

**UPPER AND LOWER SOLUTION METHOD FOR
FRACTIONAL EVOLUTION EQUATIONS WITH ORDER
 $1 < \alpha < 2$**

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ABSTRACT. In this work, we investigate the existence of the extremal solutions for a class of fractional partial differential equations with order $1 < \alpha < 2$ by upper and lower solution method. Using the theory of Hausdorff measure of noncompactness, a series of results about the solutions to such differential equations is obtained.

1. Introduction

Fractional order differential equation has broad applications in resolving real-world problems, and as such it attracted researchers' attention from different areas. In order to evaluate the behaviors of fractional order differential equation based models, one need to know the properties of such equation systems, in particular, the existence of solutions to such equations. Recently, the existence of solutions to different forms of fractional differential equation systems has been investigated [1, 2, 3, 4, 5, 9, 10, 11, 15, 16, 19, 20, 21, 23, 26, 27, 29, 30, 32, 33, 34].

Using the upper and lower solution method to study the existence of extremal solutions for fractional differential equations is an interesting topic of research, which has been gaining increasing attention recently [1, 15, 16, 20, 23, 27, 30, 32, 33, 34]). Presently, the upper and lower solution method is widely used to investigate fractional ordinary differential equations (see [1, 15, 20, 23, 30, 32, 33, 34]). However, this method is seldom used to study semilinear fractional evolution equations. [27] considered the existence of extremal solutions to the following semilinear fractional evolution equation

$$(1.1) \quad \begin{cases} {}^c D_t^\alpha u(t) + Au(t) = f(t, u(t)), & t \in J = [0, T], \quad 0 < \alpha < 1 \\ u(0) = x_0, \end{cases}$$

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where $-A$ is the infinitesimal generator of an analytic semigroup $T(t) = e^{At}|_{t \geq 0}$, and $f : I \times X \rightarrow X$ is continuous.

As is well known, a mild solution to system (1.1) satisfies the operator equation

$$u(t) = S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s, u(s)).$$

Since $0 < \alpha < 1$, we can combine the probability density function and semigroup to describe the corresponding solution operators $S_\alpha(t)$, $T_\alpha(t)$ (see [24]), i.e.,

$$T_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) t^{\alpha-1} T(t^\alpha \theta) d\theta, \quad S_\alpha(t) = \int_0^\infty \phi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

where $\phi_\alpha(\theta)$ is the probability density function defined on $(0, \infty)$ such that its Laplace transform is

$$\int_0^\infty e^{-\theta x} \phi_\alpha(\theta) d\theta = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1 + \alpha j)}, \quad x > 0.$$

Thus, it is obvious that $T(t) = e^{At}$, $T_\alpha(t)$ and $S_\alpha(t)$ are positive if $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$.

However, for the case $1 < \alpha < 2$, the properties of solution operators $S_\alpha(t)$, $T_\alpha(t)$, $K_\alpha(t)$ corresponding to fractional evolution equations are unknown (see [30]). On one hand, we do not know if $S_\alpha(t)$, $T_\alpha(t)$, $K_\alpha(t)$ are positive. On the other hand, we do not know if we can still use the probability density function together with semigroup to describe the corresponding solution operators $S_\alpha(t)$, $T_\alpha(t)$, $K_\alpha(t)$. Thus, using the upper and lower solution method to investigate such fractional order differential equation is a challenging research topic.

In this paper, we use the upper and lower solution method to investigate a class of fractional partial differential equations of the form

$$(1.2) \quad \begin{cases} D_t^\alpha u(t) = Au(t) + f(t, u(t)), & t \in J = [0, T] \\ u(0) = x_0, u'(0) = x_1, \end{cases}$$

where the superscript α is the order of fractional differentiation, $1 < \alpha < 2$. We use the properties of the Mittag-Leffler function to study the corresponding solution operators $S_\alpha(t)$, $T_\alpha(t)$, $K_\alpha(t)$. Based on the theories of accretive operators and m -accretive operators, we prove that the solution operators are positive. Then, a series of results about the solutions to such differential equations is obtained.

The rest of the paper is organized as follows. In Section 2, some notions and notations that are used throughout the paper are presented. The main results of this article are given in Section 3. Finally, in Section 4, an example is considered to illustrate the applications of the main results presented in Section 3.

2. Preliminaries

In this section, we present some notions and notations that are used throughout the paper.

2.1. Definitions and lemmas

In this work, $C(J; X)$ (resp. $C^m(J; X)$) denotes the Banach spaces of functions $f : J \rightarrow X$, which are continuous (resp. m -times continuous) and differentiable from J to X equipped with the norm $\|f\|_C = \sup_{t \in J} \|f(t)\|_X$ (resp. $\|f\|_{C^m} = \sup_{t \in J} \sum_{k=0}^m \|f^{(k)}(t)\|_X$).

For a function $\varphi(t)$ of a real variable $t \in \mathbb{R}^+$, its Laplace transform is

$$(\mathcal{L}\varphi)(\lambda) = \int_0^\infty e^{-\lambda t} \varphi(t) dt, \quad (\lambda \in \mathbb{C}).$$

The corresponding inverse Laplace transform for $x \in \mathbb{R}^+$ is then defined by

$$(\mathcal{L}^{-1}g)(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} g(s) ds.$$

In general, the Mittag-Leffer function is defined as [28]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{H_\alpha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where H_α denotes a Hankel path, a contour starting and ending at $-\infty$, and encircling the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ counterclockwise. It then follows from the above definition that

$$\begin{aligned} E_{1,1}(z) &= e^z, \\ E_{2,1}(z^2) &= \cosh(z), \\ E_{2,1}(-z^2) &= \cos(z). \end{aligned}$$

In addition, we have

$$zE_{2,2}(z^2) = \sinh(z),$$

and

$$zE_{2,2}(-z^2) = \sin(z).$$

Applying the Laplace transform to the Mittag-Leffer function yields

$$\mathcal{L}(t^{\beta-1} E_{\alpha,\beta}(-\rho^\alpha t^\alpha)) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho^\alpha}, \quad \text{Re}\lambda > \rho^{\frac{1}{\alpha}}, \rho > 0.$$

In this paper, we use the Hausdorff measure of noncompactness $\alpha(\cdot)$ on each bounded subset \mathcal{B} of Banach space X , which is expressed as

$$\alpha(\mathcal{B}) = \inf\{\varepsilon > 0; \mathcal{B} \text{ has a finite } \varepsilon\text{-net in } Y\}.$$

Now, we consider some basic properties of $\alpha(\cdot)$.

As is well known, the Hausdorff measure of noncompactness has the following properties. (See Deimling [14], Heinz [18], Lakshmikantham and Leela [22].)

- (1) For all bounded subsets \mathcal{B}, \mathcal{D} of X , if $\mathcal{B} \subseteq \mathcal{D}$, then $\alpha(\mathcal{B}) \leq \alpha(\mathcal{D})$ (monotone).
- (2) $\alpha(\{x\} \cup \mathcal{B}) = \alpha(\mathcal{B})$ for every $x \in X$ and every nonempty subset $\mathcal{B} \subset X$ (nonsingular).
- (3) \mathcal{B} is precompact if and only if $\alpha(\mathcal{B}) = 0$ (regular).
- (4) Let $\mathcal{B} + \mathcal{D} = \{x + y; x \in \mathcal{B}, y \in \mathcal{D}\}$. Then, $\alpha(\mathcal{B} + \mathcal{D}) \leq \alpha(\mathcal{B}) + \alpha(\mathcal{D})$.
- (5) $\alpha(\mathcal{B} \cup \mathcal{D}) \leq \max\{\alpha(\mathcal{B}), \alpha(\mathcal{D})\}$.
- (6) $\alpha(\lambda\mathcal{B}) \leq |\lambda|\alpha(\mathcal{B})$.

For any $W \subset C(J; X)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : \text{for all } u \in W, t \in J \right\}.$$

Lemma 2.1 ([17]). *If $W \subset C(J; X)$ is bounded and equicontinuous, then $t \rightarrow \alpha(W(t))$ is continuous on J , and*

$$(2.1) \quad \alpha(W) \leq \max_{t \in J} \alpha(W(t)),$$

$$(2.2) \quad \alpha\left(\int_0^t W(s)ds\right) \leq \int_0^t \alpha(W(s))ds \text{ for all } t \in J.$$

Lemma 2.2 ([25]). *If $\{u_n\}_1^\infty$ is a sequence of Bochner integrable functions from J into X with $\|u_n(t)\| \leq \widehat{m}(t)$ for almost every $t \in J$ and every $n \geq 1$, where $\widehat{m}(t) \in L(J; \mathbb{R}^+)$, then the function $\psi(t) = \alpha(\{u_n\}_{n=1}^\infty)$ belongs to $L(J; \mathbb{R}^+)$ and satisfies*

$$(2.3) \quad \alpha\left(\left\{\int_0^t u_n(s)ds : n \geq 1\right\}\right) \leq 2 \int_0^t \psi(s)ds.$$

Lemma 2.3 ([13]). *If W is bounded, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset W$ satisfying*

$$(2.4) \quad \alpha(W) \leq 2\alpha(\{u_n\}_{n=1}^\infty) + \varepsilon.$$

Let X be a Banach space. If there exists a positive constant $k < 1$ satisfying $\alpha(Q\mathcal{B}) \leq k\alpha(\mathcal{B})$ for any bounded closed subset $\mathcal{B} \subseteq W$, then the map $Q : W \subseteq X \rightarrow X$ is called an α -contraction.

The following lemma will be used to prove our main results.

Lemma 2.4 (See [6], Darbo. Sadovskii.). *If $W \subseteq X$ is bounded closed and convex, the continuous map $Q : W \rightarrow W$ is an α -contraction, then the map Q has at least one fixed point in W .*

Now, we recall here several definitions about fractional differential equations.

Definition 2.1 ([28]). Assume $a, \alpha \in \mathbb{R}$. A function $f : [a, \infty) \rightarrow \mathbb{R}$ is said to be in the space $C_{a,\alpha}$ if there exist a real number $p > \alpha$ and a function $g \in C([a, \infty), \mathbb{R})$ satisfying $f(t) = t^p g(t)$. In addition, assuming m is a positive integer, if $f^{(m)} \in C_{a,\alpha}$, then f is said to be in the space $C_{a,\alpha}^m$.

Definition 2.2. Suppose function $f \in C_{a,\alpha}^m$, where $m \in \mathbb{N}^+$. Its fractional derivative of order $\alpha > 0$ in the Caputo sense is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad m-1 < \alpha \leq m.$$

Definition 2.3 ([30]). Suppose $A : \mathcal{D} \subseteq X \rightarrow X$ is a closed linear operator. Then, A is called a sectorial operator of type (M, θ, α, μ) if there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbb{R}$ such that the α -resolvent of A exists outside the sector

$$\mu + S_\theta = \{\mu + \lambda^\alpha : \lambda \in \mathbb{C}, |\text{Arg}(-\lambda^\alpha)| < \theta\}$$

and

$$\|(\lambda^\alpha I - A)^{-1}\| \leq \frac{M}{|\lambda^\alpha - \mu|}, \quad \lambda^\alpha \notin \mu + S_\theta.$$

Denote $C^\alpha(J, X) = \{x \in C(J, X) : D^\alpha x \text{ exists and } D^\alpha x \in C(J, X)\}$. Obviously, $C^\alpha(J, X)$ is a Banach space whose norm is

$$\|x\| = \sup_{t \in J} \{\|x(t)\| + \|D^\alpha x(t)\|\}.$$

In fact, if $1 < \alpha < 2$, we have

$$C^\alpha(J, X) \subset C^1(J, X) \subset C(J, X).$$

Here, we denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. A function $u \in C^\alpha(J, X) \cap C(J, X_1)$ is called a classical solution of (1.2) if $u(t)$ satisfies equalities (1.2).

Definition 2.4 ([30]). A function $u \in C([0, T], X)$ is said to be a mild solution to (1.2) if it satisfies the operator equation

$$u(t) = S_\alpha(t)x_0 + K_\alpha(t)x_1 + \int_0^t T_\alpha(t-s)f(s, u(s))ds.$$

Here

$$\begin{aligned} S_\alpha(t) &= \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda, \\ K_\alpha(t) &= \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-2} R(\lambda^\alpha, A) d\lambda, \\ T_\alpha(t) &= \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda, \end{aligned}$$

where c is a suitable path satisfying $\lambda^\alpha \notin \mu + S_\theta$ for $\lambda \in c$.

Lemma 2.5 ([30]). Suppose A is a sectorial operator of type (M, θ, α, μ) . If f satisfies a uniform Hölder condition with exponent $\beta \in (0, 1]$, then the unique solution of the linear initial value problem for the fractional evolution equation

$$(2.5) \quad \begin{cases} D_t^\alpha u(t) = Au(t) + f(t), & t \in J = [0, T] \\ u(0) = x_0, u'(0) = x_1 \end{cases}$$

is

$$(2.6) \quad u(t) = S_\alpha(t)u_0 + K_\alpha(t)u_1 + \int_0^t T_\alpha(t-s)f(s)ds.$$

Next, we recall some definitions and concepts of cone.

Let P be a cone in X . Then a partial ordering in X can be defined by $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, we say that $x < y$.

P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ is the zero element of X .

Besides, if X is an ordered Banach space, then $C(J, X)$ is also an ordered Banach space with partial order “ \leq ” induced by the positive cone $K = \{x : x \in C(J, X), x(t) \geq \theta \text{ for all } t \in J\}$, and if K is normal, then there exists the same normal constant N , for all $\theta \leq x \leq y$, such that $\|x\| \leq N\|y\|$ holds. Here, we use $[u, v]$, where $u, v \in C(J, X)$ and $u \leq v$, to denote the order interval $\{\omega \in C(J, X) : u(t) \leq \omega(t) \leq v(t) \text{ for all } t \in J\}$ in $C(J, X)$.

Definition 2.5 ([27]). If a function $u_0 \in C^\alpha(J, X) \cap C(J, X_1)$ satisfies

$$(2.7) \quad \begin{cases} D_t^\alpha u_0(t) \leq Au_0(t) + f(t, u_0(t)), & t \in J = [0, T] \\ u_0(0) \leq x_0, u'(0) \leq x_1, \end{cases}$$

then u_0 is said to be a lower solution of system (1.2). If the directions of all the inequalities in (2.7) are changed, then u_0 is called an upper solution of system (1.2).

Lemma 2.6 ([8, Gronwall inequality]). *Let a and b be nonnegative constants. If continuous function $u(t)$ on $t_0 < t < T$ (some $T \leq \infty$) satisfies*

$$(2.8) \quad u(t) \leq a + b \int_{t_0}^t u(s) ds,$$

then we have

$$(2.9) \quad u(t) \leq ae^{b(t-t_0)}, \quad t_0 \leq t < T.$$

2.2. Properties of solution operators

Lemma 2.7. *If A is a sectorial operator of type (M, θ, α, μ) , then we have*

$$(2.10) \quad S_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda = E_{\alpha,1}(At^\alpha) = \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(1 + \alpha k)},$$

$$(2.11) \quad T_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1} \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha + \alpha k)},$$

and

$$(2.12) \quad K_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-2} R(\lambda^\alpha, A) d\lambda = t E_{\alpha,2}(At^\alpha) = t \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(2 + \alpha k)}.$$

Proof. We note that

$$(2.13) \quad \frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_c e^\zeta \zeta^{-s} d\zeta, \quad \text{Re } s > 0.$$

Applying the transformation $\zeta = \eta^{1/\alpha}$ to equation (2.13) gives

$$(2.14) \quad \frac{1}{\Gamma(s)} = \frac{1}{2\pi i \alpha} \int_c e^{\eta^{1/\alpha}} \eta^{-\frac{s}{\alpha} + \frac{1}{\alpha} - 1} d\eta.$$

Since A is a sectorial operator of type (M, θ, α, μ) , it follows from the inequality

$$\|(\lambda^\alpha I - A)^{-1}\| \leq \frac{M}{|\lambda^\alpha - \mu|}$$

that A is the infinitesimal generator of α -resolvent families

$$\{S_\alpha(t)\}_{t \geq 0}, \{T_\alpha(t)\}_{t \geq 0} \text{ and } \{K_\alpha(t)\}_{t \geq 0} \text{ (see [30]).}$$

Hence, using the transformation $t^{-\alpha}\eta = \lambda^\alpha$ (i.e., $t^{-\alpha}d\eta = \alpha\lambda^{\alpha-1}d\lambda$ and $e^{\eta^{1/\alpha}} = e^{t\lambda}$), we obtain

$$\begin{aligned} E_{\alpha,1}(At^\alpha) &= \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(1 + \alpha k)} \\ &= \frac{1}{2\pi i \alpha} \sum_{k=0}^{\infty} \left\{ \int_c e^{\eta^{1/\alpha}} \eta^{-k-1} \right\} (At^\alpha)^k \\ &= \frac{1}{2\pi i \alpha} \int_c e^{\eta^{1/\alpha}} \eta^{-1} \left\{ \sum_{k=0}^{\infty} (At^\alpha \eta^{-1})^k \right\} d\eta \\ &= \frac{1}{2\pi i \alpha} \int_c e^{\eta^{1/\alpha}} t^{-\alpha} (t^{-\alpha}\eta I - A)^{-1} d\eta \\ &= \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda \\ &= S_\alpha(t). \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) &= t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(\alpha + \alpha k)} \\ &= \frac{t^{\alpha-1}}{2\pi i \alpha} \sum_{k=0}^{\infty} \left\{ \int_c e^{\eta^{1/\alpha}} \eta^{-k + \frac{1}{\alpha} - 2} \right\} (At^\alpha)^k \\ &= \frac{t^{\alpha-1}}{2\pi i \alpha} \int_c e^{\eta^{1/\alpha}} \eta^{\frac{1}{\alpha} - 2} \left\{ \sum_{k=0}^{\infty} (At^\alpha \eta^{-1})^k \right\} d\eta \\ &= \frac{1}{2\pi i \alpha} \int_c e^{\eta^{1/\alpha}} t^{-1} \eta^{\frac{1}{\alpha} - 1} (t^{-\alpha}\eta I - A)^{-1} d\eta \\ &= \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda \\ &= T_\alpha(t), \end{aligned}$$

and

$$\begin{aligned}
 {}_tE_{\alpha,2}(At^\alpha) &= t \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(2 + \alpha k)} \\
 &= \frac{t}{2\pi i \alpha} \sum_{k=0}^{\infty} \left\{ \int_c e^{\eta^{1/\alpha}} \eta^{-k-\frac{1}{\alpha}-1} \right\} (At^\alpha)^k \\
 &= \frac{t}{2\pi i \alpha} \int_c e^{\eta^{1/\alpha}} \eta^{-\frac{1}{\alpha}-1} \left\{ \sum_{k=0}^{\infty} (At^\alpha \eta^{-1})^k \right\} d\eta \\
 &= \frac{1}{2\pi i \alpha} \int_c e^{\eta^{1/\alpha}} t^{-\alpha+1} \eta^{-\frac{1}{\alpha}} (t^{-\alpha} \eta I - A)^{-1} d\eta \\
 &= \frac{1}{2\pi i} \int_c \lambda^{\alpha-2} e^{\lambda t} R(\lambda^\alpha, A) d\lambda \\
 &= K_\alpha(t). \quad \square
 \end{aligned}$$

Lemma 2.8. *If A is a sectorial operator of type (M, θ, α, μ) , then we have*

$$(2.15) \quad \frac{d(K_\alpha(t))}{dt} = S_\alpha(t) \quad \text{and} \quad AT_\alpha(t) = \frac{dS_\alpha(t)}{dt}.$$

Proof. A being a sectorial operator of type (M, θ, α, μ) indicates that it is the infinitesimal generator of α -resolvent families $\{S_\alpha(t)\}_{t \geq 0}$, $\{T_\alpha(t)\}_{t \geq 0}$ and $\{K_\alpha(t)\}_{t \geq 0}$. Therefore, the series

$$t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(\alpha + \alpha k)}, \quad \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(1 + \alpha k)} \quad \text{and} \quad t \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(2 + \alpha k)}$$

are uniformly convergent on $[0, T]$, where $T > 0$. Hence, we obtain

$$\begin{aligned}
 \frac{dK_\alpha(t)}{dt} &= \left[t \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(2 + \alpha k)} \right]' = \left[\sum_{k=0}^{\infty} \frac{A^k t^{k\alpha+1}}{\Gamma(2 + \alpha k)} \right]' \\
 &= \sum_{k=0}^{\infty} \frac{A^k (1 + \alpha k) t^{\alpha k}}{(1 + \alpha k) \Gamma(1 + \alpha k)} = \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(1 + \alpha k)} = S_\alpha(t)
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{dS_\alpha(t)}{dt} &= \left[\sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(1 + \alpha k)} \right]' = At^{\alpha-1} \sum_{k=1}^{\infty} \frac{A^{k-1} t^{\alpha(k-1)}}{\Gamma(\alpha + \alpha(k-1))} \\
 &= At^{\alpha-1} \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(\alpha + \alpha k)} = AT_\alpha(t)
 \end{aligned}$$

for $t \in [0, T]$. □

Remark 2.1. Now we consider a special case of $A = -\rho$. It then follows from

$$(2.16) \quad \begin{cases} D_t^\alpha u(t) = -\rho u(t) + f(t) \\ u(0) = c_0, u'(0) = c_1 \end{cases}$$

that

$$u(t) = c_0 u_0(t) + c_1 u_1(t) + \int_0^t u_\delta(t-s)f(s)ds,$$

where $u_0(t) = E_{\alpha,1}(-\rho t^\alpha)$, $u_1(t) = tE_{\alpha,2}(-\rho t^\alpha)$, and $u_\delta(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\rho t^\alpha)$. We note that $u_0(t), u_1(t)$ and $u_\delta(t)$ satisfy

$$u_1(t) = \int_0^t u_0(s)ds, \quad \text{and} \quad u_\delta(t) = -\frac{1}{\rho}u_0'(t).$$

Definition 2.6 ([27]). Let $R(t)_{(t \geq 0)}$ be an α -resolvent solution operator in X . If $R(t)x \geq \theta$ for every $x \geq \theta$, $x \in X$ and $t \geq 0$, then $R(t)_{(t \geq 0)}$ is said to be positive.

Definition 2.7 ([7]). Let $A : D(A) \rightarrow X$ be a linear operator. $A : D(A) \rightarrow X$ is said to be nonnegative if and only if it satisfies both of the following conditions:

(i) There exists $K \geq 0$ such that, for every value of $\lambda > 0$ and every $u \in D(A)$,

$$(2.17) \quad \lambda \|u\|_X \leq K \|\lambda u + Au\|_X.$$

(ii) $R(\lambda I + A) = X$ for every value of $\lambda > 0$.

Definition 2.8 ([7]). If A is a linear operator and satisfies condition (i) in Definition 2.7 for $K = 1$, then A is said to be accretive. In addition, A is said to be m -accretive if condition (ii) is also satisfied.

Remark 2.2 ([7]). Assume that X is a Hilbert space with inner product $(\cdot; \cdot)$. Then the necessary and sufficient condition for A to be accretive is $\text{Re}(Au; u) \geq 0$ for every $u \in D(A)$. Particularly, if X is a real Hilbert space and A is positive, then we obtain $(Au; u) \geq 0$ for every $u \in D(A)$. Note that an ordered Banach space is a real space, implying that if X is an ordered Banach space and A is accretive, then $(Au; u) \geq 0$ for every $u \in D(A)$.

Remark 2.3. It follows from Lemma 2.7 and Remark 2.2 that if X is an ordered Banach space and A is a sectorial accretive operator of type (M, θ, α, μ) , then the α -resolvent families $\{T_\alpha(t)\}_{t \geq 0}$, $S_\alpha(t)_{t \geq 0}$ and $\{K_\alpha(t)\}_{t \geq 0}$ are all positive.

Lemma 2.9 ([30]). Suppose that A is a sectorial operator of type (M, θ, α, μ) . Then, for $\|S_\alpha(t)\|$, there hold the following estimates:

(i) If $\mu \geq 0$, then for $\phi \in (\max\{\theta, (1 - \alpha)\pi\}, \frac{\pi}{2}(2 - \alpha))$, we have

$$(2.18) \quad \|S_\alpha(t)\| \leq \frac{K_1(\theta, \phi) M e^{[K_1(\theta, \phi)(1 + \mu t^\alpha)]^{\frac{1}{\alpha}}} [(1 + \frac{\sin \phi}{\sin(\phi - \theta)})^{\frac{1}{\alpha}} - 1]}{\Gamma(\alpha) M^{\frac{\pi \sin^{1+\frac{1}{\alpha}} \theta}} (1 + \mu t^\alpha)} + \frac{1}{\pi(1 + \mu t^\alpha) |\cos \frac{\pi - \phi}{\alpha}|^\alpha \sin \theta \sin \phi},$$

where $t > 0$.

(ii) If $\mu < 0$, then for $\phi \in (\max\{\frac{\pi}{2}, (1 - \alpha)\pi\}, \frac{\pi}{2}(2 - \alpha))$, we have

$$(2.19) \quad \|S_\alpha(t)\| \leq \left(\frac{eM[(1 + \sin \phi)^{\frac{1}{\alpha}} - 1]}{\pi |\cos \phi|^{1 + \frac{1}{\alpha}}} + \frac{\Gamma(\alpha)M}{\pi |\cos \phi| |\cos \frac{\pi - \phi}{\alpha}|^\alpha} \right) \frac{1}{1 + |\mu|t^\alpha},$$

where $t > 0$.

Lemma 2.10 ([30]). *If A is a sectorial operator of type (M, θ, α, μ) , we have the following estimates:*

(i) If $\mu \geq 0$, for $\phi \in (\max\{\theta, (1 - \alpha)\pi\}, \frac{\pi}{2}(2 - \alpha))$, we have

$$\begin{aligned} \|T_\alpha(t)\| &\leq \frac{M[(1 + \frac{\sin \phi}{\sin(\phi - \theta)})^{\frac{1}{\alpha}} - 1]}{\pi \sin \theta} (1 + \mu t^\alpha)^{\frac{1}{\alpha}} t^{\alpha - 1} e^{[K_1(\theta, \phi)(1 + \mu t^\alpha)]^{\frac{1}{\alpha}}} \\ &\quad + \frac{Mt^{\alpha - 1}}{\pi(1 + \mu t^\alpha) |\cos \frac{\pi - \phi}{\alpha}|^\alpha \sin \theta \sin \phi}, \\ \|K_\alpha(t)\| &\leq \frac{M[(1 + \frac{\sin \phi}{\sin(\phi - \theta)})^{\frac{1}{\alpha}} - 1]K_1(\theta, \phi)}{\pi \sin \theta^{\frac{\alpha + 2}{\alpha}}} (1 + \mu t^\alpha)^{\frac{\alpha - 1}{\alpha}} t^{\alpha - 1} e^{[K_1(\theta, \phi)(1 + \mu t^\alpha)]^{\frac{1}{\alpha}}} \\ &\quad + \frac{M\alpha\Gamma(\alpha)}{\pi(1 + \mu t^\alpha) |\cos \frac{\pi - \phi}{\alpha}|^\alpha \sin \theta \sin \phi}, \end{aligned}$$

where $t > 0$ and $K_1(\theta, \phi) = \max\{1, \frac{\sin \theta}{\sin(\phi - \theta)}\}$.

(ii) If $\mu < 0$, for $\phi \in (\max\{\frac{\pi}{2}, (1 - \alpha)\pi\}, \frac{\pi}{2}(2 - \alpha))$, we have

$$\begin{aligned} \|T_\alpha(t)\| &\leq \left(\frac{eM[(1 + \sin \phi)^{\frac{1}{\alpha}} - 1]}{\pi |\cos \phi|} + \frac{M}{\pi |\cos \phi| |\cos \frac{\pi - \phi}{\alpha}|} \right) \frac{t^{\alpha - 1}}{1 + |\mu|t^\alpha}, \\ \|K_\alpha(t)\| &\leq \left(\frac{eM[(1 + \sin \phi)^{\frac{1}{\alpha}} - 1]t}{\pi |\cos \phi|^{1 + \frac{2}{\alpha}}} + \frac{\alpha\Gamma(\alpha)M}{\pi |\cos \phi| |\cos \frac{\pi - \phi}{\alpha}|} \right) \frac{1}{1 + |\mu|t^\alpha}, \end{aligned}$$

where $t > 0$.

3. Main results

In this section, we give the main results of this article, i.e., the existence of mild solutions (which is defined in Definition 2.4) to equation (1.2). We consider the mild solutions under the following assumptions:

(H₁) There exists a constant $C \geq 0$ satisfying

$$(3.1) \quad f(t, x_2) - f(t, x_1) \geq -C(x_2 - x_1)$$

for every $t \in J$, and $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$.

(H₂) For $C \geq 0$ of inequality (3.1), linear operator $(A - CI)$ is a sectorial accretive operator of type (M, θ, α, μ) and generates compact α -resolvent families $\{T_\alpha^*(t)\}_{t \geq 0}$, $S_\alpha^*(t)_{t \geq 0}$ and $\{K_\alpha^*(t)\}_{t \geq 0}$.

(H₃) Cauchy problem (1.2) has a lower solution $v_0 \in C^\alpha(J, X) \cap C(J, X_1)$ and an upper solution $w_0 \in C^\alpha(J, X)C(J, X_1)$. Notice that, $v_0, w_0 \in C(J, X)$ and $v_0 \leq w_0$ in ordered Banach space $C(J, X)$.

(H₄) There exist a constant $L \geq 0$ satisfying

$$\alpha(\{f(t, u_n)\}) \leq L\alpha(\{u_n\})$$

for every $t \in J$, and an increasing monotonic sequence $\{u_n\} \subset [v_0(t), w_0(t)] \subset C(J, X)$.

Theorem 3.1. *Assume that X is an ordered Banach space and its positive cone K is normal. If conditions $(H_1) \sim (H_4)$ are satisfied, then Cauchy problem (1.2) has minimal and maximal mild solutions that are between v_0 and w_0 . Such solutions can be obtained by using monotone iterative procedure starting from v_0 and w_0 , respectively.*

Proof. Since $C > 0$, system (1.2) can be written as

$$(3.1) \quad \begin{cases} D_t^\alpha u(t) = (A - CI)u(t) + f(t, u(t)) + Cu(t), & t \in J \\ u(t_0) = x_0, u'(t_0) = x_1. \end{cases}$$

Notice that $(A - CI)$ is an sectorial accretive operator of type (M, θ, α, μ) , and generates compact and positive α -resolvent families $\{T_\alpha^*(t)\}_{t \geq 0}, S_\alpha^*(t)_{t \geq 0}$ and $\{K_\alpha^*(t)\}_{t \geq 0}$. From Lemmas 2.9 and 2.10 we know that for $t \in J = [0, T]$, there exists a constant \widetilde{M} such that

$$(3.2) \quad \sup_{t \in [0, T]} \{\|T_\alpha^*(t)\|\} \leq \widetilde{M}, \quad \sup_{t \in [0, T]} \{\|S_\alpha^*(t)\|\} \leq \widetilde{M}, \quad \sup_{t \in [0, T]} \{\|K_\alpha^*(t)\|\} \leq \widetilde{M}.$$

By Definition 2.4, the mild solutions to Cauchy problem (3.1) are obtained as

$$(3.3) \quad u(t) = S_\alpha^*(t)x_0 + K_\alpha^*(t)x_1 + \int_0^t T_\alpha^*(t-s)[f(s, u(s)) + Cu(s)]ds.$$

Let $D = [v_0, w_0]$. Then the mapping $\Gamma : D \rightarrow C(J, X)$ can be expressed as

$$(3.4) \quad (\Gamma u)(t) = S_\alpha^*(t)x_0 + K_\alpha^*(t)x_1 + \int_0^t T_\alpha^*(t-s)[f(s, u(s)) + Cu(s)]ds.$$

We notice that $\Gamma : D \rightarrow C(J, X)$ is continuous and $u \in D$ is a mild solution of problem (3.1) or (1.2) if and only if

$$(3.5) \quad u = \Gamma u.$$

By (H₂), $f(t, x) + Cx$ is a non-decreasing function for $x \in D$. We notice that $\{T_\alpha^*(t)\}_{t \geq 0}, S_\alpha^*(t)_{t \geq 0}$ and $\{K_\alpha^*(t)\}_{t \geq 0}$ are all positive. Thus, for $u_1, u_2 \in D$, if $u_1 \leq u_2$, we have

$$(3.6) \quad \Gamma u_1 \leq \Gamma u_2.$$

Next, we show that $v_0 \leq \Gamma v_0$ and $\Gamma w_0 \leq w_0$. Let

$$D^\alpha(t)v_0(t) = (A - CI)v_0(t) + \sigma(t), \quad t \in J = [0, T].$$

Using Lemma 2.5 and Definition 2.5, the positivity of operators $\{T_\alpha^*(t)\}_{t \geq 0}$, $S_\alpha^*(t)_{t \geq 0}$ and $\{K_\alpha^*(t)\}_{t \geq 0}$ yields

$$\begin{aligned} v_0(t) &= S_\alpha^*(t)v(0) + K_\alpha^*(t)v'(0) + \int_0^t T_\alpha^*(t-s)\sigma(s)ds \\ &\leq S_\alpha^*(t)x_0 + K_\alpha^*(t)x_1 + \int_0^t T_\alpha^*(t-s)[f(s, v_0(s)) + Cv_0(s)]ds \end{aligned}$$

for every $t \in J$.

Similarly, we can prove the following inequality:

$$\begin{aligned} w_0(t) &= S_\alpha^*(t)w(0) + K_\alpha^*(t)w'(0) + \int_0^t T_\alpha^*(t-s)\varrho(s)ds \\ &\geq S_\alpha^*(t)x_0 + K_\alpha^*(t)x_1 + \int_0^t T_\alpha^*(t-s)[f(s, w_0(s)) + Cw_0(s)]ds \end{aligned}$$

for every $t \in J$. Here $\varrho(t) = D^\alpha(w_0(t)) - (A - CI)w_0(t)$. Therefore, for every $u \in D$, we have

$$(3.7) \quad v_0 \leq \Gamma v_0 \leq \Gamma u \leq \Gamma w_0 \leq w_0.$$

That is to say, $\Gamma : D \rightarrow D$ is a continuous increasing monotonic operator. We define

$$(3.8) \quad v_n = \Gamma v_{n-1}, \quad \text{and} \quad w_n = \Gamma w_{n-1}.$$

It then follows from inequality (3.6) that

$$(3.9) \quad v_0 \leq v_1 \leq v_2 \leq \dots \leq w_n \leq w_{n-1} \leq \dots \leq w_2 \leq w_1 \leq w_0.$$

Denote $\Omega = \{v_n\}$ and $\Omega_0 = \{v_n\} \cup v_0, n = 1, 2, \dots$. By (3.9) and the normality of the positive cone K , we know that Ω and Ω_0 are bounded, implying that

$$\alpha(\Omega(t)) = \alpha(\Omega_0(t)) \quad \text{for all } t \in J.$$

Let

$$(3.10) \quad \varphi(t) = \alpha(\Omega(t)) = \alpha(\Omega_0(t)), \quad t \in J.$$

Using (H_4) , (3.2), (3.3), (3.8), (3.10), Lemma 2.2 and the positivity of operators $\{T_\alpha^*(t)\}_{t \geq 0}, S_\alpha^*(t)_{t \geq 0}$ and $\{K_\alpha^*(t)\}_{t \geq 0}$, we obtain the following estimates:

$$\begin{aligned} \varphi(t) &= \alpha(\Omega(t)) = \alpha(\Gamma\Omega_0(t)) \\ &= \alpha\left(\{S_\alpha^*(t)x_0 + K_\alpha^*(t)x_1 \right. \\ &\quad \left. + \int_0^t T_\alpha^*(t-s)[f(s, v_{n-1}(s)) + Cv_{n-1}(s)]ds | n = 1, 2, \dots\}\right) \\ &= \alpha\left(\left\{\int_0^t T_\alpha^*(t-s)[f(s, v_{n-1}(s)) + Cv_{n-1}(s)]ds | n = 1, 2, \dots\right\}\right) \\ &\leq 2 \int_0^t \alpha(\{T_\alpha^*(t-s)[f(s, v_{n-1}(s)) + Cv_{n-1}(s)] | n = 1, 2, \dots\}) ds \end{aligned}$$

$$\begin{aligned}
 &\leq 2\widetilde{M} \int_0^t \alpha(\{[f(s, v_{n-1}(s)) + Cv_{n-1}(s)] | n = 1, 2, \dots\}) ds \\
 &\leq 2\widetilde{M}(L + C) \int_0^t \alpha(\Omega_0(s)) ds \\
 (3.11) \quad &= 2\widetilde{M}(L + C) \int_0^t \varphi(s) ds.
 \end{aligned}$$

It thus follows from (3.11) and Lemma 2.6 (Gronwall inequality) that $\varphi(t) \equiv 0$ for every $t \in J$. Hence,

$$\alpha(\Omega) = \alpha(\Omega_0) = \max_{t \in J} \alpha(\Omega_0(t)) = 0,$$

indicating that $\{v_n(t)\}$ ($n = 1, 2, \dots$) is precompact in X for every $t \in J$. Therefore, $\{v_n(t)\}$ has a convergent subsequence in X . Based on (3.9), it can be shown that $\{v_n(t)\}$ itself is convergent in X , i.e., there exists $\underline{u}(t) \in X$ such that $v_n(t) \rightarrow \underline{u}(t)$ as $n \rightarrow \infty$ for every $t \in J$. It then follows from (3.3), (3.4) and (3.8) that

$$(3.12) \quad v_n(t) = S_\alpha^*(t)x_0 + K_\alpha^*(t)x_1 + \int_0^t T_\alpha^*(t-s)[f(s, v_{n-1}(s)) + Cv_{n-1}(s)] ds, \quad t \in J.$$

If $n \rightarrow \infty$, using the Lebesgue-dominated convergence theorem, we obtain

$$(3.13) \quad \underline{u}(t) = S_\alpha^*(t)x_0 + K_\alpha^*(t)x_1 + \int_0^t T_\alpha^*(t-s)[f(s, \underline{u}(s)) + C\underline{u}(s)] ds, \quad t \in J.$$

Hence, we have $u \in C(J, X)$ and $u = \Gamma u$. In a similar way, we can prove that there exists $\bar{u} \in C(J, X)$ such that $\bar{u} = \Gamma \bar{u}$. Using (3.6), if $u \in D$ is a fixed point of Γ , we have

$$v_1 = \Gamma v_0 \leq \Gamma u = u \leq \Gamma w_0 = w_1.$$

Then, an easy induction implies that $v_n \leq u \leq w_n$. By (3.9), taking the limit as $n \rightarrow \infty$ yields

$$v_0 \leq \underline{u} \leq u \leq \bar{u} \leq w_0,$$

which implies that \underline{u} and \bar{u} are the minimal and maximal fixed points of Γ on $[v_0, w_0]$, respectively. It thus follows from (3.5) that they are also the minimal and maximal mild solutions of Cauchy problem (1.2) on $[v_0, w_0]$, respectively. \square

Corollary 3.1. *Suppose that X is an ordered Banach space, whose positive cone K is regular. Then conditions $(H_1) \sim (H_3)$ guarantee that Cauchy problem (1.2) has minimal and maximal mild solutions between v_0 and w_0 . Such minimal and maximal mild solutions can be obtained by a monotone iterative procedure starting from v_0 and w_0 , respectively.*

Proof. Since K is regular, any ordered-monotonic and ordered-bounded sequence in X is convergent. Suppose that $\{x_n\}$ is a monotonic sequence in

$[v_0(t), w_0(t)]$ for $t \in I$. It follows from (H_1) that $\{f(t, x_n) + Cx_n\}$ is an ordered-monotonic and ordered-bounded sequence in X , indicating that $\alpha(\{f(t, x_n) + Cx_n\}) = \alpha(\{x_n\}) = 0$. Thus, using the properties of the measure of noncompactness, we obtain

$$(3.14) \quad \alpha(\{f(t, x_n)\}) \leq \alpha(\{f(t, x_n) + Cx_n\}) = 0.$$

Hence, condition (H_4) is satisfied. Then, using Theorem 3.1, one can prove that the conclusion of Corollary 3.1 holds. \square

Remark 3.1. Since the normal cone K is regular in an ordered and weakly sequentially complete Banach space, we get the following corollary.

Corollary 3.2. *If X is an ordered and weakly sequentially complete Banach space, whose positive cone K is normal with normal constant N , then conditions $(H_1) \sim (H_3)$ guarantee that Cauchy problem (1.2) has minimal and maximal mild solutions between v_0 and w_0 . Such minimal and maximal mild solutions can be obtained by a monotone iterative procedure starting from v_0 and w_0 , respectively.*

Remark 3.2. Based on the fact that the normal cone P is regular in an ordered and reflective Banach space, we obtain the following corollary.

Corollary 3.3. *If X is an ordered and reflective Banach space, whose positive cone K is normal with normal constant N , then conditions $(H_1) \sim (H_3)$ guarantee that Cauchy problem (1.2) has minimal and maximal mild solutions between v_0 and w_0 . Such minimal and maximal mild solutions can be obtained by a monotone iterative procedure starting from v_0 and w_0 , respectively.*

4. An example

In this section, we consider an example to illustrate the applications of the main results of this article. Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. We study the initial boundary value problem of parabolic type, given by

$$(4.1) \quad \begin{cases} D_t^\alpha u(t, x) = -\Delta u(t, x) + Cu(t, x) + f(t, u(t, x)), & t \in J = [0, T], \quad x \in \Omega, \\ u|_{\partial\Omega=0}, \\ u(0, x) = \varphi(x), \\ \frac{du(t, x)}{dt}|_{t=0} = \psi(x), \end{cases}$$

where Δ is a Laplace operator, $1 < \alpha < 2$, and $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Let $X = L^2(J \times \Omega, \mathbb{R})$ and $P = \{u \in C(J, X) : u(t, x) \geq 0, \text{ a.e. } (t, x) \in J \times \Omega\}$.

Obviously, X is an ordered Banach space and P is a normal cone in $C(J, X)$. We define the operator A by

$$(4.2) \quad D(A) = H^2(\Omega) \cup H_0^1(\Omega), \quad Au = -\Delta u + Cu.$$

As indicated in [30], the operator $A - C = -\Delta : D(A) \subset X \rightarrow X$ is a sectorial operator of type (M, θ, α, μ) and generates compact and positive α -resolvent

families $\{T_\alpha^*(t)\}_{t \geq 0}$, $S_\alpha^*(t)_{t \geq 0}$ and $\{K_\alpha^*(t)\}_{t \geq 0}$. Since it was proved in [12] that $A - C = -\Delta$ is an m -accretive operator on $L^2(\Omega)$ with dense domain, then system (4.1) can be reformulated as problem (1.2). Thus condition (H_2) is satisfied. Besides, we suppose that the following conditions hold:

(A₁) $f(t, 0) \geq 0$ for $t \in [0, T]$, $\varphi(x) \geq 0, \psi(x) \geq 0$ for $x \in \Omega$.

(A₂) There exists $w(t, x) \in C^\alpha(J, X) \cap (H^2(\Omega) \cup H_0^1(\Omega))$ such that

$$(4.3) \quad \begin{cases} D_t^\alpha w(t, x) \geq -\Delta w(t, x) + Cu(t, x) + f(t, w(t, x)), & t \in J = [0, T], \quad x \in \Omega, \\ w|_{\partial\Omega} = 0, \\ w(0, x) \geq \varphi(x), \\ \frac{dw(t, x)}{dt}|_{t=0} \geq \psi(x), \end{cases}$$

where $w(t, x)$ ($(t, x) \in J \times \Omega$), $D_t^\alpha w(t, x)$ and $\Delta w(t, x)$ are continuous.

Theorem 4.1. *Conditions (H_1) , (H_4) and (A_1) - (A_2) guarantee that system (1.2) has minimal and maximal mild solutions between 0 and w .*

Proof. It follows from (A_1) and (A_2) that 0 and w are the lower and upper solutions of problem (1.2), respectively. Using Theorem 3.1, system (4.1) has minimal and maximal mild solutions between 0 and w . \square

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