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Unsolved problems in combinatorial games

RICHARD J. NOWAKOWSKI

We have sorted the problems into sections:

- A. Taking and breaking
- B. Pushing and placing pieces
- C. Playing with pencil and paper
- D. Disturbing and destroying
- E. Theory of games

The numbers in parentheses are the old numbers used in each of the lists of unsolved problems given on pp. 183–189 of AMS *Proc. Sympos. Appl. Math.* **43** (1991), called PSAM **43** below; on pp. 475–491 of *Games of No Chance*, hereafter referred to as GONC; on pp. 457–473 of *More Games of No Chance* (MGONC); and on pp. 475–500 of *Games of No Chance 3* (GONC3). Some numbers have little more than the statement of the problem if there is nothing new to be added. References [year] may be found in Fraenkel's bibliography at the end of this volume. References [#] are at the end of this article. A useful reference for the rules and an introduction to many of the specific games mentioned below is M. Albert, R. J. Nowakowski and D. Wolfe, *Lessons in Play*: *An Introduction to the Combinatorial Theory of Games*, A. K. Peters, 2007 (LIP) or Berlekamp, Conway and Guy, *Winning Ways for your Mathematical Plays*, vol. 1–4, A. K. Peters, 2000–2004 (WW).

A. Taking and breaking games

A1. **(1) Subtraction games** with finite subtraction sets are known to have periodic nim-sequences. Investigate the relationship between the subtraction set and the length and structure of the period. The same question can be asked about **partizan** subtraction games, in which each player is assigned an individual subtraction set. See Fraenkel and Kotzig [1987].

(A move in the game $S(s_1, s_2, s_3, \ldots)$ is to take a number of beans from a heap, provided that number is a member of the **subtraction-set**, $\{s_1, s_2, s_3, \ldots\}$.

Keywords: none.

Analysis of such a game and of many other heap games is conveniently recorded by a **nim-sequence**,

$$n_0 n_1 n_2 n_3 \ldots,$$

meaning that the nim-value of a heap of h beans is n_h; i.e., that the value of a heap of h beans in this particular game is the **nimber** $*n_h$.)

For examples see Table 2 in Section **4** on p. 67 of "Impartial games" in GONC.

It would now seem feasible to give the complete analysis for games whose subtraction sets have just three members, though this has so far eluded us. Several people, including Mark Paulhus and Alex Fink, have given a complete analysis for all sets $\{1, b, c\}$ and for sets $\{a, b, c\}$ with $a < b < c < 32$.

In general, period lengths can be surprisingly long, and it has been suggested that they could be superpolynomial in terms of the size of the subtraction set. However, Guy conjectures that they are bounded by polynomials of degree at most $\binom{n}{2}$ in s_n, the largest member of a subtraction set of cardinality n. It would also be of interest to characterize the subtraction sets which yield a purely periodic nim-sequence, i.e., for which there is no preperiod. Carlos Santos [6] reduced this upper bound slightly by using a dynamical system approach.

Angela Siegel [20] considered infinite subtraction sets which are the complement of finite ones and showed that the nim-sequences are always arithmetic periodic. That is, the nim-values belong to a finite set of arithmetic progressions with the same common difference. The number of progressions is the period and their common difference is called the **saltus**. For instance, the game $S\{\hat{4}, \hat{9}, \widehat{26}, \widehat{30}\}$ (in which a player may take any number of beans except 4, 9, 26 or 30) has a preperiod of length 243, period-length 13014 and saltus 4702. Marla Clusky and Danny Sleator have proved her conjecture that the class of games $S\{\hat{a}, \hat{b}, \widehat{a+b}\}$ is purely periodic with period length $3(a+b)$.

For infinite subtraction games in general there are corresponding questions about the length and purity of the period. Suppose the elements of the finite subtraction sets are not constants. It was shown by Fraenkel [2011] that the game with subtraction set

$$\big\{1, 2, \ldots, t-1, \lfloor n/t \rfloor\big\},$$

where $t \geq 2$ is a fixed integer, n the pile size, has an aperiodic Sprague–Grundy sequence. See also Fraenkel [2012] and Guo [2012].

A2. (2) Are all finite **octal games** ultimately periodic? (If the binary expansion of the k-th code digit in the game with code $\mathbf{d}_0 \cdot \mathbf{d}_1 \mathbf{d}_2 \mathbf{d}_3 \ldots$ is

$$\mathbf{d}_k = 2^{a_k} + 2^{b_k} + 2^{c_k} + \cdots,$$

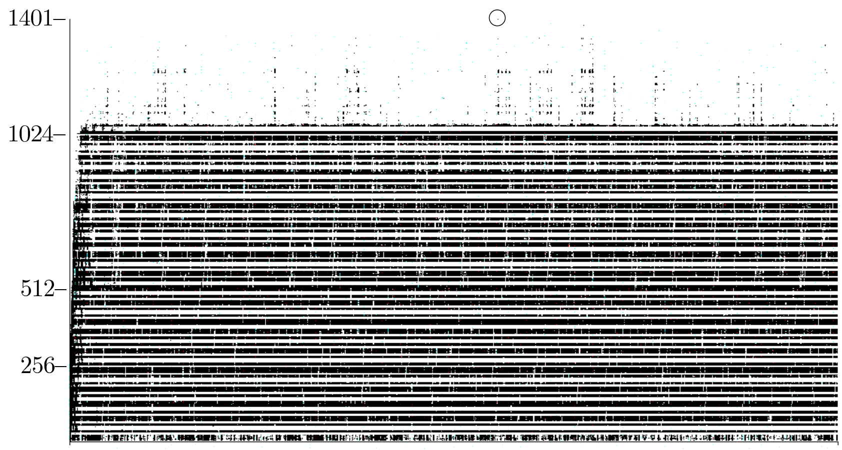

Figure 1. Plot of 11000000 nim-values of the octal game ·**007**.

where $0 \le a_k < b_k < c_k < \dots$, then it is legal to remove k beans from a heap, provided that the rest of the heap is left in exactly a_k or b_k or c_k or $\dots$ nonempty heaps. See WW, pp. 81–115. Some specimen games are exhibited in Table 3 of Section **5** of "Impartial games" in GONC.)

Resolve any number of outstanding particular cases, e.g., ·**6** (**Officers**), ·**04**, ·**06**, ·**14**, ·**36**, ·**37**, ·**64**, ·**74**, ·**76**, ·**004**, ·**005**, ·**006**, ·**007**, ·**014**, ·**015**, ·**016**, ·**024**, ·**026**, ·**034**, ·**064**, ·**114**, ·**125**, ·**126**, ·**135**, ·**136**, ·**142**, ·**143**, ·**146**, ·**162**, ·**163**, ·**164**, ·**166**, ·**167**, ·**172**, ·**174**, ·**204**, ·**205**, ·**206**, ·**207**, ·**224**, ·**244**, ·**245**, ·**264**, ·**324**, ·**334**, ·**336**, ·**342**, ·**344**, ·**346**, ·**362**, ·**364**, ·**366**, ·**371**, ·**374**, ·**404**, ·**414**, ·**416**, ·**444**, ·**564**, ·**604**, ·**606**, ·**744**, ·**764**, ·**774**, ·**776** and **Grundy's Game** (split a heap into two unequal heaps; WW, pp. 96–97, 111–112; LIP, p. 142), which has been analyzed, first by Dan Hoey, and later by Achim Flammenkamp, as far as heaps of 2^{35} beans.

J.P. Grossman (preprint) using a new approach based on the sparse space phenomenon, has analyzed ·**.6** up to heaps of size 2^{47} and has found no periodicity.

Perhaps the most notorious and deserving of attention is the game ·**007**, one-dimensional Tic-Tac-Toe, or **Treblecross**, which Flammenkamp has pushed to 2^{25}. Figure 1 shows the first 11 million nim-values, a small proportion of which are ≥ 1024; the largest, $\mathcal{G}(6193903) = 1401$ is shown circled. Will 2048 ever be reached?

Flammenkamp has settled ·**106**: it has the remarkable period and preperiod lengths of 328226140474 and 465384263797. For information on the current status of each of these games, we refer the read to Flammenkamp's web page at uni-bielefeld.de/ achim/octal.html.

A game similar to Grundy's, and which is also unsolved, is John Conway's **Couples-Are-Forever** (LIP, p. 142) where a move is to split any heap except a heap of two. The first 50 million nim-values haven't displayed any periodicity. See Caines et al. [1999]. More generally, Bill Pulleyblank suggests looking at splitting games in which you may only split heaps of size $> h$, so that $h = 1$ is She-Loves-Me-She-Loves-Me-Not and $h = 2$ is Couples-Are-Forever. David Singmaster suggested a similar generalization: you may split a heap provided that the resulting two heaps each contain at least k beans: $k = 1$ is the same as $h = 1$, while $k = 2$ is the third cousin of **Dawson's Chess**.

Explain the structure of the periods of games known to be periodic.

In *Discrete Math.*, **44** (1983), pp. 331–334, Problem 38, Fraenkel raised questions concerning the computational complexity (see Section E1 below) of octal games. In Problem 39, he and Kotzig define **partizan octal games** in which distinct octals are assigned to the two players. Mesdal [2009] show that in many cases, if the game is "all-small" (WW, pp. 229–262; LIP, pp. 183–207), then the atomic weights are arithmetic periodic. (See Section E13 for an explanation of the new name "dicot" games.) In Problem 40, Fraenkel introduces **poset games**, played on a partially ordered set of heaps, each player in turn selecting a heap and then removing a nonnegative number of beans from this heap and from each heap above it in the ordering, at least one heap being reduced in size. For posets of height one, new regularities in the nim-sequence can occur; see Horrocks and Nowakowski [2003].

Note that this includes, as particular cases, Subset Takeaway, Chomp or Divisors, and Green Hackenbush forests. Compare Problems A3, D1 and D2 below.

A3. (3) Hexadecimal games have code digits $\mathbf{d}_k$ in the interval from **0** to **f** (= **15**), so that there are options splitting a heap into three heaps. See WW, pp. 116–117.

Such games may be arithmetically periodic. Nowakowski has calculated the first 100000 nim-values for each of the 1-, 2- and 3-digit games. Richard Austin's Theorem 6.8 in his thesis [1976] and the generalization by Howse and Nowakowski [2004] suffice to confirm the arithmetic periodicity of several of these games.

Some interesting specimens are ·**28** = ·**29**, which have period 53 and saltus 16, the only exceptional value being $\mathcal{G}(0) = 0$; ·**9c**, which has period 36, preperiod 28 and saltus 16; and ·**f6** with period 43 and saltus 32, but its apparent preperiod of 604 and failure to satisfy one of the conditions of the theorem prevent us from verifying the ultimate periodicity. The game ·**205200c** is arithmetic periodic with preperiod length of 4, period length of 40, saltus 16 except that $40k + 19$ has nim-value 6 and $40k+39$ has nim-value 14. This regularity, (which also seems to be exhibited by ·**660060008** with a period length of approximately 300,000), was first reported in Horrocks and Nowakowski [2003] (see Problem A2). Grossman and Nowakowski [this volume] have shown that the nim-sequences for ·**200…0048**, with an odd number of zero code digits, exhibit "ruler function" patterns. The game ·**9** has not so far yielded its complete analysis, but, as far as analyzed (to heaps of size 100000), exhibits a remarkable fractal-like set of nim-values. See Howse and Nowakowski [2004]. Also of special interest are ·**e**; ·**7f** (which has a strong tendency to period 8, saltus 4, but, for $n \le 100,000$, has 14 exceptional values, the largest being $\mathcal{G}(94156) = 26614$); ·**b6** (which "looks octal"); ·**b33b** (where a heap of size n has nim-value n except for 27 heap sizes which appear to be random).

Other unsolved hexadecimal games are

·**1x**,	$\mathbf{x} \in \{8, 9, c, d, e, f\}$,	·**2x**,	$a \le \mathbf{x} \le f$,
·**3x**,	$8 \le \mathbf{x} \le e$,	·**4x**,	$\mathbf{x} \in \{9, b, d, f\}$,
·**5x**,	$8 \le \mathbf{x} \le f$,	·**6x**,	$8 \le \mathbf{x} \le f$,
·**7x**,	$8 \le \mathbf{x} \le f$,	·**9x**,	$1 \le \mathbf{x} \le a$,
·**9d**,		·**bx**,	$\mathbf{x} \in \{6, 9, d\}$,
·**dx**,	$1 \le \mathbf{x} \le f$,	·**fx**,	$\mathbf{x} \in \{4, 6, 7\}$.

A4. (53) *N*-heap Wythoff game. Given $N \ge 2$ heaps of finitely many tokens, whose sizes are $p_1, \dots, p_N$ with $p_1 \le \cdots \le p_N$. Players take turns removing any positive number of tokens from a *single* heap or removing $(a_1, \dots, a_N)$ from *all* the heaps — a_i from the i-th heap — subject to the conditions: (i) $0 \le a_i \le p_i$ for each i, (ii) $\sum_{i=1}^N a_i > 0$, (iii) $a_1 \oplus \cdots \oplus a_N = 0$, where $\oplus$ is nim addition. The player making the last move wins and the opponent loses. Note that the classical Wythoff game is the case $N = 2$.

For $N \ge 3$, Fraenkel makes the following conjectures.

Conjecture 1. For every fixed set $K := (A^1, \dots, A^{N-2})$ there exists an integer $m = m(K)$ (i.e, m depends only on K), such that

$$\left(A^1, \dots, A^{N-2}, A_n^{N-1}, A_n^N\right), \quad A^{N-2} \le A_n^{N-1} \le A_n^N,$$

with $A_n^{N-1} < A_{n+1}^{N-1}$ for all $n \ge 1$, is the n-th $\mathcal{P}$-position, and

$$A_n^{N-1} = \operatorname{mex}\left(\{A_i^{N-1}, A_i^N : 0 \le i < n\} \cup T\right), \quad A_n^N = A_n^{N-1} + n,$$

for all $n \ge m$, where $T = T(K)$ is a (small) set of integers.

That is, if you fix $N-2$ of the heaps, the $\mathcal{P}$-positions resemble those for the classical Wythoff game. For example, for $N = 3$ and $A^1 = 1$, we have $T = \{2, 17, 22\}$, $m = 23$.

Conjecture 2. For every fixed K there exist integers $a = a(K)$ and $M = M(K)$ such that

$$A_n^{N-1} = \lfloor n\phi \rfloor + \varepsilon_n + a \quad \text{and} \quad A_n^N = A_n^{N-1} + n,$$

for all $n \geq M$, where $\phi = (1+\sqrt{5})/2$ is the golden section, and $\varepsilon_n \in \{-1, 0, 1\}$.

In Fraenkel and Krieger [2004] the following was shown, inter alia: let $t \in \mathbb{Z}_{\geq 1}$, $\alpha = (2 - t + \sqrt{t^2+4})/2$ ($\alpha = \phi$ for $t = 1$), $T \subset \mathbb{Z}_{\geq 0}$ a finite set, $A_n = (\text{mex}\,\{A_i, B_i : 0 \leq i < n\} \cup T)$, where $B_n = A_n + nt$. Let $s_n := \lfloor n\alpha \rfloor - A_n$. Then there exist $a \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$, such that for all $n \geq m$, either $s_n = a$, or $s_n = a + \varepsilon_n$, $\varepsilon_n \in \{-1, 0, 1\}$. If $\varepsilon_n \neq 0$, then $\varepsilon_{n-1} = \varepsilon_{n+1} = 0$. Also the general structure of the ε_n was characterized succinctly.

This result was then applied to the N-heap Wythoff game. In particular, for $N = 3$ (such that $K = A^1$) it was proved that $A_n^2 = \text{mex}\left(\{A_i^2, A_i^3 : 0 \leq i < n\} \cup T\right)$, where

$$T = \{x \geq K : \text{there exists } 0 \leq k < K \text{ such that } (k, K, x) \text{ is a } P\text{-position}\} \cup \{0, \dots, K-1\}.$$

The following upper bound for A_n^3 was established: $A_n^3 \leq (K+3)A_n^2 + 2K + 2$. It was also proved that Conjecture 1 implies Conjecture 2.

In Sun and Zeilberger [2004], a sufficient condition for the conjectures to hold was given. It was then proved that the conjectures are true for the case $N = 3$, where the first heap has up to 10 tokens. For those 10 cases, the parameter values m, M, a, T were listed in a table.

Sun [2005] obtained results similar to those in Fraenkel and Krieger [2004], but the proofs are different. It was also proved that Conjecture 1 implies Conjecture 2. A method was given to compute a in terms of certain indexes of the A_i and B_j.

A5. (23) Burning-the-Candle-at-Both-Ends. Conway and Fraenkel ask us to analyze Nim played with a row of heaps. A move may only be made in the leftmost or in the rightmost heap. When a heap becomes empty, then its neighbor becomes the end heap.

Albert and Nowakowski [2001] have determined the outcome classes in impartial and partizan versions (called **End-Nim**, LIP, pp. 210, 263) with finite heaps, and Duffy, Kolpin and Wolfe [2009] extend the partizan case to infinite ordinal heaps. Wolfe asks for the actual values.

Nowakowski suggested to analyze impartial and partizan **End-Wythoff**: take from either end-pile, or the *same* number from both ends. The impartial game is solved by Fraenkel and Reisner [2009]. Fraenkel [1982] asks a similar question about a generalized Wythoff game: take from either end-pile or take $k > 0$ from one end-pile and $\ell > 0$ from the other, subject to $|k - \ell| < a$, where a is a fixed integer parameter ($a = 1$ is **End-Wythoff**).

There is also **Hub-and-Spoke Nim**, proposed by Fraenkel. One heap is the hub and the others are arranged in rows forming spokes radiating from the hub. Albert notes that this game can be generalized to playing on a forest, i.e., a graph each of whose components is a tree. The most natural variant is that beans may only be taken from a leaf (valence 1) or isolated vertex (valence 0).

The partizan game of **Red-Blue Cherries** is played on an arbitrary graph. A player picks an appropriately colored cherry from a vertex of minimum degree, which disappears at the same time. Albert, Grossman, McCurdy, Nowakowski and Wolfe [1] show that if the graph has a leaf, then the value is an integer. They ask: Is every **Red-Blue Cherries** position an integer? See also [10].

A6. (17) Extend the analysis of **Kotzig's Nim** (WW, pp. 515–517). Is the game eventually periodic in terms of the length of the circle for every finite move set? Analyze the misère version of **Kotzig's Nim**.

Let $\Gamma(S; n)$ be the outcome of the game S lists all the moves and n is the size of the circle. Recently, Ward and Xin [2010] gave more evidence of the periodicity by showing: If $n \in \{1, 3, 5, 7, 15\}$, or if $n \equiv 3 \pmod 5$ and $n \geq 23$, then $\Gamma(1, 4; n) \in \mathcal{P}$; otherwise, $\Gamma(1, 4; n) \in \mathcal{N}$. They also give several conjectures including: If n is odd and $n \notin \{9, 11, 17\}$ then $\Gamma(3, 5; n) \in \mathcal{P}$ and is in $\mathcal{N}$ otherwise. Their evidence bolsters the periodicity conjecture but also indicates that there is likely to be a lot of noise when n is small.

A7. (18) Obtain asymptotic estimates for the proportions of $\mathcal{N}$-, $\mathcal{O}$- and $\mathcal{P}$-positions in Epstein's **Put-or-Take-a-Square** game (WW, pp. 518–520).

A8. Gale's nim. This is Nim played with four heaps, but the game ends when three of the heaps have vanished, so that there is a single heap left. Brouwer and Guy have independently given a partial analysis, but the situation where the four heaps have distinct sizes greater than 2 is open. An obvious generalization is to play with h heaps and play finishes when k of them have vanished.

A9. Euclid's Nim is played with a pair of positive integers, a move being to diminish the larger by any multiple of the smaller. The winner is the player who reduces a number to zero. Analyses have been given by Cole and Davie [1969], Spitznagel [1973], Lengyel [2003], Collins [2005], Fraenkel [2005] and Nivasch [2006]. Lengyel [2003] reports that Schwartz first found that (a, b) is the sequential sum of nim-heaps given by the normal continued fraction of a/b, the last number in the continued fraction depends on the presence or absence of fibonacci numbers. Gurvich [2007] shows that the nim-value, $g^+(a, b)$ for the pair (a, b) in normal play is the same as the misère nim-value, $g^-(a, b)$

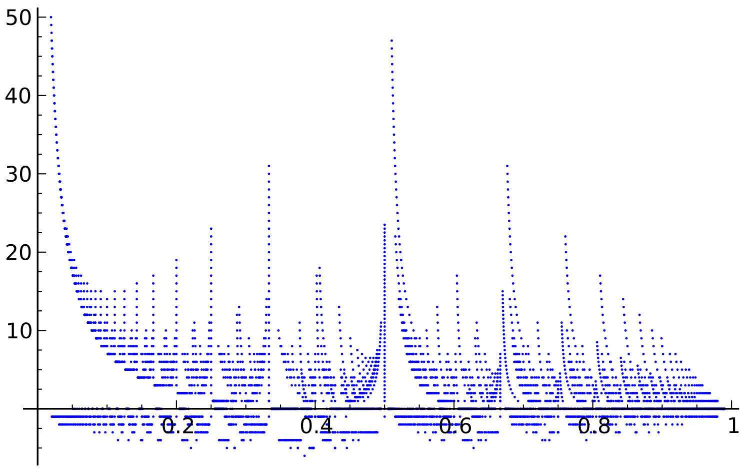

Figure 2. Partizan Euclid: mean atomic weights for $0 < q < p < 101$.

except for $(a, b) = (kF_i, kF_{i+1})$ where $k > 0$ and F_i is the i-th Fibonacci number. In this case, $g^+(kF_i, kF_{i+1}) = 0$ and $g^-(kF_i, kF_{i+1}) = 1$ if i is even and the values are reversed if i is odd. Aaron Siegel notes that this can be restated as: **Euclid** is tame, and the fickle positions have the form (kF_i, kF_{i+1}). The latter have genus 0^1 when i is even and 1^0 when i is odd.

Collins and Lengyel [392] play **Euclid's Nim** with three integers and solve some special cases.

In **Partizan Euclid**, from the position (p, q), where $p = kq + r$, $0 \le r < q$, Left moves to (q, r) and Right to $(q, (k+1)q - p)$. The outcome classes are investigated in [12]. Determining the values seems hard. The atomic weights may be easier — see Figure 2, where the *mean atomic-weights* for $0 < q < p \le 100$ are given, where the coordinates (x, y) are $x = q/p$ and y is mean atomic-weight of (p, q).

A10. (20) D.U.D.E.N.E.Y (WW, pp. 521–523) is **Nim**, but with an upper bound, Y, on the number of beans that may be taken, and with the restriction that a player may not repeat his opponent's last move. If Y is even, the analysis is easy. Some advance in the analysis, when Y is odd, has been made by Marc Wallace, Alex Fink and Kevin Saff.

We can, for example, extend the table of strings of pearls given in WW, p. 523, with the values of Y in Table 1, which have the pure periods shown, where D=$Y+2$, E=$Y+1$. The first entry corrects an error of $128r + 31$ in WW.

It seems likely that the string for $Y = 2^{2k+1} + 2^{2k} - 1$ has the simple period E for all values of k. But there is some evidence to suggest that an analysis will

$256r+31$	DEE	$512r+153$	DEE	$1024r+415$	DEE
$512r+97$	DDEDDDE	$512r+159$	DEE	$512r+425$	DE
$1024r+103$	DE	$512r+225$	DDE	$512r+487$	DEE
$128r+119$	DEE	$512r+255$	E	$1024r+521$	DDDE
$1024r+127$	DEEE	$512r+257$	DDDDE	$1024r+607$	DDE
$512r+151$	DDDEE	$512r+297$	DDEDEDE	$1024r+735$	DEEE

Table 1. Strings of "pearls" in D.U.D.E.N.E.Y for values of Y of various forms.

never be complete. Indeed, consider the following table showing the fraction, among 2^k cases, that remain undetermined:

$k=$	3	5	6	7	8	9	10	11	12	13	14	15	16	17
fraction	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{9}{32}$	$\frac{11}{64}$	$\frac{21}{128}$	$\frac{33}{256}$	$\frac{60}{512}$	$\frac{97}{1024}$	$\frac{177}{2048}$	$\frac{304}{4096}$	$\frac{556}{8192}$	$\frac{974}{16384}$	$\frac{1576}{32768}$	$\frac{2763}{65536}$

Moreover, the periods of the pearl-strings appear to become arbitrarily long.

A11. (21) Schuhstrings is the same as **D.U.D.E.N.E.Y**, except that a deduction of zero is also allowed, but cannot be immediately repeated (WW, pp. 523–524). In *Winning Ways* it was stated that it was not known whether there is any Schuhstring game in which three or more strings terminate simultaneously. Kevin Saff has found three such strings (when the maximum deduction is $Y=3430$, the three strings of multiples of 2793, 3059, 3381 terminate simultaneously) and he conjectures that there can be arbitrarily many such simultaneous terminations.

A12. (22) Analyze **Dude**, i.e., unbounded Dudeney, or Nim in which you are not allowed to repeat your opponent's last move.

Let $[h_1, h_2, \ldots, h_k; m]$, $h_i \le h_{i+1}$, be the game with heaps of size h_1 through h_k, where m is the move just made and $m=0$ denotes a starting position. Then [5], the $\mathcal{P}$-positions are

$$[(2s+1)2^{2j}; (2s+1)2^{2j}] \quad \text{for } k=1,$$
$$[(2s+1)2^{2j}, (2s+1)2^{2j}; 1] \quad \text{for } k=2,$$

and for $k \ge 3$ the heap sizes are arbitrary, the only condition being that the previous move was 1. The nim-values do not seem to show an easily described pattern.

A13. Nim with pass. David Gale suggested an analysis of Nim played with the option of a single pass by either of the players, which may be made at any time up to the penultimate move. It may not be made at the end of the game. Once a player has passed, the game is as in ordinary Nim. The game ends when all heaps have vanished. Morrison, Friedman and Landsberg [2011] have looked at this game with their renormalization techniques.

A14. Games with a Muller twist. In such games, each player specifies a condition on the set of options available to her opponent on his next move.

In **Odd-or-Even Nim**, for example, each player specifies the parity of the opponent's next move. This game was analyzed by Smith and Stănică [2002], who propose several other such games which are still open (see also Gavel and Strimling [2004]).

The game of **Blocking Nim** proceeds in exactly the same way as ordinary Nim with N heaps, except that before a given player takes his turn, his opponent is allowed to announce a **block**, $(a_1, \dots a_N)$; i.e., for each pile of counters, he has the option of specifying a positive number of counters which may not be removed from that pile. Flammenkamp, Holshouser and Reiter [2003, 2004] give the $\mathcal{P}$-positions for three-heap Blocking Nim with an incomplete block containing only one number, and ask for an analysis of this game with a block on just two heaps, or on all three. There are corresponding questions for games with more than three heaps. Larsson [2011] shows with two piles of counters where at most $k-1$ moves, for some fixed k, may be blocked off at each stage. Then the $\mathcal{P}$-positions are of the form $\{x, y\}$, where either $|y-x| < k$ and $y-x \equiv k-1 \pmod 2$ or $x+y < k$.

Let S be a set of positive integers. The **complementary subtraction** game $\hat{S}$ is played on a heap where the last act of a move is say whether the next subtraction is to be a number from S or from its complement. Horrocks and Trenton [2008] introduced this variant. They analyzed heaps up to size 8000 the case where S is the Fibonacci numbers without finding periodicity although there appears to be regularity. They also ask about the case $S = \{x \mid x \equiv a \text{ or } b \pmod c\}$. They report that the nim-values appear chaotic. The set of $(a_n, a_n + n)$ where $a_1 = 1$ and $a_n = \text{mex}\{a_i : i < n\}$ form the $\mathcal{P}$-positions of Wythoff's game. We ask what happens in this game if $S = \{a_n : n+1, 2, 3, \dots\}$?

A15. (13) Misère caternary and octal games. Misère analysis has been revolutionized by Thane Plambeck and Aaron Siegel [2008] with their concept of the **misère quotient** of a game, though the number of unsolved problems continues to increase. See Section E13 for more theoretical questions and an explanation of some of the concepts mentioned here.

Plambeck and Siegel ask the specific questions:

(1) The misère quotient of **·07** (**Dawson's Kayles**) has order 638 at heap size 33. Is it infinite at heap size 34? Even if the misère quotient is infinite at heap 34 then, by Rédei's theorem [8, p. 142; 17], it must be isomorphic to a finitely presented commutative monoid. Call this monoid D_{34}. Exhibit a monoid presentation of D_{34}, and having done that, exhibit D_{35}, D_{36}, etc, and explain what is going on in general. Given a set of games $\mathcal{A}$, describe an algorithm

to determine whether the misère quotient of $\mathscr{A}$ is infinite. Much harder: if the quotient is infinite, give an algorithm to compute a presentation for it.

(2) Give complete misère analyses for any of the (normal-play periodic) octal games that show "algebraic-periodicity" in misère play. Some examples are ·**54**, ·**261**, ·**355**, ·**357**, ·**516** and ·**724**. Give a precise definition of algebraic periodicity and describe an algorithm for detecting and generalizing it. This is a huge question: if such an algorithm exists, it would likely instantly give solutions to at least a half-dozen unsolved 2- and 3-digit octals.

Plambeck also offers prizes of US $500.00 for a complete analysis of **Dawson's Chess**, ·**137** (alias **Dawson's Kayles**, ·**07**); US $200.00 for the "wild quaternary game", ·**3102**; and US $25.00 each for ·**3122**, ·**3123** and ·**3312**.

There is a website miseregames.org which contains thousands of misère quotients for octal games.

Siegel notes that Dawson first proposed his problem in 1935, making it perhaps the oldest open problem in combinatorial game theory. [Michael Albert offers the alternative "Is chess a first player win?"] It may be of historical interest to note that Dawson showed the problem to one of the present authors around 1947. Fortunately, he forgot that Dawson proposed it as a losing game, was able to analyze the normal play version, rediscover the Sprague–Grundy theory, and get Conway interested in games.

A16. **A16** is now **E12**.

A17. Nem and mnem. **Mem** is played with heaps of tokens. Remove any number of tokens from any one heap. The number of tokens removed must be at least as large as the number that were removed on the previous move from that heap. Equivalently, a "heap" is a pair of integers (n, k), and a move is to any pair $(n-i, i)$, where $k \le i \le n$.

Mnem is exactly the same as **Mem**, with additional options: Either player may add tokens to a heap instead of removing them. If adding tokens, the number added must be strictly less than the number of tokens added or removed on the previous move. If removing tokens, the number must be at least as large as the number added or removed on the previous move. So a move from (n, k) is to $(n-i, i)$ for $k \le i \le n$, as in **Mem**, or to $(n+i, i)$ for $1 \le i < k$.

Conway and (Aaron) Siegel have investigated this game. They conjecture: *Every position in* **Mnem** *has finite nim value*. They verified this experimentally up to $n = 10,000$. They also conjecture: *For both* **Mem** *and* **Mnem**, *if* $k^2 \ge n$, *then* $(n, k) = \lfloor n/k \rfloor$. Siegel reports that they have no idea how to prove either so these may be very difficult problems.

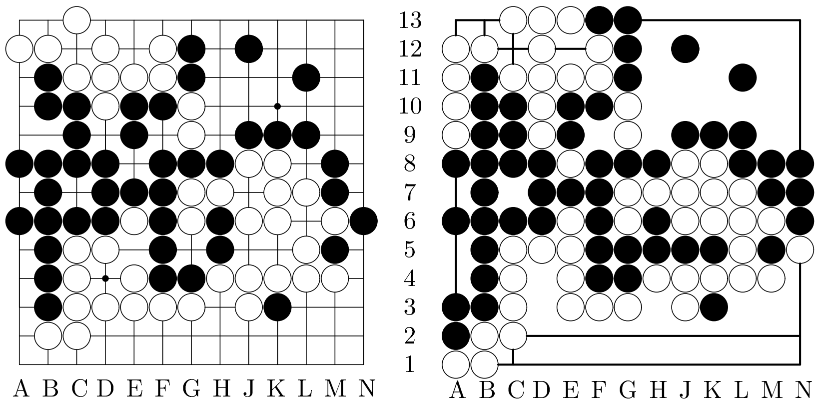

Figure 3. Jiang v. Rui, MSRI, July, 2000.

B. Pushing and placing pieces

B1. (5) Go is of particular interest, partly because of the loopiness induced by the "ko" rule, and many problems involve general theory; see E4 and E5.

Elwyn writes:

> I attach one region that has been studied intermittently over the past several years. The region occurs in the southeast corner of the board (Figure 3). At move 85 Black takes the ko at L6. What then is the temperature at N4? This position is copied from the game Jiang and Rui played at MSRI in July 2000. In 2001, Bill Spight and I worked out a purported solution by hand, assuming either Black komaster or White komaster. I've recently been trying to get that rather complicated solution confirmed by GoExplorer, which would then presumably also be able to calculate the dogmatic solution. I've been actively pursuing this off and on for the past couple weeks, and haven't gotten there yet.

Elwyn also writes:

> Nakamura (GONC3) has shown how capturing races in Go can be analyzed by treating liberties as combinatorial games. Like atomic weights, when the values are integers, each player's best move reduces his opponent's resources by one. The similarities between atomic weights and Nakamura's liberties are striking.

Theoretical problem: Either find a common formulation which includes much or all of atomic weight theory and Nakamura's theory of liberties, OR find some significant differences.

Important practical applied problem: Extend Nakamura's theory to include other complications which often arise in **Go**, such as simple kos, either internal and/or external.

B2. Woodpush (see LIP, pp. 214, 275) is a game that involves ko but that is simpler than **Go**. **Woodpush** is played on a finite strip of squares. Each square is empty or occupied by a black or white piece. A piece of the current player's color retreats: Left retreats to the left and Right to the right — to the next empty square, or off the board if there is no empty square; except, if there is a contiguous string containing an opponent's piece then it can move in the opposite direction *pushing* the string ahead of it. Pieces can be moved off the end of the strip. Immediate repetition of a global board positions is not allowed. A "ko" threat must be played first. For example:

$$\begin{array}{ccccccccc} & \text{Left} & & \text{Right} & & \text{Left} & & \text{Right} & \\ LRR\Box & \to & \Box LRR & \to & LR\Box R & \to & \text{ko-threat} & \to & R\Box\Box R \end{array}$$

Note that Right's first move to $LRR\Box$ is illegal because it repeats the immediately prior board position and Left's second move to $\Box LRR$ is also illegal so he must play a ko-threat. Also note that in $\Box LRR\Box$, Right never has to play a ko-threat since he can always push with either of his two pieces — with Left moving first,

$$\begin{array}{ccccc} & \text{Left} & & \text{Right} & \\ \Box LRR\Box & \to & \Box\Box LRR & \to & \Box LR\Box R \\ & \to & \text{ko-threat} & \to & \text{Right answers ko-threat} \\ & \to & \Box\Box LRR & \to & \Box LRR\Box \end{array}$$

Berlekamp, Plambeck, Ottaway, Aaron Siegel and Spight (work in progress) use top-down thermography to analyze the three piece positions. What about more pieces? Cazenave and Nowakowski (this volume) show that the position $L.L.R.R$ is ± 4 but that $L.L...R.R$ and $L.L.....R.R$ are draws by superko (repetition of the board position after more than 2 moves).

B3. (40) Chess. Noam Elkies [2002] has examined Dawson's chess, but played under usual Chess rules, so that capture is not obligatory.

He would still welcome progress with his conjecture that the value $*k$ occurs for all k in (ordinary chess) pawn endings on sufficiently large chessboards.

Thea van Roode has suggested **Impartial Chess**, in which the players may make moves of either color. Checks need not be responded to and Kings may be captured. The winner could be the first to promote a pawn.

B4. (30) Chess: **King and Rook.** Low and Stamp [2006] have given a strategy in which White wins the King and Rook vs. King problem within an 11×9 region. Kanungo and Low [2007] show that with the initial position WK(1, 1), WR(x, y) and BK(a, b), where $1 < x < a$, White has a winning strategy on the $(a+b+3) \times (a+b+5)$ board.

B5. Nonattacking queens. Noon and Van Brummelen [2006] alternately place queens on an $n \times n$ chessboard so that no queen attacks another. The winner is the last queen placer. They give nim-values for boards of sizes $1 \leq n \leq 10$ as 1121312310 and ask for the values of larger boards.

B6. (55) Amazons. Müller [14] has shown that the 5×5 game is a first player win and asks about the 6×6 game.

B7. Phutball, more properly, Conway's **Philosopher's Football**, is usually played on a Go board with positions (i, j), $-9 \leq i, j \leq 9$ and the ball starting at (0,0). For the rules, see WW, pp. 752–755. The game is loopy (see Section E5 below), and Nowakowski, Ottaway and Siegel (see [19]) discovered positions that contained tame cycles, i.e., cycles with only two strings, one each of Left and Right moves. Aaron Siegel asks if there are positions in such combinatorial games which are stoppers but contain a **wild cycle**, i.e., one which contains more than one alternation between Left and Right moves. Demaine, Demaine and Eppstein [2002] show that it is NP-complete to decide if a player can win on the next move. Loosen (see **directional phutball**) raises the question of whether there is any $\mathcal{P}$-position.

Phlag Phutball is a variant played on an $n \times n$ board with the initial position of the ball at $(0, 0)$ except that now only the ball may occupy the positions $(2i, 2j)$ with both coordinates even. This eliminates "tackling", and is an extension of one-dimensional **Oddish Phutball**, analyzed in Grossman and Nowakowski [2002]. The $(3, 2n+1)$ board (i.e., (i, j), $i = 0, 1, 2$ and $-n \leq j \leq n$) is already interesting and requires a different strategy from that appropriate to **Oddish Phutball**.

Loosen [2008] introduces **Directional Phutball**, a nonloopy version, which is also played on a grid. The ball starts in the bottom left corner with Left's goal-line is the right edge, Right's is the top edge. Players can only jump toward their opponent's goal-line, and win by jumping on over that goal-line. Men can only be placed ahead of the ball (i.e., in the positive quadrant with origin on the ball). She notes that there is no $\mathcal{P}$-position in this game and that making an off-parity move ("poultry") can be good. She asks for a complete analysis of the $2 \times n$ board. She shows that the atomic weights of the $m \times n$ board for $2 \leq m, n, \leq 4$ is 0 and is 0, 1, 1, 2 for $2 \times n$ for $n = 5, 6, 7, 8$ respectively.

B8. Hex. (LIP, pp. 264–265) Nash's strategy stealing argument shows that Hex is a first player win but few quantitative results are known. Arneson et al. report that 8×8 is solved as are most openings on the 9×9.

Garikai Campbell [2004] asks:

(1) For each n, what is the shortest path on an $n \times n$ board with which the first player can guarantee a win?

(2) What is the least number of moves that guarantees the first player a win? Campbell showed that this is at least n on an $n \times n$ board. Peng et al. [2010] show that it is 7 on the 5×5 board.

B9. (54) Fox and geese. Berlekamp and Siegel [19, Chapter 2] and WW, pp. 669–710, "analysed the game fairly completely, relying in part on results obtained using CGSuite." On p. 710 of WW the following open problems are given.

(1) Define a position's **span** as the maximum occupied row-rank minus its minimum occupied row-rank. Then quantify and prove an assertion such as the following: If the backfield is sufficiently large, and the span is sufficiently large, and if the separation is sufficiently small, and if the Fox is neither already trapped in a daggered position along the side of the board, nor immediately about to be so trapped, then the Fox can escape and the value is **off**.

(2) Show that any formation of three Geese near the center of a very tall board has a "critical rank" with the following property: If the northern Goose is far above, and the Fox is far below, then the value of the position is either positive, HOT, or **off**, according as the northern Goose is closer, equidistant, or further from the critical rank than the Fox.

(3) Welton asks what happens if the Fox is empowered to retreat like a Bishop, going back several squares at a time in a straight line? More generally, suppose his straight-line retreating moves are confined to some specific set of sizes. Does $\{1, 3\}$, which maintains parity, give him more or less advantage than $\{1, 2\}$?

(4) What happens if the number of Geese and board widths are changed?

In Aaron Siegel's thesis there are several other questions:

(5) In the critical position, with Geese at (we use the algebraic Chess notation of a, b, c, d, ... for the files and $1, 2, 3, \ldots, n$ for the ranks) (b, n), (d, n), $(\text{e}, n-1)$, $(\text{h}, n-1)$, and Fox at $(\text{c}, n-1)$, which has value $1 + 2^{-(n-8)}$ on an $n \times 8$ board with $n \geq 8$ in the usual game, is the value $-2n + 11$ for all $n \geq 6$ when played with "Ceylonese rules"? (Fox allowed two moves at each turn.)

(6) On an $n \times 4$ board with $n \geq 5$ and Geese at (b, n) and $(\text{c}, n-1)$ do all Fox positions have value **over**? With the Geese on (b, n) and (d, n) are only other values 0 at $(\text{c}, n-1)$ and $\{$**over** $\mid 0\}$ at $(\text{b}, n-2)$ and $(\text{d}, n-2)$?

(7) On an $n \times 6$ board with $n \geq 8$ and Geese at (b, n), (d, n) and $(\text{e}, n-1)$ do the positions $(\text{a}, n-2k+1)$, $(\text{c}, n-2k+1)$, $(\text{e}, n-2k+1)$, all have value 0,

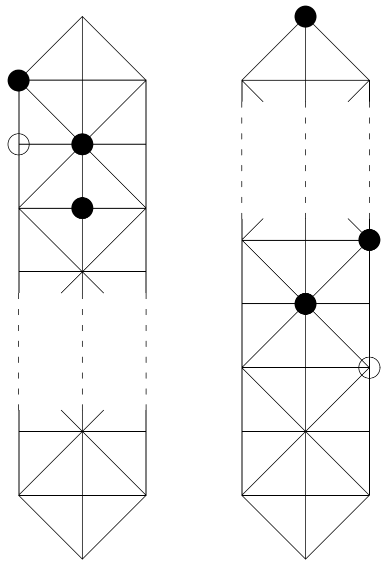

Figure 4. Sequences of hare and hounds positions.

and those at (b, $n-2k$), (d, $n-2k$), (f, $n-2k$) all have value Star? And if the Geese are at (b, n), (d, n) and (f, n) are the zeroes and Stars interchanged?

B10. Hare and Hounds. Aaron Siegel asks if the sequences of positions of increasing board length shown in Section B10 are, on the left, increasingly hot, and, on the right, have arbitrarily large negative atomic weight. He also conjectures that the starting position on a $6n+5\times 3$ board, for $n>0$, has value

$$-(n-1)+\Big\{b, c\,|0\|0\|0\Big\|0\ldots\Big\|0\Big\}$$

where there are $2n+4$ zeroes and slashes and $b=\Big\{0, a\,\|0, \{0\mid \mathbf{off}\}\Big\}$,

$$c=\Big\{0\,\Big\|\,{\downarrow_{[2]}}{*}|0\|0\Big\} \quad \text{and} \quad a=\{0, {\downarrow_{[2]}}{*}\mid 0, {\downarrow_{[2]}}{*}\}.$$

B11. (4) Domineering. (WW, pp. 119–122, 138–142; LIP pp. 1–7, 260). Extend the analysis.

(Left and Right take turns to place dominoes on a checker-board. Left orients her dominoes North–South and Right orients his East–West. Each domino exactly covers two squares of the board and no two dominoes overlap. A player unable to play loses.)

See Berlekamp [1988] and the second edition of WW, pp. 138–142, where some new values are given. For example David Wolfe and Dan Calistrate have

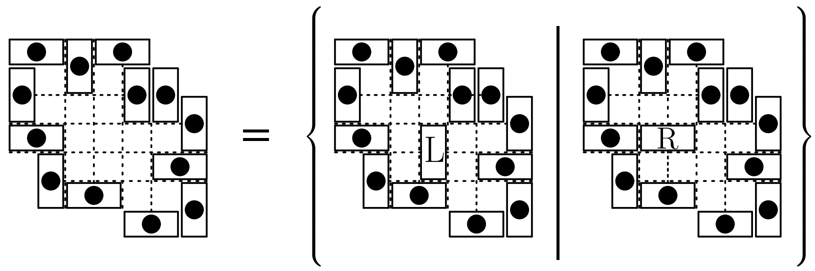

Figure 5. A **Domineering** position of value $\pm 2*$.

found the values (to within '-ish', i.e., infinitesimally shifted) of 4×8, 5×6 and 6×6 boards. The value for a 5×7 board is

$$\left\{ \tfrac{3}{2} \,\middle|\, \left\{ \tfrac{5}{4} \middle| -\tfrac{1}{2} \right\}, \left\{ \tfrac{3}{2} \,\middle|\, -\tfrac{1}{2}, \left\{ \tfrac{3}{2} \middle| -1 \right\} \,\middle\|\, -1 \middle| -3 \right\} \,\middle\|\, -1, \left\{ \tfrac{3}{2} \,\middle|\, -\tfrac{1}{2} \middle\| -1 \right\} \,\middle|\, -3 \right\}$$

Lachmann, Moore and Rapaport [2002] discover who wins on rectangular, toroidal and cylindrical boards of widths 2, 3, 5 and 7, but do not find their values. Bullock [3, p. 84] showed that 19×4, 21×4, 14×6 and 10×8 are wins for Left and that 10×10 is a first player win.

Berlekamp notes that the value of a $2 \times n$ board, for n even, is only known to within "ish", and that there are problems on $3 \times n$ and $4 \times n$ boards that are still open.

Berlekamp asks, as a hard problem, to characterize all hot **Domineering** positions to within "ish". As a possibly easier problem he asks for a **Domineering** position with a new temperature, i.e., one not occurring in Table 1 on GONC, p. 477. Gabriel Drummond-Cole [2002] found values with temperatures between 1.5 and 2. Figure 5 shows a position of value $\pm 2*$ and temperature 2.

Shankar and Sridharan [2005] have found many **Domineering** positions with temperatures other than those shown in Table 1 on p. 477 of GONC. Blanco and Fraenkel [2] have obtained partial results for the game of **Tromineering**, played with trominoes in place of (or, alternatively, in addition to) dominoes.

B12. NoGo can be found under the name "Anti-Atari Go" at the Sensei Library (see senseis.xmp.net/?AntiAtariGo) and was invented independently by Neil McKay. On a Go-board (or on any graph) pieces are placed as in Go, the only restriction is that every connected group, of each player, must be adjacent to at least one empty intersection. Kyle Burke has shown that **NoGo** is NP-hard on a graph. McKay, Nowakowski and (Angela) Siegel found positions of value 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ on a Go board. Using CGSuite, one can find the nimbers $*$, $*2$, $*4$, $*8$ on a one-dimensional board. The position $\bullet \cdot \circ \cdot \bullet \cdot \circ \ldots$, where $\bullet$ is a black piece, $\circ$ is

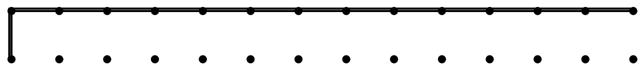

Figure 6. Starting position for "Icelandic" $1 \times n$ Dots-and-Boxes.

a white piece and '$\cdot$' is an empty space, is equivalent to the octal game .**6** (see Section A2) which is not known to be periodic. We ask for the nim-dimension of both the one- and two-dimensional board. Is there a limit to the denominator of the fractions found on the two-dimensional board?

C. Playing with pencil and paper

C1. (51) Dots-and-Boxes. Elwyn Berlekamp asks for a complete theory of the "Icelandic" $1 \times n$, i.e., with starting position as in Figure 6.

See Berlekamp's book [2000] for more problems about this popular children's (and adults') game (see also WW, pp. 541–584; LIP, pp. 21–28, 260).

C2. (25) Sprouts. Extend the analysis of this Conway–Paterson game in either the normal or misère form. (WW, pp. 564–568).

(A move joins two spots, or a spot to itself by a curve which doesn't meet any other spot or previously drawn curve. When a curve is drawn, a new spot must be placed on it. The valence of any spot must not exceed three.)

C3. (26) Sylver coinage. (WW, pp. 575–597): Extend the analysis.

(Players alternately name distinct positive integers, but may not name a number which is a sum, with repetitions allowed, of previously named integers. Whoever names 1 loses.) Sicherman [2002] contains the most recent information.

C4. (28) Extend Úlehla's or Berlekamp's analysis of **von Neumann's Game** from directed forests to directed acyclic graphs (WW, pp. 570–572; Úlehla [1980]).

(Von Neumann's game, or Hackendot, is played on one or more rooted trees. The roots induce a direction, towards the root, on each edge. A move is to delete a node, together with all nodes on the path to the root, and all edges incident with those nodes. Any remaining subtrees are rooted by the nodes that were adjacent to deleted nodes.)

C5. (43) Inverting Hackenbush. Thea van Roode has written a thesis [24] investigating both this and **Reversing Hackenbush**, but there is plenty of room for further analysis of both games.

In **Inverting Hackenbush**, when a player deletes an edge from a component, the remainder of the component is replanted with the new root being the pruning point of the deleted edge. In **Reversing Hackenbush**, the colors of the edges are all changed after each deletion. Both games are hot, in contrast to **Blue-Red Hackenbush** (WW, pp. 1–7; LIP, pp. 82, 88, 111–112, 212, 266) which is cold, and **Green Hackenbush** (WW, pp. 189–196), which is tepid.

C6. (42) Beanstalk and **Beans-Don't-Talk** are games invented respectively by John Isbell and John Conway. See Guy [1986]. **Beanstalk** is played between Jack and the Giant. The Giant chooses a positive integer, n_0. Then Jack and the Giant play alternately $n_1, n_2, n_3, \ldots$, according to the rule

$$\begin{aligned} n_{i+1} &= n_i/2 \text{ if } n_i \text{ is even,} \\ &= 3n_i \pm 1 \text{if } n_i \text{ is odd;} \end{aligned}$$

that is, if n_i is even, there's only one option, while if n_i is odd there are just two. The winner is the person moving to 1.

We still don't know if there are any $\mathcal{O}$-**positions** (positions of infinite remoteness).

C7. (63) The Erdős–Szekeres game [7] (and see [18]) was introduced by Harary, Sagan and West [1985]. From a deck of cards labeled from 1 through n, Alexander and Bridget alternately choose a card and append it to a sequence of cards. The game ends when there is an ascending subsequence of a cards or a descending subsequence of d cards.

The game appears to have a strong bias towards the first player. Albert et al., (this volume) show that for $d = 2$ and $a \le n$ the outcome is $\mathcal{N}$ or $\mathcal{P}$ according as n is odd or even, and is $\mathcal{O}$ (drawn) if $n < a$. They conjecture that for $a \ge d \ge 3$ and all sufficiently large n, it is $\mathcal{N}$ with both normal and misère play, and also with normal play when played with the rationals in place of the first n integers.

They also suggest investigating the form of the game in which players take turns naming pairs (i, π_i) subject to the constraint that the chosen values form part of the graph of some permutation of $\{1, 2, \ldots, n\}$.

D. Disturbing and destroying

D1. (27) Extend the analysis of **Chomp** (WW, pp. 598–599; LIP pp. 19, 46, 216).

David Gale offered \$300.00 for the solution of the infinite 3D version where the board is the set of all triples (x, y, z) of nonnegative integers, that is, the lattice points in the positive octant of $\mathbb{R}^3$. The problem is to decide whether it is a win for the first or second player.

Chomp (Gale [1974]) is equivalent to **Divisors** (Schuh [1952]). **Chomp** is easily solved for $2 \times n$ arrays, Sun [2002], and indeed a recent result by Steven

Byrnes [2003] shows that any poset game eventually displays periodic behavior if it has two rows, and a fixed finite number of other elements. See also the Fraenkel poset games mentioned near the end of Section A2.

Thus, most of the work in recent years has been on three-rowed **Chomp**. The situation becomes quite complicated when a third row is added, see Zeilberger [2001] and Brouwer et al. [2005]. A novel approach (renormalization) is taken by Friedman and Landsberg (GONC3). They demonstrate that three-rowed **Chomp** exhibits certain scaling and self-similarity patterns similar to chaotic systems. Is there a deterministic proof that there is a unique winning move from a $3 \times n$ rectangle? The renormalization approach is based on nonlinear dynamics techniques from physics; its results are highly suggestive but as of yet not fully mathematically rigorous.

Transfinite Chomp has been investigated by Huddleston and Shurman [2002]. An open question is to calculate the nim value of the position $\omega \times 4$ — they conjecture this to be $\omega \cdot 2$, but it could be as low as 46, or even uncomputable! Perhaps the most fascinating open question in **Transfinite Chomp** is their *Stratification Conjecture*, which states that if the number of elements taken in a move is $< \omega^i$, then the change in the nim-value is also $< \omega^i$.

D2. (33) Subset Take-away. Given a finite set, players alternately choose proper subsets subject to the rule that once a subset has been chosen no proper subset can be removed. Last player to move wins.

Many people play the dual, that is, a nonempty subset must be chosen and no proper superset of this can be chosen. We discuss this version of the game which now can be considered a poset game with the sets ordered by inclusion.

The $(n; k)$ Subset Take-away game is played using all subsets of sizes 1 through k of a n-element set. In the $(n; n)$ game one has the whole set (i.e., the set of size n) as an option, so a strategy-stealing argument shows this must be a first player win.

(1) Gale and Neyman [1982], in their original paper on the game, conjectured that the winning move in the $(n; n)$ game is to remove just the whole set. This is equivalent to the statement that the $(n; n-1)$ game is a second-player win, which has been verified only for $n \leq 5$.

(2) A stronger conjecture states that $(n; k)$ is a second player win if and only if $k+1$ divides n. This was proved in the original paper only for $k = 1$ or 2.

See also Fraenkel and Scheinerman [1991].

D3. (39) Sowing or **Mancala** games. There appears to have been no advance on the papers mentioned in MGONC, although we feel that this should be a fruitful field of investigation at several different levels.

D4. Annihilation Games. ***k*-annihilation**. Initially place tokens on some of the vertices of a finite digraph. Denote by $\rho_{\text{out}}(u)$ the outvalence of a vertex u. A move consists of removing a token from some vertex u, and "complementing" $t := \min(k, \rho_{\text{out}}(u))$ (immediate) followers of u, say $v_1, \ldots, v_t$: if there is a token on v_j, remove it; if there is no token there, put one on it. The player making the last move wins. If there is no last move, the outcome is a draw. For $k = 1$, there is an $O(n^6)$ algorithm for deciding whether any given position is in $\mathcal{P}$, $\mathcal{N}$, or $\mathcal{D}$; and for computing an optimal next move in the last 2 cases (Fraenkel and Yesha [1982]). Fraenkel asks: Is there a polynomial algorithm for $k > 1$? For an application of k-annihilation games to lexicodes; see Fraenkel and Rahat [2003].

D5. Toppling Dominoes (LIP, pp. 110–112, 274) is played with a row of vertical dominoes each of which is either blue or red. A player topples one of his/her dominoes to the left or to the right.

See Fink et al [this volume] for the proofs that every number occurs exactly once and is a palindrome; there are exactly n positions with value $*n$. Several conjectures are listed but the most intriguing seems to be: *if G is a palindrome then G's value appears uniquely.*

There are several variants of **Toppling Dominoes**. If all the dominoes must be toppled in the same direction then this is a **Hackenbush** string. **Timber** [15] is an impartial version of **Toppling Dominoes** played on a directed graph. The dominoes are on the edges. A player chooses a domino which is toppled in the direction of the edge. The dominoes on incident edges are then toppled (regardless of the underlying edge) and the process is iterated. If only outcomes are required then only trees are interesting. In [15], an algorithm is given to determine the outcome class of a tree, which surprisingly requires nim-values. Values, however, appear to be difficult to determine. The normal and misère versions on a path are related to Dyck paths and Catalan numbers; see Section E15 for more details. The partizan version, imaginatively called **Partizan Timber**, has Left and Right dominoes placed on the edges of a directed graph. Any **Toppling Dominoes** position can be transformed into a **Partizan Timber** position by subdividing each edge and directing the edge toward the new vertex. A similar algorithm to that of **Timber** can be used to determine the outcome class of a tree. The uniqueness of numbers doesn't hold even on a path. In the other direction, are there any values that do not occur in the game?

D6. Hanoi Stick-Up is played with the disks of the Towers of Hanoi puzzle, starting with each disk in a separate stack. A move is to place one stack on top of another such that the size of the bottom of the first stack is less than the size of the top of the second; the two stacks then fuse (&) into one. The only relevant information about a stack are its top and bottom sizes, and it's often possible to collapse the labeling of positions: so, for instance, starting with 8 disks and fusing 1&7 and 2&5,

we have stacks	0	1&7	2&5	3	4	6
which can be relabeled	0	1&3	1&2	1	2	3

in which the legal moves are still the same. John Conway, Alex Fink and others have found that the $\mathcal{P}$-positions of height ≤ 3 in normal Hanoi Stickup are exactly those which, after collapsing, are of the form $0^a\ 01^b\ 1^c\ 12^d\ 2^e$ with $\min(a+b+c, c+d+e, a+e)$ even, except that if $a+e \leq a+b+c$ and $a+e \leq c+d+e$ then both a and e must be even (02 can't be involved in a legal move so can be dropped).

They also found the normal and misère outcomes of all positions with up to six stacks, but there is more to be discovered.

D7. (56) Are there any draws in **Beggar-My-Neighbor**? Marc Paulhus showed that there are no cycles when using a half-deck of two suits, but the problem for the whole deck (one of Conway's "antiHilbert" problems) is still open.

E. Theory of games

E1. (49) Fraenkel updates Berlekamp's questions on computational complexity as follows:

Demaine, Demaine and Eppstein [2002] proved that deciding whether a player can win in a *single* move in **Phutball** (WW, pp. 752–755; LIP, p. 212) is NP-complete. Grossman and Nowakowski [2002] gave constructive partial strategies for one-dimensional **Phutball**. Thus, these papers do not show that **Phutball** has the required properties.

Perhaps **Nimania** (Fraenkel and Nešetřil [1985]) and **Multivision** (Fraenkel [1998]) satisfy the requirements. **Nimania** begins with a single positive integer, but after a while there is a multiset of positive integers on the table. At move k, a copy of an existing integer m is selected, and 1 is subtracted from it. If $m = 1$, the copy is deleted. Otherwise, k copies of $m-1$ are adjoined to the copy $m-1$. The player first unable to move loses and the opponent wins. It was proved: (i) The game terminates. (ii) Player I can win. In Fraenkel, Loebl and Nešetřil [1988], it was shown that the max number of moves in **Nimania** is an Ackermann function, and the min number satisfies $2^{2^{n-2}} \leq \mathrm{Min}(n) \leq 2^{2^{n-1}}$.

The game is thus intractable simply because of the length of its play. This is a *provable* intractability, much stronger than NP-hardness, which is normally only a *conditional* intractability. One of the requirements for the tractability of a game is that a winner can consummate a win in at most $O(c^n)$ moves, where

$c > 1$ is a constant, and n a sufficiently succinct encoding of the input (this much is needed for nim on 2 equal heaps of size n).

To consummate a win in **Nimania**, player I can play randomly most of the time, but near the end of play, a winning strategy is needed, given explicitly. Whether or not this is an "intricate" solution, depends on the beholder. But it seems that it's of even greater interest to construct a game with a very *simple* strategy which still has high complexity!

Also every play of **Multivision** terminates, the winner can be determined in linear time, and the winning moves can be computed linearly. But the length of play can be arbitrarily long. So let's ask the following: Is there a game which has

1. simple, playable rules,
2. a simple explicit strategy,
3. length of play at most exponential, and
4. is NP-hard or harder.

Tung [1987] proved the following:

Theorem. *Given a polynomial* $P(x, y) \in \mathbb{Z}[x, y]$, *the problem of deciding whether for all* x *there exists* $y[P(x, y) = 0]$ *holds over* $\mathbb{Z}_{\geq 0}$, *is co-NP-complete.*

Define the following game of length 2: player I picks $x \in \mathbb{Z}_{\geq 0}$, player II picks $y \in \mathbb{Z}_{\geq 0}$. Player I wins if $P(x, y) \neq 0$, otherwise player II wins. For winning, player II has only to compute y such that $P(x, y) = 0$, given x, and there are many algorithms for doing so.

Also Jones and Fraenkel [1995] produced games, with small length of play, which satisfy these conditions.

So we are led to the following reformulation of Berlekamp's question: Is there a game which has

1. simple, playable rules,
2. a finite set of options at every move,
3. a simple explicit strategy,
4. length of play at most exponential,
5. and is NP-hard or harder.

E2. Complexity closure. Aviezri Fraenkel asks: Are there partizan games G_1, G_2, G_3 such that: (i) G_1, G_2, G_3, $G_1 + G_2$, $G_2 + G_3$ and all their options have polynomial-time strategies, (ii) $G_1 + G_3$ is NP-hard?

E3. Sums of switch games. David Wolfe considers a sum of games G, each of the form $a\|b|c$ or $a|b\|c$ where a, b, and c are integers specified in unary. Is there a polynomial time algorithm to determine who wins in G, or is the problem NP-hard?

E4. (52) How does one play **sums** of games with varied overheating operators? Sentestrat and Top-down thermography (LIP, p. 214):

David Wolfe would like to see a formal proof that sentestrat works, an algorithm for top-down thermography, and conditions for which top-down thermography is computationally efficient.

Aaron Siegel asks the following generalized thermography questions.

(1) Show that the Left scaffold of a dogmatic (neutral ko-threat environment; LIP, p. 215) thermograph is decreasing as function of t. (Note, this is NOT true for komaster thermographs.) [Dogmatic thermography was invented by Berlekamp and Spight. See [21] for a good introduction.]

(2) Develop the machinery for computing dogmatic thermographs of double kos (multiple alternating 2-cycles joined at a single node).

In the same vein as (2):

(3) Develop a temperature theory that applies to all loopy games.

Siegel thinks that (3) is among the most important open problems in combinatorial game theory. The temperature theory of Go appears radically different from the classical combinatorial theory of loopy games (where infinite plays are draws). It would be a huge step forward if these could be reconciled into a "grand unified temperature theory". Problem (2) seems to be the obvious next step toward (3).

Conway asks for a natural set of conditions under which the mapping $G \mapsto \int^{*} G$ is the *unique* homomorphism that annihilates all infinitesimals.

E5. Loopy games (WW, pp. 334–377; LIP, pp. 213–214) are partizan games that do not satisfy the ending condition. A **Stopper** is a game that, when played on its own, has no ultimately alternating, Left and Right, infinite sequence of legal moves. Aaron Siegel reminds us of WW, 2nd ed., p. 369, where the authors tried hard to prove that every loopy game had stoppers, until Clive Bach found the Carousel counterexample. Is there an alternative notion of simplest form that works for *all* finite loopy games, and, in particular, for the Carousel? The simplest form theorem for stoppers is at WW, p. 351.

Siegel conjectures that, if Q is an arbitrary cycle of Left and Right moves that contains at least two moves for each player, and is not strictly alternating, then there is a stopper consisting of a single cycle that matches Q, together

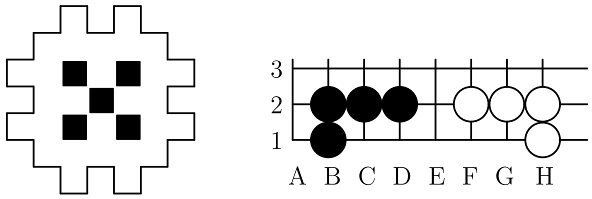

Figure 7. A **Domineering** position and a Chilled Go position of value $*2$.

with various exits to enders, i.e., games which end in a finite, though possibly unbounded, number of moves. (Note that games normally have Left and Right playing alternately, but if the game is a sum, then play in one component can have arbitrary sequences of Left and Right moves, not just alternating ones.)

A long cycle is *tame* if it alternates just once between Left and Right, otherwise it is *wild*. Aaron Siegel writes:

> I can produce wild cycles "in the laboratory," by specifying their game graphs explicitly. So the question is to detect one "in nature", i.e., in an actual game with (reasonably) playable rules such as **phutball** [Problem B7].

Siegel also asks under what conditions does a given infinitesimal have a well-defined atomic weight, and asks to specify an algorithm to calculate the atomic weight of an infinitesimal stopper g. The algorithm should succeed whenever the atomic weight is well-defined, i.e., whenever g can be sandwiched between loopfree all-smalls of equal atomic weight.

E6. (45) Elwyn Berlekamp asks for the **habitat** of $*2$, where $*2 = \{0, *|0, *\}$. Gabriel Drummond-Cole [2005] has found **Domineering** positions with this value. See, for example, Figure 7, which also shows a Go position, found by Nakamura and Berlekamp [2003], whose chilled value is $*2$. The Black and White groups are both connected to life via unshown connections emanating upwards from the second row. Either player can move to $*$ by placing a stone at E2, or to 0 by going to E1. Given a game, let n be the smallest nonnegative integer n such that if a position of the game has value $*k$ then $k < 2^n$. Carlos dos Santos calls n the *nim-dimension* of the game. He shows, this volume, that **Konane** has infinite nim-dimension and ask for the nim-dimension of other games.

E7. Partial ordering of games. David Wolfe lets $g(n)$ be the number of games born by day n, notes that an upper bound is given by $g(n+1) \le g(n) + 2^{g(n)} + 2$, and a lower bound for each $\alpha < 0$ is given by $g(n+1) \ge 2^{g(n)^\alpha}$, for n sufficiently large, and asks us to tighten these bounds.

He also asks what group is generated by the all-small games (or — much harder — of all games) born by day 3. Describe the partial order of games born by day 3, identifying all the largest "hypercubes" (Boolean sublattices) and how they are interconnected. These questions have been answered for day 2, see this volume, pp.??.

Berlekamp suggests other possible definitions for games born by day n, $\mathcal{G}_n$, depending on how one defines $\mathcal{G}_0$. Our definition is 0-based, as $\mathcal{G}_0 = \{0\}$. Other natural definitions are integer-based (where $\mathcal{G}_0$ are integers) or number-based. These two alternatives do not form a lattice, for if G_1 and G_2 are born by day k, then the games

$$H_n \;:=\; \bigl\{G_1, G_2 \;\|\; G_1, \{G_2|-n\}\bigr\}$$

form a decreasing sequence of games born by day $k+2$ exceeding any game $G \geq G_1, G_2$, and the day $k+2$ join of G_1 and G_2 cannot exist. What is the structure of the partial order given by one of these alternative definitions of birthday?

The set of all short games does not form a lattice, but Calistrate, Paulhus and Wolfe [2002] have shown that the games born by day n form a distributive lattice $\mathcal{L}_n$ under the usual partial order. They ask for a description of the exact structure of $\mathcal{L}_3$. Siegel describes $\mathcal{L}_4$ as "truly gigantic and exceedingly difficult to penetrate" but suggests that it may be possible to find its dimension and the maximum **longitude**, $\mathrm{long}_4(G)$, of a game in $\mathcal{L}_4$, which he defines as

$$\mathrm{long}_n(G) = \mathrm{rank}_n(G \vee G^\bullet) - \mathrm{rank}_n(G),$$

where $\mathrm{rank}_n(G)$ is the rank of G in $\mathcal{L}_n$ and $G^\bullet$ is the **companion** of G,

$$G^\bullet = \begin{cases} * & \text{if } G = 0, \\ \{0, (G^L)^\bullet \mid (G^R)^\bullet\} & \text{if } G > 0, \\ \{(G^L)^\bullet \mid 0, (G^R)^\bullet\} & \text{if } G < 0, \\ \{(G^L)^\bullet \mid (G^R)^\bullet\} & \text{if } G \parallel 0. \end{cases}$$

Albert and Nowakowski [2011] show that starting with any set of games, instead of just $0 = \{\cdot|\cdot\}$, then the games born on the next day form a complete but not necessarily distributive lattice. They ask which sets of games will give a distributive lattice? Is there a set which will give a nondistributive but modular lattice? Carvalho, dos Santos, Dias, Coelho, Neto, Nowakowski and Vinagre [4] answered this by showing that for any lattice L, there is a set of games which generates L on the next day.

The set of all-small games does not form a lattice, but Siegel forms a lattice $\mathcal{L}_n^0$ by adjoining least and greatest elements $\triangle$ and $\triangledown$ and asks: do the elements of $\mathcal{L}_n^0$ have an intrinsic "handedness" that distinguishes, say, $(n-1)\cdot\uparrow$ from $(n-1)\cdot\uparrow + *$?

A game is *option-closed* (Nowakowski and Ottaway [2011]) if, recursively, each G^{LL} is also $\mathcal{G}^L$ and the same for Right. For example, **Hackenbush** strings are option-closed. Nowakowski and (Angela) Siegel [16] show that the option-closed games born on day n form a planar, nondistributive lattice, but the question of *how many?* remains unanswered. Are there other natural families of games that form planar lattices?

E8. Aaron Siegel asks, given a group or monoid, $\mathcal{H}$, of games, to specify a technique for calculating the simplest game in each $\mathcal{H}$-equivalence class. He notes that some restriction on $\mathcal{H}$ might be needed; for example, $\mathcal{H}$ might be the monoid of games absorbed by a given idempotent.

E9. Siegel also would like to investigate how search methods might be integrated with a canonical-form engine.

E10. **(9)** Develop a **misère theory** for unions of partizan games (WW, p. 312).

E11. Four-outcome-games. Guy [2007] has given a brute force analysis of a parity subtraction game which didn't allow the use of Sprague–Grundy theory because it wasn't impartial, nor the Conway theory, because it was not last-player-winning. Is there a class of games in which there are four outcomes, $\mathcal{N}$ext, $\mathcal{P}$revious, $\mathcal{L}$eft and $\mathcal{R}$ight, and for which a general theory can be given?

E12. Impartial Misère Analysis. See **A16.** From the works of Plambeck and Siegel: Let $\mathcal{A}$ be some set of games (the *universe*) played under misère rules. Typically, $\mathcal{A}$ is the set of positions that arise in a particular game, such as **Dawson's Chess**. Games $H, K \in \mathcal{A}$ are said to be equivalent, denoted by $H \equiv K(\text{mod}\mathcal{A}$, if $H+X$ and $K+X$ have the same outcome for all games $X \in \mathcal{A}$. The relation $\equiv$ is an equivalence relation, and a set of representatives, one from each equivalence class, forms the **misère quotient**, $\mathfrak{Q} = \mathcal{A}/\equiv$. A **quotient map** $\Phi : \mathcal{A} \to \mathfrak{Q}$ is defined, for $G \in \mathcal{A}$, by $\Phi : \mathcal{G} = [G]_\equiv$.

(1) A quotient map $\Phi : \mathcal{A} \to \mathfrak{Q}$ is said to be **faithful** if, whenever $\Phi(G) = \Phi(H)$, then G and H have the same normal-play Grundy value. Is every quotient map faithful?

(2) Let $(\mathfrak{Q}, \mathcal{P})$ be a quotient and $\mathcal{S}$ a maximal subgroup of $\mathfrak{Q}$. Must $\mathcal{S} \cap \mathcal{P}$ be nonempty? (Note: it's easy to get a "yes" answer in the special case when $\mathcal{S}$ is the kernel.)

(3) Extend the classification of impartial misère quotients. We have preliminary results on the number of quotients of order $n \le 18$ but believe that this can be pushed far higher.

(4) Exhibit an impartial misère quotient with a period-5 element. Same question for period 8, etc. We've detected quotients with elements of periods 1, 2, 3, 4, 6, and infinity, and we conjecture that there is no restriction on the periods of quotient elements.

(5) In the flavor of both (3) and (4): What is the smallest quotient containing a period 4 (or 3 or 6) element?

E13. Inverses in Partizan misère games. Siegel [Misère Canonical Forms of Partizan Games, this volume] and Allen [this volume] extends the misère monoid concepts to partizan games. In normal play, games in both players have a move or neither does are called *all-small* because their values are infinitesimal. This is not true in misère play and the term *dicot* has been coined to refer to these games in all play conventions. (See Section E14 as well.)

Recall that $-G$ is obtained from the game G by reversing the roles of Left and Right — "turning the board around". In normal play, $G - G = 0$ which is decidedly not true in misère play. To avoid confusion, when dealing with misère play $-G$ is frequently written as $\bar{G}$ and is called the *conjugate* of G.

(1) Allen asks, in the dicot universe, $\mathcal{D}$, when is it true that $G+\bar{G} \equiv 0 (\text{mod } \mathcal{D})$? For example, $* + * \equiv 0(\text{mod } \mathcal{D})$. McKay, Milley and Nowakowski [11] show that this is true if $G = * : x$ where x is a number in canonical form (regarded as a game tree) and more generally: if $o^-(H + \bar{H}) = \mathcal{N}$ for every H of G, including G, then $G + \bar{G} \equiv 0(\text{mod } \mathcal{D})$.

(2) Rebecca Milley asks, in a universe, $\mathcal{U}$, closed under sums, conjugation and subpositions if $G+H \equiv 0(\text{mod } \mathcal{U})$ is it necessarily true that $H = \bar{G}$? Jason Brown had asked the question about arbitrary universes. Milley [13] has the counterexample: on a finite strip of squares, Left places a 1×1 piece and Right places a 1×2 (domino). In the universe of sums of strips a strip of length 1 plus a strip of length 2 is equivalent to 0.

E14. Scoring games. Instead of "the last play determines the winner", another natural way is to have "whichever player has the higher score wins". Both **Dots-and-Boxes** and **Go**, for example, are scoring games.

Milnor looked at dicot scoring games with nonnegative incentives. Milnor [1953] defines a "positional game" to be what we would call a dicot (Section E13) scoring game, but only looks at positional games with nonnegative incentives. Hanner [1959] looked at the same class of games as Milnor, and invented thermography. Ettinger [2000] looked at dicot scoring games (possibly with negative incentives), which he calls "positional games" following Milnor. Johnson [9] looked at the dicot scoring games that are "well-tempered" in some sense (a subset of Ettinger's games). Stewart [23; 22] looks at general scoring games.

Since the games are no longer dicot, a choice has to be made about how to handle the situation where the current player can't move but their opponent can.

There is much work yet to be done in this field. Johnson mentions one problem worthy of a place to start: is there a general theory that covers **Dots-and-Boxes** (Section C1)?

E15. Find a formula for the number of $\mathcal{P}$-positions for **Nim** played with $2n$ tokens. there are none with an odd number of tokens. The On-line Encyclopedia of Integer Sequences (OEIS) has the value for small values of n.

In general, given a game, enumerate the number of $\mathcal{P}$-positions of a given size. We only know of **Timber** (Section D5) where the number of $\mathcal{P}$-positions, both normal and misère are related to Dyck paths with certain properties (also to Catalan numbers and Fine numbers) and Heytei [2009; 2010] which relates the number of $\mathcal{P}$-positions to the Bernoulli numbers of the second kind.

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rjn@mathstat.dal.ca

Department of Mathematics and Statistics,
Dalhousie University, Halifax, NS B3H 3J5, Canada