(\lambda_5)][\tilde{u}(\lambda_5)]^J} [\tilde{u}_s(\lambda_5(1 - C(z)))]\ ,$$

$$\chi_3' = \frac{[1 - \tilde{u}(\lambda_5)^J]}{\lambda_5[1 - \tilde{u}(\lambda_5)][\tilde{u}(\lambda_5)]^J} \tilde{u}_l^{(1)}(-\lambda_5 C'(1))$$

$$\chi_3'' = \frac{[1 - \tilde{u}(\lambda_5)^J]}{\lambda_5[1 - \tilde{u}(\lambda_5)][\tilde{u}(\lambda_5)]^l} \left(\tilde{u}_l^{(2)}(-\lambda_5 C'(1))^2 + \tilde{u}_l^{(1)}(-\lambda_5 C''(1))\right)$$

$$\varsigma_4 = \tilde{\eta}_i(\lambda_3(1 - C(z)),\ \varsigma_4' = -\eta_i^{(1)}\lambda_3 C'(1),$$
$$\varsigma_4'' = \eta_i^{(2)}(-\lambda_3 C'(1))^2 - \eta_i^{(1)}\lambda_3 C''(1)$$

$$\varsigma_5 = \prod_{r=1}^{j-1} \tilde{g}_{i,r}(\lambda_4(1 - C(z))),\ \varsigma_5' = \prod_{r=1}^{j-1} g_{i,r}^{(1)}(-\lambda_4 C'(1)),$$

$$\varsigma_5'' = \prod_{r=1}^{j-1} g_{i,r}^{(1)}(-\lambda_4 C''(1)) + \prod_{r=1}^{j-1} g_{i,r}^{(2)}(-\lambda_4 C'(1))^2$$

$$\varsigma_6 = 1 - \tilde{\eta}_i(\lambda_3(1 - C(z))),\ \varsigma_6' = -\eta_i^{(1)}(-\lambda_3 C'(1)),$$
$$\varsigma_6'' = -\eta_i^{(1)}(-\lambda_3 C''(1)) - \eta_i^{(2)}(\lambda_3 C'(1))^2$$

$$\chi_7 = \lambda_3(1 - C(z)),\ \chi_7' = -\lambda_3 C'(1),\ \chi_7'' = -\lambda_3 C''(1)$$

$$\varsigma_7 = 1 - \tilde{u}_l(\lambda_5(1 - C(z))),\ \varsigma_7' = -u_l^{(1)}(-\lambda_5 C'(1)),$$
$$\varsigma_7'' = -u_l^{(1)}(-\lambda_5 C''(1)) - u_l^{(2)}(-\lambda_5 C'(1))^2$$

$$\chi_{10} = \lambda_5(1 - C(z)),\ \chi_{10}' = -\lambda_5 C'(1),\ \chi_{10}'' = -\lambda_5 C''(1)$$

$$\varsigma_8 = (z - \chi_1\chi_2)\chi_5,\ \varsigma_8'' = 2(1 - \chi_1' - \chi_2')\chi_5',$$
$$\varsigma_8''' = 3(z - \chi_1\chi_2)''\chi' + 3(z - \chi_1\chi_2)'\chi_5''$$

$$\varsigma_9 = \chi_2\varsigma_1\varsigma_2,\ \varsigma_9'' = 2\chi_2'\varsigma_2',$$
$$\varsigma_9''' = 3\chi_2''\varsigma_2' + 6\chi_2'\varsigma_1'\varsigma_2' + 3\chi_2'\varsigma_2'',\ \varsigma_{11} = \varsigma_8\chi_6,$$
$$\varsigma_{11}''' = 3\varsigma_8''\chi_6',\ \varsigma_{11}^{iv} = 4\varsigma_8'''\chi_6' + 6\varsigma_8''\chi_6'',$$

$$\varsigma_{10} = \varsigma_4\varsigma_5\varsigma_3,\ \varsigma_{10}' = \varsigma_3',$$
$$\varsigma_{10}'' = 2\varsigma_4'\varsigma_5'\varsigma_3' + 2\varsigma_4\varsigma_5'\varsigma_3' + \varsigma_4\varsigma_5\varsigma_3'',\ \varsigma_{12} = \varsigma_9\varsigma_{10},$$
$$\varsigma_{12}''' = 3\varsigma_9''\varsigma_{10}',\ \varsigma_{12}^{iv} = 4\varsigma_9'''\varsigma_{10}' + 6\varsigma_9''\varsigma_{10}''$$

$$\varsigma_{14} = \varsigma_8\chi_7,\ \varsigma_{14}''' = 3\varsigma_8''\chi_7',\ \varsigma_{14}^{iv} = 4\varsigma_8'''\chi_7' + 6\varsigma_8''\chi_7'',$$
$$\varsigma_{13} = \varsigma_9\varsigma_6,\ \varsigma_{13}''' = 3\varsigma_9''\varsigma_6',\ \varsigma_{13}^{iv} = 4\varsigma_9'''\varsigma_6' + 6\varsigma_9''\varsigma_6''$$

Proof The mean queue length of the retrial orbit and mean queue length of the system are obtained using

$$L_R = \lim_{z\to 1} K'(z) \text{ and } L_s = \lim_{z\to 1} L'(z)$$

Here, L Hospital rule has been used six times to evaluate the limiting value when $z \to 1$.

Theorem 7 *The exact expected waiting time for a customer in the system is obtained as*

$$W_s = \frac{L_s}{\lambda_{\text{eff}}}, \tag{66}$$

where $\lambda_{\text{eff}} = [\lambda_1 P_I + \lambda_2 P_B + \lambda_3 P_S + \lambda_4 P_R + \lambda_5 P_V]E[X]$.

Proof The exact expected waiting time W_s is obtained using Little's formula (cf. Gross and Harris 1998) as $W_s = \frac{L_s}{\lambda_{\text{eff}}}$.

Reliability measures

A mathematical model for the unreliable server queue framed analytically can be best validated by its reliability measures as reliability of the system directly affects the efficiency/availability of the system. Now, we derive some important reliability measures, namely availability and failure frequency.

Theorem 8 *The steady-state availability (A_v) and failure frequency (F_f) of the server are*

$$A_v = D_0\left[\tilde{a}(\lambda_1) + \sum_{i=1}^{k} \lambda_1 \frac{[1 + \chi_3' - r(1 - \tilde{a}(\lambda_1))]}{(1 - \chi_2' - \chi_1')\mu_i}\right] \tag{67a}$$

$$F_f = D_0 \sum_{i=1}^{k} \lambda_1 \alpha_i \frac{[1 + \chi_3' - r(1 - \tilde{a}(\lambda_1))]}{(1 - \chi_2' - \chi_1')\mu_i} \tag{67b}$$

Proof The availability and failure frequency for the system are obtained using

$$A_v = D_0 + \int_0^\infty D(w,1)\mathrm{d}w + \sum_{i=1}^{k} \int_0^\infty P_i(x,1)\mathrm{d}x$$

$$F_f = \sum_{i=1}^{k} \int_0^\infty \alpha_i P_i(x,1)\mathrm{d}x$$

Maximum entropy analysis

The principle of maximum entropy can be used for estimating probabilistic information measures which can be used further to obtain queue size distribution of the concerned queueing systems. In this section, we employ

MEP to determine the steady-state probabilities $P_{i,n}$ $(1 \le i \le k)$, $R_{i,j,n}$ $(1 \le j \le d)$, $S_{i,n}$ $(1 \le i \le k)$, $V_{l,n}$ $(1 \le l \le J)$ and D_n for the $M^{[x]}/G/1$ retrial queueing system with modified vacation policy. For the analysis purpose, we follow the following procedure (c.f. Wang et al. 2007):

(i) The construction of Lagrange's function H using the method of Lagrange's multipliers subject to a set of constraints in terms of known indices.
(ii) Partial differentiation of Lagrange's function H w.r.t. $P_{i,n}$, $R_{i,j,n}$, $S_{i,n}$, $V_{l,n}$ and D_n and setting the results to zero.
(iii) Finally, solving the equations obtained in (ii) to derive results for the required probabilities.

Maximum entropy function

The maximum entropy function Y (cf. El-Affendi and Kouvatos 1983) is formulated to evaluate the steady-state probabilities using several known constraints in terms of performance characteristics as follows:

$$Y = -\sum_{n=1}^{\infty}\sum_{i=1}^{k} P_{i,n}\log P_{i,n} - \sum_{n=1}^{\infty}\sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j,n}\log R_{i,j,n} - \sum_{n=1}^{\infty}\sum_{i=1}^{k} S_{i,n}\log S_{i,n} - \sum_{n=1}^{\infty} D_n \log D_n - \sum_{n=1}^{\infty}\sum_{l=1}^{J} V_{l,n}\log V_{l,n} \tag{68}$$

Subject to the constraints

$$\text{(i)}\ \sum_{n=1}^{\infty}\sum_{i=1}^{k} P_{i,n} = P_B,\ \sum_{n=1}^{\infty}\sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j,n} = P_R,\ \sum_{n=1}^{\infty}\sum_{i=1}^{k} S_{i,n} = P_S \tag{69}$$

$$\text{(ii)}\ \sum_{n=1}^{\infty} D_n = P_I,\ \sum_{n=1}^{\infty}\sum_{l=1}^{J} V_{l,n} = P_V \tag{70}$$

$$\text{(iii)}\ \sum_{n=1}^{\infty} n\left\{\sum_{i=1}^{k} P_{i,n} + \sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j,n} + \sum_{i=1}^{k} S_{i,n} + \sum_{l=1}^{J} V_{l,n} + D_n\right\} = L_S \tag{71}$$

Construction of Lagrange's function

To determine the maximum value of entropy function, we construct Lagrange's function $H\,(P_{i,n}, R_{i,j,n}, S_{i,n}, D_n, V_{l,n})$ by introducing the Lagrange's multipliers θ_i $(1 \le i \le k)$, δ_{ij} $(1 \le i \le k)$, ϕ_i $(1 \le i \le k)$, θ_{k+1}, ε_l $(1 \le l \le J)$ and ϕ_{k+1} corresponding to the information, i.e. constraints (69–71) available in the form of derived analytical results. Thus, we have

$$\begin{aligned} H\left(P_{i,}, R_{i,j,n}, S_{i,n,}, D_n, V_{l,n}\right) &= \sum_{n=1}^{\infty}\sum_{i=1}^{k} P_{i,n}\log P_{i,n} - \sum_{n=1}^{\infty}\sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j,n}\log R_{i,j,n} \\ &- \sum_{n=1}^{\infty}\sum_{i=1}^{k} S_{i,n}\log S_{i,n} - \sum_{n=1}^{\infty} D_n\log D_n \\ &- \sum_{n=1}^{\infty}\sum_{l=1}^{J} V_{l,n}\log V_{l,n} - \sum_{i=1}^{k}\theta_i\left[\sum_{n=1}^{\infty}\sum_{i=1}^{k} P_{i,n} - P_B\right] \\ &- \sum_{i=1}^{k}\sum_{j=1}^{d}\delta_{ij}\left[\sum_{n=1}^{\infty}\sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j,n} - P_R\right] \\ &- \sum_{i=1}^{k}\phi_i\left[\sum_{n=1}^{\infty}\sum_{i=1}^{k} S_{i,n} - P_S\right] - \theta_{k+1}\left[\sum_{n=1}^{\infty} D_n - P_I\right] \\ &- \sum_{l=1}^{J}\varepsilon_l\left[\sum_{n=1}^{\infty}\sum_{l=1}^{J} V_{l,n} - P_V\right] \\ &- \phi_{k+1}\left[\sum_{n=1}^{\infty} n\left\{\sum_{i=1}^{k} P_{i,n} + \sum_{i=1}^{k}\sum_{j=1}^{d} R_{i,j,n} + \sum_{i=1}^{k} S_{i,n}\right.\right. \\ &\left.\left. + \sum_{i=1}^{k} V_{l,n} + D_n\right\} - L_S\right] \end{aligned} \tag{72}$$

The obtained results for the approximate probabilities of different states are presented in the form of theorem as given below.

Theorem 9 *The maximum entropy solutions for the approximate values of probabilities* $P_{i,n}, R_{i,j,n}, D_n, S_{i,n}\&$ $V_{l,n}(1 \le i \le k),\ (1 \le j \le d),\ (1 \le l \le J), n \ge 1$ *subject to the constraints are*

$$\begin{aligned} &\hat{P}_{i,n} = \frac{P_B\sigma[L_S - \sigma]^{n-1}}{L_S^n}\quad \hat{R}_{i,j,n} = \frac{P_R\sigma[L_S - \sigma]^{n-1}}{L_S^n} \\ &\hat{D}_n = \frac{P_I\sigma[L_S - \sigma]^{n-1}}{L_S^n} \\ &\hat{S}_{i,n} = \frac{P_S\sigma[L_S - \sigma]^{n-1}}{L_S^n}\quad \hat{V}_{l,n} = \frac{P_V\sigma[L_S - \sigma]^{n-1}}{L_S^n} \end{aligned} \tag{73}$$

where

$$\sigma = P_B + P_R + P_V + P_s + P_I \tag{74}$$

Proof The approximate state probabilities $P_{i,n}$, $R_{i,j,n}$, $S_{i,n}$, $V_{l,n}\,\&\,D_n$ can be obtained by taking partial derivatives of H with respect to $P_{i,n}$, $R_{i,j,n}$, $S_{i,n}$, $V_{l,n}\,\&\,D_n$, and then setting the results equal to zero. Thus, we get

$$D_n = e^{-(1+\theta_{k+1})}e^{-n\phi_{k+1}},$$

$$P_{i,n} = e^{-(1+\theta_i)}e^{-n\phi_{k+1}}, (1 \le i \le k)$$
$$S_{i,n} = e^{-(1+\phi_i)}e^{-n\,\phi_{k+1}},$$
$$R_{i,j,n} = e^{-(1+\delta_{ij})}\,e^{-n\,\phi_{k+1}},\ \ (1 \le i \le k)\ ,(1 \le j \le d)$$
$$V_{l,n} = e^{-(1+\varepsilon_l)}e^{-n\phi_{k+1}}\ (1 \le i \le k),\ (1 \le j \le d),\ (1 \le l \le J) \tag{75}$$

For brevity of notations, we denote

$$e^{-(1+\phi_i)} = \psi_i(1 \le i \le k+1)$$
$$e^{-(1+\phi_i)} = \gamma_i\ ; (1 \le i \le k)$$
$$e^{-(1+\delta_{ij})} = \pi_{ij}\ (1 \le i \le k),\ (1 \le j \le d)$$
$$e^{-\phi_{k+1}} = d_{k+1}$$
$$e^{-(1+\varepsilon_l)} = \varpi_l;\ (1 \le l \le J) \tag{76}$$

Therefore, (75) reduces to

$$P_{i,n} = \psi_i d^n_{k+1}, R_{i,j,n} = \pi_{ij}d^n_{k+1}, S_{i,n} = \gamma_i d^n_{k+1}, \quad D_n = \psi_{k+1}d^n_{k+1}, V_{l,n} = \varpi_l d^n_{k+1} \tag{77}$$

Using (77) in constraints, i.e. (69–71), we get approximate results for the long-run probabilities of several states as follows:

$$P_B = \frac{\psi_i d_{k+1}}{1-d_{k+1}},\ P_R = \frac{\pi_{ij}d_{k+1}}{1-d_{k+1}},\ P_S = \frac{\gamma_i d_{k+1}}{1-d_{k+1}},\quad P_I = \frac{\psi_{k+1}d_{k+1}}{1-d_{k+1}}, P_V = \frac{\varpi_l d_{k+1}}{1-d_{k+1}} \tag{78}$$

Now using (71) and (77) we get the approximate queue length of the system as

$$\hat{L}_s = \frac{d_{k+1}\left[\sum_{i=1}^{k+1}\psi_i + \sum_{i=1}^{k}\sum_{j=1}^{d}\pi_{ij} + \sum_{i=1}^{k}\gamma_i + \sum_{l=1}^{J}\varpi_l\right]}{\left(1-d^2_{k+1}\right)} \tag{79}$$

Again denoting $\sigma = P_B + P_R + P_V + P_s + P_I$ and using (77) and (78), we have

$$L_s = \frac{\sigma}{1-d_{k+1}}\ \text{ and }\ d_{k+1} = \frac{L_s - \sigma}{L_s} \tag{80}$$

Further, using (77) and (79), we get

$$\psi_i = \frac{P_B\sigma}{L_s-\sigma},\ \pi_{ij} = \frac{P_R\sigma}{L_s-\sigma},\ \gamma_i = \frac{P_s\sigma}{L_s-\sigma},\quad \psi_{k+1} = \frac{P_I\sigma}{L_s-\sigma},\ \varpi_l = \frac{P_V\sigma}{L_s-\sigma}\ \ (0 \le i \le k) \tag{81}$$

Finally, substituting results from Eqs. (80) and (81) in Eq. (78), we get expressions given in Eq. (73).

Theorem 10 *Using the principle of maximum entropy, the approximate expected waiting time in the system is*

$$\begin{aligned}\hat{W}_s = &\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]\hat{D}_n \\ &+\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\frac{n}{\mu_i}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]\hat{P}_{i,n} \\ &+\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\sum_{j=1}^{d}\frac{g^{(2)}_{ij}}{2g^{(1)}_{ij}}+\sum_{i=1}^{k}\frac{n}{\mu_i}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]\hat{R}_{i,j,n} \\ &+\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\sum_{j=1}^{d}\frac{\eta^{(2)}_{i}}{2\eta^{(1)}_{i}}+\sum_{i=1}^{k}\frac{n}{\mu_i}+\frac{1}{\beta_{ij}}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]\hat{S}_{i,n} \\ &+\sum_{n=1}^{\infty}\left[\sum_{l=1}^{J}\frac{u^{(2)}_{l}}{2u^{(1)}_{l}}+\sum_{i=1}^{k}\frac{n}{\mu_i}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]\hat{V}_{l,n}\end{aligned} \tag{82}$$

Proof For proof see "Appendix".

Numerical simulation

The present section deals with the sensitivity analysis of the performance indices of queueing model with respect to various parameters. It is true to say that the efficiency of any mathematical model can be best deciphered by means of numerical illustration. The numerical simulation of derived analytic results seems to be an important step with regard to the validation of mathematical modeling of any queueing system. The present analysis has been divided into various subsections which are as follows:

Queue length (L_s)

To study the sensitivity of queue length towards various parameters, Figs. 1, 2, 3, 4, 5, 6 have been plotted corresponding to different service time distributions. Three service time distributions, namely Erlangian-2, exponential and gamma distributions have been considered for the service time. The vacation time, retrial process, set up process as well as repair process are assumed to follow exponential distribution. The set of default parameters assumed for simulation are as

$$\lambda = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0.5,\ \xi = 1,\quad \mu = \mu_1 = \mu_2 = 5,\ \alpha = \alpha_1 = \alpha_2 = 0.01,\ \gamma = 0.1,\quad \theta = 2,\ r = 0.1,\ \beta = \beta_1 = \beta_2 = 0.9.$$

The effect of parameters, namely breakdown rate (α) and repair rate (β), on the queue length of the system have been demonstrated in Figs. 1 and 2. It is noticed that the breakdown rate and repair rate are supposed to have contradictory effect on the queue length of the system which is very true. It is clear from Fig. 1a–c that the queue length increases as the breakdown rate increases from 0.008 to 0.01 U for all the service time distributions. The maximum

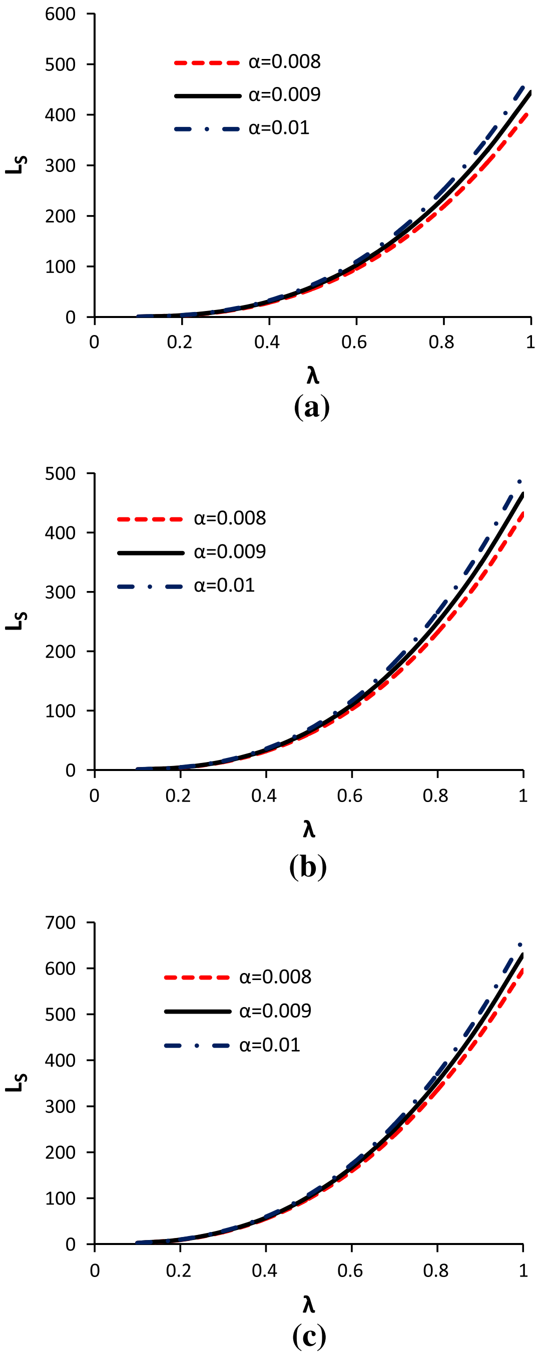

Fig. 1 Effect of α. **a** Erlangian-2. **b** Exponential. **c** Gamma distribution on L_s

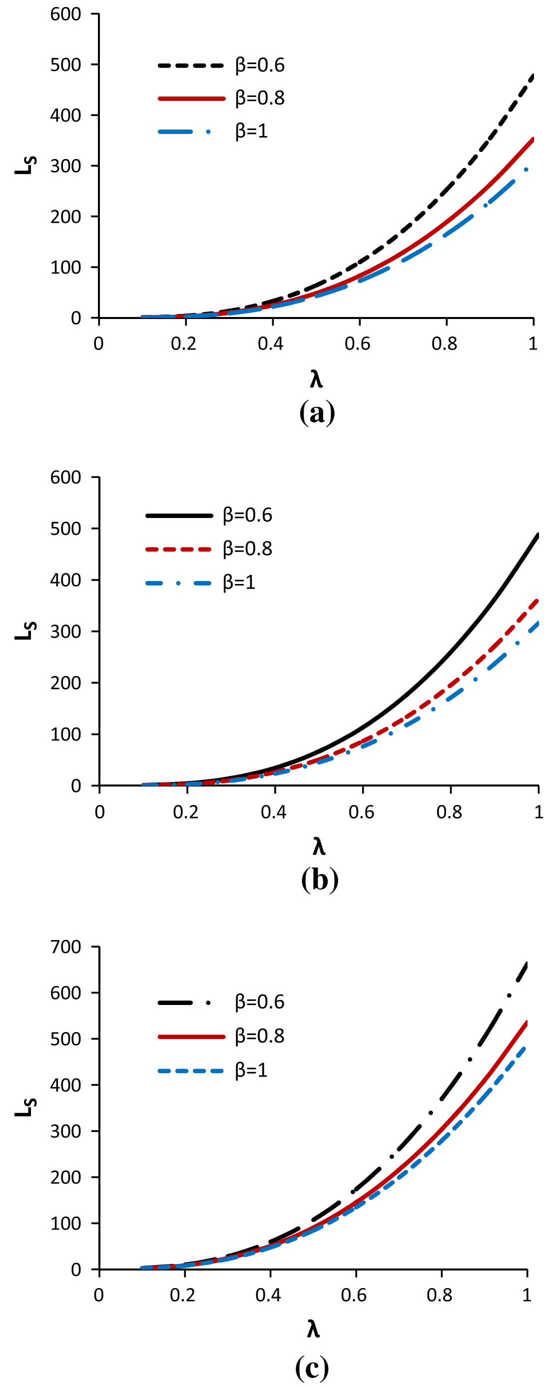

Fig. 2 Effect of β. **a** Erlangian-2. **b** Exponential. **c** Gamma distributions on L_s

number of customers or queue length is observed in the case 1(c) when the service time is supposed to be gamma distributed. On the other hand, in Fig. 2 wherein graphs are plotted for different values of repair rate β (0.6, 0.8 and 1), L_S decreases with an increase in the repair rate. This is due to the fact that an increase in breakdown rate forces the customers to accumulate in the system due to the non-working condition of the server and hence increases the queue length. However, an increase in the repair rate helps in the fast recovery of the server and thus reduces the number of customers in the system.

Figure 3 has been plotted for various values of service rate μ for (a) Erlangian-2 (b) exponential and (c) gamma service time distributions with arrival rate λ. The graphs plotted in Fig. 3 clearly demonstrate that the queue length of the system

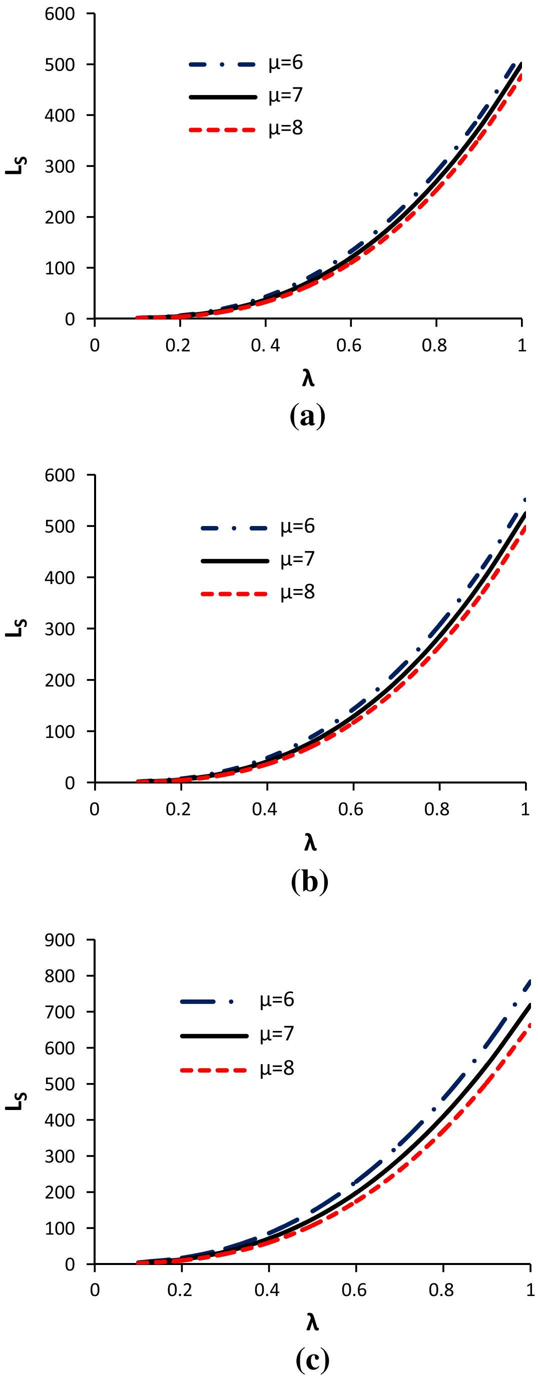

Fig. 3 Effect of μ. **a** Erlangian-2. **b** Exponential. **c** Gamma distribution on L_s

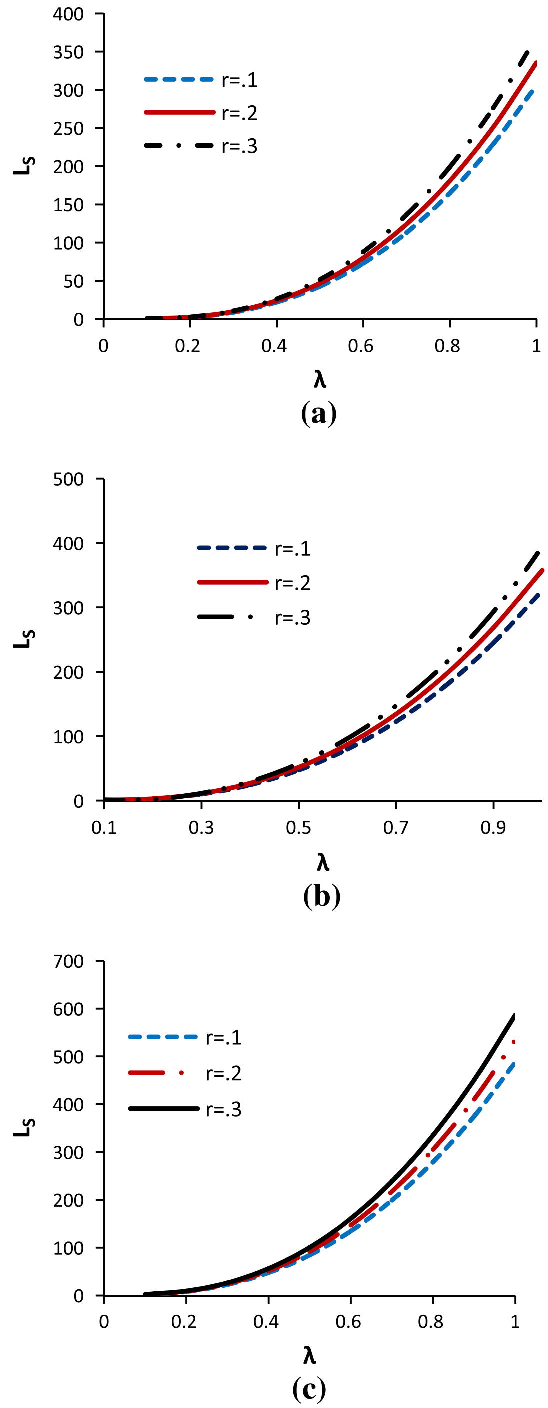

Fig. 4 Effect of r. **a** Erlangian-2. **b** Exponential. **c** Gamma distributions on L_s

decreases with an increase in the service rate for all the three service time distributions. The maximum queue length is observed in case of system with gamma service time distribution as compared to system following exponential and Erlangian-2 service time distributions. Figure 4 depicts the sensitivity of reneging probability r on the system size; the queue length of the system increases as r increases from 0.1 to 0.3 U. The variations of the queue length with the vacation rate θ and setup rate ξ are explored through Figs. 5 and 6. The queue length decreases as the vacation rate increases; this is due to the fact that a server goes for vacation only when there is no customer in the system which implies the reduction in the number of customers in the system.

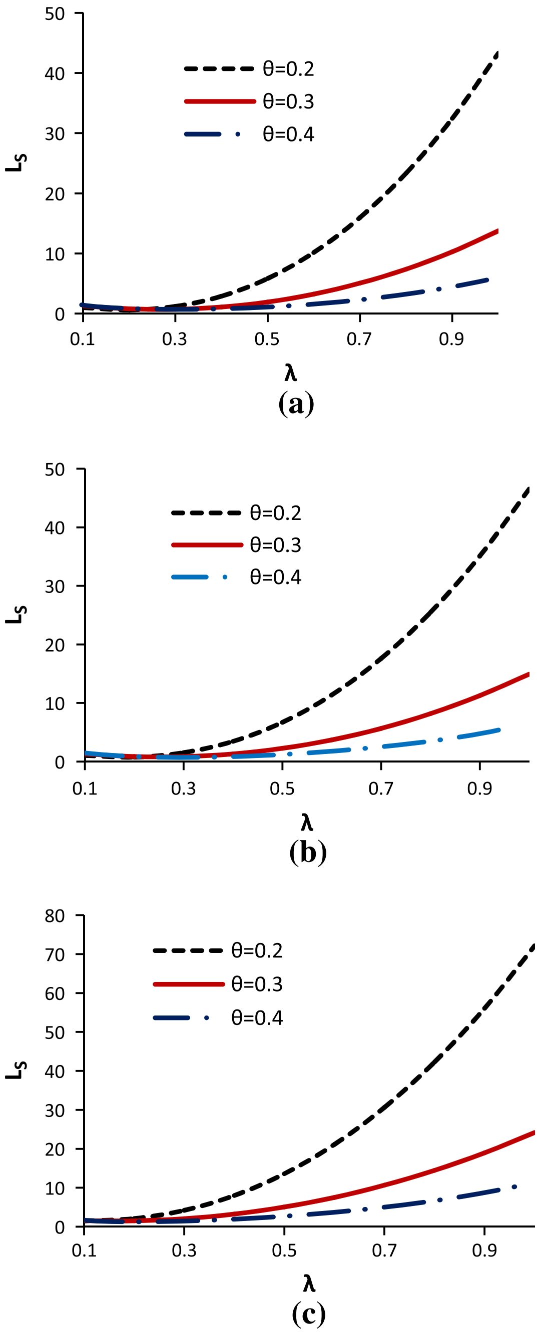

Fig. 5 Effect of θ. **a** Erlangian-2. **b** Exponential. **c** Gamma distribution on L_s

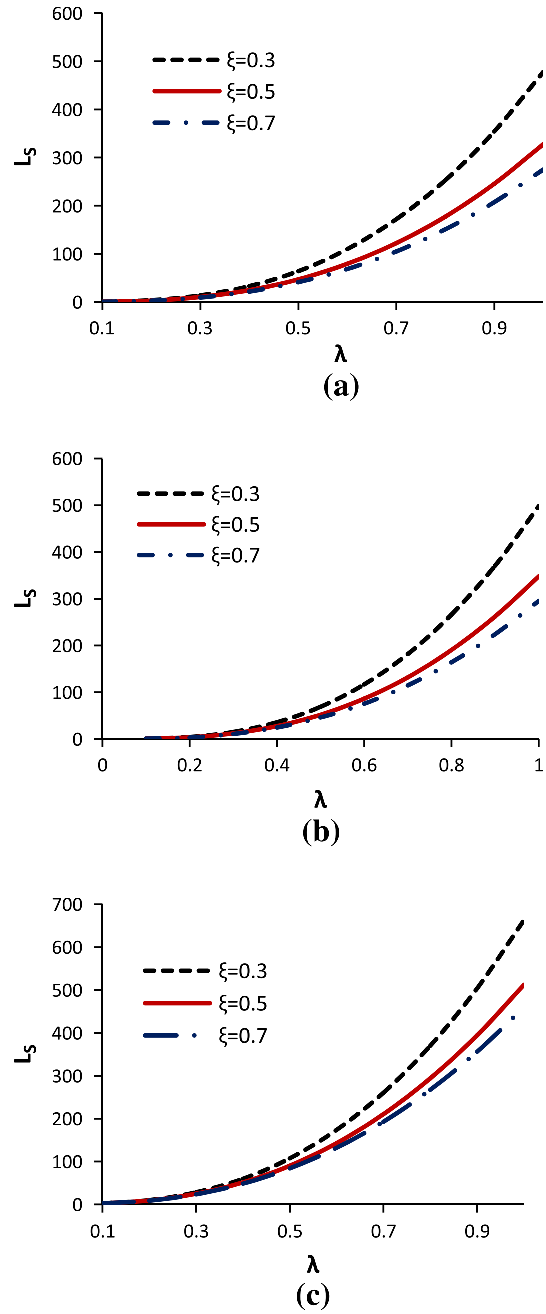

Fig. 6 Effect of ξ. **a** Erlangian-2. **b** Exponential. **c** Gamma distributions on L_s

An increase in the vacation rate clearly implies that the system becomes deprived of customers frequently. Moreover, the set up rate also affects the system size; an increase in the setup rate decreases the number of customers in the system as demonstrated by Fig. 6a–c. This pattern is due to the fact that an increase in the setup rate improves the repair process of the server which in turn increases the availability of the server and thus a reduction in the number of customers in the system is observed.

Comparison of expected and approximate average waiting time

Waiting time plays a significant role in the validation of any retrial queueing model. It is an important parameter that judges the efficiency of mathematical model. A customer always wishes to join a system where service can be availed in the minimum waiting time (either in queue or in system) and hence the importance of waiting time. In the

Table 1 Comparison of exact and approximate average waiting time for Exponential and Gamma distributed service time

	Service time as Exponential distributed			Service time as Gamma distributed		
	Case 1			Case 1		
ξ	W_q	$\hat{W}_q$	APE (%)	W_q	$\hat{W}_q$	APE (%)
1.0	4.1838	4.1969	0.3143	3.8552	3.8575	0.0592
1.8	4.1718	4.2041	0.7755	3.8460	3.8577	0.3066
2.0	4.1685	4.2098	0.9900	3.8430	3.8624	0.5062
	Case 2			Case 2		
γ	W_q	$\hat{W}_q$	APE (%)	W_q	$\hat{W}_q$	APE (%)
0.080	4.2578	4.9479	16.2072	3.9054	4.5272	15.9205
0.090	4.2195	4.5384	7.5560	3.8795	4.1622	7.2858
0.10	4.1838	4.1969	0.3143	3.8552	3.8575	0.0592
	Case 3			Case 3		
α	W_q	$\hat{W}_q$	APE (%)	W_q	$\hat{W}_q$	APE (%)
0.006	4.1146	3.6017	12.4659	3.7856	3.3029	12.7513
0.008	4.1498	3.8788	6.5305	3.8210	3.5611	6.8004
0.01	4.1838	4.1969	0.3143	3.8552	3.8575	0.0592

Table 2 Comparison of exact and approximate average waiting time for Erlangian-2 and Deterministic distributed service time

	Service time as Erlangian-2 distributed			Service time as deterministic distributed		
	Case 1			Case 1		
ξ	W_q	$\hat{W}_q$	APE (%)	W_q	$\hat{W}_q$	APE (%)
1.0	4.2248	4.2393	0.3434	4.2659	4.2818	0.3720
1.8	4.2125	4.2474	0.8290	4.2533	4.2907	0.8815
2.0	4.2092	4.2532	1.0452	4.2499	4.2966	1.0994
	Case 2			Case 2		
γ	W_q	$\hat{W}_q$	APE (%)	W_q	$\hat{W}_q$	APE (%)
0.080	4.3019	5.0005	16.2398	4.3459	5.0531	16.2716
0.090	4.2621	4.5854	7.5868	4.3046	4.6324	7.6169
0.10	4.2248	4.2393	0.3434	4.2659	4.2818	0.3720
	Case 3			Case 3		
α	W_q	$\hat{W}_q$	APE (%)	W_q	$\hat{W}_q$	APE (%)
0.006	4.1969	3.6764	12.4015	4.1558	3.6391	12.4333
0.008	4.2320	3.9582	6.4696	4.1909	3.9185	6.4998
0.01	4.2659	4.2818	0.3720	4.2248	4.2393	0.3434

present subsection, a comparison between exact expected (W_q) and approximate average waiting time has been framed in Tables 1, 2. Table 1 shows the comparison between expected and approximate average waiting time for two types of service time distributions, namely exponential and gamma distributions. The absolute percentage error (APE %) has been obtained for various varying parameters and four cases with variation in different parameters, namely (Case 1) setup rate ξ, (Case 2) retrial rate γ and (Case 3) breakdown rate $\alpha = \alpha_1 = \alpha_2$.

An increase in the set up rate ξ from 1.0 to 2.5 U affects the waiting time of the customer in the queue. APE decreases with an increase in the retrial rate for both the distributions with maximum % error as 16.20 % for the exponential distribution and 15.92 % for gamma distribution in Case 2. The data captured in Case 3 depict the effect of the breakdown rate α on the waiting time of the customer in the system. Both expected and approximate waiting times increases with the increase in breakdown rate from 0.006 to 0.01 U. This is because breakdown of the server increases the queue length of the system and thus increases the waiting time for the service.

Table 2 depicts the data for the waiting time for a queueing model with Erlangian-2 and deterministic distributed service time. With an increase in the retrial rate γ and breakdown rate α, APE as well as waiting time decreases. APE increases with an increase in the setup rate ξ for both Erlangian-2 and deterministic distributed service process. Hence, the choice of appropriate service time distribution may help in reducing the waiting time of the customers in the system.

Cost analysis

In the present section, we frame the expected total cost function (ETC) for the retrial queueing model with modified vacation policy under consideration. The cost function is formulated as

$$\mathrm{ETC} = C_h L_s + C_b P_B + C_s P_s + C_R P_R + C_V P_V + C_I P_I,$$

where

$C_h =$ Holding cost per unit customer

$C_b =$ Cost per unit time while servicing the customers

$C_s =$ Cost per unit time for making pre repair settings

$C_R =$ Cost per unit time for providing repair to the broken down server

$C_V =$ Cost per unit time in the system when the server is on vacation

$C_I =$ Cost per unit time when the customer retry for the service

The effect of various parameters on the total cost of the system has been examined so as to visualize the nature of cost function towards various parameters. The set of default cost elements are taken as $C_I = 10$, $C_R = 50$, $C_h = 5$, $C_b = 50$, $C_V = 20$, $C_s = 20$.

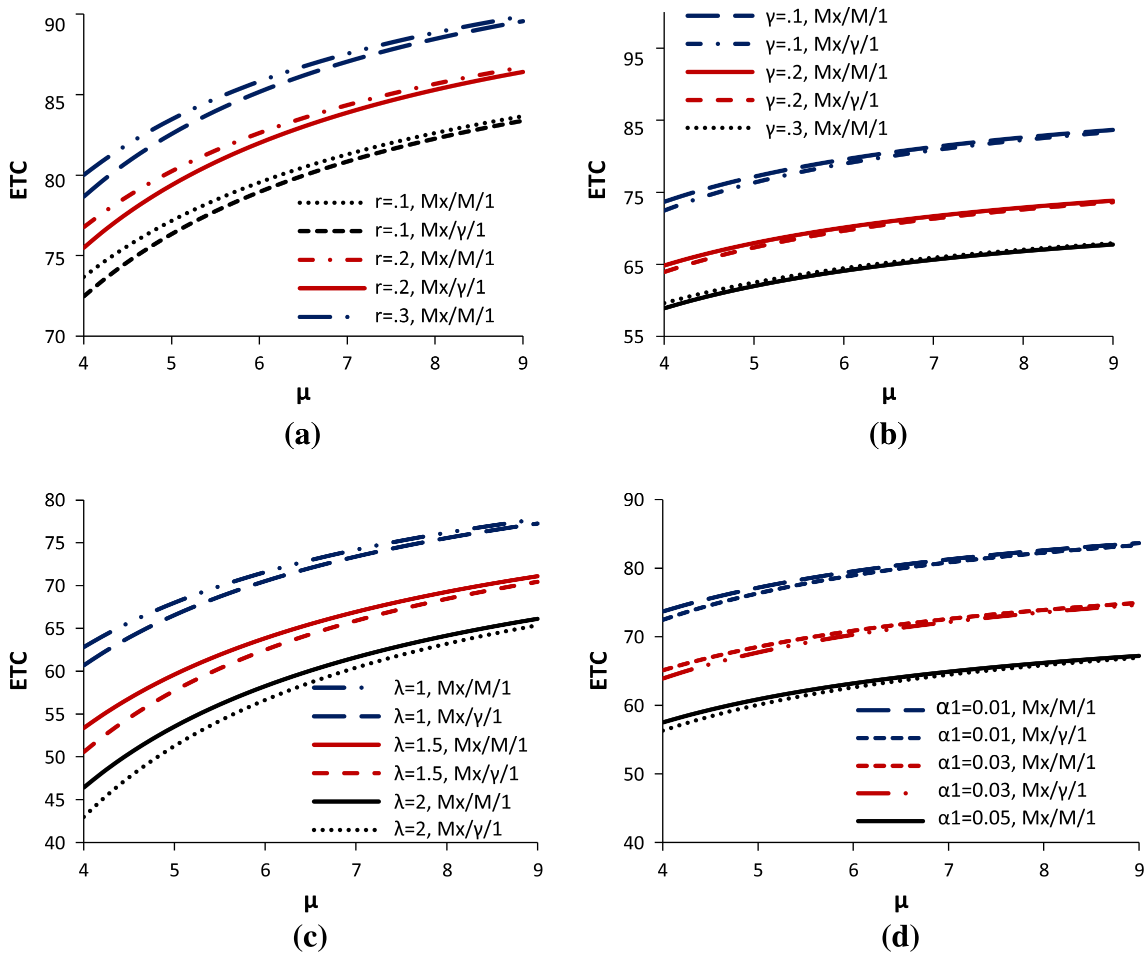

Fig. 7 Effect of **a** reneging rate r, **b** retrial rate γ, **c** arrival rate λ and **d** break down rate α_1 on ETC of the system

Figure 7 displays the effect of various parameters, namely reneging probability r, retrial rate γ, arrival rate (λ) and breakdown rate (α_1), respectively, on the total cost ETC of the system. The graphs are plotted with ETC on the y-axis and service rate μ ($\mu_1 = \mu_2$) on the x-axis. The sensitivity of cost with respect to service rate μ for different values of r is displayed in Fig. 7a. The total cost of the system increases with an increase in the reneging rate r for both exponential and gamma service time distributions. Higher costs are noticed in case of exponentially distributed service time. It is seen that the total cost of the system decreases with the increase in retrial rate (γ) from 0.1 to 0.3 U for fixed values of other parameters as displayed in Fig. 7b for both types of distributions. The higher cost values are achieved for the exponential distributed service time as compared to gamma distributed service time. The sensitivity of total cost with varying values of arrival rate (λ) and breakdown rate (α_1) has been depicted in Fig. 7c and d, respectively. It is seen that the total cost increases with an increase in the reneging probability ($1 - r$), retrial rate (γ), arrival rate (λ), breakdown rate (α_1) and service rate (μ). Figure 7d displays the effect of breakdown rate and service rate on ETC; it is quite interesting to observe that the total cost values obtained for different service time distributions, namely exponential and gamma service time, are quite close enough. In this case, ETC varies minutely till μ reaches 6.5 U; after that the total cost of the system becomes independent of service time distribution chosen. Overall, we observed that the total cost of the system is sensitive towards variation in different parameters likewise arrival rate, retrial rate, reneging rate, etc. A control over these parameters can help in controlling and reducing the total cost of the system at a permissible level.

Concluding remarks

The bulk arrival retrial queueing model with discouragement and modified vacation has been analyzed. The concepts of phase service and phase repair incorporated along with delaying repair make the present model close to many real-life queueing scenarios. The analytic results obtained for various queueing and reliability indices along with long run probabilities are also validated numerically by taking an illustration. The process of taking admission in any institute, working at bank counters, hospital formalities for a patient, etc. can be considered as suitable examples of real-life congestion situation for this particular model. The admission procedure in any institute starts from the filling of application to the completion of fees deposition wherein there is a series of essential formalities to be completed. And, if there emerges any mistake/breakdown in the server, then again it follows a phase service/repair. The person involved in the admission may go for a series of vacations with some probabilities if no more jobs are available. Hence, our study clearly relates a very common scenario of proposed model. Moreover, our model is best fitted to the transmission of messages and telecommunication processes also. The present investigation can be further extended by enriching the model with the concepts of negative customers and bulk service to make it more general and effective.

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Appendix

MEP is used to obtain the approximate expected waiting time in the queue is detailed below (cf. Wang et al. 2007).

Let us consider that a particular customer, say "Q", arrives in the system and finds n customers preceding him in the queue. The server can be in any one of the states (i) idle, (ii) busy, (iii) repair, (iv) set up and (v) vacation, when "Q" arrives. So, the following cases arise:

Idle state If on arrival, the customer "Q" finds the server in idle state then the incoming batch will be immediately served. The mean waiting time for a customer in this case includes the time taken by the additional customers in the batch preceding him to be served which is given as

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]$$

Busy state If the server is found in busy state, then the incoming batch joins the orbit and waits for its turn. For this case, the waiting time includes the serving time $\sum_{i=1}^{k}\frac{n}{\mu_i}$ of those n customers already present in the queue plus the waiting time $\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)$ of those who precedes "Q" in the batch. The total mean waiting time is given by

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\frac{n}{\mu_i}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]$$

Setup state If the server breaks down, then it is sent for the repair. But before repair, it is essential for the repairman to make some preliminary settings before starting the repair. If the incoming customer finds the server in set up state, then it has to wait for the server to complete its set up procedure with mean remaining set up time $\sum_{i=1}^{k}\frac{\eta_i^{(2)}}{2\eta_i^{(1)}}$, repair process $\frac{1}{\beta_{ij}}$, as well as $\sum_{i=1}^{k}\frac{n}{\mu_i}$, which is the servicing time of n customers already present in the system. Moreover, the customers preceding "Q" will also take time $\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)$. Thus, the mean waiting time in the set up state is given by

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\frac{\eta_i^{(2)}}{2\eta_i^{(1)}}+\frac{1}{\beta_{ij}}+\sum_{i=1}^{k}\frac{n}{\mu_i}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]$$

Repair state In case when the server is in repair state, then the incoming batch will be served after completing the repair of the server plus the servicing of those n customers already waiting in the queue. The mean remaining repair time is given by $\sum_{i=1}^{k}\sum_{j=1}^{d}\frac{g_{ij}^{(2)}}{2g_{ij}^{(1)}}$, and waiting time for the servicing of n customers is $\sum_{i=1}^{k}\frac{n}{\mu_i}$. Moreover, the customers preceding "Q" will also take some time $\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)$. Hence, the total mean waiting when the server is in repair state is $\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k}\sum_{j=1}^{d}\frac{g_{ij}^{(2)}}{2g_{ij}^{(1)}}+\sum_{i=1}^{k}\frac{n}{\mu_i}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]$.

Vacation state When the server is in vacation state, the incoming batch will be served after completing the vacation of the server plus the servicing of those n customers already waiting in the queue. The mean remaining vacation time is given by $\sum_{l=1}^{J}\frac{u_l^{(2)}}{2u_l^{(1)}}$, and waiting time for the servicing of n customers is $\sum_{i=1}^{k}\frac{n}{\mu_i}$. Moreover, the customers preceding "Q" will also take some time $\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)$. Hence, the total mean waiting when the server is in vacation state is obtained as

$$\sum_{n=1}^{\infty}\left[\sum_{l=1}^{J}\frac{u_l^{(2)}}{2u_l^{(1)}}+\sum_{i=1}^{k}\frac{n}{\mu_i}+\sum_{i=1}^{k}\frac{1}{2\mu_i}\left(\frac{E[X^2]}{E[X]}-1\right)\right]$$

The total mean waiting time in the system given by Eq. (82) is obtained by adding the (i–v) after multiplying the respective probabilities of system states.

References

Arrar NK, Djellab NV, Baillon JB (2012) On the asymptotic behavior of retrial queues with batch arrivals and impatience phenomenon. Math Comput Model 55:654–665

Artalejo JR (1999a) A classified bibliography of research on retrial queues: Progress in 1990–1999. Top 7:187–211

Artalejo JR (1999b) Accessible bibliography on retrial queues. Math Comput Model 30:1–6

Artalejo JR (2010) Accessible bibliography on retrial queues. Math Comput Model 51:1071–1081

Artalejo JR, Corral AG (2008) Retrial queueing systems: A computational approach. Springer, Berlin

Artalejo JR, Falin GI (2002) Standard and retrial queueing systems: A comparative analysis. Revista Matemática Complutense 15:101–129

Atencia I, Fortes I, Moreno P, Sanchez S (2006) An M/G/1 retrial queue with active breakdowns and Bernoulli schedule in the server. Inf Manag Sci 17:1–17

Bhagat A, Jain M (2013) Unreliable M^X/G/1 retrial queue with multi-optional services and impatient customers. Int J Oper Res 17:248–273

Chang FM, Ke JC (2009) On a batch retrial model with J vacations. J Comput Appl Math 232:402–414

Choudhury G, Ke JC (2012) A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair. Appl Math Model 36:255–269

Choudhury G, Tadj L (2011) The optimal control of an $M^{[x]}$/G/1 unreliable server queue with two phases of service and Bernoulli vacation schedule. Math Comput Model 54:673–688

Choudhury G, Tadj L, Deka K (2010) A batch arrival retrial queueing system with two phases of service and service interruption. Comput Math Appl 59:437–450

Dimitriou I (2012) A mixed priority retrial queue with negative arrivals, unreliable server and multiple vacations. Appl Math Model 37:1295–1309

Doshi BT (1986) Queueing system with vacation: A survey. QUESTA 1:29–66

El-Affendi MA, Kouvatsos DD (1983) A maximum entropy analysis of the M/G/1 and G/M/1 queueing systems in equilibrium. Acta Inf 19:339–355

Falin GI, Templeton JGC (1997) Retrial queues. Chapman and Hall

Gross D, Harris CM (1998) Fundamentals of queueing theory, 3rd edn. Wiley, New York

Jain M, Agarwal PK (2010) N-policy for state dependent batch arrival queueing system with *l*-stage service and modified Bernoulli schedule vacation. Qual Technol Quant Manag 3:215–230

Jain M, Sharma GC, Sharma R (2012) Maximum entropy approach for discrete-time unreliable server Geo^X/Geo/1 queue with working vacation. Int J Math Oper Res 4:56–77

Kahraman A, Gosavi A (2011) On the distribution of the number stranded in bulk-arrival, bulk-service queues of the M/G/1 form. Eur J Oper Res 212:352–360

Ke JC, Chang FM (2009) Modified vacation policy for M/G/1 retrial queue with balking and feedback. Comput Ind Eng 57:433–443

Ke JC, Huang KB, Pearn WL (2011) The performance measures and randomized optimization for an unreliable server $M^{[x]}$/G/1 vacation system. Appl Math Comput 217:8277–8290

Shannon CE (1948) A mathematical theory of communication. Bell Syst Tech J 27:379–423

Takagi H (1991) Queueing analysis: a foundation of Performance evaluation, vacation and priority systems, part 1. North Holland, Amsterdam

Wang KH, Huang KB (2009) A maximum entropy approach for the HP; Ni-policy M/G/1 queue with a removable and unreliable server. Appl Math Model 33(4):2024–2034

Wang J, Cao J, Li Q (2001) Reliability analysis of the retrial queue with server breakdowns and repairs. Queueing Syst 38:363–380

Wang KH, Chuang SL, Pearn WL (2002) Maximum entropy analysis of the N policy M/G/1 queueing system with a removable server. Appl Math Model 26:1151–1162

Wang KH, Chen MC, Ke JC (2007) Maximum entropy analysis of the M^x/M/1 queueing system with multiple vacations and server breakdowns. Comput Ind Eng 52:192–202

Yang T, Templeton JGC (1987) A survey on retrial queues. Queueing Syst 2:201–233