Borodzik, M.
Osaka J. Math.
56 (2019), 497–505

UNTWISTING NUMBER AND BLANCHFIELD PAIRINGS

Maciej BORODZIK

(Received November 14, 2017, revised February 1, 2018)

Abstract

In this note we use Blanchfield forms to study knots that can be turned into an unknot using a single $\bar{t}_{2k}$ move.

1. Overview

Let $K \subset S^3$ be a knot and $k \in \mathbb{Z} \setminus \{0\}$. In this paper by a *k–twisting move* we mean a move depicted in Fig. 1, that is, a full right k–twist on two strands of K going in the opposite direction (in [16] this move is called a $\bar{t}_{2k}$–move). We will call a knot *k–simple*, if it can be unknotted by a single k–untwisting move. A knot is *algebraically k–simple*, if a single k–untwisting move turns it into a knot with Alexander polynomial 1.

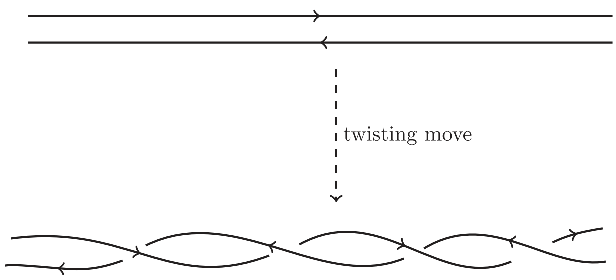

Fig. 1. A k–twisting move for $k = 2$. Note that the strands in the picture go in different directions.

Our first result gives an obstruction to the untwisting move in terms of the algebraic unknotting number [7, 15, 20].

Theorem 1.1. *Suppose K is an algebraically k–simple knot. If k is odd, then K can be turned into a knot with Alexander polynomial* 1 *using at most two crossing changes. If k is even, then at most three crossing changes are enough to turn K into a knot with Alexander polynomial* 1.

Our second result restricts the homology of the double branched cover of an algebraically k–simple knot.

Theorem 1.2. *Suppose K is an algebraically k–simple knot. Denote by $\Sigma(K)$ the double branched cover of K. Then $H_1(\Sigma(K);\mathbb{Z})$ is cyclic.*

2010 Mathematics Subject Classification. 57M25.

Both Theorem 1.1 and Theorem 1.2 follow from the following result, which is the main technical result of this paper.

Theorem 1.3. *Suppose K is an algebraically k–simple knot. Then there exists a polynomial* $\alpha(t) \in \mathbb{Z}[t, t^{-1}]$ *satisfying* $\alpha(1) = 0$, $\alpha(t^{-1}) = \alpha(t)$*, such that the matrix*

$$\begin{pmatrix} \alpha(t) & 1 \\ 1 & -k \end{pmatrix}$$

represents the Blanchfield pairing for K.

Theorem 1.3 can be regarded as a generalization of [16, Theorem 3.2(b)].

It is possible to generalize the techniques used in this paper to study knots that are untwisted with several $\bar{t}_{2k}$ moves, possibly with varying the twisting coefficients k. This generalization is straightforward, we omit it make the paper shorter and more concise.

Proof of Theorem 1.3 is given in Section 3. Proof of Theorem 1.1 is given in Section 4. Section 5 contains the proof of a stronger version of Theorem 1.2.

2. Blanchfield pairing

Let $K \subset S^3$ be a knot and let M_K denote its zero-framed surgery. Denote by $\widetilde{M}_K$ the universal abelian cover of M_K. The chain complex $C_*(\widetilde{M}_K; \mathbb{Z})$ admits the action of the deck transform and thus it has a structure of a Λ–module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$. The homology of this complex, regarded as a Λ–module, is denoted by $H_*(M_K; \Lambda)$. The module $H_1(M_K; \Lambda)$ is called the *Alexander module* of the knot K.

Remark 2.1. Usually the Alexander module is defined using knot complements instead of zero–framed surgeries, but the two definitions are equivalent; see e.g. [10].

The ring Λ has a naturally defined convolution $t \mapsto t^{-1}$. The Blanchfield pairing defined in [1] for K is a sesquilinear symmetric pairing $H_1(M_K; \Lambda) \times H_1(M_K; \Lambda) \to Q/\Lambda$, where Q is the field of fractions for Λ. We refer to [10, 13] for a precise and detailed construction of the Blanchfield pairing and [5, 6] for generalizations.

Definition 2.2. We say that an $n \times n$ matrix A with entries in Λ *represents* the Blanchfield pairing if $H_1(M_K; \Lambda) \cong \Lambda^n / A\Lambda^n$ as a Λ–module, under this identification the Blanchfield pairing has form $(a, b) \mapsto \overline{a}^T A^{-1} b$ and moreover $A(1)$ is diagonalizable over $\mathbb{Z}$.

It is known, see [14], that every Blanchfield pairing can be represented by a finite matrix. The minimal size of a matrix representing the Blanchfield pairing of a knot is denoted by $n(K)$. It is equal to the algebraic unknotting number $u_a(K)$; see [2, 4].

The invariant $n(K)$ can also be generalized for other coefficient rings. In this paper we restrict to rings that are subrings of $\mathbb{C}$. If R is such a ring, we denote by $n_R(K)$ the minimal size of a matrix over $R[t, t^{-1}]$ representing the Blanchfield pairing over $R[t, t^{-1}]$. We have that $n_R(K) \leq n_{R'}(K)$ if R' is a subring of R. Often $n_R(K)$ is easier to compute than $n(K) = n_{\mathbb{Z}}(K)$, for example the value of $n_{\mathbb{R}}$ can be calculated from the Tristram–Levine signature [3]. One motivation of this paper is to give a geometric interpretation of $n_R(K)$ for some rings R.

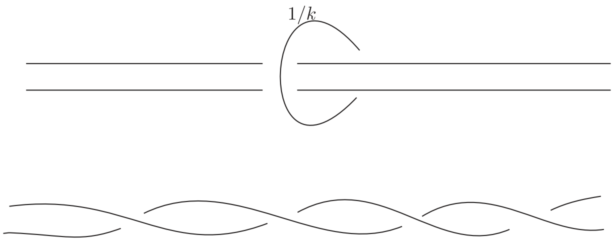

Fig. 2. The $1/k$ surgery on the circle in the top picture induces k full left twists of the two strands passing through the circle.

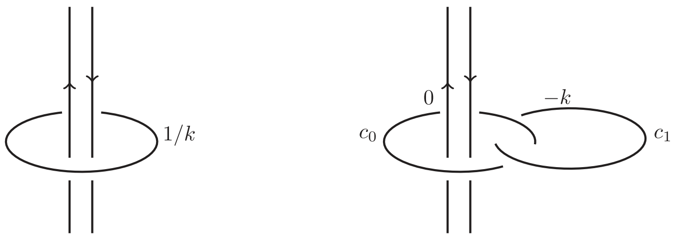

Fig.3. Changing a $1/k$ surgery on a circle to a surgery on a two-component link with framings 0 and $-k$.

3. Proof of Theorem 1.3

The main ingredient in the proof of Theorem 1.3 is the following.

Theorem 3.1 (see [4, Theorem 2.6])**.** *Suppose W_K is a topological four–manifold such that $\partial W_K = M_K$, $\pi_1(W_K) = \mathbb{Z}$ and the inclusion induced map $H_1(M_K;\mathbb{Z}) \to H_1(W_K;\mathbb{Z})$ is an isomorphism. Then $H_2(W_K;\Lambda)$ is free of rank $b_2(W_K)$. Moreover if A is matrix over Λ representing the twisted intersection form on $H_2(W_K;\Lambda)$ in some basis of $H_2(W_K;\Lambda)$, then A also represents the Blanchfield pairing on M_K (in the sense of* Definition 2.2*).*

In the light of Theorem 3.1, the proof of Theorem 1.3 consists of constructing an appropriate manifold W_K and applying Theorem 3.1. The construction begins with noticing that the twisting move can be realized by a surgery. Namely we have the following well-known fact.

Proposition 3.2. *A k–twisting move can be realized by a $-1/k$ surgery on a knot. That is, if K_2 arises from K_1 by a k–twisting move, then there is a simple closed circle C disjoint from K_1, such that C bounds a smooth disk intersecting K_2 at two points with opposite signs and such that the $-1/k$ surgery on C transforms K_1 into K_2; see* Fig. 2

REMARK 3.3. The move described in Fig. 2 is a special case of the Rolfsen twist, see [12, Figure 5.27]. It can be seen on [21, Figure 3.12] that the surgery with a positive coefficient (i.e. the $1/k$ surgery if $k > 0$) gives rise to a left k–twist and the surgery with a negative coefficient (i.e. the $-1/k$ surgery with $k > 0$) gives rise to a right k–twist.

The surgery in Fig. 2 can be changed into a surgery with integer coefficients as in Fig. 3 by a 'slam-dunk' operation, see [12, Section 5.3].

Suppose J is a knot with Alexander polynomial 1 and K is a knot resulting from J by applying a full left k–twist (so J is obtained from K by a full right k–twist). Let M_J be the zero-surgery on J and M_K the zero–surgery on K. By [11, Theorem 117B] M_J is a boundary of a topological four–manifold that is a homotopy $D^3 \times S^1$. Denote this four–manifold by W_J.

A full left k–twist on J can be realized as a surgery on a two-component link with framings 0 and $-k$ as in Fig. 3. Let c_0 and c_1 denote the components of this link. The curve c_0 has framing 0, c_1 has framing k. Both c_0 and c_1 are curves disjoint from J, so we can and will assume that they are separated from a small neighborhood of J in S^3. Performing a 0–surgery on J does not affect these curves, therefore c_0 and c_1 can also be viewed as curves on M_J. Now performing surgery on c_0 and c_1 with coefficients 0 and $-k$, respectively, produces M_K.

The trace of the surgery on c_0 and c_1 yields a cobordism between M_J and M_K. Call this cobordism W_{JK}. Define now

$$W_K = W_J \cup W_{JK}$$

so that $\partial W_K = M_K$. We have the following fact.

Lemma 3.4. *We have $\pi_1(W_K) \cong \mathbb{Z}$, $H_1(W_K;\mathbb{Z}) \cong \mathbb{Z}$ and the inclusion of M_K to W_K induces an isomorphism on the first homology. Moreover $H_2(W_K;\mathbb{Z}) \cong \mathbb{Z}^2$ and there are generators of $H_2(W_K;\mathbb{Z})$ that are represented by immersed spheres.*

Proof. The homology groups of W_K are calculated using the Mayer-Vietoris sequence. The manifold W_K is obtained from W_J by adding two-handles along null-homologous curves c_0 and c_1. This shows that $H_1(W_K;\mathbb{Z}) \cong \mathbb{Z}$ and $H_2(W_K;\mathbb{Z}) \cong \mathbb{Z}^2$.

To compute π_1 we observe that $\pi_1(W_J) \cong \mathbb{Z}$. Hence c_0, c_1 being null-homologous are also null-homotopic. Van Kampen theorem implies that $\pi_1(W_K) \cong \mathbb{Z}$.

To show that the generators of $H_2(W_K;\mathbb{Z})$ can be chosen to be spherical we again use the fact that c_0 and c_1 are null-homotopic in W_J. This implies that c_0 and c_1 bound disks D_0 and D_1 in W_J. The disk D_1 can be chosen to be the obvious disk on M_J, but D_0 is in general only an immersed disk and it cannot lie on M_J (because in general c_0 is not null-homotopic on M_J). We can form spheres Σ_0 and Σ_1 by adding to D_0 and D_1 the cores of the two-handles that are attached. It is clear that the homology classes $[\Sigma_0]$ and $[\Sigma_1]$ generate $H_2(W_K;\mathbb{Z})$. Moreover, by construction, Σ_1 is a smoothly embedded sphere and Σ_0 can be chosen to intersect Σ_1 precisely at one point.

Finally, in order to prove that the inclusion induced map $H_1(M_K;\mathbb{Z}) \to H_1(W_K;\mathbb{Z})$ is an isomorphism, invert the cobordism W_{JK}, that is, present W_{JK} as $M_K \times [0,1]$ with two two–handles attached. The attaching curves of these handles are homologically trivial (but not necessarily homotopy trivial, $\pi_1(M_K)$ can be complicated), hence the boundary inclusion induces an isomorphism $H_1(M_K;\mathbb{Z}) \cong H_1(W_{JK};\mathbb{Z})$. Clearly $H_1(W_{JK};\mathbb{Z}) \cong H_1(W_K;\mathbb{Z})$. □

Lemma 3.4 gives us two spheres $\Sigma_0, \Sigma_1 \subset W_K$, which are the generators of $H_2(W_K;\mathbb{Z})$. Choose a basepoint $x_0 = \Sigma_0 \cap \Sigma_1$. This choice allows us to consider Σ_0 and Σ_1 as elements of $\pi_2(W_K, x_0)$.

Lemma 3.5. *The group $\pi_2(W_K, x_0)$ is freely generated as a $\Lambda = Z[\pi_1(W_K, x_0)]$–module by classes of Σ_0 and Σ_1. In particular $\pi_2(W_K, x_0) \cong \Lambda^2$.*

Proof. The space W_K is obtained from W_J by attaching two two–handles along null-homotopic curves c_0 and c_1. We have that $\pi_1(W_J) = \mathbb{Z}$ and $\pi_2(W_J) = 0$ by definition. The statement follows from [18, Proposition 3.30]. □

We will use Lemma 3.5 in connection with the following well-known result.

Lemma 3.6. *We have an isomorphism of* Λ*–modules* $\pi_2(W_K, x_0) \cong \pi_2(\widetilde{W}_K, \widetilde{x}_0) \cong H_2(\widetilde{W}_K; \mathbb{Z}) \cong H_2(W_K; \Lambda)$.

Proof. The first isomorphism in the lemma is the isomorphism of higher homotopy groups under the covering map. The second is the Hurewicz isomorphism because $\widetilde{W}_K$ is simply connected. The third isomorphism is the definition of the twisted homology groups. □

In particular, Lemma 3.5 together with Lemma 3.6 gives a simple and independent argument that $H_2(W_K; \Lambda)$ is a free Λ–module, compare [4, Lemma 2.7].

Corollary 3.7. *The (classes of the) lifts of* Σ_0 *and* Σ_1 *to* $\widetilde{W}_K$ *generate* $H_2(W_K; \Lambda)$ *as a* Λ*–module.*

Recall that $A(t)$ is a matrix over Λ representing the intersection form on $H_2(W_K; \Lambda)$. The following result together with Theorem 3.1 gives the proof of Theorem 1.3 from the introduction.

Theorem 3.8. *The matrix* $A(t)$ *has form*

$$\begin{pmatrix} \alpha(t) & 1 \\ 1 & -k \end{pmatrix},$$

where $\alpha(t) \in \Lambda$ *is such that* $\alpha(1) = 0$ *and* $\alpha(t^{-1}) = \alpha(t)$.

Proof. By Corollary 3.7 the entries of $A(t)$ are twisted intersection indices of Σ_0 and Σ_1. For example, the bottom-right entry of $A(t)$ is equal to the twisted intersection index of Σ_1 and Σ_1', where Σ_1' is a small perturbation of Σ_1 intersecting Σ_1 in finitely many points.

To compute the twisted intersection index of Σ_1 and Σ_1', choose a basing for Σ_1, Σ_1', that is a path γ from x_0 to Σ_1 and a path γ' from x_0 to Σ_1'. Let x, x' be the end points of γ and γ'.

For any intersection point $y \in \Sigma_1$ and Σ_1' we choose a smooth path ρ_y from x to y on Σ_1 and a path ρ_y' from x' to y on Σ_1'; see Fig. 4.

Let θ_y be the loop $\gamma\rho_y(\rho_y')^{-1}(\gamma')^{-1}$. Define $m_y \in \mathbb{Z}$ to be the homology class of θ_y in $H_1(W_K; \mathbb{Z}) \cong \mathbb{Z}$. Finally, let ϵ_y be the sign of the intersection point y assigned in the usual way, that is, if $T_y\Sigma_1 \oplus T_y\Sigma_1' = T_yW_K$ agrees with the orientation, we set $\epsilon_y = +1$, otherwise we set $\epsilon_y = -1$.

Given these definitions, the twisted intersection index of Σ_1 and Σ_1' is equal to

$$\sum_{y \in \Sigma_1 \cap \Sigma_1'} \epsilon_y t^{m_y} \in \mathbb{Z}[t, t^{-1}]. \tag{3.9}$$

In general this sum might depend on the choice of ρ_y and ρ_y'. However if any smooth closed curve on Σ_1 and on Σ_1' is homologically trivial in W_K (in the language of [2, Section 3.2] this means that Σ_1 and Σ_1' are homologically invisible in W_K), the definition does not depend on paths ρ_y and ρ_y'. In the present situation Σ_1 and Σ_1' are immersed (and even embedded)

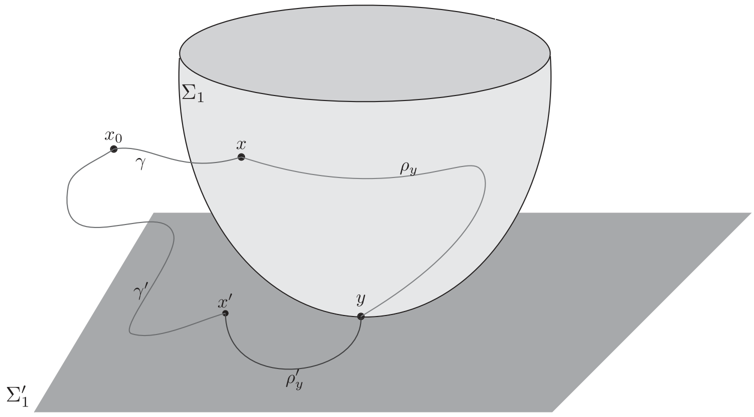

Fig. 4. Notation in proof of Theorem 3.8. In the four-dimensional situation the intersection of Σ and Σ' at y is transverse.

spheres, so they are homologically invisible, in particular (3.9) is a well-defined Laurent polynomial.

As Σ_1 and Σ'_1 are embedded spheres, we claim more, namely that m_y does not depend on y. In fact, suppose z is another intersection point of Σ_1 and Σ'_1. Consider the curve $\delta = (\rho'_y)^{-1}\rho'_z\rho_z^{-1}\rho_y$ in W_K. As Σ'_1 is a perturbation of Σ_1, the path $(\rho'_y)^{-1}\rho'_z$ can be pushed by a homotopy (in W_K) to a path $\widetilde{\rho}$ on Σ_1 having the same endpoints. Then $\widetilde{\rho}\rho_z^{-1}\rho_y$ is a loop homotopically equivalent to δ, but this is a loop on a smoothly embedded sphere Σ_1. Hence it is contractible in W_K. This shows that $m_y = m_z$.

We conclude that the twisted intersection index of Σ_1 and Σ'_1 is equal to the standard intersection number of Σ_1 and Σ'_1 (which is equal to the self-intersection of Σ_1, that is $-k$) multiplied by t^{m_y}. We can choose a basing for Σ'_1 in such a way that $m_y = 0$.

An analogous, but simpler argument shows that $\Sigma_0 \cdot \Sigma_1 = \pm 1$. Indeed by construction $\Sigma_0 \cap \Sigma_1$ consists of a single point. It follows that the twisted intersection between Σ_0 and Σ_1 is $\pm t^m$ for some m. We choose a basing for Σ_0 in such a way that $m = 0$. We can also choose an orientation of Σ_0 in such a way that the sign is positive.

It remains to discuss the properties of $\alpha(t)$, that is, the top-left entry of $A(t)$. First we notice that as the twisted intersection pairing is sesquilinear, the matrix $A(t)$ is hermitian and so $\alpha(t^{-1}) = \alpha(t)$. Moreover, it follows from the construction of $A(t)$ sketched above that $A(t = 1)$ is a matrix of classical intersection indices between Σ_0 and Σ_1. Therefore $\alpha(1)$ is the self-intersection of Σ_0. This is the same number as the framing of the curve c_0. This concludes the proof that $\alpha(1) = 0$. □

REMARK 3.10. There is an alternative calculation of the matrix A using Rolfsen's argument [19]. However one still has to make some effort proving that A represents not only the Alexander module, but also the Blanchfield pairing.

4. Proof of Theorem 1.1

We begin with proving Theorem 1.1. The following corollary deals with the first part of this theorem.

Corollary 4.1. *Suppose K is an algebraically k–simple and k is odd. Then there are at most two crossing changes that turn K into a knot with Alexander polynomial* 1.

Proof. We have $A(1) = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}$. As k is odd, this matrix is diagonalizable over $\mathbb{Z}$. By [2, Theorem 1.1] we infer that the algebraic unknotting number of K is at most 2. □

If k is even, then $A(1)$ is not diagonalizable over $\mathbb{Z}$, because it is an even symmetric form. However the block sum $A(1) \oplus (1)$ is diagonalizable. The block matrix $A(t) \oplus (1)$ is a 3×3 matrix over Λ representing the Blanchfield pairing, so the algebraic unknotting number of K is bounded from above by 3. This shows the second part of Theorem 1.1.

We have the following consequence of Theorem 1.3.

Theorem 4.2. *Suppose K can be algebraically k–simple. Let $R_k = \mathbb{Z}[\frac{1}{k}]$. Then $n_{R_k}(K) =$* 1.

Proof. By Theorem 1.3 we know that the Blanchfield pairing over $\mathbb{Z}$ can be represented by a matrix of form $\begin{pmatrix} \alpha(t) & 1 \\ 1 & -k \end{pmatrix}$. The same matrix represents the Blanchfield pairing over R_k, but over R_k this matrix is congruent to a matrix $\begin{pmatrix} \widetilde{\alpha}(t) & 0 \\ 0 & 1 \end{pmatrix}$ for $\widetilde{\alpha}(t) \in R_k[t, t^{-1}]$. By [17, Proposition 1.7.1] (see also [4, Proposition 3.1]) the matrix $(\widetilde{\alpha}(t))$ also represents the Blanchfield pairing over $R_k[t, t^{-1}]$. □

The following corollary is well known, see [16].

Corollary 4.3. *If K is algebraically k–simple, then its Alexander polynomial is equal to $\Delta_K(t) = 1 + k\alpha(t)$, where $\alpha(t) \in \mathbb{Z}[t, t^{-1}]$.*

Proof. This follows from Theorem 4.1 because if $A(t)$ represents the Blanchfield pairing of a knot K, then $\Delta_K(t) = \det A(t)$ up to multiplication by a unit in Λ. □

5. Linking forms

An abstract *linking pairing* is a pair (H, l), where H is a finite abelian group of an odd order and l is a bilinear symmetric pairing $l \colon H \times H \to \mathbb{Q}/\mathbb{Z}$, As a model example, if Y is a closed three–manifold with $b_1(Y) = 0$, there is defined a linking pairing $l(Y)$ on $H = H_1(Y; \mathbb{Z})$. If $Y = \Sigma(K)$ is the double branched cover of a knot K, we denote this pairing by $l(K)$. It is known that the linking pairing $l(K)$ is represented by $V + V^T$, where V is the Seifert matrix for K. The meaning of 'represented' is explained in the following definition.

DEFINITION 5.1. Let P be an $n \times n$ matrix with integer coefficients and such that $\det P$ is odd. The *linking form represented by* P is the pair $(H(P), l(P))$, where $H(P) = \mathbb{Z}^n / P\mathbb{Z}^n$ and $l(P)$ is the bilinear form defined by

$$\begin{aligned} \mathbb{Z}^n / P\mathbb{Z}^n \times \mathbb{Z}^n / P\mathbb{Z}^n &\to \mathbb{Q}/\mathbb{Z} \\ (a, b) &\mapsto a^T P^{-1} b \bmod 1. \end{aligned}$$

We have the following relation between the Blanchfield form for K and the linking form $l(K)$.

Proposition 5.2 (see [4, Lemma 3.3]). *If A is a matrix over* Λ *representing the Blanchfield pairing, then* $l(A(-1)) = 2l(K)$.

Here $2l(K)$ means the linking pairing with the same underlying group as $l(K)$, but the linking form is multiplied by 2; compare [4, Section 3].

We can use this result to obtain the following corollary.

Corollary 5.3. *Suppose K is algebraically k–simple. Then the linking form* $2l(K)$ *is isometric to the linking form represented by*

$$B = \begin{pmatrix} d & 1 \\ 1 & -k \end{pmatrix}, \tag{5.4}$$

where $d = \alpha(-1) \in \mathbb{Z}$ *is such that* $-(dk + 1)$ *is the (signed) determinant of K.*

As in [4, Section 5.2] we can use Corollary 5.3 to obstruct untwisting number 2.

From Corollary 5.3 we immediately recover Theorem 1.2 from the introduction.

Proposition 5.5. *If K is algebraically k–simple and* $\Sigma(K)$ *is the double branched cover, then* $H_1(\Sigma(K); \mathbb{Z})$ *is cyclic.*

Remark 5.6. It follows that Wendt's criterion for the unknotting number [22] coming from the double branched covers, does not distinguish between knots that have unknotting number 1 and knots that are algebraically k–simple for some k.

Proof of Proposition 5.5. By Corollary 5.3 we infer that $H_1(\Sigma(K); \mathbb{Z}) \cong \mathbb{Z}^2/BZ^2$, where B is as in (5.4). Subtract from the first column of B the second column multiplied by d to obtain the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1+dk & -k \end{smallmatrix}\right)$. Then add to the second row the first one multiplied by k. We obtain the matrix

$$B' = \begin{pmatrix} 0 & 1 \\ 1 + dk & 0 \end{pmatrix}.$$

Row and column operations on matrices do not affect the cokernel, hence $\mathbb{Z}^2/B'\mathbb{Z}^2 \cong \mathbb{Z}^2/B\mathbb{Z}^2$. Evidently we have $\mathbb{Z}^2/B'\mathbb{Z} \cong \mathbb{Z}/|dk + 1|\mathbb{Z}$. □

Acknowledgements. The author is grateful to A. Conway, S. Friedl, C. Livingston, W. Politarczyk and J. Przytycki for fruitful conversations. He is especially indebted to A. Ranicki for pointing out [18, Proposition 3.30]. The author would also like to thank the referee for his/hers comments. The research is supported by the National Science Center grant 2016/22/E/ST1/00040.

References

[1] R.C. Blanchfield: *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math. **65** (1957), 340–356.

[2] M. Borodzik and S. Friedl: *On the algebraic unknotting number*, Trans. London Math. Soc. **1** (2014), 57–84.
[3] M. Borodzik and S. Friedl: *Unknotting number and classical invariants. II.*, Glasg. Math. J. **56** (2014), 657–680.
[4] M. Borodzik and S. Friedl: *Unknotting number and classical invariants. I.*, Alg. Geom. Top. **15** (2015), 85–135.
[5] A. Conway: *An explicit computation of the Blanchfield pairing for arbitrary links*, preprint, arxiv: 1706.00226.
[6] A. Conway, S. Friedl and E. Toffoli: *The Blanchfield pairing of colored links*, preprint, arxiv:1609.08057.
[7] M. Fogel: *The Algebraic Unknotting Number*, PhD thesis, University of California, Berkeley 1993.
[8] T. Cochran, K. Orr and P. Teichner: *Knot concordance, Whitney towers and L^2-signatures*, Ann. Math. **157** (2003), 433–519.
[9] M. Fogel: *Knots with Algebraic Unknotting Number One*, Pacific J. Math. **163** (1994), 277–295.
[10] S. Friedl and M. Powell: *A calculation of Blanchfield pairings of 3-manifolds and knots*, preprint, arxiv 1512.04603.
[11] M. Freedman and F. Quinn: Topology of 4–manifolds, Princeton Mathematical Series **39**, Princeton University Press, Princeton, NJ, 1990.
[12] R. Gompf and A. Stipsicz: 4-manifolds and Kirby calculus, Graduate Studies in Mathematics. **20**, American Mathematical Society, Providence, RI. 1999.
[13] J. Hillman: Algebraic invariants of links, second edition, Series on Knots and Everything, **52**, World Scientific Publishing Co. Inc., River Edge, NJ, 2012.
[14] C. Kearton: *Blanchfield duality and simple knots*, Trans. Am. Math. Soc. **202** (1975), 141–160.
[15] H. Murakami: *Algebraic Unknotting Operation*, Q&A. Gen. Topology **8** (1990), 283–292.
[16] J. Przytycki: *t_k moves on links*, Braids (Santa Cruz, CA, 1986); in Contemp. Math. **78**, Amer. Math. Soc., Providence, RI, 1988, 615–656.
[17] A. Ranicki: Exact sequences in the algebraic theory of surgery, Mathematical Notes **26**, Princeton University Press, Princeton, N.J., 1981.
[18] A. Ranicki: Algebraic and geometric surgery, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, Oxford, 2002.
[19] D. Rolfsen: *A surgical view of Alexander's polynomial*; in Geometric topology (Proc. Conf., Park City, Utah, 1974), Lecture Notes in Math. **438**, Springer, Berlin, 1975, 415–423.
[20] O. Saeki: *On Algebraic Unknotting Numbers of Knots*, Tokyo J. Math. **22** (1999), 425–443.
[21] N. Saveliev: Lectures on the topology of 3-manifolds, An introduction to the Casson invariant, De Gruyter Textbook, Walter de Gruyter & Co., Berlin, 1999.
[22] H. Wendt: *Die gordische Auflösung von Knoten*, Math. Z. **42** (1937), 680–696.

Institute of Mathematics
Polish Academy of Science
ul. Śniadeckich 8
00–656 Warsaw
Poland

Institute of Mathematics
University of Warsaw
ul. Banacha 2
02–097 Warsaw
Poland
e-mail: mcboro@mimuw.edu.pl