# Unsolved Problems in Spectral Graph Theory 

Lele Liu* and Bo Ning ${ }^{\dagger}$


#### Abstract

Spectral graph theory is a captivating area of graph theory that employs the eigenvalues and eigenvectors of matrices associated with graphs to study them. In this paper, we present a collection of 20 topics in spectral graph theory, covering a range of open problems and conjectures. Our focus is primarily on the adjacency matrix of graphs, and for each topic, we provide a brief historical overview.


Key words: Eigenvalues; Spectral radius; Adjacency matrix; Spectral graph theory AMS Classifications: 05C35; 05C50; 15A18

Spectral graph theory is a beautiful branch of graph theory that utilizes the eigenvalues and eigenvectors of matrices naturally associated with graphs to study them. The primary objective in spectral graph theory is twofold: firstly, to compute or estimate the eigenvalues of these matrices and secondly, to establish links between the eigenvalues and the structural properties of graphs. As it turns out, the spectral perspective is a powerful tool in the study of graph theory.

Over the past few decades, numerous results and applications in various fields of mathematics have been obtained through spectral graph theory. However, many open problems and conjectures in spectral graph theory remain unresolved, necessitating further exploration.

In this paper, we collect 20 topics in spectral graph theory that include a range of conjectures and open problems, with a focus primarily on the adjacency matrix of graphs. Additionally, we provide a brief historical overview of each topic. Inevitably, it is a somewhat personal perspective on the choice of these problems and conjectures, and is not intended to be exhaustive; the authors apologize for any omissions.

Let us begin with some definitions and notation. Throughout this paper, we only consider graphs that are simple (there are no loops or multiple edges), undirected and unweighted. Given a graph $G$, the adjacency matrix $A(G)$ of $G$ is a $(0,1)$-matrix, where the rows and columns are indexed by the vertices in $V(G)$. The $(i, j)$-entry of $A(G)$ is equal to 1 if the

[^0]vertices $i$ and $j$ are adjacent, and 0 otherwise. Since $A(G)$ is a real and symmetric matrix, it has a full set of real eigenvalues which we will denote by
$$
\lambda_{1}(G) \geq \lambda_{2}(G) \geq \cdots \geq \lambda_{n}(G)
$$

Recall that the Laplacian matrix of $G$ is defined as $L(G):=D(G)-A(G)$, where $D(G)$ is the diagonal matrix whose entries are the degrees of the vertices of $G$. We shall write $\mu_{1}(G) \geq \mu_{2}(G) \geq \cdots \geq \mu_{n}(G)=0$ for the eigenvalues of $L(G)$. If there is no danger of ambiguity, for any $1 \leq i \leq n$ we write $\lambda_{i}$ and $\mu_{i}$ instead of $\lambda_{i}(G)$ and $\mu_{i}(G)$, respectively. We also write $\lambda(G):=\lambda_{1}(G)$ for short.

For a graph $G$, let $\omega(G)$ denote the clique number of $G$, which is defined as the number of vertices in the largest complete subgraph in $G$. Let $e(G)$ and $\bar{G}$ denote the number of edges and the complement of $G$, respectively. We use the notations $\delta(G), \Delta(G)$, and $\bar{d}(G)$ to represent, respectively, the minimum degree, maximum degree, and average degree of $G$. For two vertex-disjoint graphs $G$ and $H$, we use $G \vee H$ to denote the join of $G$ and $H$, which is obtained by adding all possible edges between $G$ and $H$. The complete split graph $S_{n, k}$ with parameters $n$ and $k$ is the graph on $n$ vertices obtained from a clique on $k$ vertices and an independent set on the remaining $n-k$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. If there is no special explanation, we use $n$ to denote the number of vertices in $G$, and $m$ to denote the number of edges in $G$. As usual, $K_{n}, K_{p, n-p}$, $P_{n}$ and $C_{n}$ denote respectively the complete graph, the complete bipartite, the path and the cycle on $n$ vertices. For graph notation and terminology undefined here, we refer the reader to [24].

## 1. Extensions of two classic inequalities

### 1.1. An extension of Hong's inequality

In 1988, Hong [83] proved that $\lambda(G) \leq \sqrt{2 m-n+1}$ if $G$ is connected. In fact, Hong's inequality holds for each graph without isolated vertices. This was emphasised in [84] later by Hong himself.

Let $s^{+}(G)\left(s^{-}(G)\right)$ be the sum of the squares of the positive (negative) eigenvalues of the adjacency matrix $A(G)$ of $G$. Elphick, Farber, Goldberg and Wocjan [54] proposed the following conjecture:

Conjecture 1 (Elphick-Farber-Goldberg-Wocjan [54]). Let $G$ be a connected graph. Then

$$
\min \left\{s^{+}(G), s^{-}(G)\right\} \geq n-1
$$

From the above conjecture, one can obtain an extension of Hong's inequality, i.e., $s^{+}(G) \leq$ $2 m-n+1$. In fact, the above conjecture has a stronger form that every graph with $\kappa$ components satisfies that $\min \left\{s^{+}(G), s^{-}(G)\right\} \geq n-\kappa$.

Till now, Conjecture 1 was confirmed for special classes of graphs, such as, bipartite graphs, regular graphs, and complete $k$-partite graphs [54], connected graphs with no more
than 4 vertices after blowing up (see [79] for details). A graph $G$ is said to be hyper-energetic if the energy $\mathcal{E}(G)=\sum_{i=1}^{n}\left|\lambda_{i}\right|>2(n-1)$. Conjecture 1 was also confirmed for hyper-energetic graphs [54, Theorem 10]. Since Nikiforov [115] proved that almost all graphs are hyperenergetic, Conjecture 1 is true for almost all graphs. For more families of graphs supporting Conjecture 1 and results on structural properties, we refer to [2].

### 1.2. An extension of Wilf's inequality

Consider a graph $G$ of order $n$ with clique number $\omega$. In 1986, Wilf [146] proved that

$$
\begin{equation*}
\lambda(G) \leq\left(1-\frac{1}{\omega}\right) n . \tag{1}
\end{equation*}
$$

Strengthening Wilf's bound, Elphick and Wocjan [56] proposed a conjecture suggesting that (1) can be improved by substituting $\lambda(G)$ with $\sqrt{s^{+}(G)}$.

Conjecture 2 (Elphick-Wocjan [56]). Let $G$ be a graph of order $n$ with clique number $\omega$. Then

$$
\sqrt{s^{+}(G)} \leq\left(1-\frac{1}{\omega}\right) n
$$

This conjecture is exact, for example, for complete regular multipartite graphs. Based on some numerical experiments, we suspect that equality holds in Conjecture 2 only when $G$ is a complete regular multipartite graph. In [56], Elphick and Wocjan proved this conjecture for various classes of graphs, including triangle-free graphs, and for almost all graphs. They also tested against the thousands of named graphs with up to 40 vertices from the Wolfram Mathematica database, but no counterexamples were found. Using SageMath software we confirmed this conjecture for graphs having at most 10 vertices.

## 2. The Bollobás-Nikiforov Conjecture

The other conjecture regarding to $\lambda_{2}$ is one due to Bollobás and Nikiforov [22], which can be seen as a spectral Turán-type conjecture. The starting point of extremal graph theory is Mantel's theorem, which says that every graph on $n$ vertices contains a triangle if $m>\left\lfloor n^{2} / 4\right\rfloor$. In 1970, Nosal [126] proved that $G$ contains a triangle if $\lambda(G)>\sqrt{m}$. Since Nosal's result can imply Mantel's theorem, it is always called the spectral Mantel Theorem.

Nosal's theorem was generalized by Nikiforov [113] to the inequality

$$
\lambda(G) \leq \sqrt{2\left(1-\frac{1}{\omega(G)}\right) m}
$$

It should be mentioned that this result was implicitly suggested by Edwards and Elphick [51]. The above inequality can imply the classic Turán theorem on cliques, and also Wilf's inequality (1). By introducing $\lambda_{2}(G)$ as a new parameter, Bollobás and Nikiforov [22] proposed a more stronger spectral inequality.

Conjecture 3 (Bollobás-Nikiforov [22]). Let $G$ be a graph on $m$ edges. Then

$$
\lambda_{1}^{2}(G)+\lambda_{2}^{2}(G) \leq 2\left(1-\frac{1}{\omega(G)}\right) m
$$

Till now, Conjecture 3 was confirmed by Lin, Ning and Wu [99] for the case $\omega(G)=2$. They proved that for any triangle-free graph $G$ on $m$ edges and without isolated vertices, if $\lambda_{1}^{2}(G)+\lambda_{2}^{2}(G)=m$ then $G$ is a blow up of some member in $\left\{P_{2}, 2 P_{2}, P_{4}, P_{5}\right\}$. Li, Sun and Yu [95] generalized this result by giving an upper bound of $\lambda_{1}^{2 k}+\lambda_{2}^{2 k}$ for $\left\{C_{2 i+1}\right\}_{i=1}^{k}$-free graphs. Additionally, by the result of Ando and Lin [10], we know that Conjecture 3 holds for weakly perfect graphs, which are graphs with equal clique number and chromatic number.

There is also another conjecture related to Conjecture 3 that strengthens it.
Conjecture 4 (Elphick-Linz-Wocjan [55]). Let $G$ be a graph on $m$ edges. Then

$$
\sum_{i=1}^{\ell} \lambda_{i}^{2}(G) \leq 2\left(1-\frac{1}{\omega(G)}\right) m
$$

where $\ell=\min \left\{n^{+}(G), \omega(G)\right\}$, and $n^{+}(G)$ is the number (counting multiplicities) of positive eigenvalues of $A(G)$.

The conjecture was confirmed by Elphick, Linz and Wocjan [55] for weakly perfect graphs, Kneser graphs, Johnson graphs and two classes of strongly regular graphs.

It is important to note that choosing only $\ell=n^{+}(G)$ would result in a counterexample to Conjecture 4. In fact, consider the cycle $C_{7}$, one can check that $n^{+}\left(C_{7}\right)=3$ and

$$
\lambda_{1}^{2}\left(C_{7}\right)+\lambda_{2}^{2}\left(C_{7}\right)+\lambda_{3}^{2}\left(C_{7}\right)>2\left(1-\frac{1}{\omega\left(C_{7}\right)}\right) m
$$

## 3. Maximum spectral radius of planar (hyper)graphs

Planar graphs have been extensively studied for a long time. Among various topics in spectral graph theory, the investigation of the spectral radius of planar graphs is particularly intriguing. This topic can be traced back at least to Schwenk and Wilson's fundamental question [134]: "What can be said about the eigenvalues of a planar graph?"

In 1988, Hong [83] established the first significant result that for a planar graph $G$, the spectral radius satisfies the inequality $\lambda(G) \leq \sqrt{5 n-11}$, using Hong's inequality as mentioned in Section 1. Subsequently, Cao and Vince [32] improved Hong's bound to $4+\sqrt{3 n-9}$, while Hong [85] himself improved it further to $2 \sqrt{2}+\sqrt{3 n-15 / 2}$, and Ellingham and Zha [53] to $2+\sqrt{2 n-6}$. Additionally, Boots and Royle [25] and independently, Cao and Vince [32], conjectured that $P_{2} \vee P_{n-2}$ attains the maximum spectral radius among all planar graphs on $n \geq 9$ vertices. Recently, significant progress has been made on the conjecture. Tait and Tobin [142] confirmed the conjecture for sufficiently large $n$. It is worth noting that Guiduli announced a proof of the conjecture for large $n$ in his Ph.D. Thesis (see [78] and comments from [142]). However, the conjecture remains open for small values of $n$.

Conjecture 5 (Boots-Royle 1991 [25] and independently Cao-Vince 1993 [32]). Among all planar graph on $n \geq 9$ vertces, $K_{2} \vee P_{n-2}$ attains the maximum spectral radius.

Extending the investigations of graph case, Ellingham, Lu and Wang [52] studied a hypergraph analog of the Cvetković-Rowlinson conjecture which states that among all outerplanar graphs on $n$ vertices, $K_{1} \vee P_{n-1}$ attains the maximum spectral radius. Given a hypergraph $H$, the shadow of $H$, denoted by $\partial(H)$, is the graph $G$ with $V(G)=V(H)$ and $E(G)=\{u v: u v \in e$ for some $e \in E(H)\}$. For a hypergraph $H$, if each edge of $H$ contains precisely $r$ vertices, then $H$ is called $r$-uniform. The spectral radius of an $r$-uniform hypergraph $H$ is defined as

$$
r!\max _{\boldsymbol{x} \in \mathbb{R}^{n},\|\boldsymbol{x}\|_{r}=1} \sum_{\left\{i_{1}, \ldots, i_{r}\right\} \in E(H)} x_{i_{1}} x_{i_{2}} \cdots x_{i_{r}}
$$

where $\mathbb{R}^{n}$ is the set of real vectors of dimension $n$ and $\|\boldsymbol{x}\|_{r}$ is the $\ell^{r}$-norm of $\boldsymbol{x}$.
A 3-uniform hypergraph $H$ is called outerplanar (planar) if its shadow has an outerplanar (planar) embedding such that each edge of $H$ is the vertex set of an interior triangular face of the shadow. In [52], Ellingham, Lu and Wang proved that for sufficiently large $n$, the $n$ vertex outerplanar 3 -uniform hypergraph of maximum spectral radius is the unique 3 -uniform hypergraph whose shadow is $K_{1} \vee P_{n-1}$. In particular, they proposed a conjecture that serves as a hypergraph counterpart to Conjecture 5 .

Conjecture 6 (Ellingham-Lu-Wang [52]). For large enough n, the n-vertex planar 3-uniform hypergraph of maximum spectral radius is the unique hypergraph whose shadow is $K_{2} \vee P_{n-2}$.

## 4. The relationship between Turán theorem and spectral Turán theorem

A graph is said to have the Hereditarily Bounded Property $P_{t, r}$ if $|E(H)| \leq t \cdot|V(H)|+r$ for any subgraph $H$ of $G$ with $|V(H)| \geq t$. In Guiduli's Ph.D. Thesis [78], he proved a tight upper bound on $\lambda(G)$ for graphs with property $P_{t, r}$.
Theorem 1 ([78]). Let $t \in \mathbb{N}$ and $r \geq-\binom{t+1}{2}$. If $G$ is a graph on $n$ vertices with property $P_{t, r}$, then

$$
\lambda(G) \leq \sqrt{t n}+\sqrt{t(t+1)+2 r}+\frac{t-1}{2}
$$

and asymptotically,

$$
\lambda(G) \leq \sqrt{t n}+\frac{t-1}{2}+o(1) .
$$

Furthermore, the asymptotic bound is tight.
A natural question is to ask for a generalization of this property $P_{t, r}$, where the hereditary bound on the number of edges is not linear.

Problem 1 (Guiduli [78]). If for a graph $G,|E(H)| \leq c \cdot|V(H)|^{2}$ holds for all subgraph $H$ of $G$, does it follows that $\lambda(G) \leq 2 c \cdot|V(G)|$ ? What can be said if the exponent of 2 is replaced by some other constant less than 2 ?

It is worth noting that $\lambda(G) \leq \sqrt{2 c} \cdot|V(G)|$ is a trivial bound from the well-known inequality $\lambda(G) \leq \sqrt{2|E(G)|}$. Moreover, the constant $2 c$ would be best possible, as seen by Wilf's inequality. If this problem were true, then the spectral Erdős-Stone-Simonovits theorem [78, 118] would be a consequence of the Erdős-Stone-Simonovits theorem [67, 66], and Wilf's inequality (1) would follow from Turán's theorem [143].

## 5. Tight spectral conditions for a cycle of given length

In spectral graph theory, it is very natural to ask the following problem: Determine tight spectral radius conditions for the existence of a cycle of length $\ell$ in a graph of order $n$ for $\ell \in[3, n]$. This problem has two aspects, i.e., the case of short cycles and the case of long cycles.

For any given graph $H$, denote by $\operatorname{spex}(n, H)$ the maximum spectral radius of $n$-vertex graph $G$ containing no the subgraph $H$, and by $\operatorname{SPEX}(n, H)$ the class of extremal graphs $G$ such that $\lambda(G)=\operatorname{spex}(n, H)$. For example, when $\ell=3$, from Nosal's theorem, one can obtain $\operatorname{spex}\left(n, C_{3}\right)=\sqrt{\left\lfloor n^{2} / 4\right\rfloor}$. When $n$ is odd, Nikiforov [114] showed $\operatorname{SPEX}\left(n, C_{4}\right)=\left\{K_{1} \vee\right.$ $\left.\left(\frac{n-1}{2} K_{2}\right)\right\}$; For the case $n$ is even, confirming a conjecture in [117], Zhai and Wang [150] showed that $\operatorname{SPEX}\left(n, C_{4}\right)=\left\{K_{1} \vee\left(K_{1} \cup \frac{n-2}{2} K_{2}\right)\right\}$. In 2010, Nikiforov [119] proposed the following conjecture: Every graph on sufficiently large order $n$ contains a $C_{2 k+2}$ if $\lambda(G) \geq \lambda\left(S_{n, k}^{+}\right)$, unless $G=S_{n, k}^{+}$where $k \geq 2$ and $S_{n, k}^{+}$is obtained from $S_{n, k}$ by adding an edge. Zhai and Lin [147] confirmed this conjecture for $k=2$. Very recently, Cioabă, Desai and Tait [39] announced a complete proof of Nikiforov's conjecture. However, the problem of determining tight spectral conditions for cycles of given length $\ell \in[3, n]$ is still wide open. A refined version of Nikiforov's conjecture can be found in [94].

Problem 2 (A refined version of Nikiforov's even cycle conjecture [94]). For any integer $k \geq 3$, determine the infimum $\alpha:=\alpha(k)$ such that every graph of order $n=\Omega\left(k^{\alpha}\right)$ (where $\Omega\left(k^{\alpha}\right)$ means there exists some constant $c$ which is not related to $k$ and $n$, such that $n \geq c k^{\alpha}$ ) satisfying $\lambda(G)>\lambda\left(S_{n, k}^{+}\right)$contains a $C_{2 k+2}$.

Recently, Zhai, Lin, and Shu [149] investigated the existence of short consecutive cycles in fixed-size graphs and put forward the following conjecture.

Conjecture 7 (Zhai-Lin-Shu [149]). Let $k$ be a fixed positive integer and $G$ be a graph of sufficiently large size $m$ without isolated vertices. If

$$
\lambda(G) \geq \frac{k-1+\sqrt{4 m-k^{2}+1}}{2}
$$

then $G$ contains a cycle of length $t$ for every $t \leq 2 k+2$, unless $G \cong S_{m / k+(k+1) / 2, k}$.
For the case $k=2$, the conjecture has been confirmed, see [149], [108] and [141] for further details.

## 6. Nikiforov's problem on consecutive cycles

A Hamilton cycle in a graph $G$ is a cycle passing through all the vertices of $G$. If it exists, then $G$ is called Hamiltonian. Maybe the most famous theorem in Hamiltonian graph theory is Dirac's theorem [49], which states that every graph on $n \geq 3$ vertices has a Hamilton cycle if every vertex has degree at least $n / 2$. In 1971, Bondy [23] introduced the concept of pancyclicity of graphs. Let $G$ be a graph on $n$ vertices. We say that $G$ is pancyclic, if $G$ contains each cycle of length $\ell$, where $\ell \in[3, n]$. Extending Ore's condition [128], Bondy [23] proved that every Hamiltonian graph on $n$ vertices is pancyclic if $e(G) \geq n^{2} / 4$, unless $G$ is isomorphic to $K_{n / 2, n / 2}$ where $n$ is even. If one drops the condition that $G$ is Hamiltonian in Bondy's theorem, the phenomenon of consecutive cycles still persists, i.e, there are all cycles of length $\ell \in[3,\lfloor(n+3) / 2\rfloor]$ in a graph $G$ if $e(G) \geq n^{2} / 4$. This theorem may be due to Woodall and independently, due to Kopylov (see also Bollobás [20, Corollary 5.4]).

In 2008, Nikiforov [116] considered a spectral analog of the above theorem.
Problem 3 (Nikiforov [116]). What is the maximum $C$ such that for all positive $\varepsilon<C$ and sufficiently large $n$, every graph $G$ of order $n$ with $\lambda(G)>\sqrt{\left\lfloor n^{2} / 4\right\rfloor}$ contains a cycle of length $\ell$ for every integer $\ell \leq(C-\varepsilon) n$.

Nikiforov [116] firstly showed $C \geq 1 / 320$ by the method of successively deleting the least component of eigenvector with respect to spectral radius of a graph. Ning and Peng [122] improved it to $C \geq 1 / 160$. Later, Zhai and Lin [148] proved some spectral result for theta graphs, and a direct main corollary is that $C \geq 1 / 7$. At the same time, Li and Ning [93] showed that $C \geq 1 / 4$, by using some ideas from Ramsey Theory [6] and theorems on parity of cycles in graphs [144]. Li and Ning's result was immediately used in [153] to attack another problem in spectra graph theory. On the other hand, Nikiforov [116] constructed the class of graphs $G=K_{s} \vee(n-s) K_{1}$ where $s=(3-\sqrt{5}) n / 4$ from which one can find $C \leq(3-\sqrt{5}) / 2$. Till now, Problem 3 is still open.

## 7. Graph toughness from Laplacian eigenvalues

Let $c(G)$ denote the number of components of a graph $G$. The toughness $t(G)$ of $G$ is defined by

$$
t(G):=\min \left\{\frac{|S|}{c(G-s)}\right\},
$$

in which the minimum is taken over all proper subsets $S \subset V(G)$ such that $c(G-S)>1$. A graph $G$ is called $t$-tough if $t(G) \geq t$.

In 1973, Chvátal [35] introduced the concept of graph toughness, which has close connections to a variety of graph properties such as connectivity, Hamiltonicity, pancyclicity, factors, and spanning trees (see [15]). The study of toughness from eigenvalues was initiated by Alon [7], who showed that for any connected $d$-regular graph $G$,

$$
t(G)>\frac{1}{3}\left(\frac{d^{2}}{\left(d+\lambda^{\prime}\right) \lambda^{\prime}}-1\right),
$$

where $\lambda^{\prime}$ is the second largest absolute eigenvalue. Brouwer [29] discovered a better bound and showed that $t(G)>d / \lambda^{\prime}-2$ for a connected $d$-regular graph $G$. He mentioned in [29] that the bound might be able to be improved to $d / \lambda^{\prime}-1$ and then proposed the exact conjecture as an open problem in [29, 30]. In 2021, the conjecture has been proved by Gu [76].

Recently, Haemers [80] proposed studying lower bounds on $t(G)$ in terms of the eigenvalues of the Laplacian matrix $L(G)$. He also made the following conjecture.

Conjecture 8 (Haemers [80]). Let $G$ be a connected graph on $n$ vertex with minimum degree $\delta$. Then

$$
t(G) \geq \frac{\mu_{n-1}}{\mu_{1}-\delta}
$$

For a connected $d$-regular graph $G$, this conjecture implies that $t(G) \geq \frac{d-\lambda_{2}}{-\lambda_{n}}$, which is stronger than Brouwer's toughness conjecture. The bound in Conjecture 8 is tight in case $G$ is the complete multipartite graph $K_{n_{1}, \ldots, n_{k}}(1<k<n)$. Indeed, assume $n_{1} \geq n_{2} \geq \cdots \geq n_{k}$ then $t(G)=\left(n-n_{1}\right) / n_{1}, \mu_{1}=n$ and $\mu_{n-1}=\delta=n-n_{1}$.

Let $S \subset V(G)$ be such that $t(G)=|S| / c(G-S)$. It was proved in [80] and [77] that Conjecture 8 is true in each of the following cases:
(1) The complement of $G$ is disconnected;
(2) All connected components of $G-S$ are singletons;
(3) The union of some components of $G-S$ has order $(n-|S|) / 2$;
(4) $c(G-S)=2$.

In [77], two tight lower bounds for $t(G)$ in terms of the Laplacian eigenvalues were presented, which provided support for Conjecture 8.

## 8. Hamilton cycles in pseudo-random graphs

Finding general conditions which ensure that a graph is Hamiltonian is a central topic in graph theory, and researchers have devoted many efforts to obtain sufficient conditions for Hamiltonicity.

There is an old and well-known conjecture related to pseudo-random graphs in this area. A pseudo-random graph with $n$ vertices of edge density $p$ is a graph that behaves like a truly random graph $G(n, p)$. Spectral techniques are a convenient way to express pseudorandomness. An $\left(n, d, \lambda^{\prime}\right)$-graph is a $d$-regular graph $G$ on $n$ vertices whose second largest eigenvalue in absolute value is at most $\lambda^{\prime}$. It is known that $\left(n, d, \lambda^{\prime}\right)$-graphs with small $\lambda^{\prime}$ compared to $d$ possess pseudo-random properties. For more details on pseudo-random graphs, we refer the reader to [92]. In this area, a well-known conjecture on Hamilton cycles in an ( $n, d, \lambda^{\prime}$ )-graph can be found in [91].

Conjecture 9 (Krivelevich-Sudakov [91]). There exists an absolute constant $C>0$ such that any ( $n, d, \lambda^{\prime}$ )-graph with $d / \lambda^{\prime} \geq C$ contains a Hamilton cycle.

In [91], Krivelevich and Sudakov proved a result that ( $n, d, \lambda^{\prime}$ )-graphs are Hamiltonian, provided

$$
\frac{d}{\lambda^{\prime}} \geq \frac{1000 \cdot \log n(\log \log \log n)}{(\log \log n)^{2}}
$$

for sufficiently large $n$. In recent work by Glock, Correia and Sudakov [74], progress has been made towards Conjecture 9 in two significant ways. Firstly, they improved the Krivelevich and Sudakov's bound above by showing that for some constant $C>0, d / \lambda^{\prime} \geq C \cdot(\log n)^{1 / 3}$ already guarantees Hamiltonicity. Secondly, they established that for any constant $\alpha>0$, there exists a constant $C>0$ such that any ( $n, d, \lambda^{\prime}$ )-graph with $d \geq n^{\alpha}$ and $d / \lambda^{\prime} \geq C$ contains a Hamilton cycle.

Let us remark that there exist three additional conjectures that are related to Conjecture 9. The first one has to do with the concept of $f$-connected which is a generalization of the traditional notion of connectedness. For a graph $G$, a pair $(A, B)$ of proper subsets of $V(G)$ is called a separation of $G$ if $A \cup B=V(G)$ and $G$ has no edge between $A \backslash B$ and $B \backslash A$. Let $f: \mathbb{N} \backslash\{0\} \rightarrow \mathbb{R}$ be a function, $G$ is called $f$-connected if every separation $(A, B)$ of $G$ with $|A \backslash B| \leq|B \backslash A|$ satisfies $|A \cap B| \geq f(|A \backslash B|)$. In 2006, Brandt, Broersma, Diestel and Kriesell [28] conjectured that there exists a function $f(k)=O(k)$ (here $f(k)=O(k)$ means there exists some absolute constant $c>0$ such that $f(k)<c k$ for large $k$ ) such that every $f$-connected graph of order $n \geq 3$ is Hamiltonian. As asserted in [28], if this conjecture were true, it would imply Conjecture 9.

The second one is related to Chvátal's toughness conjecture [35]. In 1973, Chvátal [35] conjectured that there is a constant $t$ such that every $t$-tough graph is Hamiltonian. In [28], the authors demonstrated that the Brandt-Broersma-Diestel-Kriesell's conjecture [28] can be derived from Chvátal's toughness conjecture. Therefore, if Chvátal's conjecture were proven to be true, it would imply the validity of Conjecture 9 .

The third conjecture relates to the Laplacian eigenvalues of graphs. Gu (c.f. [77]) conjectured that there exists a positive constant $C<1$ such that if $\mu_{n-1} / \mu_{1} \geq C$ and $n \geq 3$, then $G$ contains a Hamilton cycle. It is evident that for a $d$-regular graph $G$, we have $\mu_{n-i+1}(G)=d-\lambda_{i}(G)$ for $i=1,2, \ldots, n$. Moreover, it can be observed that Gu's conjecture above also implies Conjecture 9 .

## 9. A spectral problem on counting subgraphs

Mantel's theorem, a well-known result, states that a graph with $n$ vertices and $\left\lfloor n^{2} / 4\right\rfloor+1$ edges contains a triangle. Strengthening Mantel's theorem, Rademacher (c.f. [57]) proved that there are at least $\lfloor n / 2\rfloor$ copies of a triangle. Erdős [58, 59] further generalized the result by proving that if $q<c n$ for some small constant $c$, then $\left\lfloor n^{2} / 4\right\rfloor+q$ edges guarantees at least $q\lfloor n / 2\rfloor$ triangles. He also conjectured that the same result holds for $q<n / 2$, which was later proved by Lovász and Simonovits [105].

In 2010, Mubayi [112] extended these theorems to color-critical graphs, which are graphs whose chromatic number can be decreased by removing some edges. Let $T_{n, k}$ denote the

Turán graph on $n$ vertices, which is the complete $k$-partite graph with parts of size $\lfloor n / k\rfloor$ or $\lceil n / k\rceil$.

Theorem 2 ([112]). Let $k \geq 2$ and $F$ be a color-critical graph with chromatic number $\chi(F)=$ $k+1$. There exists $\delta=\delta_{F}>0$ such that if $n$ is sufficiently large and $1 \leq q<\delta n$, then every $n$-vertex graph with $e\left(T_{n, k}\right)+q$ edges contains at least $q \cdot c(n, F)$ copies of $F$, where $c(n, F)$ is the minimum number of copies of $F$ in the graph obtained from $T_{n, k}$ by adding one edge.

Motivated by Mubayi's result above, Ning and Zhai [124] proposed to study the spectral analog of Mubayi's theorem.

Problem 4 (Ning-Zhai [124]). (i) (The general case) Find a spectral version of Mubayi's result.
(ii) (The critical case) For $q=1$ (where $q$ is defined as in Mubayi's theorem), find the tight spectral versions of Mubayi's result when $F$ is some particular color-critical subgraph, such as triangle, clique, book, odd cycle or even wheel, etc.

In [124], Ning and Zhai studied the fundamental cases of triangles for Problem 4. In [123], they also studied some special bipartite case, i.e., the quadrilaterals case.

## 10. Extreme eigenvalues of nonregular graphs

Regular graphs are a well-studied class of graphs, but for nonregular graphs where not all vertices have equal degrees, it is possible to quantify how close they are to regularity using various measures. One such measure is the difference between the maximum degree $\Delta(G)$ and the largest eigenvalue $\lambda(G)$ of a graph $G$. It is a well-known fact that $\lambda(G) \leq \Delta(G)$ for any connected graph $G$, and equality holds if and only if $G$ is regular. Therefore, we can use the difference $\Delta(G)-\lambda(G)$ as a measure of irregularity of a nonregular graph $G$. It is natural to ask how small this difference can be for nonregular graphs. This question has attracted the interest of many researchers in the past few decades [36, 40, 121, 136, 137, 140, 152, 151].

Let $\mathcal{G}(n, \Delta)$ denote the set of graphs attaining the maximum spectral radius among all connected nonregular graphs with $n$ vertices and maximum degree $\Delta$, and let $\lambda(n, \Delta)$ denote the maximum spectral radius. For a graph $G \in \mathcal{G}(n, \Delta)$, Liu, Shen and Wang [102] investigated the order of magnitude of $\Delta-\lambda(G)$, and posed the following conjecture: For each fixed $\Delta$ and $G \in \mathcal{G}(n, \Delta)$, the limit of $n^{2}(\Delta-\lambda(G)) /(\Delta-1)$ exists. Furthermore,

$$
\lim _{n \rightarrow \infty} \frac{n^{2}(\Delta-\lambda(G))}{\Delta-1}=\pi^{2} .
$$

This conjecture is trivially true for $\Delta=2$, and the condition that $\Delta$ is fixed is crucial (see [103] for details). Recently, the first author of this paper gave a negative answer to the above conjecture by showing that the limit superior is at most $\pi^{2} / 2$ (see [103]).

Although Liu-Shen-Wang's conjecture is not true, we can still ask what is the exact leading term of $\Delta-\lambda(n, \Delta)$. Based on some numerical experiments and heuristic arguments, the following conjecture was presented.

Conjecture 10 (Liu [103]). Let $\Delta \geq 3$ be a fixed integer and $G \in \mathcal{G}(n, \Delta)$. Then the limit of $n^{2}(\Delta-\lambda(G))$ always exists. Furthermore,
(1) if $\Delta$ is odd, then

$$
\lim _{n \rightarrow \infty} \frac{n^{2}(\Delta-\lambda(G))}{\Delta-1}=\frac{\pi^{2}}{4}
$$

(2) if $\Delta$ is even, then

$$
\lim _{n \rightarrow \infty} \frac{n^{2}(\Delta-\lambda(G))}{\Delta-2}=\frac{\pi^{2}}{2} .
$$

For $\Delta=3$ and $\Delta=4$, the conjecture was confirmed by Liu [103]. However, for the general $\Delta$, it seems to be difficult to solve it.

On the other hand, it is intuitive that the graphs attaining the maximum spectral radius among all connected nonregular graphs with prescribed maximum degree must be close to regular graphs. In particular, Liu and $\mathrm{Li}[101]$ posed the following conjecture: Let $3 \leq \Delta \leq$ $n-2$ and $G \in \mathcal{G}(n, \Delta)$. Then $G$ has degree sequence $(\Delta, \ldots, \Delta, \delta)$, where

$$
\delta= \begin{cases}\Delta-1, & n \Delta \text { is odd } \\ \Delta-2, & n \Delta \text { is even. }\end{cases}
$$

In spite of the statement of the conjecture above seems intuitive, it is challenging to either prove or disprove it, even for small values of $\Delta$. One reason for the difficulty of this conjecture, as well as Liu-Shen-Wang's conjecture, is that graphs with bounded degree are sparse graphs whose spectral radius is bounded by a constant. Therefore, many tools from spectral graph theory are not effective.

By analyzing the structural properties of the extremal graphs, Liu [103] confirmed Liu-Li's conjecture above for small $\Delta$. However, we cannot expect an affirmative answer to Liu-Li's conjecture for general $\Delta$. In fact, there is some evidence to support the following speculation.

Conjecture 11 (Liu [103]). Let $G \in \mathcal{G}(n, \Delta)$. For each fixed $\Delta$ and sufficiently large $n, G$ has degree sequence $(\Delta, \ldots, \Delta, \delta)$, where

$$
\delta= \begin{cases}\Delta-1, & \Delta \text { is odd, } n \text { is odd } \\ 1, & \Delta \text { is odd, } n \text { is even }, \\ \Delta-2, & \Delta \text { is even }\end{cases}
$$

Liu confirmed Conjecture 11 for $\Delta=3$ and $\Delta=4$ in [103].

## 11. Graphs with a prescribed average degree

While a significant amount of research has focused on finding the maximum spectral radius of graphs under specified conditions, there has been relatively less work done on determining the minimum spectral radius.

For given $n$ and $m$, let $\mathcal{H}(n, m)$ denote the set of connected graphs on $n$ vertices and $m$ edges. In this section, we talk about the graphs in $\mathcal{H}(n, m)$ with minimum spectral radius. It is a challenging task to identify the exact graph. Nevertheless, the following two questions which, if true, would provide significant structural insights into the extremal graphs.

In 1993, Hong [84] posed a problem concerning the maximum degree and minimum degree of extremal graphs.

Problem 5 (Hong [84]). Let $G \in \mathcal{H}(n, m)$ be a connected graph minimizing $\lambda(G)-\bar{d}(G)$. Is it true that $\Delta(G)-\delta(G) \leq 1$ ?

Obviously, Problem 5 is true for $m=n-1$ (see [43, 104]); for $m=n$ (the unique extremal graph is clearly $C_{n}$ ); for $m=n+1$ (see [138]). In [38], Cioabă claimed that Problem 5 is also true when $2 m / n$ is an integer. Indeed, If $\bar{d}:=2 m / n$ is an integer, then it is always possible to construct a $\bar{d}$-regular graph which clearly minimizes the quantity $\lambda(G)-\bar{d}(G)$.

In [78], Guiduli proposed a conjecture [78, Conjecture 5.8] which provides a different perspective on the extremal graphs. It is worth mentioning that the original conjecture of Guiduli is not true for small $m$, the following is a modified version of Guiduli's conjecture.

Conjecture 12 (A modified version of Guiduli's conjecture). Let $G \in \mathcal{H}(n, m)$ be a connected graph having minimum spectral radius. Let $r$ be such that $e\left(T_{n, r-1}\right)<m \leq e\left(T_{n, r}\right)$. Then there is a constant $\alpha>0$ such that $G$ is $r$-colorable for $m=\Omega\left(n^{\alpha}\right)$.

## 12. The Bilu-Linial Conjecture

A signed graph $\Gamma=(G, \sigma)$ is a graph $G=(V, E)$ along with a function $\sigma: E \rightarrow\{+1,-1\}$ that assigns a positive or negative sign to each edge. The (unsigned) graph $G$ is said to be the underlying graph of $\Gamma$, while the function $\sigma$ is referred to as the signature of $\Gamma$. The adjacency matrix $A(\Gamma)$ of a signed graph $\Gamma$ is derived from the adjacency matrix of its underlying graph $G$ by replacing every 1 with -1 if the corresponding edge in $\Gamma$ is negative. An important feature of signed graphs is the concept of switching equivalent. Given a signed graph $\Gamma=(G, \sigma)$ and a subset $U \subseteq V(G)$, let $\Gamma_{U}$ be the signed graph obtained from $\Gamma$ by reversing the signs of the edges in the edge boundary of $U$, the set of edges joining a vertex in $U$ to one not in $U$. The signed graph $\Gamma_{U}$ is said to be switching equivalent to $\Gamma$. When we talk about the eigenvalues and spectral radius of a signed graph, we are actually referring to those of the corresponding signed adjacency matrix.

It was proved in [17] that for a signed graph $\Gamma=(G, \sigma)$, the spectral radius $\rho(G, \sigma)$ of $\Gamma$ is at most that of $G$. So, we know that, up to switching equivalence, the signature leading to the maximal spectral radius is the all-positive one. A natural question is to identify which signature leads to the minimum spectral radius. This problem has important connections and consequences in the theory of expander graphs [18].

Bilu and Linial [18] proved that every regular graph has a signature with small spectral radius.

Theorem 3 ([18]). Every connected d-regular graph has a signature with spectral radius at most $c \sqrt{d \cdot(\log d)^{3}}$, where $c>0$ is some absolute constant.

Furthermore, they posed the following conjecture.
Conjecture 13 (Bilu-Linial [18]). Every connected d-regular graph $G$ has a signature $\sigma$ with spectral radius at most $2 \sqrt{d-1}$.

If true, this conjecture would provide a way to construct or show the existence of an infinite family of Ramanujan graphs, a connected $d$-regular graph with $\max \left\{\left|\lambda_{2}\right|,\left|\lambda_{n}\right|\right\} \leq 2 \sqrt{d-1}$. In 2015, Marcus, Spielman and Srivastava [106] made significant progress towards solving the Bilu-Linial conjecture.

Theorem 4 ([106]). Let $G$ be a connected d-regular graph. Then there exists a signature $\sigma$ of $G$ such that the largest eigenvalue of $\Gamma=(G, \sigma)$ is at most $2 \sqrt{d-1}$.

Remark 1. In general, the largest eigenvalue $\lambda_{1}(\Gamma)$ of $\Gamma$ may not be equal to its spectral radius because the Perron-Frobenius Theorem is valid only for nonnegative matrices. To put it simply, the Bilu-Linial conjecture aims to limit all eigenvalues of $\Gamma=(G, \sigma)$ between $-2 \sqrt{d-1}$ and $2 \sqrt{d-1}$, while the Marcus-Spielman-Srivastava result shows the existence of a signature where all the eigenvalues of $\Gamma=(G, \sigma)$ are at most $2 \sqrt{d-1}$.

If the regularity assumption on $G$ is dropped, Gregory ${ }^{1}$ considered the following variant of Conjecture 13.

Conjecture 14 (Gregory, c.f. [17]). Let $G$ be a nontrivial graph with maximum degree $\Delta$. Then there exists a signature $\sigma$ such that $\rho(G, \sigma)<2 \sqrt{\Delta-1}$.

Furthermore, Belardo, Cioabă, Koolen and Wang [17] posed the following question whose affirmative answer would imply Conjecture 14.

Problem 6 (Belardo-Cioabă-Koolen-Wang [17]). Let $G$ be a connected graph. Is there a signature $\sigma$ such that $\rho(G, \sigma)<2 \sqrt{\rho(G)-1}$ ?

## 13. Isoperimetric problem in hypercube

The $d$-dimensional hypercube, denoted by $Q_{d}$, is a $d$-regular graph on $2^{d}$ vertices, with each vertex corresponding to a binary string of length $d$. The adjacency between two vertices in $Q_{d}$ occurs if and only if they differ in exactly one bit position. Thus, each vertex is connected to $d$ other vertices, which correspond to the vertices obtained by flipping each of its bits in turn.

In [21], Bollobás, Lee and Letzter studied the maximum eigenvalue of subgraphs of the hypercube $Q_{d}$. To be precise, they gave a partial answer to the following question posed by Fink (c.f. [21]) and by Friedman and Tillich [70].

[^1]Problem 7 (Fink, c.f. [21], Friedman-Tillich [70]). What is the maximum of the largest eigenvalue of $Q_{d}[U]$, where $|U|=m$ and $1 \leq m \leq 2^{d}$ ?

Bounding the maximum eigenvalue of $Q_{d}[U]$ is closely related to the size of the edge boundary of $U$. Indeed, since the largest eigenvalue of a graph is at least its average degree, for $Q_{d}[U]$, an induced subgraph of $Q_{d}$, we have $e\left(Q_{d}[U]\right) \leq \lambda\left(Q_{d}[U]\right) \cdot|U| / 2$. Since $Q_{d}$ is $d$-regular, the edge boundary of $U$ is at least $\left(d-\lambda\left(Q_{d}[U]\right)\right) \cdot|U|$. Hence, if we denote the maximum of the largest eigenvalue of $Q_{d}[U]$ with $|U|=m$ by $\lambda(m)$, then for every set of $m$ vertices of the hypercube $Q_{d}$, the edge boundary has size at least $(d-\lambda(m)) m$. In this sense, Problem 7 can be viewed as a variant of the isoperimetric problem in hypercube.

In [21], several theorems were proved regarding this problem, but there are still many open problems.

## 14. Minimum spectral radius of $\boldsymbol{K}_{r+1^{-}}$-saturated graphs

A graph $G$ is $F$-saturated if $G$ does not contain $F$ as a subgraph but the addition of any new edge to $G$ creates at least one copy of $F$. In other words, $G$ is $F$-saturated if and only if it is a maximal $F$-free graph. The maximum possible number of edges in a graph $G$ that is $F$-saturated is known as the Turán number of $F$. The study of Turán numbers for various families of graphs is a cornerstone of extremal combinatorics.

On the other hand, the minimum number of edges in an $F$-saturated graph with $n$ vertices, denoted by $\operatorname{sat}(n, F)$, is called the saturation number of $F$. Saturation numbers were first studied by Erdős, Hajnal and Moon [65]. They determined the saturation number of $K_{r+1}$ and characterized the unique extremal graph, which is $S_{n, r-1}$. For a thorough account of the results known about saturation numbers, we refer the reader to a nice dynamic survey [46].

Similarly to the spectral Turán-type problems for clique, one can naturally ask whether the spectral radius version of the Erdős-Hajnal-Moon theorem is true. In 2020, Kim, Kim, Kostochka and O [89] made a first progress on this problem, and posed the following conjecture.

Conjecture 15 (Kim-Kim-Kostochka-O [89]). Let $G$ be a $K_{r+1}$-saturated graph on $n$ vertices. Then $\lambda(G) \geq \lambda\left(S_{n, r-1}\right)$, with equality if and only if $G \cong S_{n, r-1}$.

For the cases $r=2$ and $r=3$, the conjecture was confirmed in [89] and [90] respectively. Generally, it would be interesting to consider the following problem.

Problem 8. Given a graph $F$, what is the minimum spectral radius of an $F$-saturated graph on $n$ vertices?

## 15. Brouwer's Laplacian spectrum conjecture

For a graph $G$ on $n$ vertices and $1 \leq k \leq n$, let $S_{k}(G)$ denote the sum of the $k$ largest Laplacian eigenvalues of $G$, that is,

$$
S_{k}(G):=\sum_{i=1}^{k} \mu_{i}(G) .
$$

As a variation of the Grone-Merris theorem [12], Brouwer [31] proposed the following conjecture.

Conjecture 16 (Brouwer's conjecture [31]). Let $G$ be a graph of order n. Then

$$
S_{k}(G) \leq e(G)+\binom{k+1}{2}, \quad k=1,2, \ldots, n .
$$

The progress made on Brouwer's conjecture is worth mentioning. Brouwer himself used computers to verify the conjecture for all graphs with at most 10 vertices [31]. For $k=1$, the conjecture follows from the well-known inequality $\mu_{1}(G) \leq n$. Haemers, Mohammadian and Tayfeh-Rezaie [81] showed that the conjecture is true for all graphs when $k=2$, and recently the equality was characterized by Li and Guo [96]. The cases $k=n-1$ and $k=n$ are straightforward due to the fact that $S_{n-1}(G)=S_{n}(G)=2 e(G) \leq e(G)+\binom{n}{2}$.

Chen [34] showed that if Conjecture 16 holds for all graphs when $k=p$, then it holds for all graphs when $k=n-p-1$ as well, where $1 \leq p \leq(n-1) / 2$. Thus, Conjecture 16 also holds for all graphs when $k=n-2$ and $k=n-3$. Rocha and Trevisan [132] proved that the conjecture is true for all $k$ with $1 \leq k \leq\lfloor g / 5\rfloor$, where $g$ is the girth of the graph $G$ (the length of the smallest cycle in $G$ ). They also showed that the conjecture is true for a connected graph $G$ having maximum degree $\Delta, p$ pendant vertices and $c$ cycles with $\Delta \geq c+p+4$.

In addition, it has been proved that Brouwer's conjecture is true for several classes of graphs (for all $k$ ) such as trees [81], unicyclic graphs [50, 145], bicyclic graphs [50], threshold graphs [81], regular graphs [107] and split graphs [107]. For more progress on Brouwer's Conjecture, we refer to $[19,33,44,72,71,73,131]$ and the references therein. However, Conjecture 16 remains open at large.

Recently, Li and Guo [96] proposed the following full Brouwer's conjecture. Before continuing, we introduce some notation. For $1 \leq k \leq n-1$, let $G_{k, r, s}(r \geq 1, s \geq 0)$ be a graph of order $n=k+r+s$ consisting of a clique of size $k$ and two independent sets $\bar{K}_{r}$ and $\bar{K}_{s}$, where each vertex of $K_{k}$ is adjacent to all vertices in $\bar{K}_{r}$, and for each vertex $v_{i} \in V\left(\bar{K}_{s}\right)=\left\{v_{1}, v_{2}, \ldots, v_{s}\right\}$, $N\left(v_{i}\right) \subsetneq V\left(K_{k}\right)(i=1,2, \ldots, s)$ and $N\left(v_{i+1}\right) \subseteq N\left(v_{i}\right)(i=1,2, \ldots, s-1)$. Obviously, if $G_{k, r, s}$ is connected, then for $k=1, G_{1, r, s}$ is the star $K_{1, n-1}$; for $k=n-1, G_{n-1, r, s}$ is the complete graph $K_{n}$.

Conjecture 17 (Li-Guo [96], The full Brouwer's conjecture). Let $G$ be a graph of order $n$. Then

$$
S_{k}(G) \leq e(G)+\binom{k+1}{2}, \quad k=1,2, \ldots, n
$$

with equality if and only if $G \cong G_{k, r, s}(r \geq 1, s \geq 0)$.
In [96], the authors confirmed Conjecture 17 for $k \in\{1,2, n-3, n-2, n-1\}$.

## 16. The Spectral Gap conjecture

Given a graph $G$, the spectral gap of $G$ is defined as $\lambda_{1}(G)-\lambda_{2}(G)$. Obviously, if $G$ is connected, then $\lambda_{1}(G)-\lambda_{2}(G)>0$. The spectral gap is primarily investigated for the
class of regular graphs, as it is established that regular graphs with large spectral gap possess high connectivity properties. This property renders them significant in numerous branches of theoretical computer science (see [47, pp. 392-394]).

To the contrary, Stanić [139] suggested studying graphs with small spectral gap, which can be viewed as an adjacency matrix version of Aldous and Fill's problem about maximizing the relaxation time of a random walk on a connected graph (see [4] and [5] for details). In particular, Stanić conjectured that the minimum spectral gap is attained for the double kite graphs. A double kite graph $D K(r, s)$ can be constructed by taking a $(s+2)$-vertex path $P_{s+2}$, two copies of the $r$-vertex complete graph $K_{r}$, and identifying one terminal vertex of $P_{s+2}$ with a vertex of one copy of $K_{r}$ and the other terminal vertex with a vertex of the other copy of $K_{r}$ (see Fig. 1 for an illustration).


Figure 1: The double kite graph $D K(8,5)$

Conjecture 18 (Stanić [139]). The spectral gap is minimized for some double kite graph over all connected graphs with given number of vertices.

The conjecture has been confirmed by Stanić [139] for connected graphs with up to 10 vertices.

The spectral gap and the algebraic connectivity of graphs exhibit certain similarities. Specifically, for regular graphs, the algebraic connectivity coincides with the spectral gap, and connected regular graphs of degree 3 and 4 with minimum algebraic connectivity (and therefore, minimum spectral gap) are determined in [26] and [1] respectively.

If we restrict our study on trees, Jovović, Koledin and Stanić [87] conjectured that the spectral gap is minimized for some double comet among all trees. The double comet $C(k, \ell)$ is a tree obtained by attaching $k$ pendant vertices at one end of the path $P_{\ell}$ and $k$ pendant vertices at the other end of the same path.

Conjecture 19 (Jovović-Koledin-Stanić [87]). Among all trees of order $n$, the spectral gap is minimized for some double comet.

The conjecture is confirmed by computer search for trees with at most 20 vertices [87]. There exists a unique tree that achieves the minimum spectral gap in all cases, and the corresponding trees are listed in the following table.

| $n$ | $n \leq 8$ | $9 \leq n \leq 11$ | $12 \leq n \leq 15$ | $16 \leq n \leq 20$ |
| :---: | :---: | :---: | :---: | :---: |
| The unique tree | $C(1, n-2)\left(\cong P_{n}\right)$ | $C(2, n-4)$ | $C(3, n-6)$ | $C(4, n-8)$ |

## 17. Two lower bounds on graph energy

The graph energy is a graph-spectrum-based quantity, introduced by Ivan Gutman in the 1970s. For a graph $G$ on $n$ vertices, the energy $\mathcal{E}(G)$ of $G$ is defined to be the sum of the absolute values of the eigenvalues of $A(G)$, that is,

$$
\mathcal{E}(G):=\sum_{i=1}^{n}\left|\lambda_{i}(G)\right| .
$$

This graph invariant is very closely connected to a chemical quantity known as the total $\pi$ electron energy of conjugated hydrocarbon molecules. For results on graph energy, we refer the reader to [98], which is a monograph summarizing the main theorems, applications and methods regarding the adjacency energy of a graph.

The following conjecture comes from the Written on the Wall (c.f. [11]).
Conjecture 20 ([11]). Let $G$ be a graph on $n$ vertices with independence number $\alpha$. Then

$$
\sum_{\lambda_{i}(G)>0} \lambda_{i}(G) \geq n-\alpha
$$

Note that if Conjecture 20 is proven to be true, it would provide us with a concise lower bound on the energy of graph $G$, as the left-hand side of the above inequality is precisely equal to $\mathcal{E}(G) / 2$. We utilized computer computations to verify Conjecture 20 for all graphs having at most 10 vertices, and we did not find any counterexamples.

The well-known inertia bound [75, Lemma 9.6.3] due to Cvetković states that

$$
\alpha(G) \leq \min \left\{n-n^{+}, n-n^{-}\right\},
$$

where $\alpha(G)$ is the independence number of $G, n^{+}$and $n^{-}$are the numbers (counting multiplicities) of positive and negative eigenvalues of $A(G)$ respectively. Hence, if Conjecture 20 is true, we can promptly derive that

$$
\begin{equation*}
\sum_{\lambda_{i}(G)>0} \lambda_{i}(G) \geq \max \left\{n^{+}, n^{-}\right\}, \tag{2}
\end{equation*}
$$

which is also a conjecture in [11]. On the ther hand, Elphick, Farber, Goldberg and Wocjan [54, Lemma 5] proved that

$$
s^{+}(G) \geq \frac{\mathcal{E}(G)^{2}}{4 n^{+}} \text {and } s^{-}(G) \geq \frac{\mathcal{E}(G)^{2}}{4 n^{-}}
$$

Thus, Elphick pointed to us that if Conjecture 20 holds true, it would imply $s^{+}(G) \geq n^{+}$ and $s^{-}(G) \geq n^{-}$by (2), which is a slightly weaker, yet analogous statement, compared to Conjecture 1 .

The other conjecture regarding to $\mathcal{E}(G)$ is one due to Akbari and Hosseinzadeh [3].
Conjecture 21 (Akbari-Hosseinzadeh [3]). For every graph with maximum degree $\Delta$ and minimum degree $\delta$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta+\delta$ and the equality holds if and only if $G$ is a complete graph.

In [9], Al-Yakoob, Filipovski and Stevanović have demonstrated the validity of Conjecture 21 for various non-singular graphs, specifically those that satisfy either $n \geq \Delta+\delta$ or $|\operatorname{det}(A(G))| \geq \lambda(G)$ or $2 m+n(n-1) \geq(\Delta+\delta)^{2}$ or $\lambda(G)-\ln \lambda(G) \geq \delta$ or $\Delta \leq(n-1)^{1-1 / n}$.

The condition that the adjacency matrix of $G$ is non-singular is somewhat wondrous, Akbari and Hosseinzadeh [3] did not provide any motivation for the condition. However, as mentioned in [9] there are obvious counterexamples of Conjecture 21 among the graphs that have zero eigenvalues. For example, the complete bipartite graph $K_{\Delta, \delta}$ has the adjacency spectrum: $\sqrt{\Delta \delta},-\sqrt{\Delta \delta}$ and 0 (multiplicity $\Delta+\delta-2$ ). Hence, $\mathcal{E}\left(K_{\Delta, \delta}\right)=2 \sqrt{\Delta \delta}<\Delta+\delta$ whenever $\Delta>\delta$.

## 18. Maximal $\boldsymbol{\lambda}_{1}+\lambda_{n}$ of $\boldsymbol{K}_{r+1}$-free graphs

Erdős put forth a conjecture that any triangle-free graph $G$ on $n$ vertices must contain a set of $n / 2$ vertices that span at most $n^{2} / 50$ edges, which is one of his favourite conjectures [60, 61]. Significant advancements have been made in various directions regarding this conjecture [16, 63, 88, 125]. Most recently, Razborov [130] proved that every triangle-free graph on $n$ vertices has an induced subgraph on $n / 2$ vertices with at most $(27 / 1024) n^{2}$ edges.

A problem with similar motivation is to determine $D_{2}(G)$, the minimum number of edges which has to be removed to make $G$ bipartite, for a triangle-free graph $G$ on $n$ vertices. A long-standing conjecture of Erdős is that at most $n^{2} / 25$ edges need to be deleted [60]. This conjecture has been studied by several researchers [8, 62, 64, 135] , and the most recent result was obtained by Balogh, Clemen and Lidický [13], who proved that $D_{2}(G) \leq n^{2} / 23.5$.

According to the Perron-Frobenius theorem, we know that $\lambda_{1}(G) \geq-\lambda_{n}(G)$. If $G$ is connected, then equality holds if and only if $G$ is bipartite. Hence, the difference between $\lambda_{1}$ and $-\lambda_{n}$ can be viewed as a measure, how far a graph is from being bipartite. On the other hand, Brandt [27] found a surprising connection between these two conjectures of Erdős and the eigenvalues of regular graphs $G$. It was proved that

$$
\frac{\lambda_{1}(G)+\lambda_{n}(G)}{4} \cdot n \leq D_{2}(G)
$$

Brandt [27] also conjectured that

$$
\lambda_{1}(G)+\lambda_{n}(G) \leq \frac{4}{25} n
$$

for regular triangle-free graphs $G$ on $n$ vertices. If either of the two conjectures of Erdős were true, it would imply Brandt's conjecture (see [14] for details). Towards Brandt's conjecture, it was proved that $\lambda_{1}(G)+\lambda_{n}(G) \leq(3-2 \sqrt{2}) n$ for regular triangle-free graphs [27], which was shown to hold also in the non-regular setting by Csikvári [45]. Obviously, the quantity $\lambda_{1}(G)+\lambda_{n}(G)$ coincides with the smallest signless Laplacian eigenvalue of $G$ if $G$ is regular. Very recently, Balogh, Clemen, Lidický, Norin and Volec [14] proved that for a triangle-free $n$-vertex graph $G$, the smallest signless Laplacian eigenvalue of $G$ is at most $15 n / 94$, which confirms Brandt's conjecture in strong form.

In general, the following problem is of interest.

Problem 9 (Brandt [27]). Let $r \geq 2$. How large can $\lambda_{1}(G)+\lambda_{n}(G)$ be if $G$ is a $K_{r+1-f r e e ~}^{\text {f }}$ graph of order $n$ ?

One can also consider a similar problem for the smallest signless Laplacian eigenvalue of graphs, see [48] and [127] for details.

## 19. Problems on other adjacency eigenvalues of graphs

In this section, our attention will be directed towards the adjacency eigenvalues of graphs, excluding the largest and smallest eigenvalues.

### 19.1. The third and fourth eigenvalues of graphs

Let $G$ be a connected graph on $n$ vertices. In 1989, Powers [129] presented an upper bound for $\lambda_{i}(G)(1 \leq i \leq n / 2)$ of $G$, just in terms of the order of $G$, i.e.,

$$
\begin{equation*}
\lambda_{i}(G) \leq\left\lfloor\frac{n}{i}\right\rfloor . \tag{3}
\end{equation*}
$$

The inequality above is notable for its simplicity and elegance. The validity of Inequality (3) for $i \leq 2$ is clear (for $\lambda_{2}$, see [82]), but unfortunately, it does not hold for $i \geq 5$ (c.f. [120]). Currently, the upper bound of Powers for $i \in\{3,4\}$ remains unknown.

Problem 10 (Powers [129]). Let $G$ be a graph of order n. Is it true that

$$
\lambda_{i}(G) \leq\left\lfloor\frac{n}{i}\right\rfloor
$$

for $i=3,4$ ?
Recently, Linz [100] provided a counterexample to Problem 10 for $i=4$ by constructing a class of graphs with $\lambda_{4}>n / 4$. Nevertheless, it is still worth exploring the best possible upper bound for Problem 10, as well as considering the general question posed by Hong [84].

Problem 11 (Hong [84]). Find the best possible lower and upper bounds for the $i$-th eigenvalue of graphs with $n$ vertices.

Let us remark that Hong's problem is related to other areas of combinatorics other than spectral graph theory, like the existence of symmetric Hadamard matrices, Ramsey theory and etc. For progress on Hong's problem we refer the reader to [120].

### 19.2. HL-index Conjecture

Fowler and Pisanski [69,68] introduced the notion of the HL-index of a graph that is related to the HOMO-LUMO separation studied in theoretical chemistry. This is the gap between the Highest Occupied Molecular Orbital (HOMO) and Lowest Unoccupied Molecular Orbital (LUMO). Let $G$ be a graph of order $n$. The $H L$-index $R(G)$ of $G$ is defined as $R(G)=\max \left\{\left|\lambda_{H}(G)\right|,\left|\lambda_{L}(G)\right|\right\}$, where

$$
H=\left\lfloor\frac{n+1}{2}\right\rfloor, \quad L=\left\lceil\frac{n+1}{2}\right\rceil .
$$

Several bounds for this index have been proposed for some classes of graphs in [42, 69, 68, 86, 97].

A graph $G$ is said to be subcubic if its maximum degree is at most 3. Fowler and Pisanski [69, 68] proved that every subcubic graph $G$ satisfies $0 \leq R(G) \leq 3$ and that if $G$ is bipartite, then $R(G) \leq \sqrt{3}$. In 2015, Mohar [110] showed that $R(G) \leq \sqrt{2}$ for each subcubic graph $G$, which improved the results of Fowler and Pisanski. This result is best possible since the Heawood graph (the bipartite incidence graph of points and lines of the Fano plane) has HL-index $\sqrt{2}$. In the same paper, Mohar also proposed the following conjecture.

Conjecture 22 (Mohar [110]). If $G$ is a planar subcubic graph, then $R(G) \leq 1$.
The conjecture has been verified for planar bipartite graphs in [109]. Furthermore, Mohar [111] confirmed Conjecture 22 for all bipartite subcubic graphs with a single exception of the Heawood graph (or a disjoint union of copies of it).

## 20. Principal eigenvectors of graphs

In the last section, we collect two conjectures on the principal eigenvectors of graphs. For a connected graph $G$, the Perron-Frobenius theorem implies that $A(G)$ has a unique unit positive eigenvector corresponding to $\lambda(G)$, which is usually called the principal eigenvector of $G$.

In 2010, Cioabă [37] presented a necessary and sufficient condition for a connected graph to be bipartite in terms of its principal eigenvector.
Theorem 5 ([37]). Let $S$ be an independent set of a connected graph $G$. Then

$$
\sum_{v \in S} x_{v}^{2} \leq \frac{1}{2}
$$

with equality if and only if $G$ is bipartite having $S$ as one color class.
Strengthening Cioabă's result, Gregory posed the following conjecture in 2010.
Conjecture 23 (Gregory, c.f. [38]). Let $G$ be a connected graph with chromatic number $k \geq 2$ and $S$ be an independent set. Then

$$
\sum_{v \in S} x_{v}^{2} \leq \frac{1}{2}-\frac{k-2}{\sqrt{(k-2)^{2}+4(k-1)(n-k+1)}} .
$$

One can check that $S_{n, k-1}$ attains equality. Let $P_{r} \cdot K_{s}$ denote the graph of order $(r+s-1)$ attained by identifying an end vertex of the path $P_{r}$ to any vertex of the complete graph $K_{s}$. This graph $P_{r} \cdot K_{s}$ is called a kite graph or a lollipop graph.

The following conjecture appears in several papers [41, 133, 38], which was presented in different backgrounds.

Conjecture 24 (Rücker-Rücker-Gutman [133]). Among all connected graphs on $n$ vertices, the graph $P_{n-3} \cdot K_{4}$ minimizes $\ell^{1}$-norm $\|\boldsymbol{x}\|_{1}$ of principal eigenvectors.

The conjecture has been verified to be true for connected graphs with at most 10 vertices using SageMath software.

## Acknowledgment

This paper is an invited paper of ORSC. The second author is indebted to ORSC for inviting him to submit a paper to the Operations Research Transactions (ORT). The authors would like to thank Clive Elphick for his useful comments on the connection between Conjecture 1 and Conjecture 20. The authors express sincere gratitude to Xiaofeng Gu for drawing attention to a related conjecture in [77] that connects to Conjecture 9, and William Linz for sharing with us the reference [100]. Furthermore, the authors thank Dragan Stevanović for bringing to our attention a related conjecture in [3].

## References

[1] M. Abdi, E. Ghorbani, Quartic graphs with minimum spectral gap, J. Graph Theory, 102: 205-233, 2023.
[2] A. Abiad, L. de Lima, D.N. Desai, K. Guo, L. Hogben, J. Madrid, Positive and negative square energies of graphs, Electron. J. Linear Algebra, 39: 307-326, 2023.
[3] S. Akbari, M. A. Hosseinzadeh, A short proof for graph energy is at least twice of minimum degree, MATCH Commun. Math. Comput. Chem. 83: 631-633, 2020.
[4] S.G. Aksoy, F. Chung, M. Tait, J.Tobin, The maximum relaxation time of a random walk. Adv. in Appl. Math., 101: 1-14, 2018.
[5] D. Aldous, J. Fill, Reversible Markov Chains and Random Walks on Graphs, 2002. Available at https://www.stat.berkeley.edu/ aldous/RWG/book.pdf
[6] P. Allen, T. łuczak, J. Polcyn, Y.B. Zhang, The Ramsey number of a long even cycle versus a star, J. Combin. Theory Ser. B, 162: 144-153, 2023.
[7] N. Alon, Tough Ramsey graphs without short cycles. J. Algebraic Combin., 4: 189-195, 1995.
[8] N. Alon, Bipartite subgraphs, Combinatorica, 16(3): 301-311, 1996.
[9] S. Al-Yakoob, S. Filipovski, D. Stevanović, Proofs of a few special cases of a conjecture on energy of non-singular graphs, MATCH Commun. Math. Comput. Chem. 86: 577586, 2021.
[10] T. Ando, M. Lin, Proof of a conjectured lower bound on the chromatic number of a graph, Linear Algebra Appl., 485: 480-484, 2015.
[11] M. Aouchiche, P. Hansen, A survey of automated conjectures in spectral graph theory, Linear Algebra Appl., 432: 2293-2322, 2010.
[12] H. Bai, The Grone-Merris conjecture, Trans. Amer. Math. Soc., 363: 4463-4474, 2011.
[13] J. Balogh, F.C. Clemen, B. Lidický, Max cuts in triangle-free graphs, In: Trends in Mathematics (Extended Abstracts EuroComb), Birkhäuser, Cham, 2021.
[14] J. Balogh, F.C. Clemen, B. Lidický, S. Norin, J. Volec, The spectrum of triangle-free graphs, SIAM J. Discrete Math., 37 (2): 1173-1179, 2023.
[15] D. Bauer, H. Broersma, E. Schmeichel, Toughness in graphs-a survey, Graphs Combin., 22(1): 1-35, 2006.
[16] W. Bedenknecht, G.O. Mota, C. Reiher, M. Schacht, On the local density problem for graphs of given odd-girth, J. Graph Theory, 90(2): 137-149, 2019.
[17] F. Belardo, S.M. Cioabă, J. Koolen, J. Wang, Open problems in the spectral theory of signed graphs, Art Discr. Appl. Math., 1, \# P2.10, 2018.
[18] Y. Bilu, N. Linial, Lifts, discrepancy and nearly optimal spectral gap, Combinatorica, 26: 495-519, 2006.
[19] V. Blinovsky, L.D. Speranca, Proof of Brouwer's conjecture, arXiv: 1908.08534, 2022.
[20] B. Bollobás, Extremal Graph Theory, London Mathematical Society Monographs, Academic Press Inc., New York, 1978.
[21] B. Bollobás, J. Lee, S. Letzter, Eigenvalues of subgraphs of the cube, European J. Combin., 70: 125-148, 2018.
[22] B. Bollobás, V. Nikiforov, Cliques and the spectral radius, J. Combin. Theory Ser. B, 97: 859-865, 2007.
[23] J.A. Bondy, Pancyclic graphs I, J. Combin. Theory Ser. B, 11: 80-84, 1971.
[24] J.A. Bondy, U.S.R Murty, Graph Theory, Graduate Texts in Mathematics 244. SpringerVerlag, London, 2008.
[25] B.N. Boots, G.F. Royle, A conjecture on the maximum value of the principal eigenvalue of a planar graph, Geogr. Anal., 23(3): 276-282, 1991.
[26] C. Brand, B. Guiduli, W. Imrich, Characterization of trivalent graphs with minimal eigenvalue gap, Croat. Chem. Acta, 80: 193-201, 2007.
[27] S. Brandt, The local density of triangle-free graphs, Discrete Math., 183(1-3): 17-25, 1998.
[28] S. Brandt, H. Broersma, R. Diestel, M. Kriesell, Global connectivity and expansion: long cycles and factors in $f$-connected graphs, Combinatorica, 26: 17-36, 2006.
[29] A.E. Brouwer, Toughness and spectrum of a graph, Linear Algebra Appl., 226/228: 267-271, 1995.
[30] A.E. Brouwer, Spectrum and connectivity of graphs, CWI Quarterly, 9: 37-40, 1996.
[31] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Springer-Verlag, New York, 2012.
[32] D. Cao, A. Vince, The spectral radius of a planar graph, Linear Algebra Appl., 187: 251-257, 1993.
[33] X. Chen, Improved results on Brouwer's conjecture for sum of the Laplacian eigenvalues of a graph, Linear Algebra Appl., 557: 327-338, 2018.
[34] X. Chen, On Brouwer's conjecture for the sum of $k$ largest Laplacian eigenvalues of graphs, Linear Algebra Appl., 578: 402-410, 2019.
[35] V. Chvátal, Tough graphs and Hamiltonian circuits, Discrete Math., 5: 215-228, 1973.
[36] S.M. Cioabă, The spectral radius and the maximum degree of irregular graphs, Electron. J. Comb., 14, \# R38, 2007.
[37] S.M. Cioabă, A necessary and sufficient eigenvector condition for a connected graph to be bipartite, Electron. J. Linear Algebra, 20: 351-353, 2010.
[38] S.M. Cioabă, The principal eigenvector of a connected graph, in: Spectral Graph Theory Online Conference, 2021.
[39] S.M. Cioabă, D.N. Desai, M. Tait, The spectral even cycle problem, arXiv: 2205.00990, 2022.
[40] S.M. Cioabă, D.A. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, J. Comb. Theory Ser. B, 97: 483-486, 2007.
[41] G.J. Clark, Comparing eigenvector and degree dispersion with theprincipal ratio of a graph, Linear Multilinear Algebra, 2022. DOI: 10.1080/03081087.2022.2158171.
[42] G.P. Clemente, A. Cornaro, Bounding the HL-index of a graph: a majorization approach, J. Inequal. Appl., \# 285, 2016.
[43] L. Collatz, U. Sinogowitz, Spektren endlicher grafen, Abh. Math. Sem. Univ. Hamburg, 21: 63-77, 1957.
[44] J.N. Cooper, Constraints on Brouwer's Laplacian spectrum conjecture, Linear Algebra Appl., 615: 11-27, 2021.
[45] P. Csikvári, Note on the sum of the smallest and largest eigenvalues of a triangle-free graph, Linear Algebra Appl., 650: 92-97, 2022.
[46] B.L. Currie, J.R. Faudree, R.J. Faudree, J.R. Schmitt, A survey of minimum saturated graphs, Electron. J. Combin., \# DS19, 2021.
[47] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs: Theory and Application, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.
[48] L. de Lima, V. Nikiforov, C. Oliveira, The clique number and the smallest $Q$-eigenvalue of graphs, Discrete Math., 339: 1744-1752, 2016.
[49] A.G. Dirac, Some theorems on abstract graphs. Proc. Lond. Math. Soc., 2: 69-81, 1952.
[50] Z. Du, B. Zhou, Upper bounds for the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl., 436: 3672-3683, 2012.
[51] C. Edwards, C. Elphick, Lower bounds for the clique and the chromatic number of a graph, Discrete Appl. Math., 5: 51-64, 1983.
[52] M.N. Ellingham, L. Lu, Z. Wang, Maximum spectral radius of outerplanar 3-uniform hypergraphs, J. Graph Theory, 100(4): 671-685, 2022.
[53] M.N. Ellingham, X. Zha, The spectral radius of graphs on surfaces, J. Combin. Theory, Ser. B, 78(1): 45-56, 2000.
[54] C. Elphick, M. Farber, F. Goldberg, P. Wocjan, Conjectured bounds for the sum of squares of positive eigenvalues of a graph, Discrete Math., 339(9): 2215-2223, 2016.
[55] C. Elphick, W. Linz, P. Wocjan, Generalising a conjecture due to Bollobás and Nikiforov, arXiv: 2101.05229, 2021.
[56] C. Elphick, P. Wocjan, Conjectured lower bound for the clique number of a graph, arXiv: 1804.03752, 2018.
[57] P. Erdős, Some theorems on graphs, Riv. Lemat., 9: 13-17, 1955.
[58] P. Erdős, On a theorem of Rademacher-Turán, Illinois J. Math., 6: 122-127, 1962.
[59] P. Erdős, On the number of complete subgraphs contained in certain graphs, Magy. Tud. Acad. Mat. Kut. Int. Közl., 7: 459-464, 1962.
[60] P. Erdős, Problems and results in graph theory and combinatorial analysis, in: Proceedings of the fifth british combinatorial conference (univ. aberdeen, aberdeen, 1975), Congr. Numer., 15: 169-192, 1976.
[61] P. Erdős, Some old and new problems in various branches of combinatorics, Discrete Math., 165/166: 227-231, 1997.
[62] P. Erdős, R. Faudree, J. Pach, J. Spencer, How to make a graph bipartite, J. Combin. Theory Ser. B, 45(1): 86-98, 1988.
[63] P. Erdős, R.J. Faudree, C.C. Rousseau, R.H. Schelp, A local density condition for triangles, Discrete Math., 127: 153-161, 1994.
[64] P. Erdős, E. Győri, M. Simonovits, How many edges should be deleted to make a triangle-free graph bipartite? In: Sets, Graphs and Numbers 60, Colloq. Math. Soc. János Bolyai, Budapest, 1991.
[65] P. Erdős, A. Hajnal, J. Moon, A problem in graph theory, Amer. Math. Monthly, 71: 1107-1110, 1964.
[66] P. Erdős, M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hungar., 1: 51-57, 1966.
[67] P. Erdős, A.H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc., 52: 1087-1091, 1946.
[68] P.W. Fowler, T. Pisanski, HOMO-LUMO maps for chemical graphs, MATCH Commun. Math. Comput. Chem., 64: 373-390, 2010.
[69] P.W. Fowler, T. Pisanski, HOMO-LUMO maps for fullerenes, Acta Chim. Slov., 57: 513-517, 2010.
[70] J. Friedman, J.P. Tillich, Generalized Alon-Boppana theorems and error-correcting codes, SIAM J. Discrete Math., 19(3): 700-718, 2005.
[71] H.A. Ganie, A.M. Alghamdi, S. Pirzada, On the sum of the Laplacian eigenvalues of a graph and Brouwer's conjecture, Linear Algebra Appl., 501: 376-389, 2016.
[72] H.A. Ganie, S. Pirzada, Corrigendum to "on the sum of the laplacian eigenvalues of a graph and Brouwer's conjecture", Linear Algebra Appl., 538: 228-230, 2018.
[73] H.A. Ganie, S. Pirzada, B.A. Rather, V. Trevisan, Further developments on Brouwer's conjecture for the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl., 588: 1-18, 2020.
[74] S. Glock, D.M. Correia, B. Sudakov, Hamilton cycles in pseudorandom graphs, arXiv: 2303.05356, 2023.
[75] C. Godsil, G. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics 207. Springer-Verlag, New York, 2001.
[76] X. Gu, A proof of Brouwer's toughness conjecture, SIAM J. Discrete Math., 35(2): 948-952, 2021.
[77] X. Gu, W.H. Haemers, Graph toughness from Laplacian eigenvalues, Algebraic Combinatorics, 5(1): 53-61, 2022.
[78] B. Guiduli, Spectral Extrema for Graphs, Ph.D. Thesis, University of Chicago, 1996.
[79] J. Guo, C. Wang, The estimation of the bound of the sum of squares of positive (negative) eigenvalues of a graph, Advances in Mathematics (China), 51(1): 69-75, 2022.
[80] W.H. Haemers, Toughness conjecture, 2020. Available at https://www.researchgate.net/.
[81] W.H. Haemers, A. Mohammadian, B. Tayfeh-Rezaie, On the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl., 432: 2214-2221, 2010.
[82] Y. Hong, Bound of eigenvalues of a graph, Acta Math. Appl. Sinica, 432(4): 165-168, 1988.
[83] Y. Hong, A bound on the spectral radius of graphs, Linear Algebra Appl., 108: 135-140, 1988.
[84] Y. Hong, Bounds on eigenvalues of graphs, Discrete Math., 123: 65-74, 1993.
[85] Y. Hong, On the spectral radius and the genus of graphs, J. Combin. Theory, Ser. B, 65(2): 262-268, 1995.
[86] G. Jaklič, P.W. Fowler, T. Pisanski, HL-index of a graph, Ars Math. Contemp., 5: 99-115, 2012.
[87] I. Jovović, T. Koledin, Z. Stanić, Trees with small spectral gap, Ars Math. Contemp., 14: 197-207, 2018.
[88] P. Keevash, B. Sudakov, Sparse halves in triangle-free graphs, J. Combin. Theory Ser. B, 96(4): 614-620, 2006.
[89] J. Kim, S. Kim, A.V. Kostochka, S. O, The minimum spectral radius of $K_{r+1}$-saturated graphs, Discrete Math., 343, Paper No. 112068, 2020.
[90] J. Kim, A.V. Kostochka, S. O, Y. Shi, Z. Wang, A sharp lower bound for the spectral radius in $K_{4}$-saturated graphs, Discrete Math., 346, Paper No. 113231, 2023.
[91] M. Krivelevich, B. Sudakov, Sparse pseudo-random graphs are Hamiltonian, J. Graph Theory, 42(1): 17-33, 2003.
[92] M. Krivelevich, B. Sudakov, Pseudo-random Graphs, In: More Sets, Graphs and Numbers, pages 199-262. Springer, Berlin, 2006.
[93] B. Li, B. Ning, Eigenvalues and cycles of consecutive lengths, J. Graph Theory, 103 (3): 486-492, 2023.
[94] B. Li, B. Ning, Stability of Woodall's theorem and spectral conditions for large cycles, Electron. J. Combin., 30(1), \# P1.39, 2023.
[95] S. Li, W. Sun, Y. Yu, Adjacency eigenvalues of graphs without short odd cycles, Discrete Math., 345, Paper No. 112633, 2022.
[96] W. Li, J. Guo, On the full Brouwer's Laplacian spectrum conjecture, Discrete Math., 345, Paper No. 113078, 2022.
[97] X. Li, Y. Li, Y. Shi, I. Gutman, Note on the HOMO-LUMO index of graphs, MATCH Commun. Math. Comput. Chem., 70: 85-96, 2013.
[98] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.
[99] H. Lin, B. Ning, B. Wu, Eigenvalues and triangles in graphs, Combin. Probab. Comput., 30(2): 258-270, 2021.
[100] W. Linz, Improved lower bounds on the extrema of eigenvalues of graphs, Graphs Combin., 39, Paper No. 82, 2023.
[101] B. Liu, G. Li, A note on the largest eigenvalue of non-regular graphs, Electron. J. Linear Algebra, 17: 54-61, 2008.
[102] B. Liu, J. Shen, X. Wang, On the largest eigenvalue of non-regular graphs, J. Combin. Theory Ser. B, 97: 1010-1018, 2007.
[103] L. Liu, Extremal spectral radius of nonregular graphs with prescribed maximum degree, arXiv: $2203.10245,2022$.
[104] L. Lovász, J. Pelikán, On the eigenvalues of trees, Period. Math. Hungar., 3: 175-182, 1973.
[105] L. Lovász, M. Simonovits, On the number of complete subgraphs of a graph II, In: Studies in Pure Math, pages 459-495. Birkhäuser, 1983.
[106] A.W. Marcus, D.A. Spielman, N. Srivastava, Interlacing families I: Bipartite Ramanujan graphs of all degrees, Ann. of Math., 182: 307-325, 2015.
[107] Mayank, On Variants of the Grone-Merris Conjecture, PhD Thesis, Eindhoven University of Technology, 2010.
[108] G. Min, Z. Lou, Q. Huang, A sharp upper bound on the spectral radius of $C_{5}$-free/ $C_{6^{-}}$ free graphs with given size, Linear Algebra Appl., 640: 162-178, 2022.
[109] B. Mohar, Median eigenvalues of bipartite planar graphs, MATCH Commun. Math. Comput. Chem., 70: 79-84, 2013.
[110] B. Mohar, Median eigenvalues and the HOMO-LUMO index of graphs, J. Combin. Theory Ser. B, 112: 78-92, 2015.
[111] B. Mohar, Median eigenvalues of bipartite subcubic graphs, Combin. Probab. Comput., 25: 768-790, 2016.
[112] D. Mubayi, Counting substructures I: color critical graphs, Adv. Math., 225(5): 27312740, 2010.
[113] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, Combin. Probab. Comput., 11: 179-189, 2002.
[114] V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra Appl., 427: 183-189, 2007.
[115] V. Nikiforov, The energy of graphs and matrices, J. Math. Anal. Appl., 326: 1472-1475, 2007.
[116] V. Nikiforov, A spectral condition for odd cycles in graphs, Linear Algebra Appl., 428(7): 1492-1498, 2008.
[117] V. Nikiforov, The maximum spectral radius of $C_{4}$-free graphs of given order and size, Linear Algebra Appl., 430: 2898-2905, 2009.
[118] V. Nikiforov, A spectral Erdős-Stone-Bollobás theorem, Combin. Probab. Comput., 18(3): 455-458, 2009.
[119] V. Nikiforov, The spectral radius of graphs without paths and cycles of specified length, Linear Algebra Appl., 432: 2243-2256, 2010.
[120] V. Nikiforov, Extrema of graph eigenvalues, Linear Algebra Appl., 482: 158-190, 2015.
[121] V. Nikiforov, Spectral radius and maximum degree of connected graphs, arXiv: 0602028, 2018.
[122] B. Ning, X. Peng, Extensions of the Erdős-Gallai theorem and Luo's theorem, Combin. Probab. Comput., 29(1): 128-136, 2020.
[123] B. Ning, M. Zhai, Counting substructures and eigenvalues II: quadrilaterals, arXiv: $2112.15279,2022$.
[124] B. Ning, M. Zhai, Counting substructures and eigenvalues I: triangles, European J. Combin., 110, Paper No. 103685, 2023.
[125] S. Norin, L. Yepremyan, Sparse halves in dense triangle-free graphs, J. Combin. Theory Ser. B, 115: 1-25, 2015.
[126] E. Nosal, Eigenvalues of Graphs, Master Thesis, University of Calgary, 1970.
[127] M.R. Oboudi, On a conjecture related to the smallest signless Laplacian eigenvalue of graphs, Linear Multilinear Algebra, 70(19): 4425-4431, 2022.
[128] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly, 67, Paper No. 55, 1960.
[129] D.L. Powers, Bounds on graph eigenvalues, Linear Algebra Appl., 117: 1-6, 1989.
[130] A.A. Razborov, More about sparse halves in triangle-free graphs, Sbornik: Mathematics, 213(1): 109-128, 2022.
[131] I. Rocha, Brouwer's conjecture holds asymptotically almost surely, Linear Algebra Appl., 597: 198-205, 2020.
[132] I. Rocha, V. Trevisan, Bounding the sum of the largest Laplacian eigenvalues of graphs, Discrete Appl. Math., 170: 95-103, 2014.
[133] G. Rücker, C. Rücker, I. Gutman, On kites, comets, and stars. sums of eigenvector coefficientsin (molecular) graphs, Zeitschrift für Naturforschung A, 57(3-4): 143-153, 2002.
[134] A.J. Schwenk, R.J. Wilson, On the eigenvalues of a graph, In: Selected Topics in Graph Theory, pages 307-336. Academic Press, London, 1978.
[135] J.B. Shearer, A note on bipartite subgraphs of triangle-free graphs, Random Structures E Algorithms, 3(2): 223-226, 1992.
[136] L. Shi, Bounds on the (Laplacian) spectral radius of graphs, Linear Algebra Appl., 422: 755-770, 2007.
[137] L. Shi, The spectral radius of irregular graphs, Linear Algebra Appl., 431: 189-196, 2009.
[138] S.K. Simić, On the largest eigenvalue of bicyclic graphs, Publ. Inst. Math. (Beograd), 46(60): 101-106, 1989.
[139] Z. Stanić, Graphs with small spectral gap, Electron. J. Linear Algebra, 26: 417-432, 2013.
[140] D. Stevanović, The largest eigenvalue of nonregular graphs, J. Combin. Theory Ser. B, 91: 143-146, 2004.
[141] W. Sun, S. Li, W. Wei, Extensions on spectral extrema of $C_{5} / C_{6}$-free graphs with given size, Discrete Math., 346 (12), Paper No. 113591, 2023.
[142] M. Tait, J. Tobin, Three conjectures in extremal spectral graph theory, J. Combin. Theory Ser. B, 126: 137-161, 2017.
[143] P. Turán, Research problems, Magyar Tud. Akad. Mat. Kutató Int. Közl., 6: 417-423, 1961.
[144] H. Voss, C. Zuluaga, Maximale gerade und ungerade kreise in graphen I (German), Wiss. Z. Tech. Hochsch. Ilmenau., 23: 57-70, 1977.
[145] S. Wang, Y. Huang, B. Liu, On a conjecture for the sum of Laplacian eigenvalues, Math. Comput. Model., 56: 60-68, 2012.
[146] H. Wilf, Spectral bounds for the clique and independence numbers of graphs, J. Combin. Theory Ser. B, 40: 113-117, 1986.
[147] M. Zhai, H. Lin, Spectral extrema of graphs: forbidden hexagon, Discrete Math., 343, Paper No. 112028, 2020.
[148] M. Zhai, H. Lin, A strengthening of the spectral chromatic critical edge theorem: Books and theta graphs, J. Graph Theory, 102(3): 502-520, 2023.
[149] M. Zhai, H. Lin, J. Shu, Spectral extrema of graphs with fixed size: cycles and complete bipartite graphs, European J. Combin., 95, Paper No. 103322, 2021.
[150] M. Zhai, B. Wang, Proof of a conjecture on the spectral radius of $C_{4}$-free graphs, Linear Algebra Appl., 437: 1641-1647, 2012.
[151] W. Zhang, A new result on spectral radius and maximum degree of irregular graphs, Graphs Combin., 37: 1103-1119, 2021.
[152] X. Zhang, Eigenvectors and eigenvalues of nonregular graphs, Linear Algebra Appl., 409: 79-86, 2005.
[153] Z. Zhang, Y. Zhao, A spectral condition for the existence of cycles with consecutive odd lengths in non-bipartite graphs, Discrete Math., 346(6), Paper No. 113365, 2023.


[^0]:    *School of Mathematical Sciences, Anhui University, Hefei 230601, P.R. China. E-mail: liu@ahu.edu.cn (L. Liu). Supported by the National Nature Science Foundation of China (No. 12001370)
    ${ }^{\dagger}$ Corresponding author. College of Computer Science, Nankai University, Tianjin 300350, P.R. China. Email: bo.ning@nankai.edu.cn (B. Ning). Partially supported by the National Nature Science Foundation of China (No. 11971346).

[^1]:    ${ }^{1}$ The original link to Gregory's work is not available. Here we cite the description from [17].

