

Research Article

Upper and Lower Solutions Method for a Class of Singular Fractional Boundary Value Problems with p -Laplacian Operator

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The upper and lower solutions method is used to study the p -Laplacian fractional boundary value problem $D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t))) = f(t, u(t))$, $0 < t < 1$, $u(0) = 0$, $u(1) = au(\xi)$, $D_{0+}^{\alpha}u(0) = 0$, and $D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta)$, where $1 < \alpha, \gamma \leq 2$, $0 \leq a, b \leq 1$, $0 < \xi, \eta < 1$. Some new results on the existence of at least one positive solution are obtained. It is valuable to point out that the nonlinearity f can be singular at $t = 0, 1$ or $u = 0$.

1. Introduction

It is well known that the upper and lower solutions method is a powerful tool for proving the existence results for boundary value problem. It has been used to deal with many multipoint boundary value problem of integer ordinary differential equations (see, e.g., [1–3] and references therein).

Recently, boundary value problems of nonlinear fractional differential equations have aroused considerable attention. Many people pay attention to the existence and multiplicity of solutions or positive solutions for boundary value problems of nonlinear fractional differential equations by means of some fixed point theorems, such as the Krasnosel'skii fixed-point theorem, the Leggett-Williams fixed-point theorem, and the Schauder fixed-point theorem (see [4–8]). To the best of our knowledge, the upper and lower solutions method is seldom considered in the literatures, and there are few papers devoted to investigate p -Laplacian fractional boundary value problems.

In this paper, we deal with the following p -Laplacian fractional boundary value problem:

$$\begin{aligned} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t))) &= f(t, u(t)), & 0 < t < 1, \\ u(0) &= 0, & u(1) = au(\xi), & D_{0+}^{\alpha}u(0) = 0, & D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta), \end{aligned} \quad (1.1)$$

where $1 < \alpha, \gamma \leq 2, 0 \leq a, b \leq 1, 0 < \xi, \eta < 1$, and D^α is the standard Riemann-Liouville fractional differential operator of order α , $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $(1/p) + (1/q) = 1$. By using upper and lower solutions method, the existence results of at least one positive solution for the above fractional boundary value problem are established, and an example is given to show the effectiveness of our results. It is valuable to point out that the nonlinearity f can be singular at $t = 0, 1$ or $u = 0$.

The remaining part of the paper is organized as follows. In Section 2, we will present some definitions and lemmas. In Section 3, some results are given. In Section 4, we present an example to demonstrate our results.

2. Basic Definitions and Preliminaries

In this section, we present some necessary definitions and lemmas.

Definition 2.1 (see [9]). The integral

$$I_{0+}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \quad (2.1)$$

where $\alpha > 0$, is called the Riemann-Liouville fractional integral of order α .

Definition 2.2 (see [10]). For a function $y(t)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds, \quad (2.2)$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of number α , is called the Riemann-Liouville fractional derivative of order α .

Remark 2.3. From the definition of the Riemann-Liouville fractional derivative, we quote for $\mu > -1$, then

$$D_{0+}^\alpha t^\mu = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha}. \quad (2.3)$$

In particular, $D_{0+}^\alpha t^{\alpha-m} = 0$, $m = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α .

Lemma 2.4 (see [4]). Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}, \quad (2.4)$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α .

Lemma 2.5 (see [6]). *Let $y \in C[0, 1]$ and $1 < \alpha \leq 2$, the unique solution of*

$$\begin{aligned} D_{0+}^{\alpha}u(t) + y(t) &= 0, \quad 0 < t < 1, \\ u(0) &= 0, \quad u(1) = au(\xi), \end{aligned} \tag{2.5}$$

is

$$u(t) = \int_0^1 G(t, s)y(s)ds, \tag{2.6}$$

where

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \xi, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 < \xi \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq \xi \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \xi \leq s. \end{cases} \tag{2.7}$$

Lemma 2.6. *Let $y \in C[0, 1]$, $1 < \alpha, \gamma \leq 2$, $0 < \xi, \eta < 1$, $0 \leq a$, and $b \leq 1$. The fractional boundary value problem*

$$\begin{aligned} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t))) &= y(t), \quad 0 < t < 1, \\ u(0) &= 0, \quad u(1) = au(\xi), \quad D_{0+}^{\alpha}u(0) = 0, \quad D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta), \end{aligned} \tag{2.8}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)\phi_q\left(\int_0^1 H(s, \tau)y(\tau)d\tau\right)ds, \tag{2.9}$$

where

$$H(t, s) = \begin{cases} \frac{[t(1-s)]^{\gamma-1} - b_1 t^{\gamma-1} (\eta-s)^{\gamma-1} - (t-s)^{\gamma-1} (1-b_1 \eta^{\gamma-1})}{(1-b_1 \eta^{\gamma-1}) \Gamma(\gamma)}, & 0 \leq s \leq t \leq 1, s \leq \eta, \\ \frac{[t(1-s)]^{\gamma-1} - (t-s)^{\gamma-1} (1-b_1 \eta^{\gamma-1})}{(1-b_1 \eta^{\gamma-1}) \Gamma(\gamma)}, & 0 < \eta \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\gamma-1} - b_1 t^{\gamma-1} (\eta-s)^{\gamma-1}}{(1-b_1 \eta^{\gamma-1}) \Gamma(\gamma)}, & 0 \leq t \leq s \leq \eta \leq 1, \\ \frac{[t(1-s)]^{\gamma-1}}{(1-b_1 \eta^{\gamma-1}) \Gamma(\gamma)}, & 0 \leq t \leq s \leq 1, \eta \leq s, \end{cases} \quad (2.10)$$

$b_1 = b^{\nu-1}$ and $G(t, s)$ is defined by (2.7).

Proof. At first, by Lemma 2.4, the (2.8) is equivalent to the integral equation

$$\phi_p(D_{0+}^\alpha u(t)) = I_{0+}^\gamma y(t) + C_1 t^{\gamma-1} + C_2 t^{\gamma-2}, \quad C_1, C_2 \in \mathbb{R}, \quad (2.11)$$

that is,

$$\phi_p(D_{0+}^\alpha u(t)) = \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau + C_1 t^{\gamma-1} + C_2 t^{\gamma-2}. \quad (2.12)$$

By the boundary conditions $D_{0+}^\alpha u(0) = 0, D_{0+}^\alpha u(1) = b D_{0+}^\alpha u(\eta)$, we have

$$C_2 = 0, \quad C_1 = - \int_0^1 \frac{(1-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-b_1 \eta^{\gamma-1})} y(\tau) d\tau + \int_0^\eta \frac{b_1 (\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-b_1 \eta^{\gamma-1})} y(\tau) d\tau. \quad (2.13)$$

Therefore, the solution $u(t)$ of boundary value problem (2.8) satisfies

$$\begin{aligned} \phi_p(D_{0+}^\alpha u(t)) &= \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau - \int_0^1 \frac{t^{\gamma-1} (1-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-b_1 \eta^{\gamma-1})} y(\tau) d\tau \\ &\quad + \int_0^\eta \frac{b_1 t^{\gamma-1} (\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-b_1 \eta^{\gamma-1})} y(\tau) d\tau \\ &= - \int_0^1 H(t, \tau) y(\tau) d\tau. \end{aligned} \quad (2.14)$$

Consequently, $D_{0+}^\alpha u(t) + \phi_q(\int_0^1 H(t, \tau)y(\tau)d\tau) = 0$. So, fractional boundary value problem (2.8) is equivalent to the following problem:

$$D_{0+}^\alpha u(t) + \phi_q\left(\int_0^1 H(t, \tau)y(\tau)d\tau\right) = 0, \quad 0 < t < 1, \tag{2.15}$$

$$u(0) = 0, \quad u(1) = au(\xi).$$

Lemma 2.5 implies that fractional boundary value problem (2.8) has a unique solution

$$u(t) = \int_0^1 G(t, s)\phi_q\left(\int_0^1 H(s, \tau)y(\tau)d\tau\right)ds. \tag{2.16}$$

The proof is completed. □

Lemma 2.7. *Let $0 \leq a, b \leq 1$, $0 < \xi, \eta < 1$, $1 < \alpha$, and $\gamma \leq 2$. The functions $G(t, s)$ and $H(t, s)$ defined by (2.7) and (2.10), respectively, are continuous on $[0, 1] \times [0, 1]$ and satisfy*

- (1) $G(t, s) \geq 0, H(t, s) \geq 0$, for $t, s \in [0, 1]$;
- (2) $G(t, s) \leq G(s, s), H(t, s) \leq H(s, s)$ for $t, s \in [0, 1]$;
- (3) $G(t, s)/G(1, s) \geq t^{\alpha-1}$, for $t, s \in (0, 1)$.

Proof. The proofs of part (1) and part (2) can be obtained in [6]. Here, we only prove part (3). If $0 < s \leq t < 1, s \leq \xi$, then

$$\frac{G(t, s)}{G(1, s)} = \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{a\xi^{\alpha-1}(1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}} \tag{2.17}$$

$$= t^{\alpha-1} \cdot \frac{(1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1} - (1-(s/t))^{\alpha-1}(1-a\xi^{\alpha-1})}{a\xi^{\alpha-1}(1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

If $0 < \xi \leq s \leq t < 1$, then

$$\frac{G(t, s)}{G(1, s)} = \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{a\xi^{\alpha-1}(1-s)^{\alpha-1}} \tag{2.18}$$

$$= t^{\alpha-1} \cdot \frac{(1-s)^{\alpha-1} - (1-(s/t))^{\alpha-1}(1-a\xi^{\alpha-1})}{a\xi^{\alpha-1}(1-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

If $0 < t \leq s \leq \xi < 1$, then

$$\frac{G(t, s)}{G(1, s)} = \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1}}{a\xi^{\alpha-1}(1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}} \tag{2.19}$$

$$= t^{\alpha-1} \cdot \frac{(1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}}{a\xi^{\alpha-1}(1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1}} \geq t^{\alpha-1}.$$

If $0 < t \leq s < 1, \xi \leq s$, then

$$\frac{G(t, s)}{G(1, s)} = \frac{[t(1-s)]^{\alpha-1}}{a\xi^{\alpha-1}(1-s)^{\alpha-1}} = t^{\alpha-1} \cdot \frac{(1-s)^{\alpha-1}}{a\xi^{\alpha-1}(1-s)^{\alpha-1}} \geq t^{\alpha-1}. \quad (2.20)$$

The above inequalities imply that part (3) of Lemma 2.7 holds. \square

From Lemmas 2.5 and 2.7, it is easy to obtain the following lemma.

Lemma 2.8. *Let $0 \leq a \leq 1, 0 < \xi < 1, 1 < \alpha \leq 2$. If $y(t) \in C[0, 1]$, and $y(t) \geq 0$, then fractional boundary value problem (2.5) has a unique solution $u(t) \geq 0, t \in [0, 1]$.*

Let $E = \{u : u, \phi_p(D_{0+}^\alpha u) \in C^2[0, 1]\}$. Now one introduces the following definitions about the upper and lower solutions of fractional boundary value problem (1.1).

Definition 2.9. A function $\alpha(t)$ is called a lower solution of fractional boundary value problem (1.1), if $\alpha(t) \in E$ and $\alpha(t)$ satisfies

$$\begin{aligned} D_{0+}^\gamma(\phi_p(D_{0+}^\alpha \alpha(t))) &\leq f(t, \alpha(t)), \quad 0 < t < 1, \quad 1 < \alpha, \quad \gamma \leq 2, \\ \alpha(0) &\leq 0, \quad \alpha(1) \leq a\alpha(\xi), \\ D_{0+}^\alpha \alpha(0) &\geq 0, \quad D_{0+}^\alpha \alpha(1) \geq bD_{0+}^\alpha \alpha(\eta). \end{aligned} \quad (2.21)$$

Definition 2.10. A function $\beta(t)$ is called an upper solution of fractional boundary value problem (1.1), if $\beta(t) \in E$ and $\beta(t)$ satisfies

$$\begin{aligned} D_{0+}^\gamma(\phi_p(D_{0+}^\alpha \beta(t))) &\geq f(t, \beta(t)), \quad 0 < t < 1, \quad 1 < \alpha, \quad \gamma \leq 2, \\ \beta(0) &\geq 0, \quad \beta(1) \geq a\beta(\xi), \\ D_{0+}^\alpha \beta(0) &\leq 0, \quad D_{0+}^\alpha \beta(1) \leq bD_{0+}^\alpha \beta(\eta). \end{aligned} \quad (2.22)$$

3. Main Result

In this section, our objective is to give and prove our main results.

For the sake of simplicity, we assume that

- (H₁) $f(t, u) \in C[(0, 1) \times (0, +\infty), [0, +\infty]]$ and $f(t, u)$ is nonincreasing relative to u ;
- (H₂) For any constant $\mu > 0, 0 < \int_0^1 H(s, s) f(s, \mu s^{\alpha-1}) ds < +\infty$;
- (H₃) There exist a continuous function $m(t)$ and a positive number λ such that $m(t) \geq \lambda t^{\alpha-1}, t \in [0, 1]$, and

$$\begin{aligned} \int_0^1 G(t, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, m(\tau)) d\tau \right) ds &= n(t) \geq m(t), \\ \int_0^1 G(t, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, n(\tau)) d\tau \right) ds &\geq m(t). \end{aligned} \quad (3.1)$$

We define $P = \{u \in C[0, 1] : \text{there exists a number } \lambda_u > 0 \text{ such that } u(t) \geq \lambda_u t^{\alpha-1}, t \in [0, 1]\}$. And define an operator T by

$$Tu(t) = \int_0^1 G(t, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds, \quad \forall u \in P. \tag{3.2}$$

Theorem 3.1. *Suppose that conditions $(H_1) \sim (H_3)$ are satisfied, then the boundary value problem (1.1) has at least one positive solution $\rho(t)$, which satisfies $\rho(t) \geq \mu t^{\alpha-1}$ for some positive number μ .*

Proof. We will divide our proof into four steps.

Step 1. We prove that $T(P) \subseteq P$.

Firstly, from Lemma 2.7 and conditions $(H_1) \sim (H_2)$, for any $u \in P$, there exists $\lambda_u > 0$, such that

$$\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \leq \int_0^1 H(\tau, \tau) f(\tau, \lambda_u \tau^{\alpha-1}) d\tau < +\infty. \tag{3.3}$$

Thus,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ &\leq \int_0^1 G(s, s) ds \cdot \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, \lambda_u \tau^{\alpha-1}) d\tau \right) \\ &< +\infty. \end{aligned} \tag{3.4}$$

Secondly, it follows from Lemma 2.7 that

$$\begin{aligned} Tu(t) &= \int_0^1 \frac{G(t, s)}{G(1, s)} G(1, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ &\geq t^{\alpha-1} \int_0^1 G(1, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ &= \lambda_{Tu} t^{\alpha-1}, \quad \forall t \in [0, 1]. \end{aligned} \tag{3.5}$$

Consequently, T is well defined and $T(P) \subseteq P$.

In the meanwhile, by direct computations, we can obtain

$$D_{0+}^\gamma (\phi_p(D_{0+}^\alpha(Tu)(t))) = f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha, \quad \gamma \leq 2, \tag{3.6}$$

$$(Tu)(0) = 0, \quad (Tu)(1) = a(Tu)(\xi), \tag{3.7}$$

$$D_{0+}^\alpha(Tu)(0) = 0, \quad D_{0+}^\alpha(Tu)(1) = bD_{0+}^\alpha(Tu)(\eta).$$

Step 2. We will prove that the functions $\alpha(t) = Tn(t)$, $\beta(t) = Tm(t)$ are lower and upper solutions of fractional boundary value problem (1.1), respectively.

From (H_1) , we know that T is nonincreasing relative to u . Combining (H_3) , we have

$$m(t) \leq n(t) = Tm(t), \quad Tn(t) \leq n(t) = Tm(t), \quad t \in [0, 1]. \quad (3.8)$$

Therefore, $\alpha(t) \leq \beta(t)$.

By Step 1, we know $\alpha(t), \beta(t) \in P$. And by (3.6)~(3.8), we obtain

$$\begin{aligned} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}(\alpha)(t))) - f(t, \alpha(t)) &\leq D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}(Tn)(t))) - f(t, n(t)) = 0, \\ \alpha(0) = 0, \quad \alpha(1) = a\alpha(\xi), \quad D_{0+}^{\alpha}\alpha(0) = 0, \quad D_{0+}^{\alpha}\alpha(1) &= bD_{0+}^{\alpha}\alpha(\eta), \\ D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}(\beta)(t))) - f(t, \beta(t)) &\geq D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}(Tm)(t))) - f(t, m(t)) = 0, \\ \beta(0) = 0, \quad \beta(1) = a\beta(\xi), \quad D_{0+}^{\alpha}\beta(0) = 0, \quad D_{0+}^{\alpha}\beta(1) &= bD_{0+}^{\alpha}\beta(\eta), \end{aligned} \quad (3.9)$$

that is, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of fractional boundary value problem (1.1), respectively.

Step 3. We will show that the fractional boundary value problem

$$\begin{aligned} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t))) &= g(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha, \quad \gamma \leq 2, \\ u(0) = 0, \quad u(1) = au(\xi), \quad D_{0+}^{\alpha}u(0) = 0, \quad D_{0+}^{\alpha}u(1) &= bD_{0+}^{\alpha}u(\eta), \end{aligned} \quad (3.10)$$

has a positive solution, where

$$g(t, u(t)) = \begin{cases} f(t, \alpha(t)) & \text{if } u(t) < \alpha(t), \\ f(t, u(t)) & \text{if } \alpha(t) \leq u(t) \leq \beta(t), \\ f(t, \beta(t)) & \text{if } u(t) > \beta(t). \end{cases} \quad (3.11)$$

Thus, we consider the operator $A : C[0, 1] \rightarrow C[0, 1]$ defined as follows:

$$Au(t) = \int_0^1 G(t, s)\phi_q\left(\int_0^1 H(s, \tau)g(\tau, u(\tau))d\tau\right)ds, \quad (3.12)$$

where $G(t, s)$ is defined as (2.7), $H(t, s)$ is defined as (2.10). It is clear that $Au \geq 0$, for all $u \in P$, and a fixed point of the operator A is a solution of the fractional boundary value problem (3.10).

Since $\alpha(t) \in P$, there exists a positive number λ_α such that $\alpha(t) \geq \lambda_\alpha t^{\alpha-1}, t \in [0, 1]$. It follows from (H_2) that

$$\begin{aligned} \int_0^1 H(\tau, \tau)g(\tau, \alpha(\tau))d\tau &\leq \int_0^1 H(\tau, \tau)f(\tau, \alpha(\tau))d\tau \\ &\leq \int_0^1 H(\tau, \tau)f(\tau, \lambda_\alpha \tau^{\alpha-1})d\tau < +\infty. \end{aligned} \tag{3.13}$$

Consequently, for all $u(t) \in C[0, 1]$, we have

$$\begin{aligned} Au(t) &= \int_0^1 G(t, s)\phi_q\left(\int_0^1 H(s, \tau)g(\tau, u(\tau))d\tau\right)ds \\ &\leq \int_0^1 G(s, s)\phi_q\left(\int_0^1 H(\tau, \tau)g(\tau, u(\tau))d\tau\right)ds \\ &\leq \int_0^1 G(s, s)ds \cdot \phi_q\left(\int_0^1 H(\tau, \tau)f(\tau, \alpha(\tau))d\tau\right) < +\infty, \end{aligned} \tag{3.14}$$

which implies that the operator A is uniformly bounded.

On the other hand, since $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$, it is uniformly continuous on $[0, 1] \times [0, 1]$. So, for fixed $s \in [0, 1]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that any $t_1, t_2 \in [0, 1]$ and $|t_1 - t_2| < \delta$,

$$|G(t_1, s) - G(t_2, s)| < \frac{\varepsilon}{\phi\left(\int_0^1 H(\tau, \tau)f(\tau, \lambda_\alpha \tau^{\alpha-1})d\tau\right)}. \tag{3.15}$$

Then, for all $u(t) \in C[0, 1]$,

$$\begin{aligned} |Au(t_2) - Au(t_1)| &\leq \int_0^1 |G(t_2, s) - G(t_1, s)|\phi_q\left(\int_0^1 H(\tau, \tau)g(\tau, u(\tau))d\tau\right)ds \\ &\leq \int_0^1 |G(t_2, s) - G(t_1, s)|\phi_q\left(\int_0^1 H(\tau, \tau)f(\tau, \alpha(\tau))d\tau\right)ds \\ &\leq \int_0^1 |G(t_2, s) - G(t_1, s)|ds \cdot \phi_q\left(\int_0^1 H(\tau, \tau)f(\tau, \lambda_\alpha \tau^{\alpha-1})d\tau\right) \\ &< \varepsilon, \end{aligned} \tag{3.16}$$

that is to say, A is equicontinuous. Thus, from the Arzela-Ascoli Theorem, we know that A is a compact operator. By the Schauder's fixed-point theorem, the operator A has a fixed point; that is, the fractional boundary value problem (3.10) has a positive solution.

Step 4. We will prove that fractional boundary value problem (1.1) has at least one positive solution.

Suppose that $\rho(t)$ is a solution of (3.10), we only need to prove that $\alpha(t) \leq \rho(t) \leq \beta(t), t \in [0, 1]$. \square

Let $\rho(t)$ be a solution of (3.10). We have

$$\rho(0) = 0, \quad \rho(1) = a\rho(\xi), \quad D_{0+}^{\alpha}\rho(0) = 0, \quad D_{0+}^{\alpha}\rho(1) = bD_{0+}^{\alpha}\rho(\eta). \quad (3.17)$$

In addition, the function $f(t, u)$ is nonincreasing in u , we know that

$$f(t, \beta(t)) \leq g(t, \rho(t)) \leq f(t, \alpha(t)), \quad t \in [0, 1]. \quad (3.18)$$

It follows from (3.8) and (H_3) that

$$f(t, n(t)) \leq g(t, \rho(t)) \leq f(t, m(t)), \quad t \in [0, 1]. \quad (3.19)$$

By (3.6) and $m(t) \in P$, we obtain

$$D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\beta(t))) = D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}(Tm)(t))) = f(t, m(t)), \quad t \in [0, 1]. \quad (3.20)$$

Together with (3.7), (3.17)–(3.20), we obtain

$$\begin{aligned} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\beta(t))) - D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\rho(t))) &= f(t, m(t)) - g(t, \rho(t)) \geq 0, \quad t \in [0, 1], \\ (\beta - \rho)(0) &= 0, \quad (\beta - \rho)(1) = a(\beta - \rho)(\xi), \\ D_{0+}^{\alpha}(\beta - \rho)(0) &= 0, \quad D_{0+}^{\alpha}(\beta - \rho)(1) = bD_{0+}^{\alpha}(\beta - \rho)(\eta). \end{aligned} \quad (3.21)$$

Let $z(t) = \phi_p(D_{0+}^{\alpha}\beta(t)) - \phi_p(D_{0+}^{\alpha}\rho(t))$, we obtain

$$D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\beta(t))) - D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\rho(t))) \geq f(t, \beta(t)) - g(t, \rho(t)) \geq 0, \quad t \in [0, 1] \quad (3.22)$$

and $z(0) = 0, z(1) - \phi_p(b)z(\eta) = 0$.

By Lemma 2.8, we know $z(t) \leq 0, t \in [0, 1]$, which implies that

$$\phi_p(D_{0+}^{\alpha}\beta(t)) \leq \phi_p(D_{0+}^{\alpha}\rho(t)), \quad t \in [0, 1]. \quad (3.23)$$

Since ϕ_p is monotone increasing, we have $D_{0+}^{\alpha}\beta(t) \leq D_{0+}^{\alpha}\rho(t)$, that is, $D_{0+}^{\alpha}(\beta - \rho)(t) \leq 0$. By Lemma 2.8 and (3.21), we have $(\beta - \rho)(t) \geq 0$. Therefore, $\beta(t) \geq \rho(t), t \in [0, 1]$.

In the similar way, we can prove that $\alpha(t) \leq \rho(t), t \in [0, 1]$. Consequently, $\rho(t)$ is a positive solution of fractional boundary value problem (1.1). This completes the proof.

Theorem 3.2. *If $f(t, u) \in C([0, 1] \times [0, +\infty), [0, +\infty))$ is nonincreasing relative to u and $f(t, u)$ does not vanish identically for any $u > 0$, then the fractional boundary value problem (1.1) has at least one positive solution $\rho(t)$, which satisfies $\rho(t) \geq \mu t^{\alpha-1}$ for some positive number μ .*

The proof is similar to Theorem 3.1, we omit it here.

4. Example

Example 4.1. As an example, we consider the fractional boundary value problem

$$\begin{aligned}
 D_{0+}^{4/3}(\phi_p(D_{0+}^{3/2}u(t))) &= t^2 + \frac{1}{\sqrt{u}}, & 0 < t < 1, \\
 u(0) = 0, \quad u(1) &= \frac{1}{3}u\left(\frac{1}{2}\right), & D_{0+}^{3/2}u(0) = 0, & \quad D_{0+}^{3/2}u(1) = \frac{1}{2}D_{0+}^{3/2}u\left(\frac{2}{3}\right).
 \end{aligned} \tag{4.1}$$

Proof. It is clear that (H_1) holds. For any $\lambda > 0$,

$$\int_0^1 H(\tau, \tau) f\left(\tau, \lambda \tau^{(3/2)-1}\right) d\tau = \int_0^1 H(\tau, \tau) \left(\tau^2 + \frac{1}{\sqrt{\lambda \tau^{1/4}}}\right) d\tau < +\infty, \tag{4.2}$$

which implies that (H_2) holds.

For $0 < r < 1$, $f(t, u) = t^2 + (1/\sqrt{u}) \leq t^2 + (1/\sqrt{ru}) \leq r^{-1/2}f(t, u)$. Let $a(t) = t^{1/2}$, by (3.5), we have

$$b(t) =: Ta(t) = \int_0^1 G(t, s) \phi_q\left(\int_0^1 H(s, \tau) f(\tau, a(\tau)) d\tau\right) ds \in P, \tag{4.3}$$

and $Tb(t) = T^2a(t) \in P$, that is, there exist positive numbers λ_1, λ_2 , such that $Ta(t) \geq \lambda_1 a(t)$, $T^2a(t) \geq \lambda_2 a(t)$.

Choose a positive number $\lambda_0 \leq \{1, \lambda_1, \lambda_2^{4/3}\}$ and combining the monotonicity of T , we have

$$\begin{aligned}
 T(\lambda_0 a(t)) &\geq Ta(t) \geq \lambda_1 a(t) \geq \lambda_0 a(t), \\
 T^2(\lambda_0 a(t)) &\geq \lambda_0^{1/4} T^2 a(t) \geq \lambda_0^{1/4} \lambda_2 a(t) \geq \lambda_0 a(t).
 \end{aligned} \tag{4.4}$$

Taking $m(t) = \lambda_0 t^{1/2}$, then,

$$\begin{aligned} n(t) &= Tm(t) = \int_0^1 G(t,s)\phi_q \left(\int_0^1 H(s,\tau) f(\tau, \lambda_0 \tau^{1/2}) d\tau \right) ds \\ &\geq \lambda_0 t^{1/2} = m(t), \\ Tn(t) &= T^2m(t) = \int_0^1 G(t,s)\phi_q \left(\int_0^1 H(s,\tau) f(\tau, T(\lambda_0 \tau^{1/2})) d\tau \right) ds \\ &\geq \lambda_0 t^{1/2} = m(t), \end{aligned} \tag{4.5}$$

that is to say, the condition (H_3) holds. Theorem 3.1 implies that the fractional boundary value problem (4.1) has at least one positive solution. \square

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