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UPPER AND LOWER FREDHOLM SPECTRA

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ABSTRACT. Joint upper and lower Fredholm spectra are defined for n-tuples of bounded linear operators, and the upper Fredholm spectrum is represented both as the simultaneous eigenvalues and as the simultaneous approximate eigenvalues of an n-tuple of operators obtained by a Berberian-Quigley construction.

Introduction. A bounded linear operator T: $X \to Y$ between Banach spaces is said to be *upper Fredholm* if it has finite dimensional null space ($T^{-1}0$) and closed range $T(X)$, and is said to be *lower Fredholm* if its range $T(X)$ is closed and has finite codimension. T is *Fredholm* iff it is upper and lower Fredholm. In this note we show that T is upper Fredholm iff a related operator $P(T)$: $\mathcal{P}(X) \to \mathcal{P}(Y)$ is bounded below, where $\mathcal{P}(X)$ is obtained as a certain quotient of the space $l_\infty(X)$ of all bounded X-valued sequences. It is then shown that T is Fredholm iff $P(T)$ is invertible. This construction is obtained in §1, the basic two theorems are established in §2, some immediate consequences in §3, and some consequences for joint upper and lower Fredholm spectra are obtained in §4.

1. *If X is a complex Banach space then let $l_\infty(X)$ denote the Banach space obtained from the space of all bounded sequences $x = (x_n)$ in X by imposing term-by-term linear combination and the supremum norm* $\|x\| = \sup_n \|x_n\|$.

Berberian [1] and Quigley [11, Theorem 1.5.11] have, essentially, considered the quotient

$$\mathcal{Q}(X) = l_\infty(X)/c_0(X), \tag{1.1}$$

where $c_0(X)$ is the subspace of all sequences converging to 0. By applying T term-by-term to elements of $x = (x_n) \in l_\infty(X)$, it induces a well defined operator $Q(T)$: $\mathcal{Q}(X) \to \mathcal{Q}(Y)$. For given X and Y the correspondence $T \to Q(T)$ has the following fundamental property ([3] and [5]):

(1.2) $\quad Q(T)$ *one-one* $\Leftrightarrow$ *T bounded below* $\Leftrightarrow$ *$Q(T)$ bounded below.*

This is the essence of the space $\mathcal{Q}(X)$: it is a space in which "approximate eigenvectors" of an operator are represented as true eigenvectors.

Lotz [10, Theorem 2.4] also observes

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(1.3) $$Q(T) \text{ dense} \Rightarrow T \text{ dense}$$

and hence

(1.4) $$Q(T) \text{ invertible} \Leftrightarrow T \text{ invertible}.$$

In the present paper we are concerned with the quotient of $l_\infty(X)$ by a much larger subspace:

DEFINITION 1. *If X is a Banach space then $m(X)$ denotes the space of those sequences $x = (x_n)$ of which every subsequence x' has a convergent subsequence x''.*

Equivalently, x is in $m(X)$ iff its range $\{x_n\}$ is totally bounded.

THEOREM 1. *If X is a Banach space then $m(X)$ is a closed linear subspace of $l_\infty(X)$. If $T \in B(X, Y)$ is bounded then there is inclusion $Tm(X) \subseteq m(Y)$. If and only if $T \in K(X, Y)$ is compact there is inclusion $Tl_\infty(X) \subseteq m(Y)$.*

PROOF. Clear.

We use $m(X)$ in place of $c_0(X)$ in our analogue of the Berberian-Quigley construction:

DEFINITION 2. *If X is a Banach space then $\mathcal{P}(X)$ is the quotient $l_\infty(X)/m(X)$, and if $T: X \to Y$ is bounded linear then $P(T)$ is the operator induced from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.*

Theorem 1 makes it clear that $P(T)$ is well defined, and is evidently bounded and linear, with norm less than or equal to that of T. Also $P(T) = P(0)$ if and only if T is compact; thus there is a well defined mapping.

(1.5) $$T \to P(T): B(X, Y)/K(X, Y) \to B(\mathcal{P}(X), \mathcal{P}(Y)).$$

This is one-one and norm-decreasing; however, it is an open question whether or not it is also isometric, or at least bounded below [9, Theorem 3.6].

2. Our main result is the analogue of the logical equivalence (1.2):

THEOREM 2. *If $T: X \to Y$ is a bounded linear operator between Banach spaces then the following are equivalent:*

(2.1) $$P(T): \mathcal{P}(X) \to \mathcal{P}(Y) \text{ is one-one};$$

(2.2) $$T: X \to Y \text{ is upper Fredholm};$$

(2.3) $$P(T): \mathcal{P}(X) \to \mathcal{P}(Y) \text{ is bounded below}.$$

PROOF. If (2.2) fails then either $T^{-1}0$ is infinite dimensional or not. If $T^{-1}0$ is infinite dimensional then repeated applications of the Riesz lemma yield a sequence $x = (x_n)$ in X for which $x_n \in T^{-1}0$, $\|x_n\| = 1$ and $m \neq n \Rightarrow \|x_n - x_m\| > \frac{1}{2}$. Evidently x is in $l_\infty(X)$ but not in $m(X)$, while trivially $Tx \in m(Y)$. On the other hand, if $T^{-1}0$ is finite dimensional then the range $T(X)$ cannot be closed, so there exists a bounded projection $E: X \to X$ with its null space $E^{-1}0 = T^{-1}0$. The restriction of T to the range $E(X)$ of E does not have closed range, therefore cannot be bounded below. Thus there exists $x = (x_n)$ for which $x_n = E$, $x_n \in E(X)$, $\|x_n\| = 1$ and $Tx_n \to 0$ $(n \to$

∞). We claim that x cannot have any convergent subsequence x'. For if $x'_\infty = \lim_n x'_n \in E(X)$, then $x'_\infty \in E(X)\|x'_\infty\| = 1$ and $Tx'_\infty = 0$; but this contradicts the fact that T is one-one on $E(X)$. It follows that the sequence x is again not in $m(X)$, while Tx is in $m(Y)$. Thus the operator $P(T)$ is not one-one, violating (2.1).

If (2.2) holds then again there is a projection $E = E^2 \in B(X)$ with $E^{-1}0 = T^{-1}0$, and now restricted to EX the operator T is one-one with closed range, therefore bounded below. Thus there is $k > 0$ for which $(\forall x \in X)\|Tx\| = \|TEx\| \geqslant k\|Ex\|$. We claim that

$$\text{(2.4)} \qquad \forall x \in l_\infty(X), \quad \operatorname{dist}(Tx, m(Y)) \geqslant \tfrac{1}{2} k \operatorname{dist}(x, m(X)).$$

To see this, suppose that $\delta > \operatorname{dist}(Tx, m(Y))$ and select any $\varepsilon > 0$. Then there must be $y \in m(Y)$ with $\|Tx - y\| < \delta$ where $x = \{x_n\}$ and $y = \{y_n\}$. Now there exists a finite ε-net S for $\{y_n\}$ in Y. Evidently the set S is also a $(\delta + \varepsilon)$-net for $\{Tx_n\}$. Thus to each n corresponds a vector $s_n \in S$ such that $\|Tx_n - s_n\| < \delta + \varepsilon$. Now select a finite subset W of terms from the sequence $\{x_n\}$ in such a way that for each n there is an $x'_n \in W$ with $\|Tx'_n - s_n\| < \delta + \varepsilon$. The sequence $x' = (x'_n)$ is evidently in $m(X)$, while $\|Tx - Tx'\| < 2(\delta + \varepsilon)$. Finally

$$\begin{aligned} \operatorname{dist}(x, m(X)) &= \operatorname{dist}(Ex, m(X)) < \|Ex - Ex'\| \\ &\leqslant \frac{1}{k}\|Tx - Tx'\| < \frac{2}{k}(\delta + \varepsilon). \end{aligned}$$

From the choice of δ and ε we obtain (2.4), so that (2.3) holds.

We have proved that (2.1) $\Rightarrow$ (2.2) and that (2.2) $\Rightarrow$ (2.3); the implication (2.3) $\Rightarrow$ (2.1) is of course trivial.

The argument for the implication (2.1) $\Rightarrow$ (2.2) is taken from Lebow and Schechter [9, Corollary 4.12]; see also Vernon Williams [13]. An alternative version of the argument for the implication (2.2) $\Rightarrow$ (2.3) has been shown to us by Marc de Wilde of Liege. Similar arguments also show that, on the space of operators $B(\mathcal{P}(X), \mathcal{P}(Y))$, an equivalent norm is obtained by the expression $\sup_{q(\Omega)\leqslant 1}(q(T(\Omega)))$, where $q(\Omega)$ denotes the "measure of noncompactness" of a bounded subset Ω of a Banach space. This means that the equivalence (2.2) $\Leftrightarrow$ (2.3) is obtained implicitly by Lebow and Schechter [9, Theorem 4.11]. The equivalence (2.1) $\Leftrightarrow$ (2.2) is also a theorem of Yood [2, Theorem 1.3.2]; [9, Corollary 4.11].

We can also obtain the analogue for $P(T)$ of Lotz's result (1.4):

THEOREM 3. *The following are equivalent*:

(2.5) *T is Fredholm*;
$P(T)$ is invertible.

PROOF. If T is Fredholm then by Atkinson's theorem ([2, Lemma 3.2.6], [9, Lemma 4.2]) T has an *essential inverse*, T': $Y \to X$ for which $I - T'T$ and $I - TT'$ are compact on X and Y respectively. It then follows immediately that $P(T)$ is invertible, with inverse $P(T')$. Conversely, if $P(T)$ is invertible then by Theorem 2, the operator T is upper Fredholm. In particular, T has closed range. We claim that

(2.6) *$T(X)$ is closed and $P(T)$ onto $\Rightarrow$ T is lower Fredholm,*

which will complete the argument.

To this end, assume T is not lower Fredholm; that is, $T(X)$ is not finite codimension. Then by repeated application of the Riesz lemma, there exists a sequence $y = (y_n)$ in Y with

$$\|y_n\| \leqslant 2 \quad \text{and} \quad \operatorname{dist}\left(y_{n+1}, T(X) + \sum_{j=1}^{n} c_j y_j\right) \geqslant 1. \tag{2.7}$$

Thus $y \in l_\infty(X)$. However, we shall show that y is not in the subspace $T(l_\infty(X)) + m(Y)$ which will contradict the fact that $P(T)$ is onto and complete the proof. If to the contrary there existed $x \in l_\infty(X)$ for which $y - Tx \in m(Y)$ then $y - Tx$ would have a Cauchy subsequence, and in particular there would be n and $m > n$ for which

$$\|Tx_n - Tx_m - y_n + y_m\| < \tfrac{1}{2}$$

which contradicts (2.7).

It is familiar [9, Lemma 4.5] that the Fredholm operators, and the upper and the lower Fredholm operators, form open subsets of $B(X, Y)$. From Theorem 2 and Theorem 3 we find that relative to the upper Fredholm operators, the Fredholm operators are closed.

3. As an immediate application of the "easy part" of Theorem 2, the equivalence between T upper semi-Fredholm and $P(T)$ one-one, we have the implication, for $T\colon X \to Y$ and $S\colon Y \to Z$ [2, Corollary 1.3.3, 1.3.4], that

(3.1) *S, T upper Fredholm $\Rightarrow$ ST upper Fredholm $\Rightarrow$ T upper Fredholm.*

For some further applications recall that $T\colon X \to Y$ has an *essential left inverse* if there is $S\colon Y \to X$ for which $I - ST$ is compact on X, and an *essential right inverse* if $I - TS$ is compact on Y. We shall call $T\colon X \to Y$ *essentially one-one* if there is implication, for bounded $U\colon X \to X$,

(3.2) *TU compact $\Rightarrow$ U compact,*

and *essentially dense* if there is implication, for bounded $U\colon Y \to Y$,

(3.3) *UT compact $\Rightarrow$ U compact.*

Theorem 4. *Let T be a bounded operator, $T\colon X \to Y$, then the following implications hold:*

(3.4) *essentially left invertible $\Rightarrow$ upper Fredholm $\Rightarrow$ essentially one-one*

and

(3.5) *essentially right invertible $\Rightarrow$ lower Fredholm $\Rightarrow$ essentially dense.*

Proof. If T has an essential left inverse S then $P(S)$ is a left inverse for

$P(T)$. Since $P(T)$ is one-one, then by Theorem 2, T is upper Fredholm. This proves the first implication of (3.4). Now, given that T is upper Fredholm then again by Theorem 2 $P(T)$ is one-one. So suppose that TU is compact. It must be shown that U is compact. However, for any $x \in l_\infty(X)$, we have that $TUx \in m(Y)$ and, hence, $U_x \in m(X)$ ($P(T)$ is one-one). Thus U is compact. This then shows that T is essentially one-one completing (3.4). Now (3.5) follows by taking adjoints and recalling that T is lower Fredholm iff T^* is upper Fredholm and that T is compact iff T^* is compact.

If X and Y are Hilbert spaces then the implications in Theorem 4 all hold in reverse. Indeed if X is a Hilbert space then it is very easy to reverse the first implication of (3.5), while the second implication of (3.4) can be reversed for the more general "sub-projective" space [9, Theorem 6.8]. Similarly if Y is a Hilbert space then the first implication of (3.4) reverses, and if more generally Y is "super-projective" then so does the second implication of (3.5) [9, Theorem 6.10]. Dash [4, Theorem 5] has obtained this version of (3.4); compare also [6, Theorem 4.1] and [9, Corollary 6.11].

4. We recall the *joint spectrum* $\sigma_A(a)$ and [8, Definitions 1.1, 1.2, 1.3] its various subsets $\omega_A(a)$, for $\omega = \sigma^{\text{left}}, \sigma^{\text{right}}, \pi^{\text{left}}, \pi^{\text{right}}, \tau^{\text{left}}$ and τ^{right}, where $a = (a_1, a_2, \ldots, a_n)$ is an n-tuple of elements of a Banach algebra A. If in particular $A = B(X)$ then [8, Theorem 2.4] these concepts can be expressed, for an n-tuple $A = (A_1, A_2, \ldots, A_n)$ of operators, in terms of the two auxiliary operators

$$\text{(4.1)} \qquad \text{Col}(A)\colon x \to (A_1x, A_2x, \ldots A_nx), \textit{from } X \textit{ to } X^n,$$

and

$$\text{(4.2)} \quad \text{Row}(A)\colon (x_1, x_2, \ldots x_n) \to A_1x_1 + A_2x_2 + \cdots + A_nx_n, \textit{from } X^n \textit{ to } X.$$

We use them here:

DEFINITION 3. *The upper Fredholm spectrum of A is the set*

$$\text{(4.3)} \qquad \sigma^+_{\text{ess}}(A) = \{s \in C^n\colon \text{Col}(A - sI) \textit{ is not upper Fredholm}\}$$

and the lower Fredholm spectrum is the set

$$\text{(4.4)} \qquad \sigma^-_{\text{ess}}(A) = \{s \in C^n\colon \text{Col}(A - sI) \textit{ is not lower Fredholm}\}.$$

We shall write $\omega(A) = \omega_{B(X)}(A)$ and $\omega_{\text{ess}}(A) = \omega_{B(X)/K(X)}(A^\wedge)$, for each of the ω introduced above. Thus for example an operator A is bounded below if and only if 0 is not in $\tau^{\text{left}}(a)$, and is essentially one-one (3.2) iff 0 is not in $\pi^{\text{left}}_{\text{ess}}(A)$.

With these conventions, Theorems 2 and 3 give the following information about the upper Fredholm spectrum:

THEOREM 5. *If $A \in B(X)^n$ is arbitrary there is equality*

$$\text{(4.5)} \qquad \pi^{\text{left}}(P(A)) = \sigma^+_{\text{ess}}(A) = \tau_{\text{ess}}(A)$$

and if $n = 1$ then

$$\text{(4.6)} \qquad \sigma(P(A)) = \sigma^+_{\text{ess}}(A) \cup \sigma^-_{\text{ess}}(A) = \sigma_{\text{ess}}(A).$$

The last part of (4.6) is of course Atkinson's theorem. From Theorem 4 we get

THEOREM 6. *If $A \in B(X)^n$ is arbitrary there is inclusion*

(4.7) $\pi_{\text{ess}}^{\text{left}}(A) \subseteq \sigma_{\text{ess}}^{+}(A) \subseteq \sigma_{\text{ess}}^{\text{left}}(A)$ and $\pi_{\text{ess}}^{\text{right}}(A) \subseteq \sigma_{\text{ess}}^{-}(A) \subseteq \sigma_{\text{ess}}^{\text{right}}(A)$

with equality throughout if X is a Hilbert space.

Theorem 6 lends support for an affirmative answer to the following open question [7, page 22]:

PROBLEM 1. *If $A \in B(X)^n$ is arbitrary, is there equality $\sigma_{\text{ess}}^{+}(A) = \tau_{\text{ess}}^{\text{left}}(A)$?*

There is obviously an affirmative answer to Problem 1 if X is a Hilbert space (Theorem 6), or under more general assumptions about the Banach space X, i.e., see [9].

Theorem 5 shows that if $n = 1$ then the upper and lower essential spectrum are nonempty, and contain the topological boundary of the essential spectrum. More generally, using the spectral mapping theorem for the approximate point spectrum [3], [5], [12], we obtain

THEOREM 7. *If $A \in B(X)^n$ is a commuting system of operators then its upper and lower Fredholm spectra are nonempty, and there is equality for systems of polynomials*

$$f\colon C^n \to C^m,$$

$$f(\sigma_{\text{ess}}^{+}(A)) = \sigma_{\text{ess}}^{+} f(A) \quad \textit{and} \quad f(\sigma_{\text{ess}}^{-}(A)) = \sigma_{\text{ess}}^{-}(f(A)). \tag{4.8}$$

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