

UNSTABLE SIMPLE MODES OF THE NONLINEAR STRING

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Abstract. We prove instability of high-energy simple modes for the nonlinear vibrating string equation

$$\begin{cases} u_{tt} - (a + b \int_0^\pi u_x^2(t, x) dx) u_{xx} = 0, \\ u|_{\partial\Omega} = 0, \end{cases}$$

in $\Omega = (0, \pi)$, where $a \geq 0, b > 0$.

Résumé. On montre l'instabilité des modes simples d'énergie assez grande pour l'équation des cordes vibrantes non-linéaire

$$\begin{cases} u_{tt} - (a + b \int_0^\pi u_x^2(t, x) dx) u_{xx} = 0, \\ u|_{\partial\Omega} = 0, \end{cases}$$

dans $\Omega = (0, \pi)$, où $a \geq 0, b > 0$.

1. Introduction. A classical model of the nonlinear vibrating string is given by the equation

$$\begin{cases} u_{tt} - (a + b \int_0^\pi u_x^2(t, x) dx) u_{xx} = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1.1)$$

in $\Omega = (0, \pi)$, where $a \geq 0, b > 0$. This model has been extensively studied, for example, by Carrier [2], Bernstein [1], and Narashimha [8]. See also Dickey [4], Medeiros and Milla Miranda [7] and the references therein. Our main interest in this model is the stability of simple modes, i.e., solutions of the form $u(t, x) = \omega(t) \sin(jx)$, such solutions necessarily being periodic in time. Dickey [4] showed that small amplitude simple modes are stable. Here, we prove that simple modes of sufficiently large energy are unstable, i.e., that they have nontrivial unstable manifolds.

We study equation (1.1) using Fourier series. Setting

$$u_j(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi u(t, x) \sin(jx) dx,$$

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we obtain for the u_j 's the following system of ODE's:

$$u_j'' + j^2 \left(a + b \sum_{i=1}^{\infty} i^2 u_i^2 \right) u_j = 0. \tag{1.2}$$

It follows that, given any integer $j \geq 1$, Eq. (1.1) has the particular (simple mode) solution $u(t, x) = \sqrt{\frac{2}{\pi}} \omega(t) \sin(jx)$ where ω solves the equation $\omega'' + j^2(a + bj^2\omega^2)\omega = 0$. To prove that simple modes are always unstable once they have sufficiently large energy, we consider a two-mode solution of (1.1) (or (1.2)), i.e., a solution of the system

$$\begin{cases} u_j'' + j^2(a + bj^2u_j^2 + bk^2u_k^2)u_j = 0, \\ u_k'' + k^2(a + bj^2u_j^2 + bk^2u_k^2)u_k = 0, \end{cases} \tag{1.3}$$

where j, k are integers such that $k > j \geq 1$. If (u_j, u_k) solves system (1.3), then $u(t, x) = \sqrt{\frac{2}{\pi}}(u_j(t) \sin(jx) + u_k(t) \sin(kx))$ solves Eq. (1.1). We set

$$\gamma = \frac{k^2}{j^2}, \quad \sigma = j\sqrt{b}, \quad \nu = \frac{a}{b},$$

so that (1.3) is equivalent to

$$\begin{cases} u'' + (\nu + u^2 + v^2)u = 0, \\ v'' + \gamma(\nu + u^2 + v^2)v = 0, \end{cases} \tag{1.4}$$

where $ju_j(t) = u(\sigma t)$ and $ku_k(t) = v(\sigma t)$. The system (1.4) is a Hamiltonian system, whose conserved energy E , expressed in terms of u, v, u', v' , is given by

$$E(u, v, u', v') = \frac{u'^2}{2} + \frac{v'^2}{2\gamma} + \nu \frac{u^2 + v^2}{2} + \frac{(u^2 + v^2)^2}{4}. \tag{1.5}$$

Our main result is the following.

THEOREM 1.1. Assume that $\nu > 0$ and let

$$\gamma \in ((m + 1)(2m + 1), (m + 1)(2m + 3)),$$

for some nonnegative integer m . There exists $C > 0$ such that if $E_0 \geq C$, then there exists a two-dimensional submanifold \mathcal{M} of the (three-dimensional) manifold $\{E(u, v, u', v') = E_0\}$ with the following property. If $(u_0, v_0, u'_0, v'_0) \in \mathcal{M}$, and if (u, v) is the solution of (1.4) with initial data (u_0, v_0, u'_0, v'_0) , then v and v' converge exponentially to 0 as $t \rightarrow \infty$, and there exists a solution w of the equation $w'' + \nu w + w^3 = 0$ with energy $\frac{w'^2}{2} + \nu \frac{w^2}{2} + \frac{w^4}{4} = E_0$ such that $u - w$ and $u' - w'$ converge exponentially to 0 as $t \rightarrow \infty$.

Note that, given an integer $j \geq 1$, there exists $k > j$ such that $\gamma = k^2/j^2$ falls in one of the intervals defined in Theorem 1.1. Thus, since the system (1.3) is time reversible, Theorem 1.1 indeed proves that a simple mode $\sqrt{\frac{2}{\pi}}\omega(t) \sin(jx)$ of sufficiently large energy has indeed a nontrivial unstable manifold.

In fact, more can be said. The first two intervals in Theorem 1.1 are $(1, 3)$ and $(6, 10)$. If we set $k = 3j, \forall j \geq 1$, then $\gamma = k^2/j^2 = 9$ independent of j . Theorem 1.1 thus implies the existence of an energy level above which all simple modes are unstable.

The reader may therefore wonder why, if $\gamma = 9$ suffices to prove instability, the complete sequence of intervals in Theorem 1.1 was analyzed. To answer this question, we remark that, while the proof of Theorem 1.1 for just the interval $(1, 3)$ is less technical than the full result, it does not suffice to show instability of all simple modes (consider $j = 1$). Once the second interval $(6, 10)$ is included, however, the rest requires little additional work. Moreover, the sequence of intervals arises in a natural way using Jacobi polynomials.

Theorem 1.1 is proved by showing that a periodic solution of system (1.4) with sufficiently large energy and such that $v \equiv 0$ gives rise to a hyperbolic fixed point of the induced Poincaré map. The details of the proof follow the same outline as in our previous paper [3]; however, several new ingredients are needed to study the linearization of the Poincaré map. In particular, the limiting form (for high energy) of the linearized Poincaré map is defined using the system

$$\begin{cases} w'' + w^3 = 0, \\ z'' + \gamma w^2 z = 0. \end{cases} \quad (1.6)$$

(See Sec. 3 below.) The second equation in (1.6) is an example of Hill's equation. In Chapter 2 of [6], the equation

$$z'' + (\gamma + Q)z = 0, \quad (1.7)$$

where γ is a parameter, is studied extensively. In particular, the real line is partitioned into a sequence of intervals, and the behavior of solutions to (1.7) depends on which interval contains γ . Even though γ in (1.6) appears in a different part of the equation than in (1.7), our results concerning (1.6) are of the same type as in [6], except that we can also specify the intervals exactly.

The paper is organized as follows. In Sec. 2, we reduce the problem to the study of the linearized Poincaré map, which we do in Sec. 3 by considering the limiting system (1.6). Section 4 is devoted to several extensions of Theorem 1.1.

The authors are indebted to M. Balabane for some interesting remarks about polynomials, as well as to A. Haraux who suggested to us that the ideas in [3] could apply to the model (1.1).

2. Reduction to the linearized system. Throughout this section, $E_0 > 0$ is fixed. We define the mapping $T : \mathcal{U} \rightarrow \mathbf{R}^2$, where

$$\mathcal{U} = \left\{ (a, b) \in \mathbf{R}^2; \frac{b^2}{2\gamma} + \nu \frac{a^2}{2} + \frac{a^4}{4} < E_0 \right\}.$$

Given $(a, b) \in \mathcal{U}$, we consider the solution (u, v) of (1.4) with initial data $(u, v, u', v')(0) = (0, a, u'_0, b)$, where $u'_0 > 0$ is defined by

$$\frac{u_0'^2}{2} + \frac{b^2}{2\gamma} + \nu \frac{a^2}{2} + \frac{a^4}{4} = E_0.$$

Since $u(0) = 0$ and $u'(0) > 0$, it follows that $u(t) > 0$ for $t > 0$ and small. On the other hand, multiplying the equation for u by $\sin(t/\sqrt{\nu})$ and integrating twice by parts on $(0, \pi/\sqrt{\nu})$, one sees easily that u must have a zero on $(0, \pi/\sqrt{\nu})$. We denote by τ the first positive zero of u . Observe that τ itself depends on (a, b) . Finally, we define the mapping T by

$$T(a, b) = -(v(\tau), v'(\tau)), \tag{2.1}$$

for all $(a, b) \in \mathcal{U}$. It is clear that T is of class C^1 and that

$$T(0, 0) = (0, 0). \tag{2.2}$$

We next define the linear operator $L \in \mathcal{L}(\mathbf{R}^2)$ as follows. Let $w'_0 > 0$ be defined by

$$\frac{w_0'^2}{2} = E_0,$$

i.e., $w'_0 = \sqrt{2E_0}$, and let w be the solution of the equation

$$w'' + \nu w + w^3 = 0,$$

with initial data $w(0) = 0$ and $w'(0) = w'_0$. Since $w(0) = 0$ and $w'(0) > 0$, it follows that $w(t) > 0$ for $t > 0$ and small. On the other hand, by the same reasoning as above, w must have a zero on $(0, \pi/\sqrt{\nu})$; we let ρ denote the first positive zero of w . Given $(a, b) \in \mathbf{R}^2$, let z be the solution of the (linear) equation

$$z'' + \gamma(\nu + w^2)z = 0,$$

with initial data $z(0) = a$ and $z'(0) = b$. We define $L \in \mathcal{L}(\mathbf{R}^2)$ by

$$L(a, b) = -(z(\rho), z'(\rho)). \tag{2.3}$$

for all $(a, b) \in \mathbf{R}^2$. T and L are related as follows.

PROPOSITION 2.1. $DT(0, 0) = L$.

Proof. We know that $DT(0, 0)$ exists. To prove it equals L , it suffices (since $T(0, 0) = (0, 0)$) to show that

$$\lim_{\varepsilon \downarrow 0} \frac{T(\varepsilon(a, b))}{\varepsilon} = L(a, b), \tag{2.4}$$

for all $(a, b) \in \mathbf{R}^2$. Fix $(a, b) \in \mathbf{R}^2$, and for $\varepsilon > 0$ small enough, let $(u_\varepsilon, v_\varepsilon)$ be the solution of (1.4) with initial data $(u_\varepsilon, v_\varepsilon, u'_\varepsilon, v'_\varepsilon)(0) = (0, \varepsilon a, c_\varepsilon, \varepsilon b)$, where $c_\varepsilon > 0$ is defined by

$$\frac{c_\varepsilon^2}{2} + \frac{\varepsilon^2 b^2}{2\gamma} + \nu \frac{\varepsilon^2 a^2}{2} + \frac{\varepsilon^4 a^4}{4} = E_0, \tag{2.5}$$

and let τ_ε be the first positive zero of u_ε (see the definition of T). Set $z_\varepsilon(t) = v_\varepsilon(t)/\varepsilon$. Then

$$\begin{cases} u_\varepsilon'' + (\nu + u_\varepsilon^2 + \varepsilon^2 z_\varepsilon^2)u_\varepsilon = 0, \\ z_\varepsilon'' + \gamma(\nu + u_\varepsilon^2 + \varepsilon^2 z_\varepsilon^2)z_\varepsilon = 0, \\ u_\varepsilon(0) = 0, \quad u'_\varepsilon(0) = c_\varepsilon, \quad z_\varepsilon(0) = a, \quad z'_\varepsilon(0) = b. \end{cases}$$

It follows from (2.5) that $\lim_{\varepsilon \downarrow 0} c_\varepsilon = \sqrt{2E_0}$. Also,

$$T(\varepsilon(a, b)) = -(v_\varepsilon(\tau_\varepsilon), v'_\varepsilon(\tau_\varepsilon)) = -\varepsilon(z_\varepsilon(\tau_\varepsilon), z'_\varepsilon(\tau_\varepsilon));$$

so,

$$\frac{T(\varepsilon(a, b))}{\varepsilon} = -(z_\varepsilon(\tau_\varepsilon), z'_\varepsilon(\tau_\varepsilon)).$$

By continuous dependence, it is clear that $(u_\varepsilon, z_\varepsilon)$ converges to the solution (w, z) of

$$\begin{cases} w'' + \nu w + w^3 = 0, \\ z'' + \gamma(\nu + w^2)z = 0, \\ w(0) = 0, \quad w'(0) = \sqrt{2E_0}, \quad z(0) = a, \quad z'(0) = b, \end{cases}$$

in $C^1([0, T])$ for every $T > 0$. On the other hand, $L(a, b) = -(z(\rho), z'(\rho))$, where ρ is the first positive zero of w . Therefore, to establish (2.4), it suffices to show that $\tau_\varepsilon \rightarrow \rho$, as $\varepsilon \downarrow 0$. This, however, is obvious since $w'(0) \neq 0, w'(\rho) \neq 0, w > 0$ on $(0, \rho)$, and $(u_\varepsilon, z_\varepsilon) \rightarrow (w, z)$ in $C^1([0, \rho + 1])$. This concludes the proof. \square

The main result of this section is the following.

THEOREM 2.2. Fix $\gamma \in ((m + 1)(2m + 1), (m + 1)(2m + 3))$ for some nonnegative integer m . If E_0 is sufficiently large, then there exist a nontrivial C^1 -parametrized curve \mathcal{C} in \mathcal{U} passing through $(0, 0)$ that is invariant under the action of T and a constant $\delta \in (0, 1)$ such that $|T(P)| \leq \delta|P|$ for all $P \in \mathcal{C}$.

The proof of Theorem 2.2 relies on the following result, which will be proved in Sec. 3.

THEOREM 2.3. Assume that $\gamma \in ((m + 1)(2m + 1), (m + 1)(2m + 3))$ for some nonnegative integer m . If E_0 is sufficiently large, then the eigenvalues of L are of the form λ and λ^{-1} for some $\lambda \in \mathbf{R}$ such that $0 < |\lambda| < 1$.

Proof of Theorem 2.2 (assuming Theorem 2.3). Using coordinates on \mathbf{R}^2 in which L is diagonal, we have $L(x, y) = (\lambda x, \lambda^{-1}y)$. Proposition 2.1 then implies that $T(x, y) = (\lambda x, \lambda^{-1}y) + F(x, y)$, for all (x, y) in a neighborhood of $(0, 0)$, where F is C^1 and $F(0, 0) = 0, DF(0, 0) = 0$. The result now follows from Lemma 5.1, p. 234 of Hartman [5]. \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $P = (v_0, v'_0) \in \mathcal{C}$, the curve constructed in Theorem 2.2. It follows that

$$|T^n(P)| \leq \delta^n |P|, \tag{2.6}$$

for all integers $n \geq 1$. Let $u'_0 > 0$ be defined by

$$\frac{u_0^2}{2} + \frac{v_0'^2}{2\gamma} + \nu \frac{v_0^2}{2} + \frac{v_0^4}{4} = E_0, \tag{2.7}$$

and let (u, v) be the solution of (1.4) with initial data $(u, v, u', v')(0) = (0, v_0, u'_0, v'_0)$. We denote by $(\tau_n)_{n \geq 1}$ the sequence of positive zeroes of u . It follows that $T(v_0, v'_0) = -(v(\tau_1), v'(\tau_1)) \in \mathcal{C}$, and $u'(\tau_1) < 0$ is given by

$$\frac{u'(\tau_1)^2}{2} + \frac{v'(\tau_1)^2}{2\gamma} + \nu \frac{v(\tau_1)^2}{2} + \frac{v(\tau_1)^4}{4} = E_0.$$

If $\tilde{u}(t) = -u(\tau_1 + t)$ and $\tilde{v}(t) = -v(\tau_1 + t)$, then (\tilde{u}, \tilde{v}) solves (1.4), and $\tau_2 - \tau_1$ is the first positive zero of \tilde{u} . It follows easily that $T(-v(\tau_1), -v'(\tau_1)) = (v(\tau_2), v'(\tau_2))$. An obvious iteration argument shows that

$$T^n(v_0, v'_0) = (-1)^n(v(\tau_n), v'(\tau_n)), \tag{2.8}$$

and that $u'(\tau_n)$ satisfies both $(-1)^n u'(\tau_n) > 0$ and

$$\frac{u'(\tau_n)^2}{2} + \frac{v'(\tau_n)^2}{2\gamma} + \nu \frac{v(\tau_n)^2}{2} + \frac{v(\tau_n)^4}{4} = E_0. \tag{2.9}$$

Furthermore, successive zeroes of u differ by less than $\pi/\sqrt{\nu}$ (multiply the equation by $\sin(t/\sqrt{\nu})$ and integrate by parts), i.e.,

$$\tau_{n+1} - \tau_n \leq \pi/\sqrt{\nu}. \tag{2.10}$$

By conservation of energy, $u^2 + v^2$ is a bounded function of t ; so it follows from (1.4) and (2.10) that there exists a constant C , independent of n , such that

$$v(t)^2 + v'(t)^2 \leq C(v(\tau_n)^2 + v'(\tau_n)^2), \tag{2.11}$$

for all $t \in [\tau_n, \tau_{n+1}]$. Applying now (2.6), (2.8), and (2.11), we get

$$v(t)^2 + v'(t)^2 \leq C\delta^n, \tag{2.12}$$

for all $t \in [\tau_n, \tau_{n+1}]$. Let $\varepsilon > 0$ be such that $e^{-\varepsilon\pi/\sqrt{\nu}} = \delta$. It follows from (2.10) that $\tau_n \leq n\pi/\sqrt{\nu}$; so, for $t \in [\tau_n, \tau_{n+1}]$ we have

$$e^{-\varepsilon t} \geq e^{-\varepsilon\tau_{n+1}} \geq e^{-\varepsilon(n+1)\pi/\sqrt{\nu}} = e^{-\varepsilon\pi/\sqrt{\nu}}\delta^n.$$

Therefore, it follows from (2.12) that there exists C such that

$$v(t)^2 + v'(t)^2 \leq Ce^{-\varepsilon t}, \tag{2.13}$$

for all $t \geq 0$. This, together with (2.9), implies that there exists a constant C such that

$$|(-1)^n u'(\tau_n) - \sqrt{2E_0}| \leq Ce^{-\varepsilon\tau_n}. \tag{2.14}$$

Let now w be the solution of the equation

$$w'' + \nu w + w^3 = 0, \tag{2.15}$$

with the initial data $w(0) = 0$ and $w'(0) = \sqrt{2E_0}$. Comparing (2.15) with the first equation in (1.4), and using the estimates (2.13) and (2.14) one can prove that there exists C such that

$$\|(-1)^n u(\tau_n + \cdot) - w(\cdot)\|_{C^1([0, 2\pi])} \leq Ce^{-\varepsilon\tau_n}, \tag{2.16}$$

for all $n \geq 1$. If we denote by ρ the first positive zero of w , it follows from (2.16) that there exists C such that $|\tau_{n+1} - \tau_n - \rho| \leq Ce^{-\varepsilon\tau_n}$. Also, (2.16) implies that $\tau_{n+1} - \tau_n \geq \rho/2$ for n large enough; so $\tau_n \geq nc$ for some constant $c > 0$ and for n large. Therefore, since $\tau_n - n\rho = \sum_{j=0}^{n-1} (\tau_{j+1} - \tau_j - \rho)$, where $\tau_0 = 0$, it follows that

$$|\tau_n - n\rho - \theta| \leq Ce^{-\varepsilon\tau_n}, \tag{2.17}$$

where $\theta = \sum_{j=0}^{\infty} (\tau_{j+1} - \tau_j - \rho)$. Since w is clearly ρ anti-periodic, i.e., $w(n\rho + \cdot) = (-1)^n w(\cdot)$, it follows from (2.17) and continuous dependence of the solutions of Eq. (2.15) on the initial values that

$$\|w(\cdot) - (-1)^n w(\tau_n - \theta + \cdot)\|_{C^1([0, 2\pi])} = \|w(n\rho + \cdot) - w(\tau_n - \theta + \cdot)\|_{C^1([0, 2\pi])} \leq Ce^{-\varepsilon\tau_n}.$$

Therefore, by (2.16),

$$\|u(\tau_n + \cdot) - w(\tau_n - \theta + \cdot)\|_{C^1([0, 2\pi])} \leq Ce^{-\varepsilon\tau_n},$$

for all $n \geq 1$. This implies that

$$\|u(t + \cdot) - w(t - \theta + \cdot)\|_{C^1([0, 2\pi])} \leq Ce^{-\varepsilon t},$$

for all $t \geq 0$. This estimate, together with (2.13), implies that (u, v) has the asymptotic properties specified in Theorem 1.1. The theorem therefore follows by setting

$$\mathcal{M} = \bigcup_{(a,b) \in \mathbf{C}} \bigcup_{t \in \mathbf{R}} (u(t), v(t), u'(t), v'(t)),$$

where (u, v) is the solution of (1.1) with initial data $(u, v, u', v')(0) = (0, a, u'_0, b)$ and $u'_0 > 0$ is determined by (2.7). \square

3. Analysis of the linearized system. This section is devoted to the proof of Theorem 2.3. In fact, we will prove a more precise result, giving a complete description of the operator B_γ defined as follows. We denote by w the solution of

$$w'' + w^3 = 0, \tag{3.1}$$

with the initial values $w(0) = 0$ and $w'(0) = 1$. We denote by θ the first positive zero of w . w is odd and symmetric about $\theta/2$, i.e., $w(t) = w(\theta - t)$. Given $\gamma \geq 0$, we consider the equation

$$z'' + \gamma w^2 z = 0. \tag{3.2}$$

For $(a, b) \in \mathbf{R}^2$, let z be the solution of (3.2) with the initial values $z(0) = a$ and $z'(0) = b$. The mapping $B_\gamma \in \mathcal{L}(\mathbf{R}^2)$ is defined by

$$B_\gamma(a, b) = -(z(\theta), z'(\theta)).$$

We define the four sequences $(\mu_n^-)_{n \geq 0}$, $(\mu_n^+)_{n \geq 0}$, $(\gamma_n^-)_{n \geq 0}$, and $(\gamma_n^+)_{n \geq 0}$ by

$$\begin{aligned} \mu_n^- &= 2n(4n + 1), & \gamma_n^+ &= (2n + 1)(4n + 1), \\ \mu_n^+ &= (2n + 1)(4n + 3), & \gamma_n^- &= 2(n + 1)(4n + 3); \end{aligned} \tag{3.3}$$

we observe that $\mu_n^- < \gamma_n^+ < \mu_n^+ < \gamma_n^- < \mu_{n+1}^-$ for all $n \geq 0$. The main result of this section is the following.

THEOREM 3.1. The mapping B_γ has the following properties.

- (i) If $\gamma \in (\gamma_n^+, \mu_n^+)$ for some $n \geq 0$, then the eigenvalues of B_γ are λ and $1/\lambda$ for some $\lambda \in (0, 1)$.
- (ii) If $\gamma \in (\gamma_n^-, \mu_{n+1}^-)$ for some $n \geq 0$, then the eigenvalues of B_γ are $-\lambda$ and $-1/\lambda$ for some $\lambda \in (0, 1)$.
- (iii) If $\gamma \in (\mu_n^-, \gamma_n^+) \cup (\mu_n^+, \gamma_n^-)$ for some $n \geq 0$, then the eigenvalues of B_γ are $e^{i\omega}$ and $e^{-i\omega}$ for some $\omega \in (0, \pi)$.
- (iv) If $\gamma = \mu_n^\pm$ for some $n \geq 0$, then $B_\gamma = \begin{pmatrix} \pm 1 & \pm a \\ 0 & \pm 1 \end{pmatrix}$ for some $a > 0$.
- (v) If $\gamma = \gamma_n^\pm$ for some $n \geq 0$, then $B_\gamma = \begin{pmatrix} \pm 1 & 0 \\ \pm a & \pm 1 \end{pmatrix}$ for some $a > 0$.

Proof of Theorem 2.3 (assuming Theorem 3.1). Let γ be as in the statement of Theorem 2.3. Given $\varepsilon > 0$, let w_ε be the solution of the equation

$$w_\varepsilon'' + \varepsilon w_\varepsilon + w_\varepsilon^3 = 0,$$

with the initial data $w_\varepsilon(0) = 0$ and $w_\varepsilon'(0) = 1$, and let θ_ε be the first positive zero of w_ε . For $(a, b) \in \mathbf{R}^2$, let z_ε be the solution of the equation

$$z_\varepsilon'' + \gamma(\varepsilon + w_\varepsilon^2)z_\varepsilon = 0,$$

with the initial data $z_\varepsilon(0) = a$ and $z_\varepsilon'(0) = b$. We define the operator $B_{\gamma,\varepsilon} \in \mathcal{L}(\mathbf{R}^2)$ by $B_{\gamma,\varepsilon}(a, b) = -(z_\varepsilon(\theta_\varepsilon), z_\varepsilon'(\theta_\varepsilon))$. One verifies easily that $B_{\gamma,\varepsilon} \rightarrow B_\gamma$ as $\varepsilon \downarrow 0$. Therefore, it follows from Theorem 3.1 that there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, $B_{\gamma,\varepsilon}$ has a real eigenvalue $\lambda_{\gamma,\varepsilon}$ with $|\lambda_{\gamma,\varepsilon}| \in (0, 1)$; so, there exists $(a, b) \in \mathbf{R}^2$ such that $(z_\varepsilon(\theta_\varepsilon), z_\varepsilon'(\theta_\varepsilon)) = -\lambda_{\gamma,\varepsilon}(a, b)$. Let now

$$w(t) = \sqrt{\frac{\nu}{\varepsilon}} w_\varepsilon \left(t \sqrt{\frac{\nu}{\varepsilon}} \right), \quad z(t) = z_\varepsilon \left(t \sqrt{\frac{\nu}{\varepsilon}} \right).$$

It follows that

$$\begin{cases} w'' + \nu w + w^3 = 0, \\ z'' + \gamma(\nu + w^2)z = 0. \end{cases}$$

Furthermore, $w(0) = 0$ and $w'(0) = \nu/\varepsilon$, and if we denote by ρ the first positive zero of w , then $\rho = \sqrt{\varepsilon/\nu}\theta_\varepsilon$. A direct calculation shows that $(z(\rho), z'(\rho)) = -\lambda_{\gamma,\varepsilon}(z(0), z'(0))$. Therefore, if $\sqrt{2E_0} \geq \nu/\varepsilon_0$, then L has the eigenvalue $\lambda_{\gamma,\varepsilon}$ with $\varepsilon = (2E_0)^{-1/2}\nu$. Since \tilde{z} defined by $\tilde{z}(t) = z(\rho - t)$ solves the same equation as does z , it follows that $1/\lambda_{\gamma,\varepsilon}$ is the second eigenvalue of L . This completes the proof of the theorem. \square

The rest of this section is devoted to the proof of Theorem 3.1, which is long and requires several lemmas. We introduce four sequences of functions $(v_n^\pm)_{n \geq 0}$ and $(u_n^\pm)_{n \geq 0}$ as follows. For $n \geq 0$, we set

$$\begin{aligned} v_n^-(t) &= P_n(x(t)), & u_n^+(t) &= w(t)Q_n(x(t)), \\ v_n^+(t) &= w'(t)R_n(x(t)), & u_n^-(t) &= w(t)w'(t)S_n(x(t)); \end{aligned} \tag{3.4}$$

where

$$x(t) = w(t)^{-1} - 1,$$

and P_n, Q_n, R_n , and S_n are the Jacobi polynomials (see [9, §2.2 and §2.4])

$$\begin{aligned} P_n(x) &= a_n P_n^{(-\frac{1}{2}, -\frac{1}{4})}(x), & Q_n(x) &= b_n P_n^{(-\frac{1}{2}, \frac{1}{4})}(x), \\ R_n(x) &= c_n P_n^{(\frac{1}{2}, -\frac{1}{4})}(x), & S_n(x) &= d_n P_n^{(\frac{1}{2}, \frac{1}{4})}(x), \end{aligned}$$

and where the constants a_n, b_n, c_n , and d_n are chosen so that

$$P_n(-1) = Q_n(-1) = R_n(-1) = S_n(-1) = 1.$$

The introduction of these sequences of functions is justified by the following result.

LEMMA 3.2. Let the functions $(v_n^\pm)_{n \geq 0}$ and $(u_n^\pm)_{n \geq 0}$ be defined by (3.4) and the numbers $(\mu_n^\pm)_{n \geq 0}$ and $(\gamma_n^\pm)_{n \geq 0}$ be defined by (3.3).

- (i) $z = v_n^\pm$ is a solution of the equation $z'' + \mu_n^\pm w^2 z = 0$.
- (ii) For every $n \geq 0$, $v_n^\pm(0) = \mp v_n^\pm(\theta) = 1$ and $(v_n^\pm)'(0) = (v_n^\pm)'(\theta) = 0$. Moreover, v_n^- has exactly $2n$ zeroes on $(0, \theta)$ and v_n^+ has exactly $2n + 1$ zeroes on $(0, \theta)$.
- (iii) $z = u_n^\pm$ is a solution of the equation $z'' + \gamma_n^\pm w^2 z = 0$.
- (iv) For every $n \geq 0$, $u_n^\pm(0) = u_n^\pm(\theta) = 0$ and $(u_n^\pm)'(0) = \mp (u_n^\pm)'(\theta) = 1$. Moreover, u_n^+ has exactly $2n$ zeroes on $(0, \theta)$ and u_n^- has exactly $2n + 1$ zeroes on $(0, \theta)$.

Proof. Properties (i) and (iii) follow from a direct calculation, since the Jacobi polynomial $y = P_n^{(a,b)}(x)$ satisfies the following ODE (see [9, Theorem 4.2.1, p. 60]):

$$(1 - x^2)y'' + [b - a - (a + b + 2)x]y' + n(n + a + b + 1)y = 0.$$

Since w is symmetric about $\theta/2$ and since $w(\theta/2)^4 = 2$, it follows that each real number in $(-1, 1)$ is achieved exactly twice by the function $w(t)^4 - 1$ in the interval $(0, \theta)$. Therefore, since the Jacobi polynomial $P_n^{(a,b)}$ has exactly n zeroes which are located in $(-1, 1)$ (see [9, Theorem 3.3.1, p. 44]), it follows that the functions $P_n(w^4 - 1), Q_n(w^4 - 1), R_n(w^4 - 1)$, and $S_n(w^4 - 1)$ have exactly $2n$ zeroes in $(0, \theta)$. Properties (ii) and (iv) follow immediately. \square

LEMMA 3.3. The mapping B_γ has the following properties:

- (i) $\det B_\gamma = 1$.
- (ii) If λ is an eigenvalue of B_γ corresponding to the eigenvector (a, b) , then $(a, -b)$ is also an eigenvector, corresponding to the eigenvalue λ^{-1} .
- (iii) If B_γ has the eigenvalue $\lambda = \pm 1$, then at least one of the vectors $(0, 1)$ and $(1, 0)$ is an eigenvector corresponding to the eigenvalue λ .

Proof. Let z_0 and z_1 be the solutions of (3.2) with the initial data $(a, b) = (0, 1)$ and $(a, b) = (1, 0)$, respectively, and set $(\alpha, \beta) = -(z_0(\theta), z_0'(\theta))$ and $(\sigma, \delta) = -(z_1(\theta), z_1'(\theta))$. In other words, $(\alpha, \beta) = B_\gamma(0, 1)$ and $(\sigma, \delta) = B_\gamma(1, 0)$. It follows that

$$B_\gamma = \begin{pmatrix} \sigma & \alpha \\ \delta & \beta \end{pmatrix}.$$

Note that if z solves (3.2), then $z(\theta - t)$ also solves (3.2). Therefore, $B_\gamma(z(\theta), -z'(\theta)) = (-a, b)$. Choosing successively $(a, b) = (0, 1)$ and $(a, b) = (1, 0)$, we obtain

$$\begin{cases} \beta^2 = \sigma^2 = 1 + \alpha\delta; \\ \alpha(\beta - \sigma) = \delta(\beta - \sigma) = 0. \end{cases}$$

It follows easily that either

$$B_\gamma = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}, \tag{3.5}$$

or else $\det B_\gamma = 1$. We first prove (i) by contradiction, and so we assume that $\det B_\gamma \neq 1$. It follows from what precedes that (3.5) holds. Suppose first that

$$B_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows that $z_0(\theta) = 0, z'_0(\theta) = 1, z_1(\theta) = -1$, and $z'_1(\theta) = 0$. This implies that $z_0(\theta - t) = -z_0(t)$ and that $z_1(\theta - t) = -z_1(t)$; so, $z_0(\theta/2) = z_1(\theta/2) = 0$. Since z_0 and z_1 solve the same linear equation (i.e., (3.2)), it follows that z_0 and z_1 are linearly dependent, which is absurd. By an analogous argument, one excludes

$$B_\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};$$

so (i) is verified. Next, assume $B_\gamma(a, b) = \lambda(a, b)$. It follows from (i) that $\lambda \neq 0$. Let z be the solution of (3.2) with the initial data $(z(0), z'(0)) = (a, b)$, and set $u(t) = z(\theta - t)$. Since $u(0) = z(\theta) = -\lambda a, u'(0) = -z'(\theta) = \lambda b, u(\theta) = z(0) = a$, and $u'(\theta) = -z'(0) = -b$, it follows that $B_\gamma(-\lambda a, \lambda b) = (-a, b)$. i.e., $B_\gamma(a, -b) = \lambda^{-1}(a, -b)$. This proves property (ii). Assume now that $\lambda = \pm 1$ is an eigenvalue of B_γ and let (a, b) be an eigenvector corresponding to the eigenvalue λ . It follows from properties (i) and (ii) that (since $\lambda = \lambda^{-1}$) $B_\gamma(a, -b) = \lambda(a, -b)$. If (a, b) and $(a, -b)$ are linearly independent, then $B_\gamma = \lambda I$ and the conclusion of (iii) follows. If (a, b) and $(a, -b)$ are linearly dependent, then either $a = 0$ or else $b = 0$, and the conclusion also follows. \square

We now study nodal properties of the solutions of Eq. (3.2). We begin with the following result.

LEMMA 3.4. Let $\gamma, \tilde{\gamma} \geq 0$ and let $u, \tilde{u} \not\equiv 0$ be such that $u'' + \gamma w^2 u = 0$ and $\tilde{u}'' + \tilde{\gamma} w^2 \tilde{u} = 0$.

- (i) If $\gamma > \tilde{\gamma}$ and if $\sigma_1 < \sigma_2$ are two consecutive zeroes of \tilde{u} , then u has at least one zero on (σ_1, σ_2) .
- (ii) If $\gamma = \tilde{\gamma}$ and if $\sigma_1 < \sigma_2$ are two consecutive zeroes of \tilde{u} , then either u and \tilde{u} are linearly dependent, or else u has exactly one zero on (σ_1, σ_2) .
- (iii) If $\gamma > \tilde{\gamma}$ and if $\sigma_1 < \sigma_2$ are such that $\tilde{u}'(\sigma_1) = 0, \tilde{u}(\sigma_2) = 0$, and $\tilde{u} \neq 0$ on (σ_1, σ_2) , and if $u(\sigma_1) > 0$ and $u'(\sigma_1) \leq 0$ then u has at least one zero on (σ_1, σ_2) .

Proof. (i) After possibly changing \tilde{u} to $-\tilde{u}$, we may assume $\tilde{u} > 0$ on (σ_1, σ_2) . We argue by contradiction and we assume (after possibly changing u to $-u$) that $u > 0$ on (σ_1, σ_2) . We have

$$(u\tilde{u}' - u'\tilde{u})' = (\gamma - \tilde{\gamma})w^2 u\tilde{u} > 0,$$

almost everywhere on (σ_1, σ_2) (“almost everywhere” due to the possible zeroes of w on (σ_1, σ_2)). After integration over (σ_1, σ_2) , it follows that

$$u(\sigma_2)\tilde{u}'(\sigma_2) - u(\sigma_1)\tilde{u}'(\sigma_1) > 0,$$

which is absurd since $\tilde{u}'(\sigma_1) > 0, \tilde{u}'(\sigma_2) < 0, u(\sigma_1) \geq 0$, and $u(\sigma_2) \geq 0$. This proves (i).

(ii) If u and \tilde{u} are not linearly dependent, then in particular $u(\sigma_1) \neq 0$ and $u(\sigma_2) \neq 0$. After possibly changing \tilde{u} to $-\tilde{u}$, we may assume $\tilde{u} > 0$ on (σ_1, σ_2) . We argue by contradiction and we assume (after possibly changing u to $-u$) that $u > 0$ on $[\sigma_1, \sigma_2]$. Arguing as in the proof of (i), we obtain

$$u(\sigma_2)\tilde{u}'(\sigma_2) - u(\sigma_1)\tilde{u}'(\sigma_1) = 0,$$

which is absurd since $\tilde{u}'(\sigma_1) > 0$, $\tilde{u}'(\sigma_2) < 0$, $u(\sigma_1) > 0$, and $u(\sigma_2) > 0$. Therefore, u has a zero on (σ_1, σ_2) . If u had two zeroes on (σ_1, σ_2) then, by exchanging the roles of u and \tilde{u} , we would obtain that \tilde{u} would have a zero on (σ_1, σ_2) , which is absurd. This proves (ii).

(iii) We may assume that $u(\sigma_1) = \tilde{u}(\sigma_1) = 1$ (since if $\tilde{u}(\sigma_1) = 0$, then $\tilde{u} \equiv 0$). We argue by contradiction and we assume that $u > 0$ on (σ_1, σ_2) . Arguing as in the proof of (i), we find

$$u(\sigma_2)\tilde{u}'(\sigma_2) + u'(\sigma_1)\tilde{u}(\sigma_1) > 0,$$

which is absurd. This completes the proof. \square

COROLLARY 3.5. Let $\gamma, \tilde{\gamma} \geq 0$ and let $u, \tilde{u} \not\equiv 0$ be such that $u'' + \gamma w^2 u = 0$ and $\tilde{u}'' + \tilde{\gamma} w^2 \tilde{u} = 0$. Let $a < b$ and assume that u has exactly n zeroes on (a, b) and that \tilde{u} has exactly \tilde{n} zeroes on (a, b) . If either $u(a) = u(b) = \tilde{u}(a) = \tilde{u}(b) = 0$ or $u'(a) = u'(b) = \tilde{u}'(a) = \tilde{u}'(b) = 0$, then the following properties hold.

- (i) If $n > \tilde{n}$, then $\gamma > \tilde{\gamma}$.
- (ii) If $n = \tilde{n}$, then $\gamma = \tilde{\gamma}$ and u and \tilde{u} are linearly dependent.
- (iii) If $\gamma > \tilde{\gamma}$, then $n > \tilde{n}$.
- (iv) If $\gamma = \tilde{\gamma}$, then $n = \tilde{n}$ and u and \tilde{u} are linearly dependent.

Proof. Assume first that $u(a) = u(b) = \tilde{u}(a) = \tilde{u}(b) = 0$. Let $a = \sigma_1 < \dots < \sigma_{\tilde{n}+2} = b$ be the sequence of zeroes of \tilde{u} on $[a, b]$. If $\gamma > \tilde{\gamma}$, then by applying Lemma 3.4(i) on each interval of the form (σ_j, σ_{j+1}) , we obtain that u has at least $\tilde{n} + 1$ zeroes on (a, b) . Hence (iii). If $\gamma = \tilde{\gamma}$, then u and \tilde{u} solve the same equation. Since this equation is linear and since $u(a) = \tilde{u}(a) = 0$, it follows that u and \tilde{u} are linearly dependent, and in particular that $\tilde{n} = n$. Hence (iv). (i) and (ii) follow immediately. Assume now that $u'(a) = u'(b) = \tilde{u}'(a) = \tilde{u}'(b) = 0$. Since $\tilde{u} \not\equiv 0$, we have $\tilde{u}(a) \neq 0$. Therefore, without loss of generality, we may assume $\tilde{u}(a) = 1$. We first consider the case $\tilde{n} \geq 1$. Let $(\sigma_i)_{1 \leq i \leq \tilde{n}}$ be the sequence of zeroes of \tilde{u} on (a, b) . If $\gamma > \tilde{\gamma}$, then by applying Lemma 3.4(i) on each interval of the form (σ_j, σ_{j+1}) (if $\tilde{n} \geq 2$) and Lemma 3.4(iii) on the intervals (a, σ_1) and $(\sigma_{\tilde{n}}, b)$, we obtain that u has at least $\tilde{n} + 1$ zeroes on (a, b) . Hence (iii). In the case $\tilde{n} = 0$, we have $\tilde{\gamma} = 0$. Indeed, otherwise we have $\tilde{u} > 0$ on (a, b) , which implies that \tilde{u} is strictly concave. This is absurd since $\tilde{u}'(a) = \tilde{u}'(b) = 0$. If $n = 0$, then the same argument shows that $\gamma = 0$, and (iii) follows in this case also. If $\gamma = \tilde{\gamma}$, then u and \tilde{u} solve the same equation. Since this equation is linear and since $u'(0) = \tilde{u}'(0) = 0$, it follows that u and \tilde{u} are linearly dependent, and in particular that $\tilde{n} = n$. Hence (iv). (i) and (ii) follow immediately. \square

Before stating the next result, we introduce some notation. Given $\gamma \geq 0$, we denote by z_γ the solution of (3.2) such that

$$z_\gamma(0) = 0, \quad z'_\gamma(0) = 1;$$

and we denote by \tilde{z}_γ the solution of (3.2) such that

$$\tilde{z}_\gamma(0) = 1, \quad \tilde{z}'_\gamma(0) = 0.$$

With this notation, we have the following result.

LEMMA 3.6. For every $n \geq 0$, there exist $\sigma_n^- \in (\mu_n^-, \gamma_n^+)$ and $\sigma_n^+ \in (\mu_n^+, \gamma_n^-)$ such that

- (i) if $\gamma \in (\mu_n^-, \sigma_n^-)$, then $z_\gamma(\theta) > 0$, $z'_\gamma(\theta) > 0$, $\tilde{z}_\gamma(\theta) > 0$, and $\tilde{z}'_\gamma(\theta) < 0$;
- (ii) if $\gamma \in (\sigma_n^-, \gamma_n^+)$, then $z_\gamma(\theta) > 0$, $z'_\gamma(\theta) < 0$, $\tilde{z}_\gamma(\theta) < 0$, and $\tilde{z}'_\gamma(\theta) < 0$;
- (iii) if $\gamma \in (\gamma_n^+, \mu_n^+)$, then $z_\gamma(\theta) < 0$, $z'_\gamma(\theta) < 0$, $\tilde{z}_\gamma(\theta) < 0$, and $\tilde{z}'_\gamma(\theta) < 0$;
- (iv) if $\gamma \in (\mu_n^+, \sigma_n^+)$, then $z_\gamma(\theta) < 0$, $z'_\gamma(\theta) < 0$, $\tilde{z}_\gamma(\theta) < 0$, and $\tilde{z}'_\gamma(\theta) > 0$;
- (v) if $\gamma \in (\sigma_n^+, \gamma_n^-)$, then $z_\gamma(\theta) < 0$, $z'_\gamma(\theta) > 0$, $\tilde{z}_\gamma(\theta) > 0$, and $\tilde{z}'_\gamma(\theta) > 0$;
- (vi) if $\gamma \in (\gamma_n^-, \mu_{n+1}^-)$, then $z_\gamma(\theta) > 0$, $z'_\gamma(\theta) > 0$, $\tilde{z}_\gamma(\theta) > 0$, and $\tilde{z}'_\gamma(\theta) > 0$.

Proof. We consider the functions $f, \tilde{f}, g, \tilde{g} \in C^\infty([0, \infty), \mathbf{R})$ defined by

$$\begin{aligned} f(\gamma) &= z_\gamma(\theta), & g(\gamma) &= z'_\gamma(\theta), \\ \tilde{f}(\gamma) &= \tilde{z}_\gamma(\theta), & \tilde{g}(\gamma) &= \tilde{z}'_\gamma(\theta). \end{aligned}$$

It follows from Lemma 3.2 that

$$\begin{aligned} f(\gamma_n^\pm) &= \tilde{g}(\mu_n^\pm) = 0, \\ g(\gamma_n^\pm) &= \tilde{f}(\mu_n^\pm) = \mp 1, \end{aligned} \tag{3.6}$$

for all $n \geq 0$. We claim that the γ_n^\pm are the only zeroes of f , and that the μ_n^\pm are the only zeroes of \tilde{g} . Indeed, if γ is a zero of f , then $z_\gamma(0) = z_\gamma(\theta) = 0$, and it follows from Corollary 3.5 that γ must be equal to one of the γ_n^\pm 's. Similarly, if γ is a zero of \tilde{g} , then $\tilde{z}'_\gamma(0) = \tilde{z}'_\gamma(\theta) = 0$, and it follows from Corollary 3.5 that γ must be equal to one of the μ_n^\pm 's. Next, set

$$u_\gamma = \frac{\partial z_\gamma}{\partial \gamma}, \quad \tilde{u}_\gamma = \frac{\partial \tilde{z}_\gamma}{\partial \gamma}.$$

It follows that u_γ solves the equation $u''_\gamma + \gamma w^2 u_\gamma + w^2 z_\gamma = 0$ with the initial data $u_\gamma(0) = u'_\gamma(0) = 0$ and that \tilde{u}_γ solves the equation $\tilde{u}''_\gamma + \gamma w^2 \tilde{u}_\gamma + w^2 \tilde{z}_\gamma = 0$ with the initial data $\tilde{u}_\gamma(0) = \tilde{u}'_\gamma(0) = 0$. Since $(u_\gamma z'_\gamma - u'_\gamma z_\gamma)' = w^2 z_\gamma^2$ and $(\tilde{u}_\gamma \tilde{z}'_\gamma - \tilde{u}'_\gamma \tilde{z}_\gamma)' = w^2 \tilde{z}_\gamma^2$, it follows that

$$\begin{aligned} u_\gamma(\theta) z'_\gamma(\theta) - u'_\gamma(\theta) z_\gamma(\theta) &= \int_0^\theta w^2 z_\gamma^2 > 0, \\ \tilde{u}_\gamma(\theta) \tilde{z}'_\gamma(\theta) - \tilde{u}'_\gamma(\theta) \tilde{z}_\gamma(\theta) &= \int_0^\theta w^2 \tilde{z}_\gamma^2 > 0. \end{aligned}$$

This means that

$$\begin{aligned} f'(\gamma)g(\gamma) &> f(\gamma)g'(\gamma), \\ \tilde{f}'(\gamma)\tilde{g}(\gamma) &> \tilde{f}(\gamma)\tilde{g}'(\gamma). \end{aligned} \tag{3.7}$$

In particular, f and f' cannot vanish together, and similarly for g, \tilde{f} , and \tilde{g} . Therefore, the zeroes of f, g, \tilde{f} , and \tilde{g} are isolated; and these functions always change sign at their

zeroes. It also follows from (3.7) that the zeroes of f and g alternate, i.e., that between two consecutive zeroes of f there is a zero of g and between two consecutive zeroes of g there is a zero of f . The same property obviously holds for the zeroes of \tilde{f} and \tilde{g} . Therefore, g must have exactly one zero on every interval of the form (γ_n^+, γ_n^-) and on every interval of the form $(\gamma_n^-, \gamma_{n+1}^+)$. Similarly, \tilde{f} must have exactly one zero on every interval of the form (μ_n^-, μ_n^+) and on every interval of the form (μ_n^+, μ_{n+1}^-) . Furthermore, we claim that the zeroes of g and \tilde{f} coincide. Indeed, if $g(\gamma) = 0$, then $z'_\gamma(\theta) = 0$. Setting

$$v(t) = \frac{1}{z_\gamma(\theta)} z_\gamma(\theta - t)$$

($z_\gamma(\theta) \neq 0$ since $z_\gamma \not\equiv 0$) it follows that v solves Eq. (3.2), that $v(0) = 1, v'(0) = 0$, and $v(\theta) = 0$; so, $\tilde{f}(\gamma) = 0$. The converse statement is proved in the same way. We denote by σ_n^- the unique zero of \tilde{f} , and therefore of g , in (μ_n^-, μ_n^+) and by σ_n^+ the unique zero of \tilde{f} , and therefore of g , in (μ_n^+, μ_{n+1}^-) . We claim that $\sigma_0^- < \gamma_0^+$. Indeed, since $\mu_0^- = 0$, we know that σ_0^- is the first positive zero of g . On the other hand, since $z_0(t) \equiv t$, it follows that $g(0) = 1$. Next, since $z_1 = w$ and $\gamma_0^+ = 1$, it follows that $g(\gamma_0^+) = -1$; so, g has a first zero in $(0, \gamma_0^+)$, and this first zero must be σ_0^- . This proves that $\sigma_0^- < \gamma_0^+$. Since σ_n^- is the unique zero of g in (μ_n^-, μ_n^+) and σ_n^+ the unique zero of g in (μ_n^+, μ_{n+1}^-) , and since g has a unique zero on every interval (γ_n^+, γ_n^-) and on every interval $(\gamma_n^-, \gamma_{n+1}^+)$, it follows easily that

$$\mu_n^- < \sigma_n^- < \gamma_n^+ < \mu_n^+ < \sigma_n^+ < \gamma_n^- < \mu_{n+1}^-.$$

Finally, observe that $z_0(t) \equiv t$ and that $\tilde{z}_0(t) \equiv 1$, so that $f(0) = \theta > 0, g(0) = 1, \tilde{f}(0) = 1$, and $\tilde{g}(0) = 0$. On the other hand, it follows from (3.7) that $\tilde{g}'(0) < 0$. Therefore, for $\gamma > 0$ and small, we have $f(\gamma) > 0, g(\gamma) > 0, \tilde{f}(\gamma) > 0$, and $\tilde{g}(\gamma) < 0$. Since $0 = \mu_0^-$, (i) through (vi) follow now immediately from the property that the zeroes of f are the γ_n^\pm , that the zeroes of \tilde{g} are the γ_n^\pm , and that the zeroes of g and \tilde{f} are the σ_n^\pm . \square

Proof of Theorem 3.1. We use the notation of the proof of Lemma 3.6, and we observe that

$$B_\gamma = - \begin{pmatrix} \tilde{f}(\gamma) & f(\gamma) \\ \tilde{g}(\gamma) & g(\gamma) \end{pmatrix}.$$

It follows that

$$B_{\gamma_n^\pm} = \begin{pmatrix} -\tilde{f}(\gamma_n^\pm) & 0 \\ -\tilde{g}(\gamma_n^\pm) & \pm 1 \end{pmatrix}.$$

In particular, ± 1 is an eigenvalue of $B_{\gamma_n^\pm}$. Since the other eigenvalue of $B_{\gamma_n^\pm}$ is also ± 1 by Lemma 3.3, it follows that $-\tilde{f}(\gamma_n^\pm) = \pm 1$. Finally, it follows easily from Lemma 3.6 that $\mp \tilde{g}(\gamma_n^\pm) > 0$; so

$$B_{\gamma_n^\pm} = \begin{pmatrix} \pm 1 & 0 \\ \pm a_n^\pm & \pm 1 \end{pmatrix},$$

where $a_n^\pm > 0$. This shows property (v), and one proves property (iv) in exactly the same way. Next, since $\det B_\gamma = 1$, it follows that the eigenvalues of B_γ are either $\{\pm 1, \pm 1\}$, or $\{\lambda, \lambda^{-1}\}$ for some $\lambda \in (-1, 1) \setminus \{0\}$, or else $(e^{i\omega}, e^{-i\omega})$ for some $\omega \in (0, \pi)$. Furthermore, by Lemmas 3.3 and 3.6, the only values of γ for which the eigenvalues of B_γ are $\{\pm 1, \pm 1\}$ are the γ_n^\pm and the μ_n^\pm . Therefore, in the interval (μ_n^-, γ_n^+) , the eigenvalues

of B_γ are either of the form $\{\lambda(\gamma), \lambda(\gamma)^{-1}\}$ for some $\lambda(\gamma) \in (-1, 1) \setminus \{0\}$ or else of the form $\{e^{i\omega(\gamma)}, e^{-i\omega(\gamma)}\}$ for some $\omega \in (0, \pi)$. Since the eigenvalues depend continuously on γ and since for μ_n^- they are $\{-1, -1\}$ and for γ_n^+ they are $\{1, 1\}$, they must be of the form $\{e^{i\omega(\gamma)}, e^{-i\omega(\gamma)}\}$ for some $\omega \in (0, \pi)$. This proves the first statement of property (iii). The second statement is proved in exactly the same way. Let now $\gamma \in (\gamma_n^+, \mu_n^+)$. It follows from property (iii) of Lemma 3.6 that

$$B_\gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

for some $a, b, c, d > 0$ such that $ad - bc = 1$. This implies that the eigenvalues of B_γ are λ and λ^{-1} for some $\lambda \in (0, 1)$. This proves property (i). The proof of property (ii) is quite similar. \square

4. Further results. In this section, we present two extensions of Theorem 1.1. First, we consider Eq. (1.1) with $a = 0$, in which case we have an analogue of Theorem 1.1, without the energy condition. Next, we consider a nonlinearity more general than in Eq. (1.1), and for which we still can prove an analogue of Theorem 1.1. We begin with the case $a = 0$, which corresponds to $\nu = 0$ in Eq. (1.4), and for which the following improvement of Theorem 1.1 can be proved.

THEOREM 4.1. Let $\nu = 0$ in Eq. (1.4). Assume that

$$\gamma \in ((m+1)(2m+1), (m+1)(2m+3)),$$

for some nonnegative integer m . For every $E_0 > 0$, there exists a two-dimensional submanifold \mathcal{M} of the (three-dimensional) manifold $\{E(u, v, u', v') = E_0\}$ with the following property. If $(u_0, v_0, u'_0, v'_0) \in \mathcal{M}$, and if (u, v) is the solution of (1.4) (with $\nu = 0$) with initial data (u_0, v_0, u'_0, v'_0) , then v and v' converge exponentially to 0 as $t \rightarrow \infty$, and there exists a solution w of the equation $w'' + w^3 = 0$ with energy $\frac{w'^2}{2} + \frac{w^4}{4} = E_0$ such that $u - w$ and $u' - w'$ converge exponentially to 0 as $t \rightarrow \infty$.

Proof. The proof is a variation of the proof of Theorem 1.1, and we only indicate the modifications. In this case, a scaling argument, similar to the one in the proof of Theorem 2.3, shows that

$$L = JB_\gamma J^{-1},$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{w'_0} \end{pmatrix},$$

and B_γ is defined in Sec. 3. In particular, the eigenvalues of L are the eigenvalues of B_γ ; so, they are given by Theorem 3.1. Therefore, in the case $\nu = 0$, the conclusion of Theorem 2.3, hence the conclusion of Theorem 2.2 hold without the energy condition. The proof is now the same as the proof of Theorem 1.1 (see the end of Sec. 2), except that inequality (2.10) is not anymore meaningful, and so one must show that there exists a constant T , independent of n , such that

$$\tau_{n+1} - \tau_n \leq T. \tag{4.1}$$

This implies, instead of (2.16), the following inequality:

$$\|(-1)^n u(\tau_n + \cdot) - w(\cdot)\|_{C^1([0, 2\eta])} \leq C e^{-\varepsilon \tau_n},$$

and the rest of the proof is unchanged. Therefore, we need only prove estimate (4.1). First of all, one verifies quite easily that u has infinitely many zeroes, i.e., that the τ_n 's are well defined. Next, by (2.6) and (2.8), it follows that $v(\tau_n)^2 + v'(\tau_n)^2 \rightarrow 0$; so by (2.9)

$$|u'(\tau_n)| \geq \eta, \quad (4.2)$$

for all $n \geq 0$. Also, by conservation of energy, there exists $M < \infty$ such that

$$u'(t)^2 + u(t)^2 + v(t)^2 \leq M^2 \quad (4.3)$$

for all $t \geq 0$. Now fix $n \geq 0$ and assume, for example, that $u'(\tau_n) \geq 0$. It follows that u is positive and concave on (τ_n, τ_{n+1}) and thus has a unique maximum in (τ_n, τ_{n+1}) , achieved at $\sigma_n \in (\tau_n, \tau_{n+1})$. Clearly, we only need to estimate $\sigma_n - \tau_n$ in terms of M and η , since then an estimate for $\tau_{n+1} - \sigma_n$ will follow by changing t to $\tau_{n+1} - t$. Let $s \in (\tau_n, \sigma_n)$ be such that

$$M \geq u'(s) = \frac{u'(\tau_n)}{2} \geq \frac{\eta}{2}.$$

Since $u' \geq \frac{\eta}{2}$ on $[\tau_n, s]$, we have

$$u(t) \geq \frac{\eta}{2}(t - \tau_n); \quad (4.4)$$

so,

$$s - \tau_n \leq \frac{2M}{\eta}. \quad (4.5)$$

In addition, since $u'' \geq -M^3$, it follows that $u'(t) \geq \eta - M^3(t - \tau_n)$. Choosing $t = s$, we obtain

$$s - \tau_n \geq \frac{\eta}{2M^3}. \quad (4.6)$$

It follows from (4.4) and (4.6) that

$$u(s) \geq \frac{\eta^2}{4M^3}. \quad (4.7)$$

Next, since $u(s) \leq u(t) \leq M$ on $[s, \sigma_n]$, we have

$$u''(t) \leq -u(s)^3$$

on $[s, \sigma_n]$; so,

$$0 \leq u'(t) \leq M - (t - s)u(s)^3,$$

from which it follows, by applying (4.7), that

$$\sigma_n - s \leq 64 \frac{M^{10}}{\eta^6}.$$

The result now follows from the above inequality and (4.5). \square

Next, in order to extend Theorem 3.1 to include more general nonlinearities, we begin by generalizing Theorem 3.1. Let $\alpha > 0$ and consider the solution w of

$$w'' + |w|^\alpha w = 0, \tag{4.8}$$

with the initial values $w(0) = 0$ and $w'(0) = 1$. We denote by θ the first positive zero of w . w is odd and symmetric about $\theta/2$, i.e., $w(t) = w(\theta - t)$. Given $\gamma \geq 0$, we consider the equation

$$z'' + \gamma|w|^\alpha z = 0. \tag{4.9}$$

For $(a, b) \in \mathbf{R}^2$, let z be the solution of (4.9) with the initial values $z(0) = a$ and $z'(0) = b$. We define the mapping $B_\gamma \in \mathcal{L}(\mathbf{R}^2)$ by

$$B_\gamma(a, b) = -(z(\theta), z'(\theta)).$$

We define the four sequences $(\mu_n^-)_{n \geq 0}$, $(\mu_n^+)_{n \geq 0}$, $(\gamma_n^-)_{n \geq 0}$, and $(\gamma_n^+)_{n \geq 0}$ by

$$\begin{aligned} \mu_n^- &= n((2(\alpha + 2)n + \alpha)), & \gamma_n^+ &= (2n + 1)((\alpha + 2)n + 1), \\ \mu_n^+ &= (2n + 1)((\alpha + 2)n + \alpha + 1), & \gamma_n^- &= (n + 1)(2(\alpha + 2)n + \alpha + 4), \end{aligned}$$

and we observe that $\mu_n^- < \gamma_n^+ < \mu_n^+ < \gamma_n^- < \mu_{n+1}^-$ for all $n \geq 0$. Note that if $\alpha = 2$, then B_γ and the various constants reduce to those defined in Sec. 3. Moreover, Theorem 3.1 is still valid in the present, more general context.

THEOREM 4.2. Let $\alpha > 0$. If B_γ, μ_n^\pm , and γ_n^\pm are as just defined, then B_γ has all the properties specified in Theorem 3.1.

Proof. The proof is quite similar to the proof of Theorem 3.1, except that the four sequences of functions $(v_n^\pm)_{n \geq 0}$ and $(u_n^\pm)_{n \geq 0}$ are defined by (3.4) where now

$$x(t) = \frac{4}{\alpha + 2} w(t)^{\alpha+2} - 1,$$

and $P_n, Q_n, R_n,$ and S_n are the Jacobi polynomials (see [9, §2.2 and §2.4])

$$\begin{aligned} P_n(x) &= a_n P_n^{(-\frac{1}{2}, -\frac{1}{\alpha+2})}(x), & Q_n(x) &= b_n P_n^{(-\frac{1}{2}, \frac{1}{\alpha+2})}(x), \\ R_n(x) &= c_n P_n^{(\frac{1}{2}, -\frac{1}{\alpha+2})}(x), & S_n(x) &= d_n P_n^{(\frac{1}{2}, \frac{1}{\alpha+2})}(x), \end{aligned}$$

and where the constants $a_n, b_n, c_n,$ and d_n are chosen so that

$$P_n(-1) = Q_n(-1) = R_n(-1) = S_n(-1) = 1.$$

With this notation, the proof of Theorem 4.1 follows the proof of Theorem 3.1 with obvious modifications. In particular, the conclusions of Lemma 3.2 are valid as stated in the present context, for all $\alpha > 0$. \square

REMARK. The functions $(v_n^\pm)_{n \geq 0}$ and $(u_n^\pm)_{n \geq 0}$ can be calculated directly by using the following recurrence formulas:

$$\begin{aligned}
 v_0^- &\equiv 1, & u_0^+ &\equiv w(t), \\
 v_n^+ &= \frac{(v_n^-)'w}{n(\alpha + 2) + \frac{\alpha}{2}} + v_n^-w', & v_{n+1}^- &= \frac{(v_n^+)'w}{n(\alpha + 2) + \alpha + 1} + v_n^+w', \\
 u_n^- &= \frac{(u_n^+)'w + (2n + 1)(\frac{\alpha}{2} + 1)u_n^+w'}{(2n + 1)(\frac{\alpha}{2} + 1) + 1}, & u_{n+1}^+ &= \frac{(u_n^-)'w + (n + 1)(\alpha + 2)u_n^-w'}{(n + 1)(\alpha + 2) + 1}.
 \end{aligned}$$

It is straightforward to check that these recurrence relations generate solutions to the appropriate equations with the appropriate initial data. By uniqueness, these functions are the same as those defined in the proof of Theorem 4.2.

Let now $f : [0, \infty) \rightarrow \mathbf{R}$ satisfy the following properties:

$$\begin{aligned}
 f &\in C([0, \infty)) \cap C^1((0, \infty)), & f(0) &= 0, & sf'(s) &\xrightarrow{s \uparrow 0} 0, \\
 \inf_{s \geq 0} f(s) &> -1, & f(s) &\xrightarrow{s \rightarrow \infty} +\infty, & \frac{f(tx)}{f(t)} &\xrightarrow{t \rightarrow \infty} |x|^\alpha,
 \end{aligned} \tag{4.10}$$

where α is a positive number and the last limit is uniform for bounded values of x . Note that nonlinearities of the form $f(x) = bx^\alpha(\log(1 + x))^q$, for $b > 0, \alpha > 0$, and $q \geq 0$ verify assumption (4.10). We consider the equation

$$\begin{cases} u_{tt} - a(1 + f(\int_0^\pi u_x^2(t, x) dx))u_{xx} = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{4.11}$$

in $\Omega = (0, \pi)$, where $a > 0$. Expanding u in a Fourier series, i.e., setting

$$u_j(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi u(t, x) \sin(jx) dx,$$

we obtain for the u_j 's the following system of ODE's:

$$u_j'' + aj^2 \left(1 + f \left(\sum_{i=1}^\infty i^2 u_i^2 \right) \right) u_j = 0. \tag{4.12}$$

It follows that, given any integer $j \geq 1$, Eq. (4.11) has the particular (simple mode) solution $u(t, x) = \sqrt{\frac{2}{\pi}}\omega(t) \sin(jx)$ where ω solves the equation $\omega'' + j^2(a + f(j^2\omega^2))\omega = 0$. We consider a two-mode solution of (4.11) (or (4.12)), i.e., a solution of the system

$$\begin{cases} u_j'' + aj^2(1 + f(j^2u_j^2 + k^2u_k^2))u_j = 0, \\ u_k'' + ak^2(1 + f(j^2u_j^2 + k^2u_k^2))u_k = 0, \end{cases} \tag{4.13}$$

where j, k are integers such that $k > j \geq 1$. If (u_j, u_k) solves system (4.13), then $u(t, x) = \sqrt{\frac{2}{\pi}}(u_j(t) \sin(jx) + u_k(t) \sin(kx))$ solves Eq. (4.11). We set

$$\gamma = \frac{k^2}{j^2}, \quad \sigma = j\sqrt{a},$$

so that (4.13) is equivalent to

$$\begin{cases} u'' + (1 + f(u^2 + v^2))u = 0, \\ v'' + \gamma(1 + f(u^2 + v^2))v = 0, \end{cases} \quad (4.14)$$

where $ju_j(t) = u(\sigma t)$ and $ku_k(t) = v(\sigma t)$. The system (4.14) is a Hamiltonian system, whose conserved energy, expressed in terms of u, v, u', v' , is given by

$$E(u, v, u', v') = \frac{1}{2} \left(u'^2 + \frac{v'^2}{\gamma} + u^2 + v^2 + F(u^2 + v^2) \right),$$

where

$$F(x) = \int_0^x f(s) ds.$$

We have the following analogue of Theorem 1.1.

THEOREM 4.3. Let f verify (4.10), and assume that

$$\gamma \in \left((m+1) \left(\frac{\alpha+2}{2} m + 1 \right), (m+1) \left(\frac{\alpha+2}{2} m + 1 + \alpha \right) \right),$$

for some nonnegative integer m . There exists $C > 0$ such that if $E_0 \geq C$, then there exists a two-dimensional submanifold \mathcal{M} of the (three-dimensional) manifold $\{E(u, v, u', v') = E_0\}$ with the following property. If $(u_0, v_0, u'_0, v'_0) \in \mathcal{M}$, and if (u, v) is the solution of (4.14) with initial data (u_0, v_0, u'_0, v'_0) , then v and v' converge exponentially to 0 as $t \rightarrow \infty$, and there exists a solution w of the equation $w'' + w + f(w^2)w = 0$ with energy $\frac{1}{2}(w'^2 + w^2 + F(w^2)) = E_0$ such that $u - w$ and $u' - w'$ converge exponentially to 0 as $t \rightarrow \infty$.

Proof. The proof is an adaptation of the proof of Theorem 1.1 with the following modifications. The linearized system is

$$\begin{cases} w'' + w + f(w^2)w = 0, \\ z'' + \gamma(1 + f(w^2))z = 0, \end{cases}$$

with the initial conditions $w(0) = 0, w'(0) = \sqrt{2E_0}, z(0) = a$, and $z'(0) = b$. Let $g : [0, \infty) \rightarrow \mathbf{R}$ be a positive, continuous function such that

$$g(t) = \sqrt{f(t^2)},$$

for t large. It follows from (4.10) that such a function exists and that

$$\frac{f(\lambda^2 x^2)}{g(\lambda)^2} = \frac{f(\lambda^2 x^2)}{f(\lambda^2)} \frac{f(\lambda^2)}{g(\lambda)^2} \xrightarrow{\lambda \rightarrow \infty} |x|^{2\alpha},$$

uniformly for x in a bounded set. For every $E_0 > 0$, set $\lambda = \sup\{t > 0; tg(t) = \sqrt{2E_0}\}$. It follows from assumption (4.10) that λ is well defined, that

$$\lambda g(\lambda) = \sqrt{2E_0} = w'(0),$$

and that

$$\lambda \xrightarrow{E_0 \rightarrow \infty} \infty, \quad g(\lambda) \xrightarrow{E_0 \rightarrow \infty} \infty.$$

Define x and y by $w(t) = \lambda x(g(\lambda)t)$ and $z(t) = y(g(\lambda)t)$. It follows that (x, y) solves the system

$$\begin{cases} x'' + \frac{1}{g(\lambda)^2}x + \frac{f(\lambda^2 x^2)}{g(\lambda)^2}x = 0, \\ y'' + \frac{\gamma}{g(\lambda)^2}y + \gamma \frac{f(\lambda^2 x^2)}{g(\lambda)^2}y = 0, \end{cases}$$

with the initial conditions $x(0) = 0$, $x'(0) = 1$ (by (4.3)), $y(0) = a$, and $y'(0) = bg(\lambda)^{-1}$. By applying Theorem 4.2 instead of Theorem 3.1, one obtains an analogue of Theorem 2.3. The rest of the proof is completely analogous to the proof of Theorem 1.1. \square

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