Untangling a Polygon[*]

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Abstract. The following problem was raised by M. Watanabe. Let P be a self-intersecting closed polygon with n vertices in general position. How manys steps does it take to untangle P, i.e., to turn it into a simple polygon, if in each step we can arbitrarily relocate one of its vertices. It is shown that in some cases one has to move all but at most $O((n \log n)^{2/3})$ vertices. On the other hand, every polygon P can be untangled in at most $n - \Omega(\sqrt{n})$ steps. Some related questions are also considered.

1. Introduction

Suppose we have a self-intersecting closed polygon P on the screen of our computer, whose vertices are $p_1, p_2, \ldots, p_n$ in this order, and no three vertices are collinear. We are allowed to modify P so that in each step we can grab a vertex and move it to an arbitrary new position. (For simplicity, we assume that the screen is very large, so we are not limited by its size.) At the 5th Czech–Slovak Symposium on Combinatorics in Prague in 1998, Mamoru Watanabe asked the following question. Is it true that every polygon P can be *untangled*, i.e., turned into a noncrossing polygon, in at most εn steps, for some absolute constant $\varepsilon < 1$?

The aim of this note is to answer this question in the negative.

Given another closed polygon Q with vertices $q_1, q_2, \ldots, q_n$ (in this order), let $f(P, Q)$ denote the number of "common points" of P and Q, i.e., the number of indices i, for which $q_i = p_i$. Let $f(P)$ denote the largest number of points that can be kept fixed

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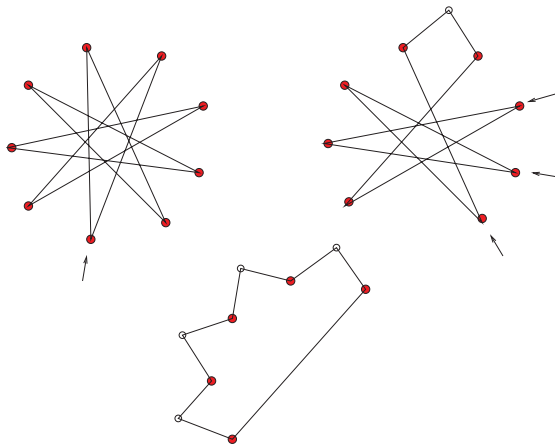
Fig. 1. For a star-polygon P with n vertices, we have $f(P) = (n+1)/2$.

when we untangle P. Using our notation,

$$f(P) = \max_{Q} f(P, Q),$$

where the maximum is taken over all noncrossing closed polygons with n vertices. See Fig. 1.

It is easy to see that every polygon P can be untangled in at most $n - \sqrt{n}$ moves. That is, we have

Proposition 1. *For every polygon P with n vertices, we have $f(P) > \sqrt{n}$.*

Proof. Assume without loss of generality that p_n is a vertex of the convex hull of $\{p_1, p_2, \ldots, p_n\}$, and let $p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(n-1)}$ be the other points of P, listed in clockwise order of visibility around p_n. According to a well-known lemma of Erdős and Szekeres [ES], every sequence of length k has a monotone subsequence of length $\lceil \sqrt{k} \rceil$. Therefore, there is a sequence $1 \leq i_1 < i_2 < i_3 < \cdots \leq n-1$ of length $\lceil \sqrt{n-1} \rceil$ such that either $\sigma(i_1) < \sigma(i_2) < \sigma(i_3) < \cdots$ or $\sigma(i_1) > \sigma(i_2) > \sigma(i_3) > \cdots$ is true. In either case, the points $p_n, p_{\sigma(i_1)}, p_{\sigma(i_2)}, \ldots$ induce a noncrossing closed polygon Q_0. Let Q denote the n-gon obtained from Q_0 by subdividing its sides with as many points as necessary, to achieve that the index of every point $p_n, p_{\sigma(i_1)}, p_{\sigma(i_2)}, \ldots$ be the same in P as in Q. Clearly, we have $f(P, Q) \geq \lceil \sqrt{n-1} \rceil + 1 > \sqrt{n}$, as required. $\qquad\square$

Our main result can now be formulated as follows.

Theorem 2. *For every sufficiently large n, there exists a closed polygon P with n vertices, which cannot be untangled in fewer than $n - c(n \log n)^{2/3}$ moves. That is, we have $f(P) \leq c(n \log n)^{2/3}$, where c is a constant.*

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, respectively. A *drawing* of G is a representation of G in the plane such that every vertex corresponds to a point, and every edge is represented by a Jordan arc connecting the corresponding two points without passing through any vertex other than its endpoints. Two edges are said to *cross* each other if they have an interior point in common. The *crossing number* $\mathrm{cr}(G)$ of G is defined as the minimum number of crossing pairs of arcs in a drawing of G.

For any partition of the vertex set of G into two disjoint parts, V_1 and V_2, let $E(V_1, V_2) \subseteq E(G)$ denote the set of edges with one endpoint in V_1 and the other in V_2. Define the *bisection width* of G as

$$b(G) = \min |E(V_1, V_2)|,$$

where the minimum is taken over all partitions $V(G) = V_1 \cup V_2$ such that $|V_1|, |V_2| \leq 2n/3$.

Theorem 2 is established by a random construction. The proof is based on the following consequence of a weighted version of the Lipton–Tarjan separator theorem for planar graphs.

Lemma 3 [PSS], [SV]. *Let G be a graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Then*

$$b^2(G) \leq (1.58)^2 \left(16 \, \mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2 \right),$$

where $b(G)$ and $\mathrm{cr}(G)$ denote the bisection width and the crossing number of G, respectively.

Corollary 4. *Let G be a graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Then, for any edge disjoint subgraphs $G_1, G_2, \ldots, G_j \subseteq G$, we have*

$$\sum_{i=1}^{j} b(G_i) \leq 1.58 \, j^{1/2} \left(16 \, \mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2 \right)^{1/2}.$$

Proof. Let d_{ik} denote the degree of the kth vertex in G_i. Corollary 4 immediately follows from Lemma 3. Indeed, applying Lemma 3 to each G_i separately and adding up the resulting inequalities, we obtain

$$\sum_{i=1}^{j} b^2(G_i) \leq (1.58)^2 \left(16 \sum_{i=1}^{j} \mathrm{cr}(G_i) + \sum_{i=1}^{j} \sum_{k=1}^{n} d_{ik}^2 \right)$$

$$\leq (1.58)^2 \left(16 \, \mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2 \right).$$

Therefore, we have

$$\left(\sum_{i=1}^{j} b(G_i) \right)^2 \leq j \sum_{i=1}^{j} b^2(G_i) \leq (1.58)^2 j \left(16 \, \mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2 \right),$$

as required. $\qquad\square$

Corollary 5.   *Let G be a graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Then, for any $1 < s \le n$, one can remove at most*

$$8.6 \left(\frac{n}{s}\right)^{1/2} \left(16\,\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2\right)^{1/2}$$

edges from G so that every connected component of the resulting graph has fewer than s vertices.

Proof.   Partition G by subsequently subdividing each of its large components into two roughly equal parts as follows. Start the procedure by deleting $b(G)$ edges of G so that it falls into two parts, each having at most $\frac{2}{3}|V(G)| = \frac{2}{3}n$ vertices. As long as there is a component $H \subset G$ whose size is at least s, by the removal of $b(H)$ edges, *cut* it into two smaller components, each of size at most $\frac{2}{3}|V(H)|$. When there are no such components left, stop.

Let $\mathcal{H}$ denote the family of all components arising at *any* level of the above procedure (e.g., we have $G \in \mathcal{H}$ if G is connected). Define the *order* of any element $H \in \mathcal{H}$ as the largest integer k, for which there is a chain

$$H_0 \underset{\ne}{\subset} H_1 \underset{\ne}{\subset} \cdots \underset{\ne}{\subset} H_k \tag{1}$$

in $\mathcal{H}$ such that $H_k = H$. For any k, let $\mathcal{H}_k$ denote the set of all elements of $\mathcal{H}$ of order k. Thus, $\mathcal{H}_0$ is the set of the components in the final decomposition.

For any fixed k, the elements of $\mathcal{H}_k$ are pairwise (vertex) disjoint. Recall that in a chain (1) we have $|V(H_1)| \ge s$ and the ratio of the sizes of any two consecutive members is at least $3/2$. Therefore, the number of vertices in any element of $\mathcal{H}_k$ is at least $(3/2)^{k-1}s$, which in turn implies that for $k \ge 1$,

$$j_k := |\mathcal{H}_k| \le \frac{n}{(3/2)^{k-1}s} = \frac{(2/3)^{k-1}n}{s}.$$

Applying Corollary 4 to the subgraphs in $\mathcal{H}_k$, we obtain that the total number of edges removed, when they are first subdivided during our procedure, is at most

$$1.58 \cdot (2/3)^{(k-1)/2} \left(\frac{n}{s}\right)^{1/2} \left(16\,\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2\right)^{1/2}.$$

Summing up over all $k \ge 1$, we conclude that the total number of edges deleted during the whole procedure does not exceed the number claimed. $\square$

2.   Proof of Theorem 2

We start with an auxiliary lemma.

Lemma 6.   *Let H_b denote the graph (cycle) defined on the vertex set $V = \{1, 2, \ldots, t\}$, whose edges are $(1, 2), (2, 3), \ldots, (t, 1)$. Let H_r be a randomly selected Hamilton cycle*

on the same vertex set, *i.e.*, *let*

$$E(H_r) = \{(\sigma(1), \sigma(2)), (\sigma(2), \sigma(3)), \ldots, (\sigma(t), \sigma(1))\},$$

where σ is a random permutation of V. Then, for every $s < t$ and K, the probability that the crossing number of $H = H_b \cup H_r$ is at most K satisfies

$$\text{Prob}[\text{cr}(H) \leq K] \leq \binom{t}{D}^2 \left(\frac{2t}{s}\right)^D \frac{s^{t-D}}{(t-D)!},$$

where $D = \lfloor 35\sqrt{t(K+t)/s} \rfloor$.

Proof.   We refer to the edges of H_b and H_r as *black* and *red* edges, respectively.

Let s be a positive integer. The degree of every vertex in H is at most 4, so by Corollary 5 we can delete at most $8.7[t(16\,\text{cr}(H)+16t)/s]^{1/2} = 34.8[t(\text{cr}(H)+t)/s]^{1/2} \leq D$ from H so that all components of the resulting graph have fewer than s vertices.

Consequently, if we want to give an upper bound on $\text{Prob}[\text{cr}(H) \leq K]$, it is sufficient to bound the probability that H can be decomposed into sets of size smaller than s by the deletion of precisely D black and precisely D red edges. In what follows, we estimate this probability.

If we succeed in deleting the edges as required, then we can (greedily) group the components of the remaining graph into at most $2t/s$ parts, each having fewer than s vertices. Let $\mathcal{H}$ denote the resulting partition of V. The D black edges that are deleted can be chosen in $\binom{t}{D}$ different ways. The remaining black edges form D paths. Each set in $\mathcal{H}$ is the union of the vertex sets of a few of these paths, thus there are at most $(2t/s)^D$ possibilities for the partition $\mathcal{H}$, once the deleted black edges have been chosen.

We consider the red Hamiltonian cycle to be picked with an *orientation*. There are $\binom{t}{D}$ different ways in which to pick the *starting points* of the D deleted red edges.

The probability that a randomly selected red Hamiltonian cycle "respects" a fixed partition of V into parts of size smaller than s, except for the edges originating at a fixed set of size D, is at most $s^{t-D}/(t-D)!$. Indeed, when we start drawing H_r randomly at a point, and we reach a vertex x which is not the starting point of a deleted red edge, then the probability that the endpoint of the red edge starting at this point belongs to the part of the partition which contains x is less than s divided by the number of vertices in V not yet visited by the initial portion of H_r. Summarizing: the probability that H can be decomposed into sets of size smaller than s by the deletion of D black and D red edges is at most

$$\binom{t}{D}^2 \left(\frac{2t}{s}\right)^D \frac{s^{t-D}}{(t-D)!},$$

and the lemma follows. $\square$

Now we are in a position to establish Theorem 2.

Consider a regular n-gon and let $p_1, p_2, \ldots, p_n$ be a *random permutation* of its vertices. Let P denote the closed polygon obtained by connecting the p_i's in this order. We claim that with high probability $f(P) \leq c(n \log n)^{2/3}$, where c is a constant.

For any positive integer t, we have $f(P) \geq t$ if and only if there is a t-element subset $T \subseteq \{p_1, p_2, \ldots, p_n\}$ such that there is a noncrossing closed polygon Q with vertices $q_1, q_2, \ldots, q_n$, in this order, with $q_i = p_i$ whenever $p_i \in T$.

To estimate the probability of this event for a fixed t-element set T, define two Hamilton cycles, H_b and H_r, on the vertex set T as follows. Let H_b consist of all edges of the convex hull of T. These edges are called *black*. A vertex $p_i \in T$ is connected to another vertex $p_j \in T$ by an edge of H_r, if p_i and p_j are consecutive in the cyclic order induced on T by the random permutation. That is, if $i < j$, then there is no index k with $i < k < j$ such that $p_k \in T$ or there is no index k with $k < i$ or $j < k$ with $p_k \in T$. The edges in H_r are said to be *red*. Let $H = H_b \cup H_r$.

Suppose now that there is a noncrossing closed polygon Q with vertices $q_1, q_2, \ldots, q_n$, such that $q_i = p_i$ whenever $p_i \in T$. By slightly changing the positions of its vertices not belonging to T, if necessary, we may achieve that no three vertices of Q are collinear.

Consider the drawing of H, in which every vertex is represented by itself, every black edge is represented by a straight line segment, and every red edge by the corresponding portion of Q. In this drawing, there is no crossing between edges of the same color. Since every edge of Q can cross the black cycle (the boundary of the convex hull of T) in at most two points, we obtain the number of crossings, and hence $\text{cr}(H)$, are at most $2n$. Thus, we have

$$\text{Prob}[f(P) \geq t] \leq \binom{n}{t} \text{Prob}[\text{cr}(H) \leq 2n].$$

Notice that any fixed set T uniquely determines H_b, but H_r is a uniformly distributed random Hamiltonian cycle on T determined by the random permutation $p_1, \ldots, p_n$. After substituting $t = 150(n \log n)^{2/3}$ and applying Lemma 6 with $K = 2n$ and $s = 101 n^{1/3} \log^{4/3} n$, Theorem 2 follows by computation:

$$\text{Prob}[f(P) \geq t] \leq \binom{n}{t}\binom{t}{D}^2 \left(\frac{3t}{s}\right)^D \frac{s^{t-D}}{(t-D)!}.$$

Here $D = \lfloor 35\sqrt{t(K+t)/s} \rfloor < t/\log n$ and hence we get

$$\text{Prob}[f(P) \geq t] \leq 2^{-t}.$$

3. Related Problems and Remarks

1. Proposition 1 (with a weaker constant) also follows from the main result in [PW]: Every planar graph with m vertices admits a crossing-free drawing in the plane such that its vertices are mapped into arbitrarily prespecified points and each of its edges are represented by a polygonal curve with fewer than Cm bends, where C is a constant. (Apply this result to the cycle with vertices $p_{\lceil\sqrt{Cn}\rceil}, p_{\lceil 2\sqrt{Cn}\rceil}, p_{\lceil 3\sqrt{Cn}\rceil}, \ldots$, where each of these points has to be mapped into itself.)

2. It is easy to see that any polygon of n vertices and only one crossing pair of edges can be untangled in $\lceil n/4 \rceil$ moves. Indeed, deleting the two crossing edges, the polygon falls into two disjoint paths. Let $p_1, \ldots, p_m, m \leq n/2$, denote the vertices of one of these
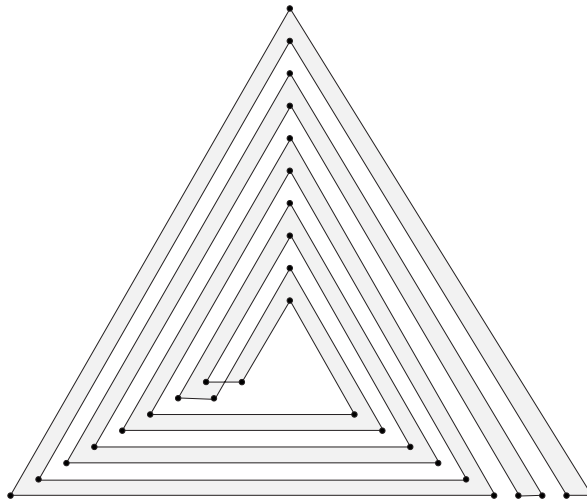
Fig. 2.   A polygon with one crossing which cannot be untangled in few moves.

paths in their natural order. We move p_1 close to the crossing of the two deleted edges and we move p_i close to p_{m+2-i} for $i = 2, \ldots, \lceil m/2 \rceil$. One can do this in a way to obtain a simple polypon. Figure 2 shows an example of a polygon with a single crossing that cannot be untangled with $o(n)$ moves and it seems that one cannot untangle it moving substantially fewer than $n/4$ vertices.

Obviously, if a polygon has c crossings, then it can be untangled without moving the vertices of its longest crossing-free section, whose length is at least $\lceil n/(2c) \rceil$. This bound is naturally far from being optimal.

More generally, we can raise the following:

Problem 1.   Let P be a polygon of n vertices with the property that every edge of P crosses at most k other edges. Is it true that P can be untangled so that at least $c_k n$ vertices remain fixed, for a suitable constant $c_k > 0$ depending only on k?

The answer to this question is in the affirmative in the special case when every edge e of the polygon is disjoint from all other edges, whose distances from e along P are larger than a constant k.

3. One can ask similar questions for straight-line drawings of *planar graphs* rather than closed polygons. Now we are allowed to relocate any vertex, keeping all of its connections straight. Our goal is to eliminate all crossings, moving as few vertices as possible.

Problem 2.   Let P be a (not necessarily crossing-free) straight-line drawing of a planar graph with n vertices. Can P be untangled leaving n^ε vertices fixed, for an absolute constant $\varepsilon > 0$?

4. Corollary 5 is of some independent interest. In a forthcoming paper, Djidjev and Vrto [DV] establish a similar relation between the *cutwidth* and the crossing number of a graph, but neither of the two statements implies the other.

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