Communications in Mathematical Physics manuscript No.
(will be inserted by the editor)

Upper bound on the decay of correlations in a general class of $O(N)$-symmetric models

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Abstract: We consider a general class of two-dimensional spin systems, with continuous but not necessarily smooth, possibly long-range, $O(N)$-symmetric interactions, for which we establish algebraically decaying upper bounds on spin-spin correlations under all infinite-volume Gibbs measures.

As a by-product, we also obtain estimates on the effective resistance of a (possibly long-range) resistor network in which randomly selected edges are shorted.

Key words. Spin systems – continuous symmetry – decay of correlations – McBryan-Spencer bound – Mermin-Wagner theorem

Contents

1. Introduction, statement of results

1.1. Settings and earlier results. We consider the following class of lattice spin systems. To each site $i \in \mathbb{Z}^2$, we associate a spin $S_i \in \mathbb{R}^N$ such that $\|S_i\|_2 = 1$. Given $\Lambda \Subset \mathbb{Z}^2$, we define the Hamiltonian

$$H_\Lambda(S) = -\sum_{(u,v)\in\mathcal{E}_\Lambda} J_{u,v} V(S_u, S_v), \tag{1}$$

where $\mathcal{E}_\Lambda = \{(x,y) : \{x,y\} \cap \Lambda \neq \varnothing\}$ is the set of all (unoriented) edges intersecting Λ.

The coupling constants are assumed to satisfy $J_{u,v} = J_{u-v} = J_{v-u} \geq 0$ and $\sum_{x\in\mathbb{Z}^2} J_x < \infty$; we shall actually assume, without loss of generality, that $\sum_{x\in\mathbb{Z}^2} J_x = 1$.

The interaction V is assumed to be invariant under simultaneous rotation of its two arguments; in other words, it is assumed that $V(S_u, S_v)$ depends only on the scalar product $S_u \cdot S_v$.

The corresponding finite-volume Gibbs measure in Λ, with boundary condition $\bar{S}$ and at inverse temperature β is then defined by

$$\mu^{\bar{S}}_{\Lambda;\beta}(\mathrm{d}^\Lambda S) = \begin{cases} \frac{1}{Z^{\bar{S}}_{\Lambda;\beta}} e^{-\beta H_\Lambda(S)} \, \mathrm{d}^\Lambda S & \text{if } S_y = \bar{S}_y \text{ for all } y \notin \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where we used the notation $\mathrm{d}^\Lambda S = \prod_{x\in\Lambda} \mathrm{d}S_x$, with $\mathrm{d}S_x$ denoting the Haar measure on $\mathbb{S}^{N-1}$. The expectation with respect to this measure will be denoted $\langle\cdot\rangle^{\bar{S}}_{\Lambda,\beta}$ or $\langle\cdot\rangle_\mu$. The conditions we will impose on V in Section 1.2 will guarantee the existence of the Gibbs measure above (that is, that $\infty > Z^{\bar{S}}_{\Lambda;\beta} > 0$).

It is well-known that in this setting, under some mild conditions, all infinite-volume Gibbs measures associated to such a system are necessarily rotation-invariant. Since the classical result of Mermin and Wagner [20], numerous works have been devoted to strengthening the claims and weakening the hypotheses. In the present context, the strongest statement to date is proven in [11]. In the latter paper, rotation invariance of the infinite-volume Gibbs measures was established under the following assumptions:

- the random walk on $\mathbb{Z}^2$ with transition probabilities from u to v given by $J_{u,v}$ is recurrent;
- the interaction V is continuous (actually, a weaker integral condition is also given there).

The recurrence assumption is known to be optimal in general, as there are versions of the two-dimensional $O(N)$ model for which rotation invariance is spontaneously broken at sufficiently low temperatures, whenever the random walk is transient (see [9, Theorem 20.15] and references therein). The absence of any smoothness assumption on V was the main contribution of [11], such assumptions having played a crucial role in earlier approaches (see, for example, [4, 23, 7, 14, 2, 12]).

A particular consequence of the rotation-invariance of the infinite-volume Gibbs measures is the fact that spin-spin correlations $\langle S_0 \cdot S_x\rangle$ vanish as $\|x\| \to \infty$. A natural problem is then to quantify the speed of decay of these quantities.

The first class of systems that have been studied had finite-range (usually nearest-neighbor) interactions. An upper bound on the decay of the form $\langle S_0 \cdot S_x\rangle \leq (\log\|x\|)^{-c}$, $c > 0$, was first derived by Fisher and Jasnow [6] for the $O(N)$ models (which correspond to $V(S_u, S_v) = S_u \cdot S_v$), $N \geq 2$. Their result was then extended by McBryan and Spencer [18], who obtained an algebraically decaying upper bound of the form $\langle S_0 \cdot S_x\rangle \leq \|x\|^{-c/\beta}$, which is best possible in general. Indeed, Fröhlich and Spencer have proved an algebraically decaying *lower* bound of that type for the two-dimensional XY model ($O(2)$ model) at low temperatures [8]. Building on [4], Shlosman managed to obtain upper bounds of the same type for a much larger class of interactions [26], under some smoothness assumption on V. A similar, but less general, result was later obtained by Naddaf [22], using an adaptation of the McBryan-Spencer approach. More recently, Ioffe, Shlosman and Velenik showed how to dispense with the smoothness assumption, extending Shlosman's result to very general interactions V [11].

The first results for models with infinite-range interactions provided an upper bound *à la* Fisher-Jasnow [2,12] for models with J_x such that the corresponding random walk is recurrent. An algebraically decaying upper bound was obtained by Shlosman [26] for a general class of models (with a smoothness assumption on V) in the case of exponentially decaying coupling constants. Algebraically decaying upper bounds were obtained for $O(N)$ models by Messager, Miracle-Sole and Ruiz [21], when the coupling constants satisfy $J_x \leq C\|x\|^{-\alpha}$ with $\alpha > 4$.

1.2. Assumptions and results. Let us now turn to the results contained in the present paper. Since, as is explained in [18], the case $N \geq 3$ can be reduced to the case $N = 2$, we restrict our attention to the latter. In that case, it is convenient to parametrize the spins by their angle, that is, to associate to each vertex $u \in \mathbb{Z}^2$ the angle $\theta_u \in [0, 2\pi)$ such that $S_u = (\cos\theta_u, \sin\theta_u)$. One can then rewrite the interaction as

$$\beta V(S_u, S_v) = f(\theta_u - \theta_v),$$

which we shall do from now on. (Of course, addition is mod 2π and f is even.) Our analysis will rely on two assumptions [1]:

A1. There exist $\alpha > 4$ and $J \geq 0$ such that $J_x \leq J\|x\|_1^{-\alpha}$, for all $x \in \mathbb{Z}^d$.
A2. The function f is continuous.

The main result of this paper is the following claim, which substantially increases the range of models for which algebraically decreasing upper bounds on spin-spin correlations can be established.

Theorem 1. *Under Assumptions A1 and A2, there exist C and c (depending on the interaction f and the coupling constants $(J_x)_{x\in\mathbb{Z}^d}$) such that, for any infinite-volume Gibbs measure μ associated to the Hamiltonian* (1)*,*

$$|\langle\cos(\theta_0 - \theta_x)\rangle_\mu| \leq c\|x\|^{-C}, \qquad \forall x \in \mathbb{Z}^d.$$

Remark 1. *The above result does not specify the dependence of the constant C on the inverse temperature β. To obtain such an information, the method of proof we use requires some assumptions on the smoothness of f. For example, as explained in Remark 6, if f is assumed to be s-Hölder, for some $s > 3$, then we obtain that $|\langle\cos(\theta_0 - \theta_x)\rangle_\mu| \leq c\|x\|^{-C/\beta}$ for large enough values of β.*

Remark 2. *We have assumed above that the coupling constants $(J_x)_{x\in\mathbb{Z}^d}$ were nonnegative, but as can easily be checked from the proof, this assumption can be removed, as long as $\sum_x |J_x| < \infty$.*

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[1] In all this paper, we use the notation $a \lesssim b$ when there exists a constant C, depending maybe on the value of α below, but not on any other parameters, such that $a \leq Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \sim b$.

Remark 3. *Note that, when $J_x \sim \|x\|^{-\alpha}$ with $\alpha < 4$, the result is actually not true in general, as long-range order has been established in that case in some models [15].*

The proof of Theorem 1 is given in Section 2. It is based on a suitable combination of the McBryan-Spencer approach, in the form derived in [21], with the expansion technique developed in [11] in order to deal with not necessarily smooth interactions.

1.3. A result on a random resistor network. The proof of Theorem 1 is closely linked to the properties of a random resistor network, and can be used to extract some information on the latter. We refer to [5] for a gentle introduction to resistor networks.

We now interpret $1/J_{x,y}$ as the resistance of a wire between the vertices x and y. The quantity

$$\mathcal{E}_{\text{eff}} = \min\Big\{\frac{1}{2}\sum_{u,v\in\mathbb{Z}^2} J_{u,v}\big(g(u)-g(v)\big)^2 \,:\, g(0)=1, g(x)=0\Big\}$$

represents the energy dissipated by the network when a voltage of 1 volt is imposed between 0 and x. It is related in a simple way to the effective resistance $\mathcal{R}(0,x)$ of the resistor network:

$$\mathcal{R}(0,x) = \big(\mathcal{E}_{\text{eff}}\big)^{-1}.$$

It is classical in random walk theory (see, for example, [16, Theorem 4.4.6]) that under the Assumption **A1** on the coupling constants $(J_x)_{x\in\mathbb{Z}^d}$, the effective resistance satisfies

$$\mathcal{R}(0,x) \sim \log\|x\|.$$

Note that an assumption of type **A1** is necessary: there are examples for which $\mathcal{R}(0,x)$ grows more slowly than $\log\|x\|$. It is the case if the transition probabilities $J_{x,y}$ are proportional to $1/\|x-y\|^4$, as can be shown by a direct computation [2]. An adaptation of the proof of Theorem 1 yields an extension of this result to a random resistor model, in which the resistance $R_{x,y}$ between two vertices x and y is taken to be

$$R_{x,y} = \begin{cases} 1/J_{x,y} & \text{with probability } 1-\epsilon J_{x,y},\\ 0 & \text{with probability } \epsilon J_{x,y}, \end{cases}$$

[2] This is ultimately due to the change of behavior of the characteristic function of the increments of the random walk associated to the coupling constant $(J_x)_{x\in\mathbb{Z}^d}$: assume that $J_x = C\|x\|^s$, then [1, (3.11)], as $k \to 0$,

$$1-\sum_x J_x e^{\mathrm{i}k\cdot x} \sim \begin{cases} \|k\|^2 & \text{if } s>4,\\ \|k\|^2\log(1/\|k\|) & \text{if } s=4,\\ \|k\|^{s-2} & \text{if } s<4. \end{cases}$$

This implies, in particular, that the random walk is transient when $s < 4$, and that the potential kernel $a(x)$ satisfies $a(x) \sim \log\|x\|$ when $s > 4$, but $a(x) \sim \log\log\|x\|$ when $s = 4$.

the choice being done independently for each pair (x, y). In other words, randomly selected resistors are "shorted" (which amounts to identifying endpoints). This, of course, can only lower the effective resistance, compared to the above deterministic case. The following result shows that, when ϵ is small enough, this decrease typically does not change the qualitative behavior of $\mathcal{R}(0, x)$.

Theorem 2. *Under Condition **A1**, we have that, for all ϵ small enough, there exist C, κ and $\tilde{c} > 0$ such that, uniformly in $x \in \mathbb{Z}^2$,*

$$\mathbb{P}\big(\mathcal{R}(0, x) \geq \tilde{c} \log \|x\|\big) \geq 1 - \frac{C}{\|x\|^\kappa}.$$

Remark 4. *Of course, the upper bound*

$$\mathbb{P}\big(\mathcal{R}(0, x) \geq \tilde{c} \log \|x\|\big) \leq 1 - \epsilon J_x,$$

always holds, since $\mathcal{R}(0, x) = 0$ whenever $R_{0,x} = 0$.

1.4. Open problems.

- We have only obtained an explicit dependence of the decay exponent on β when f is sufficiently smooth, as described in Remark 1. It would be interesting to determine whether such a behavior can be established for any continuous interaction.
- Our results above are restricted to $\alpha > 4$. It may seem more natural to assume only recurrence of the random walk with transition probabilities J_x (note that, as already mentioned, if $J_x \sim \|x\|^{-\alpha}$ with $\alpha < 4$, then the corresponding random walk is transient). Indeed, recurrence is the optimal condition for the validity of Mermin-Wagner type theorems. However, it seems unlikely (for the reason outlined in Footnote 2) that the correlations admit an algebraically decaying upper bound in the whole recurrence regime. Some upper bounds with slower decay have been obtained under weaker conditions than **A1** (in particular, when $\alpha = 4$) in [21] in the case of the $O(N)$ model. It would be interesting to clarify these issues.
- As mentioned above, negative coupling constants can be accommodated in our approach, but only in a very rough way. In general, situations in which the sign of the coupling constants is not constant should allow an extension of this type of results to interactions with slower decay; see [3] for a discussion of the XY and Heisenberg models with oscillatory interactions of the form $J_x \sim \cos(a\|x\| + b)/\|x\|^\alpha$. Alternatively, one can consider disordered $O(N)$-models in which the coupling constants are i.i.d. random variables with zero mean (satisfying suitable moment conditions); algebraic decay of correlations has been established in this setting in [24] and [27]. It would be interesting to determine whether our approach also applies in such situations.
- Of course, by far the most important open problem in this area is the proof of the conjecture stating that the spin-spin correlations in short-range $O(N)$ models with $N \geq 3$ decay exponentially fast at any temperature [25].

Acknowledgement. The authors are grateful to the two referees for a very careful reading, several corrections and useful suggestions. This research was partially supported by the Swiss National Science Foundation.

2. Proof of Theorem 1

We consider the model in a large box $\Lambda_M = \{-M, \ldots, M\}^2$, with fixed, but arbitrary, boundary condition $\bar{\theta}$. Our goal in this section is to prove that there exist C and c such that

$$\big|\langle\cos(\theta_0 - \theta_x)\rangle^{\bar{\theta}}_{\Lambda_M}\big| \le c\|x\|^{-C}, \tag{2}$$

uniformly in $M, \bar{\theta}$ and $x \in \Lambda_M$. The main claim easily follows from (2) and the DLR equations. Indeed, for an arbitrary infinite-volume Gibbs measure μ,

$$\big|\langle\cos(\theta_0 - \theta_x)\rangle_\mu\big| = \big|\big\langle\langle\cos(\theta_0 - \theta_x)\rangle^{\cdot}_{\Lambda_M}\big\rangle_\mu\big| \le \big\langle\big|\langle\cos(\theta_0 - \theta_x)\rangle^{\cdot}_{\Lambda_M}\big|\big\rangle_\mu \le c\|x\|^{-C}.$$

The remainder of this section is devoted to the proof of (2). As we shall always work in the fixed big box Λ_M, with the fixed boundary condition $\bar{\theta}$ there won't be any ambiguity if we use the following lighter notations: $\langle\cdot\rangle \equiv \langle\cdot\rangle^{\bar{\theta}}_{\Lambda_M}$, $H \equiv H_{\Lambda_M}$, $Z = Z^{\bar{\theta}}_{\Lambda_M}$ and $\mathcal{E} \equiv \mathcal{E}_{\Lambda_M}$. Also, we shall write $\|x\| \equiv \|x\|_\infty$.

2.1. Warm-up: f given by a trigonometric polynomial. We first consider an easier case, in which the interaction f can be written as a trigonometric polynomial,

$$f(x) = \sum_{k=1}^{K} c_k \cos(kx).$$

Although this case can be treated by a straightforward adaptation of the arguments in [21], we include the complete argument here, as we shall need it when considering the general case in Subsection 2.2.

2.1.1. McBryan and Spencer's complex rotation. In [18], McBryan and Spencer used a complex rotation of the spin variables in order to reduce the analysis of the correlations of the nearest-neighbor $O(N)$ model to a variational problem. The same can be done here. First, observe that

$$|\langle\cos(\theta_0 - \theta_x)\rangle| \le |\langle\cos(\theta_0 - \theta_x) + \mathrm{i}\sin(\theta_0 - \theta_x)\rangle| = \big|\frac{1}{Z}\int e^{\mathrm{i}(\theta_0 - \theta_x) - H(\theta)}\mathrm{d}^\Lambda\theta\big|.$$

Now, thanks to the periodicity and analyticity of the function $\mathrm{i}(\theta_0 - \theta_x) - H(\theta)$, applying the (inhomogeneous) complex rotation $\theta_z \to \theta_z + \mathrm{i}a_z$ leaves the integral unchanged as long as $a_z = 0$ for all $z \notin \Lambda$. Therefore,

$$\begin{aligned}\Big|\frac{1}{Z}\int e^{\mathrm{i}(\theta_0 - \theta_x) - H(\theta)}d\theta\Big| &= \Big|\frac{1}{Z}\int e^{\mathrm{i}(\theta_0 + \mathrm{i}a_0 - \theta_x - \mathrm{i}a_x) - H(\theta + \mathrm{i}a)}\mathrm{d}^\Lambda\theta\Big| \\ &\le e^{a_x - a_0}\frac{1}{Z}\int \exp\Big\{\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K} c_k\cos(k\nabla_{u,v}\theta)\cosh(k\nabla_{u,v}a)\Big\}\mathrm{d}^\Lambda\theta,\end{aligned}$$

where we used the identity $\cos(x+ia)=\cos(x)\cosh(a)-i\sin(x)\sinh(a)$ and the notation $\nabla_{u,v}a=a_v-a_u$. We now reconstruct the original Gibbs measure by adding and subtracting $H(\theta)$:

$$\begin{aligned} e^{a_x-a_0}\frac{1}{Z}\int \exp\Big\{\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K} c_k\cos(k\nabla_{u,v}\theta)\cosh(k\nabla_{u,v}a)\Big\}\mathrm{d}^{\Lambda}\theta \\ = e^{a_x-a_0}\frac{1}{Z}\int \exp\Big\{\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K} c_k\cos(k\nabla_{u,v}\theta) \\ \times\big(\cosh(k\nabla_{u,v}a)-1\big)-H(\theta)\Big\}\mathrm{d}^{\Lambda}\theta \\ = e^{a_x-a_0}\big\langle\exp\Big\{\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K} c_k\cos(k\nabla_{u,v}\theta)\big(\cosh(k\nabla_{u,v}a)-1\big)\Big\}\big\rangle \\ \le \exp\Big\{a_x-a_0+\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K}|c_k|\big(\cosh(k\nabla_{u,v}a)-1\big)\Big\}. \end{aligned} \tag{3}$$

2.1.2. Choice of a. The problem is now reduced to obtaining a good upper bound on

$$\mathcal{F}(a)=\exp\Big\{a_x-a_0+\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K}|c_k|\big(\cosh(k\nabla_{u,v}a)-1\big)\Big\}. \tag{4}$$

As in [21], we define a particular function a and check that it gives the desired decay of correlation. Let us first define the function $\bar{a}:\mathbb{N}\to\mathbb{R}$ by $\bar{a}(0)=0$ and

$$\bar{a}_i=-\delta\sum_{j=1}^{i}\frac{1}{j},\qquad i\ge 1, \tag{5}$$

where the parameter δ will be chosen (small) later.

It will be convenient to decompose $\mathbb{Z}^2$ into layers: $L_i=\{z\in\mathbb{Z}^2 : \|z\|=i\}$. Note that $|L_i|=8i$ for all $i\ge 1$.

Let $R=\|x\|$. Our choice of the function a will be radially symmetric:

$$a_z=\begin{cases}\bar{a}_{\|z\|}-\bar{a}_R & \text{if } \|z\|\le R,\\ 0 & \text{otherwise.}\end{cases}$$

Let us now return to the derivation of an upper bound on $\mathcal{F}(a)$. Note that $a_u=a_v=\bar{a}_i$ whenever u,v belong to the same layer L_i, $i\le R$; in particular, in such a case, $\cosh(k\nabla_{u,v}a) - 1 = 0$. Therefore,

$$\begin{aligned}
\mathcal{F}(a) &\le \exp\Big\{\bar{a}_R + \sum_{i=0}^{R}\sum_{u\in L_i}\sum_{j\ge 1}\sum_{v\in L_{i+j}} J_{u,v}\sum_{k=1}^{K}|c_k|\big(\cosh(k(\nabla_{u,v}a)) - 1\big)\Big\} \\
&= \exp\Big\{\bar{a}_R + \sum_{i=0}^{R}\sum_{j\ge 1}\sum_{k=1}^{K}|c_k|\big(\cosh(k(\bar{a}_i - \bar{a}_{i+j})) - 1\big)\sum_{u\in L_i}\sum_{v\in L_{i+j}} J_{u,v}\Big\} \\
&\le \exp\Big\{\bar{a}_R + 8J2^{\alpha}\sum_{i=0}^{R}|L_i|\sum_{j\ge 1} j^{1-\alpha}\sum_{k=1}^{K}|c_k|\big(\cosh(k(\bar{a}_i - \bar{a}_{i+j})) - 1\big)\Big\},
\end{aligned}$$

since

$$\sum_{u\in L_i}\sum_{v\in L_{i+j}} J_{u,v} \le |L_i| 8\sum_{\ell\ge 0} J_{j+\ell} \le 8|L_i|\sum_{\ell\ge 0}\frac{J}{(j+\ell)^{\alpha}} \le 8J2^{\alpha}\frac{|L_i|}{j^{\alpha-1}}.$$

To estimate the sum over j, we treat separately the cases $j > i$ and $j \le i$. Let

$$C_K = \sum_{k=1}^{K}|c_k| \quad \text{and} \quad D_K = \sum_{k=1}^{K}|c_k|k^2.$$

We start with the case $j > i$. Since $\cosh(z) - 1 \le e^z$ for all $z \ge 0$,

$$\begin{aligned}
\sum_{j>i} j^{1-\alpha}\sum_{k=1}^{K}|c_k|\big\{\cosh(k(\bar{a}_i - \bar{a}_{i+j})) - 1\big\} &\le C_K\sum_{j>i} j^{1-\alpha}\big\{\cosh(K(\bar{a}_i - \bar{a}_{i+j})) - 1\big\} \\
&\le C_K\sum_{j>i} j^{1-\alpha}\Big\{\cosh\Big(K\delta\log\big((i+j)/i\big)\Big) - 1\Big\} \\
&\le C_K\sum_{j>i} j^{1-\alpha}\exp\big\{K\delta\log\big((i+j)/i\big)\big\} \\
&\le C_K\sum_{j>i} j^{1-\alpha}\exp\big\{K\delta\log(2j/i)\big\} \\
&\le C_K\sum_{j>i} j^{1-\alpha}\Big(\frac{2j}{i}\Big)^{K\delta} \\
&\le 2C_K i^{2-\alpha}.
\end{aligned}$$

Note that we need δ small enough to ensure that $\alpha - 1 - K\delta > 1$; this can be done, for example, by choosing $\delta \le 2/K$ (remember that $\alpha > 4$). It will actually be convenient to assume the stronger condition $K\delta \le 1$, which we already used above to make the constants in the last line a bit simpler.

To treat the case $j \leq i$ we use the fact that $\cosh(t) - 1 \leq \frac{2}{3}t^2$ when $|t| < 1$:

$$\begin{aligned}\sum_{j=1}^{i} j^{1-\alpha} \sum_{k=1}^{K} |c_k| \big(\cosh(k\delta \sum_{\ell=i+1}^{i+j} 1/\ell\,) - 1\big) &\leq \sum_{j=1}^{i} j^{1-\alpha} \sum_{k=1}^{K} |c_k| \big(\cosh(k\delta j/i) - 1\big) \\ &\leq \sum_{j=1}^{i} j^{1-\alpha} \sum_{k=1}^{K} |c_k| \tfrac{2}{3} \big(k\delta j/i\big)^2 \\ &\leq \tfrac{2}{3} D_K \delta^2 i^{-2} \sum_{j\geq 1} j^{3-\alpha} \\ &= \tfrac{2}{3} D_K \delta^2 i^{-2} \zeta(\alpha - 3),\end{aligned}$$

where $\zeta(\cdot)$ denotes the Riemann zeta function (notice that $\zeta(\alpha-3) < \infty$ since $\alpha > 4$). Bringing all the pieces together, we see that

$$\begin{aligned}\mathcal{F}(a) &\leq \exp\Big\{\bar{a}_R + 64J2^\alpha \sum_{i=1}^{R} i\big(2C_K i^{2-\alpha} + \tfrac{2}{3} D_K \delta^2 \zeta(\alpha-3) i^{-2}\big) + 8J2^\alpha C_K \zeta(\alpha-2)\Big\} \\ &\leq \exp\Big\{\big(-\delta + \delta^2 \tfrac{2}{3} D_K 64 J 2^\alpha \zeta(\alpha-3)\big)\log R + 72 C_K J 2^\alpha \zeta(\alpha-3)\Big\}, \qquad (6)\end{aligned}$$

provided that we choose δ such that the factor multiplying $\log R$ be negative (and $K\delta \leq 1$). This can always be done. With such a choice, we conclude that there exist $c, C < \infty$, uniform in $x, M, \bar{\theta}$, such that

$$|\langle \cos(\theta_0 - \theta_x)\rangle| \leq c\|x\|^{-C}.$$

Remark 5. *Note that, if we write explicitly the dependence on β in the above expressions, then D_k is actually βD_K. This yields the classical bound*

$$|\langle \cos(\theta_0 - \theta_x)\rangle| \leq c(\beta)\|x\|^{-C/\beta}.$$

2.2. General case. We now turn our attention to the general case of functions satisfying assumption **A2.**

2.2.1. Measure decomposition. In order to treat general interactions, we proceed as in [11]. Namely, we first fix $\epsilon > 0$ (which will be assumed to be small enough later on). Trigonometric polynomials being dense (w.r.t. the sup-norm) in the set of continuous functions on $\mathbb{S}^1$, it is possible to find a number $K = K(\epsilon)$ such that

$$\begin{aligned} f(x) &= \sum_{k=0}^{K} c_k \cos(kx) + \bar{\epsilon}(x) \\ &= \tilde{f}(x) \qquad\qquad\quad + \bar{\epsilon}(x),\end{aligned}$$

with $\bar{\epsilon}$ satisfying the conditions

$$\|\bar{\epsilon}\|_\infty \leq \epsilon \quad \text{and} \quad \bar{\epsilon} \geq 0.$$

Note that the constant c_0 doesn't affect the Gibbs measure and can thus be assumed to be equal to 0, which we will do from now on.

Following [11], we then make a high-temperature expansion of the singular part:

$$\begin{aligned} Z = \int e^{-H(\theta)} \mathrm{d}^\Lambda \theta &= \int \exp\Big\{ \sum_{(u,v)\in\mathcal{E}} J_{u,v} \big(\bar{\epsilon}(\nabla_{u,v}\theta) + \tilde{f}(\nabla_{u,v}\theta) \big) \Big\} \mathrm{d}^\Lambda \theta \\ &= \int \exp\Big\{ \sum_{(u,v)\in\mathcal{E}} J_{u,v} \tilde{f}(\nabla_{u,v}\theta) \Big\} \prod_{(u,v)\in\mathcal{E}} \Big(e^{J_{u,v}\bar{\epsilon}(\nabla_{u,v}\theta)} - 1 + 1 \Big) \mathrm{d}^\Lambda \theta \\ &= \sum_{A\in\mathcal{A}} \int \exp\Big\{ \sum_{(u,v)\in\mathcal{E}} J_{u,v} \tilde{f}(\nabla_{u,v}\theta) \Big\} \prod_{(u,v)\in A} \Big(e^{J_{u,v}\bar{\epsilon}(\nabla_{u,v}\theta)} - 1 \Big) \mathrm{d}^\Lambda \theta \\ &= \sum_{A\in\mathcal{A}} Z_A, \end{aligned}$$

where we have introduced the set $\mathcal{A} = \{A \subset \mathcal{E}\}$. This allows us to decompose the original measure as the convex combination

$$\langle g(\theta) \rangle = \sum_{A\in\mathcal{A}} \pi(A) \, \langle g(\theta) \rangle_A,$$

where $\pi(A) = Z_A/Z$ and

$$\langle g(\theta) \rangle_A = \frac{1}{Z_A} \int g(\theta) \exp\Big\{ \sum_{(u,v)\in\mathcal{E}} J_{u,v} \tilde{f}(\nabla_{u,v}\theta) \Big\} \prod_{(u,v)\in A} \Big(e^{J_{u,v}\bar{\epsilon}(\nabla_{u,v}\theta)} - 1 \Big) \mathrm{d}^\Lambda \theta.$$

If we take a look at the quantity of interest, $|\langle \cos(\theta_x - \theta_0) \rangle|$, we can split the sum over $A \in \mathcal{A}$ into two: a set $\mathcal{G}$ of "good" configurations, and a set $\mathcal{B}$ of "bad" configurations, thus leading to an upper bound

$$\begin{aligned} |\langle \cos(\theta_x - \theta_0) \rangle| &= \Big| \sum_{A\in\mathcal{G}} \langle \cos(\theta_x - \theta_0) \rangle_A \pi(A) + \sum_{A\in\mathcal{B}} \langle \cos(\theta_x - \theta_0) \rangle_A \pi(A) \Big| \\ &\leq \sum_{A\in\mathcal{G}} \pi(A) \, | \, \langle \cos(\theta_x - \theta_0) \rangle_A \, | + \sum_{A\in\mathcal{B}} \pi(A). \end{aligned}$$

We will choose $\mathcal{G}$ and $\mathcal{B}$ in such a way that the quantities $|\langle \cos(\theta_x - \theta_0) \rangle_A|$ can be estimated appropriately, while keeping the probability $\pi(\mathcal{B})$ sufficiently small.

2.2.2. Working with $\langle \cdot \rangle_A$. The above decomposition is very helpful because it fixes the set A. We shall see how the complex rotation method can also be used in this case.

$$\begin{aligned} |\langle \cos(\theta_0 - \theta_x) \rangle_A| \leq \Big| \frac{1}{Z_A} \int \exp\Big\{ \mathrm{i}(\theta_0 - \theta_x) + \sum_{(u,v)\in\mathcal{E}} J_{u,v} \tilde{f}(\nabla_{u,v}\theta) \Big\} \\ \times \prod_{(u,v)\in A} \Big(e^{J_{u,v}\bar{\epsilon}(\nabla_{u,v}\theta)} - 1 \Big) \mathrm{d}^\Lambda \theta \Big|. \end{aligned}$$

Of course, the fact that one has absolutely no information on the smoothness of the function $\bar{\epsilon}$ has to be addressed now. Since this function is not analytic in general, one cannot directly apply the translation $\theta \to \theta + \mathrm{i}a$. The key observation is that the interaction can be factorized into an interaction on each cluster of A and another interaction between different clusters of A.

Let $\mathbf{C}$ be one of the clusters of A, and let us denote its vertices by $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. Assume first that $0, x \notin \mathbf{C}$ and $\mathbf{C} \subset \Lambda$. We can then factorize the above integral as follows:

$$\Big| \frac{1}{Z_A} \int \mathrm{d}^{\Lambda \setminus \mathbf{C}} \theta \, (\dots) \int \mathrm{d}^{\mathbf{C}} \theta \, F(\theta, \mathbf{C}) \prod_{\substack{u \in \mathbf{C} \\ v \notin \mathbf{C}}} e^{J_{u,v} \tilde{f}(\nabla_{u,v}\theta)} \Big|$$

where $(\dots)$ represents the terms depending only on the variables $(\theta_i, i \notin \mathbf{C})$, and

$$F(\theta, \mathbf{C}) = \prod_{\substack{u,v \in \mathbf{C} \\ (u,v) \in A}} \big(e^{J_{u,v} \bar{\epsilon}(\nabla_{u,v}\theta)} - 1 \big) \prod_{u,v \in \mathbf{C}} e^{J_{u,v} \tilde{f}(\nabla_{u,v}\theta)}.$$

For $i = 2, \dots, n$, let us define $\eta_i = \theta_{\mathbf{c}_i} - \theta_{\mathbf{c}_1}$. Since the function F depends only on the values of the gradients of θ inside the cluster $\mathbf{C}$ and is therefore a function of $\eta = (\eta_i, i = 2, \dots, n)$, changing variables from $(\theta_{\mathbf{c}_i}, i = 1, \dots, n)$ to $(\theta_{\mathbf{c}_1}, \eta_2, \dots, \eta_n)$ yields

$$\int \mathrm{d}^{\mathbf{C}} \theta \, F(\theta, \mathbf{C}) \prod_{\substack{u \in \mathbf{C} \\ v \notin \mathbf{C}}} e^{J_{u,v} \tilde{f}(\nabla_{u,v}\theta)} = \int \prod_{i=2}^{n} \mathrm{d}\eta_i \, F(\eta, \mathbf{C}) \int \mathrm{d}\theta_{\mathbf{C}_1} \prod_{\substack{u \in \mathbf{C} \\ v \notin \mathbf{C}}} e^{J_{u,v} \tilde{f}(\nabla_{u,v}\theta)}.$$

The function in the last integral is now analytic and periodic in $\mathbf{c}_1$. We are thus entitled to make the complex shift $\theta_{\mathbf{c}_1} \mapsto \theta_{\mathbf{c}_1} + \mathrm{i}a_{\mathbf{c}_1}$. In terms of the original variables, this shift corresponds to $\theta_{\mathbf{c}_i} \mapsto \theta_{\mathbf{c}_1} + \mathrm{i}a_{\mathbf{c}_1} + \eta_i$, which implies that the whole cluster $\mathbf{C}$ is shifted by the same amount $a_{\mathbf{c}_1}$. This procedure can be reproduced on each cluster of A (including isolated vertices) to obtain a global shift $\theta \mapsto \theta + \mathrm{i}a$ with the constraint that $a_u = a_v$ whenever u and v belong to the same cluster of A.

In the preceding discussion, we have made the hypothesis that $0, x \notin \mathbf{C}$. The case where either $0, x$ or both belong to $\mathbf{C}$ is treated in exactly the same way. The only change is that we apply the complex shift to $e^{\mathrm{i}(\theta_0 - \theta_x)} \prod e^{J_{u,v} \tilde{f}(\nabla_{u,v}\theta)}$ which is also an analytic function.

Finally, the assumption that $\mathbf{C} \subset \Lambda$ cannot be discarded. If a point of the boundary condition is in $\mathbf{C}$ then this point "fixes" the whole cluster, which cannot be shifted. We thus add the constraint that $a \equiv 0$ on the connected component of the exterior of Λ.

In order to emphasize the above constraints, we shall henceforth write a^A instead of a. We thus have, proceeding as in (3),

$$
\begin{aligned}
&|\langle\cos(\theta_0-\theta_x)\rangle_A| \\
&\qquad \le e^{a^A_x-a^A_0}\frac{1}{Z_A}\int \exp\Big\{\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K} c_k\cos(k\nabla_{u,v}\theta)\cosh(k\nabla_{u,v}a^A)\Big\} \\
&\qquad\qquad \times \prod_{(u,v)\in A}\Big(e^{J_{u,v}\bar\epsilon(\nabla_{u,v}\theta)}-1\Big)\mathrm{d}^\Lambda\theta \\
&\qquad \le e^{a^A_x-a^A_0}\exp\Big\{\sum_{(u,v)\in\mathcal{E}} J_{u,v}\sum_{k=1}^{K}|c_k|\big\{\cosh(k\nabla_{u,v}a^A)-1\big\}\Big\} = \mathcal{F}(a^A).
\end{aligned}
$$

We thus get the exact same result as in Subsection 2.1 (but, of course, with additional constraints on admissible a^A).

2.2.3. Good and Bad sets of edges. In order to proceed, we must now provide a suitable definition of the sets $\mathcal{G}$ and $\mathcal{B}$ and prove that they have the required properties. To this end, we need some terminology.

Definition 1. *Given $A \in \mathcal{A}$, we say that u and v are connected if there exists a sequence $x_0, x_1, \dots, x_n$ such that $x_0 = u$, $x_n = v$ and $(x_i, x_{i+1}) \in A$ for all $i = 0, \dots, n-1$; we denote this by $u \leftrightarrow v$. We need the following three quantities: for any $u \in \mathbb{Z}^2$, set*

$m_A(u) = \max\{\|v\| : v \leftrightarrow u\}$, the norm of the furthest point connected to u;
$n_A(u) = |\{v : v \leftrightarrow u\}|$, the number of points connected to u;
$r_A(u) = m_A(u) - \|u\|$, the "outwards radius" of the cluster of u.

Definition 2. *Let $\Delta_R = \Lambda_R \setminus \Lambda_{R^{1/2}}$. We say that a configuration of open edges A is c_1-*good *if:*

1. *For all $u \in \Lambda_{R^{1/2}}$, we have that $u \not\leftrightarrow \Lambda^c_{2R^{1/2}}$, where $\Lambda^c_{2R^{1/2}} = \mathbb{Z}^2 \setminus \Lambda_{2R^{1/2}}$.*
2. *For all $u \in \Delta_R$, we have that $m_A(u) \le 2\|u\|$ (equivalently, $r_A(u) \le \|u\|$).*
3. $\displaystyle\sum_{u\in\Delta_R}\frac{r_A(u)^2}{\|u\|^2} \le c_1 \log R.$

We then define $\mathcal{G} = \{A \in \mathcal{A} : A \text{ is } c_1\text{-good}\}$ and $\mathcal{B} = \mathcal{A} \setminus \mathcal{G}$.

2.3. Estimate on the good set $\mathcal{G}$. Let us now see how the approach from [21] can be used to obtain the same estimate we had in equation (6), when $A \in \mathcal{G}$. As we have seen above, the complex rotation argument leads to the following bound:

$$
|\langle\cos(\theta_0-\theta_x)\rangle_A| \le \exp\Big\{a^A_x - a^A_0 + \sum_{k=1}^{K}|c_k|\sum_{(u,v)\in\mathcal{E}} J_{u,v}\big\{\cosh\big(k(a^A_u-a^A_v)\big)-1\big\}\Big\}. \tag{7}
$$

We now have to make a choice for the function $a^A : \mathbb{Z}^2 \to \mathbb{R}$, compatible with our two requirements that $a^A_u = a^A_v$ whenever $u \leftrightarrow v$, and $a^A \equiv 0$ outside Λ. To make that choice, we first modify the function used in Subsection 2.1, making a^A grow only for points sufficiently far from the origin. Namely, we first define, for $i \in \mathbb{N}$, (remember (5))

$$\tilde{a}_i = \begin{cases} 0 & \text{if } i \le 2R^{1/2}, \\ \bar{a}_i - \bar{a}_{2R^{1/2}} & \text{if } i \in \{2R^{1/2}, \dots, R\}, \\ \bar{a}_R - \bar{a}_{2R^{1/2}} & \text{if } i \ge R. \end{cases}$$

The actual rotation a_u^A is then defined similarly to what we did in Subsection 2.1, using $\tilde{a}$ instead of $\bar{a}$, but taking its value on the furthest point v to which u is connected, thus ensuring that it remains constant on each cluster:

$$a_u^A = \tilde{a}_{m_A(u)} - \tilde{a}_R. \tag{8}$$

Now, since $f(x+y+z) \le f(3x) + f(3y) + f(3z)$ for any nonnegative increasing function f, we can write

$$\begin{aligned} &\cosh\bigl(k(\tilde{a}_{m_A(u)} - \tilde{a}_{m_A(v)})\bigr) - 1 \\ &\quad \le \bigl\{\cosh\bigl(3k(\tilde{a}_{m_A(u)} - \tilde{a}_{\|u\|})\bigr) - 1\bigr\} + \bigl\{\cosh\bigl(3k(\tilde{a}_{\|u\|} - \tilde{a}_{\|v\|})\bigr) - 1\bigr\} \\ &\qquad + \bigl\{\cosh\bigl(3k(\tilde{a}_{\|v\|} - \tilde{a}_{m_A(v)})\bigr) - 1\bigr\}. \end{aligned}$$

After multiplying them by $J_{u,v}$ and summing over $u, v \in \mathbb{Z}^2$, the contributions from the first and third terms above will be the same. Since $\sum_{v\in\mathbb{Z}^2} J_{u,v} = 1$, we get

$$\begin{aligned} \sum_{u,v\in\mathbb{Z}^2} J_{u,v}\bigl\{\cosh\bigl(k(a_u^A - a_v^A)\bigr) - 1\bigr\} &\le 2\sum_{u\in\mathbb{Z}^2}\bigl\{\cosh\bigl(3k(\tilde{a}_{m_A(u)} - \tilde{a}_{\|u\|})\bigr) - 1\bigr\} \\ &\quad + \sum_{u,v\in\mathbb{Z}^2} J_{u,v}\bigl\{\cosh\bigl(3k(\tilde{a}_{\|u\|} - \tilde{a}_{\|v\|})\bigr) - 1\bigr\}. \end{aligned} \tag{9}$$

The contribution from the second term in the right-hand side of (9) is bounded as in Subsection 2.1:

$$\sum_{u,v\in\mathbb{Z}^2}\sum_{k=1}^{K} |c_k|\bigl\{\cosh\bigl(3k(\tilde{a}_{\|u\|} - \tilde{a}_{\|v\|})\bigr) - 1\bigr\} \le c_2 C_K + c_3 D_K \delta^2 \log R.$$

To estimate the first term in the right-hand side of (9), we rely on the fact that A is assumed to be good: choosing δ small enough so that $K\delta < 1/3$,

$$\begin{aligned}\sum_{u\in\mathbb{Z}^2}\sum_{k=1}^{K}|c_k|\big\{\cosh\big(3k(\tilde a_{m_A(u)}-\tilde a_{\|u\|})\big)-1\big\} &\le \sum_{u\in\Delta_R}\sum_{k=1}^{K}|c_k|\big\{\cosh\big(3k(\bar a_{m_A(u)}-\bar a_{\|u\|})\big)-1\big\}\\ &= \sum_{u\in\Delta_R}\sum_{k=1}^{K}|c_k|\big\{\cosh\big(3k\delta\sum_{\ell=\|u\|+1}^{m_A(u)}\frac{1}{\ell}\big)-1\big\}\\ &\le \sum_{u\in\Delta_R}\sum_{k=1}^{K}|c_k|\big(3k\delta\frac{r_A(u)}{\|u\|}\big)^2\\ &\le 9D_K\delta^2c_1\log R.\end{aligned}$$

The last piece of (7) left to estimate is $a_x^A - a_0^A$, with $x\in L_R$,

$$a_x^A - a_0^A = -\delta\Big(\sum_{i=1}^{R}\frac{1}{i}-\sum_{i=1}^{2R^{1/2}}\frac{1}{i}\Big)\le -\tfrac{1}{8}\delta\log R.$$

(The factor $\frac{1}{8}$ in the last expression is just here to get rid of the constant additional terms, a better bound is $\delta+\delta\log(2)-\delta\frac{1}{2}\log(R)$.) We are now ready to bring the pieces together,

$$|\langle\cos(\theta_0-\theta_x)\rangle_A|\le\exp\big\{-\tfrac{1}{8}\delta\log R+(c_2C_K+c_3D_K\delta^2\log R+9D_K\delta^2c_1\log R)\big\}.$$

Again taking δ sufficiently small for the constant multiplying $\log R$ to be negative.

$$|\langle\cos(\theta_0-\theta_x)\rangle_A|\le cR^{-C}.$$

Remark 6. *We now explain how additional information on the smoothness of f can be used in order to obtain an explicit dependence of the constant C above on the inverse temperature β.*

The main place such an information is useful is when we approximate f by the trigonometric polynomial $\tilde f$ and the error term $\bar\epsilon$. Indeed, in order for the percolation arguments of Appendix A.1 to work, we need ϵ to be small enough (which ensures a sufficiently subcritical percolation model). If the β dependence is spelled out explicitly, this means that we need $\epsilon\lesssim 1/\beta$. To ensure this, we need to let the number K of terms appearing in $\tilde f$ grow. The question is: how fast?

Let us assume that f is s-Hölder for some $s>3$. In that case, we can draw two conclusions [13, 10]: If $\hat f(x)=\sum_{k=1}^{\infty}\hat c_k\cos(kx)$ is the Fourier series associated to f, then

- $\sup_x|f(x)-\sum_{k=1}^{K}\hat c_k\cos(kx)|\lesssim K^{-s}\log K$.
- $|\hat c_k|\lesssim k^{-s}$, *and thus* $\sum_{k=1}^{\infty}|\hat c_k|k^2=D_\infty<\infty$.

We can thus set $\tilde f=\sum_{k=1}^{K}\hat c_k\cos(kx)$ with $K\lesssim\sqrt{\beta}$ (any $o(\beta)$ would work). We then choose $\delta=\delta_1/\beta$, for δ_1 small enough. In that case:

- *the condition $K\delta\le 1/3$ is automatically satisfied for large β.*
- *Rewriting explicitly the dependance of D_K in β, $D_K\lesssim\beta D_\infty$ yields the classical upper bound $|\langle\cos(\theta_0-\theta_x)\rangle_A|\le c(\beta)R^{-C/\beta}$.*

2.4. The bad set $\mathcal{B}$ has small probability. Of course, not all configurations are good. The rest of this section is devoted to showing that bad configurations have an appropriately small probability. The following observation is very useful.

Proposition 1. *On the set $\mathcal{A}$, with the natural partial ordering, the measure π (defined in Section 2.2.1) is dominated by the independent Bernoulli percolation process $\mathbb{P}$ in which an edge (x,y) is opened with probability $2\epsilon J_{x,y}$.*

Proof. To establish this domination, we will show that the probability for an edge (x,y) to be open is bounded by $2\epsilon J_{x,y}$ uniformly in the states of all the other edges.
Let $D \in \mathcal{A}$ such that $(x,y) \notin D$. It suffices [17] to show that

$$\frac{\pi(D \cup (x,y))}{\pi(D) + \pi(D \cup (x,y))} \leq 2\epsilon J_{x,y},$$

which will clearly follow if we prove that $\pi(D \cup (x,y)) \leq 2\epsilon J_{x,y} \pi(D)$. This is straightforward using the definition of π:

$$\begin{aligned} \pi(D \cup (x,y)) = \frac{1}{Z} \int \mathrm{d}^{\Lambda}\theta \exp\{\sum_{x,y \in \mathbb{Z}^2} J_{x,y} f(\nabla_{x,y}\theta)\} \times \\ \times \prod_{(u,v)\in D} \big(e^{J_{u,v}\bar{\epsilon}(\theta_u - \theta_v)} - 1\big)\Big(e^{J_{x,y}\bar{\epsilon}(\theta_x - \theta_y)} - 1\Big) \\ \leq 2\epsilon J_{x,y} \pi(D), \end{aligned}$$

where we have used that $0 \leq \bar{\epsilon}(x) \leq \epsilon$, for all x. This concludes the proof. □

The three properties characterizing the set $\mathcal{B}$ are bounds on increasing functions on $\mathcal{A}$. Proposition 1 thus implies that $\pi(\mathcal{B}) \leq \mathbb{P}(\mathcal{B})$, which means that we can henceforth work with the measure $\mathbb{P}$ instead of π.

We shall consider the three conditions characterizing $\mathcal{B}$ one at a time.

Condition 1. For $u \in \Lambda_{R^{1/2}}$, Proposition 2 (see Appendix A.1) yields the following estimate:

$$\mathbb{P}(u \leftrightarrow \Lambda^c_{2R^{1/2}}) \leq \sum_{k \geq R^{1/2}} \mathbb{P}(r_A(u) = k) \leq \sum_{k \geq R^{1/2}} \frac{c_6}{k^{\alpha-1}} \lesssim R^{-(\alpha-2)/2}.$$

Hence,

$$\mathbb{P}(\Lambda_{R^{1/2}} \leftrightarrow \Lambda^c_{2R^{1/2}}) \leq \sum_{u \in \Lambda_{R^{1/2}}} \mathbb{P}(u \leftrightarrow \Lambda^c_{2R^{1/2}}) \lesssim R \cdot R^{-(\alpha-2)/2} = R^{-(\alpha-4)/2},$$

which decreases algebraically with R, since $\alpha > 4$.

Condition 2. The proof is similar. For $u \in L_i$, we have that

$$\mathbb{P}(m_A(u) \geq 2\|u\|) = \sum_{k \geq i} \mathbb{P}(m_A(u) = i + k) \leq \sum_{k \geq i} \frac{c_6}{k^{\alpha-1}} \lesssim i^{2-\alpha}.$$

Thus,

$$\mathbb{P}(\exists u \notin \Lambda_{R^{1/2}} \,:\, m_A(u) \geq 2\|u\|) \lesssim \sum_{i \geq R^{1/2}} \sum_{u \in L_i} i^{2-\alpha} \lesssim R^{-(\alpha-4)/2},$$

which is also algebraically decreasing.

Condition 3. In order to control $\sum_{u\in\Delta_R}\frac{r_A(u)^2}{\|u\|^2}$, it is convenient to introduce a new family $(N(u),R(u))_{u\in\Delta_R}$ of i.i.d. random variables with the same distribution as $(n_A(0),r_A(0))$; their joint law will be denoted by $\mathbb{Q}$. It is proven in Proposition 3 of Appendix A.2 that the following holds:

$$\mathbb{P}\Big(\sum_{u\in\Delta_R}\frac{r_A(u)^2}{\|u\|^2}>c_1\log R\Big)\le\mathbb{Q}\Big(\sum_{u\in\Delta_R}\frac{N(u)R(u)^2}{\|u\|^2}>c_1\log R\Big). \tag{10}$$

A first indication that the latter probability is small is given by the expectation of the sum. Thus,

$$\begin{aligned}\mathbb{E}\Big[\sum_{u\in\Delta_R}\frac{N(u)R(u)^2}{\|u\|^2}\Big]&=\sum_{u\in\Delta_R}\frac{1}{\|u\|^2}\mathbb{E}\Big[N(u)R(u)^2\Big]\le\sum_{i=1}^{R}\frac{1}{i^2}\sum_{u\in L_i}\mathbb{E}\Big[n_A(0)r_A(0)^2\Big]\\&\le\sum_{i=1}^{R}\frac{8}{i}\sum_{k\ge0}\mathbb{P}(n_A(0)r_A(0)^2>k).\end{aligned}$$

The latter probability can be bounded using Proposition 2 and Lemma 1:

$$\begin{aligned}\mathbb{P}(n_A(0)r_A(0)^2>k)&\le\mathbb{P}(n_A(0)>\log k)+\mathbb{P}\big(r_A(0)>\sqrt{k/\log k}\big)\\&\lesssim\epsilon^{(1/2)\log k}+\sum_{t>\sqrt{k/\log k}}c_6\,t^{1-\alpha}\\&\lesssim\epsilon^{(1/2)\log k}+(k/\log k)^{(2-\alpha)/2}\\&\lesssim(k/\log k)^{(2-\alpha)/2},\end{aligned}\tag{11}$$

provided that ϵ be chosen small enough. The latter expression is summable, since $\alpha>4$, and we obtain

$$\mathbb{E}\Big[\sum_{u\in\Delta_R}\frac{N(u)R(u)^2}{\|u\|^2}\Big]\le c_4\log R.$$

Let us now define the event $\mathcal{S}=\{N(u)R(u)^2\le\|u\|^2,\forall u\in\Delta_R\}$. The probability of $\mathcal{S}^c$ is easily bounded using the estimate in equation (11):

$$\begin{aligned}\mathbb{Q}(S^c)&\le\sum_{u\in\Delta_R}\mathbb{Q}(N(u)R(u)^2>\|u\|^2)\lesssim\sum_{u\in\Delta_R}(\|u\|/\sqrt{\log\|u\|})^{2-\alpha}\\&\lesssim\sum_{i=R^{1/2}}^{R}i^{3-\alpha}(\log i)^{(\alpha-2)/2}\lesssim R^{(4-\alpha)/2}(\log R)^{(\alpha-2)/2}.\end{aligned}$$

Note that this probability is algebraically decreasing in R. Then, choosing $c_1 = 2c_4$, we have

$$
\begin{aligned}
\mathbb{Q}\Big(\sum_{u\in\Delta_R}\frac{N(u)R(u)^2}{\|u\|^2} > c_1\log R\Big) &\le \mathbb{Q}\Big(\Big\{\sum_{u\in\Delta_R}\frac{N(u)R(u)^2}{\|u\|^2} > c_1\log R\Big\}\cap\mathcal{S}\Big)\\
&\quad + \mathbb{Q}\Big(\sum_{u\in\Delta_R}\frac{N(u)R(u)^2}{\|u\|^2} > c_1\log R \;\Big|\; \mathcal{S}^c\Big)\,\mathbb{Q}(\mathcal{S}^c)\\
&\le \mathbb{Q}\Big(\sum_{u\in\Delta_R}\big(\frac{N(u)R(u)^2}{\|u\|^2}\wedge 1\big) > c_1\log R\Big)\\
&\quad + \mathbb{Q}(\mathcal{S}^c).
\end{aligned}
$$

We have already bounded the second term in the right-hand side. To bound the first term, we use Lemma 3 (note that truncating the summands can only decrease the expectation):

$$
\mathbb{Q}\Big(\sum_{u\in\Delta_R}\big(\frac{N(u)R(u)^2}{\|u\|^2}\wedge 1\big) > c_1\log R\Big) \le R^{-c_5}.
$$

This concludes the bound for Condition 3.

The desired upper bound on the probability of $\mathcal{B}$ follows.

3. Proof of Theorem 2

Let us denote by A the (random) set of all edges with 0 resistance. In view of our target estimate and our bound on the probability of $\mathcal{B}$, we can assume that $A \in \mathcal{G}$. In this case, a variant of the computations done in Subsection 2.1 yields

$$
\begin{aligned}
\sum_{u,v\in\mathbb{Z}^2} J_{u,v}\big(a_u^A - a_v^A\big)^2 &= \sum_{u,v\in\mathbb{Z}^2} J_{u,v}\big(\tilde a_{m_A(u)} - \tilde a_{m_A(v)}\big)^2\\
&\le 6\sum_{u,v\in\Delta_R} J_{u,v}\big(\tilde a_{m_A(u)} - \tilde a_{\|u\|}\big)^2 + 3\sum_{u,v\in\mathbb{Z}^2} J_{u,v}\big(\tilde a_{\|u\|} - \tilde a_{\|v\|}\big)^2\\
&\le 6\sum_{u\in\Delta_R}\big(\delta\frac{r_A(u)}{\|u\|}\big)^2 + 6\sum_{i=1}^{R}\sum_{j\ge1}\sum_{u\in L_i}\sum_{v\in L_{i+j}} J_{u,v}(\delta j/i)^2\\
&\le 6c_1\delta^2\log R + 6\sum_{i=1}^{R}\sum_{j\ge1} 64J\frac{i}{j^{\alpha-1}}(\delta j/i)^2\\
&\le 6c_1\delta^2\log R + 384\delta^2 J\sum_{i=1}^{R}\frac{1}{i}\sum_{j\ge1} j^{3-\alpha}\\
&\lesssim \delta^2\log R.
\end{aligned}
$$

Now, in order to have $a(x) = 0$ and $a(0) = 1$, δ must satisfy $\delta^2 \sim \log(R^{1/2})^2 = \log(R)^2/4$. This leads to the bound

$$
\mathcal{E}_{\mathrm{eff}} \lesssim \frac{1}{\log R},
$$

and the result follows.

A. Appendix

A.1. Percolation estimates. In this subsection, we collect a number of elementary results on long-range percolation, that are needed in the proof of Theorem 1. Below, A always denotes a configuration of the Bernoulli bond percolation process $\mathbb{P}$ on $\mathbb{Z}^2$, in which an edge (x,y) is open with probability $\epsilon J_{x,y}$. The quantities of interest are

$n_A(x) = |\{y : y \leftrightarrow x\}|$, the cardinality of the cluster of x;
$m_A(x) = \max\{\|y\| : y \leftrightarrow x\}$, the norm of the furthest vertex connected to x;
$r_A(x) = m_A(x) - \|x\|$, the "radius" of the cluster of x.

Lemma 1. *For any* $\epsilon < \frac{1}{2}$,

$$\mathbb{P}(n_A(0) > k) \lesssim \epsilon^{k/2}.$$

Proof. For any finite connected graph $G = (V,E)$, it is possible to find a path γ of length $2|E|$ crossing each edge of G exactly twice, starting from any vertex of G. This implies that

$$\begin{aligned}\mathbb{P}(n_A(0) > k) &\le \sum_{n\ge k/2} \sum_{\substack{G=(V,E)\\ V\ni 0, |E|=n}} \mathbb{P}(e \text{ is open}, \forall e \in E) = \sum_{n\ge k/2} \sum_{\substack{G=(V,E)\\ V\ni 0, |E|=n}} \prod_{e\in E} \epsilon J_e \\ &\le \sum_{n\ge k/2} \sum_{\substack{\gamma:\\ \gamma(0)=0, |\gamma|=2n}} \prod_{e\in\gamma} \sqrt{\epsilon J_e} \le \sum_{n\ge k/2} \epsilon^n \Big(\sum_{x\in\mathbb{Z}^2} \sqrt{J_x}\Big)^{2n},\end{aligned}$$

and the conclusion follows since $\sum_{x\in\mathbb{Z}^2} \sqrt{J_x} \lesssim \sum_{i\ge 1} i^{(\alpha-2)/2} = \zeta((\alpha-2)/2)$. □

Proposition 2. *For any* ϵ *small enough, there exists* c_6 *such that, for* $x \in \mathbb{Z}^2$ *and* $k > \|x\|$, *we have the following bound*

$$\mathbb{P}(m_A(x) = k) \le \frac{c_6}{(k - \|x\|)^{\alpha-1}}.$$

Proof. Let $D_m(x,k)$ be the event that there exists an edge-self-avoiding path of length m from x to L_k, staying inside Λ_{k-1} and using only edges from A:

$$\begin{aligned}D_m(x,k) = \big\{&\exists\, x_0, \dots, x_m : (x_{i-1}, x_i) \in A \ \forall i = 1, \dots, m; \\ &(x_{i-1}, x_i) \neq (x_{j-1}, x_j) \text{ if } i \neq j; x_0 = x; x_1, \dots, x_{m-1} \in \Lambda_{k-1}; \|x_m\| = k\big\}.\end{aligned}$$

Obviously, $\{m_A(x) = k\} \subset \bigcup_{m\ge 1} D_m(x,k)$. We will prove below that $\mathbb{P}(D_m(x,k)) \le (\epsilon 16 c_7)^m/(k - \|x\|)^{\alpha-1}$, which will conclude our proof, since

$$\mathbb{P}(m_A(x) = k) \le \sum_{m\ge 1} \mathbb{P}(D_m(x,k)) \le \sum_{m\ge 1} \frac{(\epsilon 16 c_7)^m}{(k - \|x\|)^{\alpha-1}} \le \frac{c_6}{(k - \|x\|)^{\alpha-1}}.$$

We are left with the proof of the bound on $\mathbb{P}(D_m(x,k))$, which is done by induction. For $m=1$,

$$\sum_{x_1\in L_k}\mathbb{P}((x,x_1)\in A)\le 8\sum_{j\ge k-\|x\|}\frac{\epsilon J}{j^\alpha}\le\frac{\epsilon 16J}{(k-\|x\|)^{\alpha-1}}.$$

Assuming now that the claim is true for any $m\le M-1$, let us prove it for $m=M$:

$$\begin{aligned}
\mathbb{P}(D_m(x,k)) &\le \sum_{y\in\Lambda_{k-1}}\mathbb{P}((x,y)\in A)\mathbb{P}(D_{m-1}(y,k))\\
&=\sum_{\ell=0}^{k-1}\sum_{y\in L_\ell}\mathbb{P}((x,y)\in A)\mathbb{P}(D_{m-1}(y,k))\\
&\le\sum_{\ell=0}^{k-1}\frac{(\epsilon 16c_7)^{m-1}}{(k-\ell)^{\alpha-1}}\sum_{y\in L_\ell}\mathbb{P}((x,y)\in A)\\
&\le\Big(\sum_{\ell=0}^{\|x\|-1}+\sum_{\ell=\|x\|+1}^{k-1}\Big)\frac{(\epsilon 16c_7)^{m-1}}{(k-\ell)^{\alpha-1}}\frac{\epsilon 16J}{|\ell-\|x\||^{\alpha-1}}\\
&\qquad+\frac{(\epsilon 16c_7)^{m-1}}{(k-\|x\|)^{\alpha-1}}\sum_{y\in L_{\|x\|}}\mathbb{P}((x,y)\in A)\\
&\le(\epsilon 16c_7)^{m-1}\epsilon 16J\Big\{\sum_{\ell=0}^{\|x\|-1}\frac{1}{(k-\|x\|)^{\alpha-1}}\frac{1}{(\|x\|-\ell)^{\alpha-1}}\\
&\qquad+\sum_{\ell=\|x\|+1}^{k-1}\frac{1}{(k-\ell)^{\alpha-1}}\frac{1}{(\ell-\|x\|)^{\alpha-1}}\\
&\qquad+\frac{\zeta(\alpha-1)}{(k-\|x\|)^{\alpha-1}}\Big\}\\
&\le(\epsilon 16c_7)^{m-1}\epsilon 16J\Big\{\frac{2\zeta(\alpha-1)}{(k-\|x\|)^{\alpha-1}}\\
&\qquad+\sum_{\ell=1}^{k-\|x\|-1}\frac{1}{(k-\|x\|-\ell)^{\alpha-1}}\frac{1}{\ell^{\alpha-1}}\Big\}\\
&\le(\epsilon 16c_7)^{m-1}\epsilon 16J\zeta(\alpha-1)(2+2^\alpha)\frac{1}{(k-\|x\|)^{\alpha-1}},
\end{aligned}$$

where we have used Lemma 2 of Appendix A.2 in the second to last inequality. This ends the proof with $c_7=J\zeta(\alpha-1)(2+2^\alpha)$. □

Proposition 3. *The random variable* $\sum_{x\in\Delta_R}\frac{r_A(x)^2}{\|x\|^2}$ *is stochastically dominated by* $\sum_{x\in\Delta_R}\frac{N(x)R(x)^2}{\|x\|^2}$, *where the random variables* $(N(x),R(x))_{x\in\Lambda_R}$ *are independent and have the same distribution as* $(n_A(0),r_A(0))$.

Proof. The proof follows the line of the construction of the percolation process cluster by cluster.

STEP 1: Let $(A^x)_{x\in\Lambda_R}$ be independent realizations of the percolation process. To each $x\in\Lambda_R$, we associate its cluster C_x in the configuration A^x. Let also $\bar C_x$ be the set of all edges of $\mathcal{E}$ with at least one endpoint in common with C_x, and set $\partial C_x=\bar C_x\setminus C_x$.

STEP 2: Choose an ordering $(x_1,x_2,\ldots,x_{|\Lambda_R|})$ of the vertices in Λ_R such that $\|x_i\|\geq\|x_j\|$ whenever $i\geq j$. We want to construct a percolation configuration A; we thus need to decide for each edge (x,y) whether it is open or closed in A.

STEP 3: We start with $x_1=0$. Each edge in C_{x_1} is declared open in A and each edge in ∂C_{x_1} is declared to be closed in A. Set $k=2$ and let E_{exp} be the set of all edges the state of which has already been decided (those of $\bar C_{x_1}$).

STEP 4: If $k>|\Lambda_R|$, stop the procedure. Otherwise, let $\bar C'_{x_k}$ be the connected component of x_k in $\bar C_{x_k}\setminus E_{\mathrm{exp}}$. Note that $\bar C'_{x_k}$ may be empty and that it is not the same as $\bar C'_{x_k}\setminus E_{\mathrm{exp}}$; indeed E_{exp} could separate $\bar C'_{x_k}$ into several connected component and we only want to keep the one containing x_k. Declare all edges $e\in\bar C'_{x_k}$ to be open in A if they belong to C_{x_k} and closed if they belong to ∂C_{x_k}. Let E_{exp} be the set of all edges the state of which has already been defined. Increment k and return to STEP 4.

Let A_k be the cluster of x_k in A produced using the above procedure, and ∂A_k its boundary. Since each edge $e\in\bigcup_{k=1}^{|\Lambda_R|}(A_k\cup\partial A_k)$ has been examined exactly once, and set to be open or closed independently with probabilities ϵJ_e and $(1-\epsilon)J_e$, respectively, the joint law of all these clusters is identical to that under $\mathbb{P}$.

We shall need the following three quantities:

$N(x)$, the number of points of C_x;
$r(x)=\max\{\|y\|\,:\,y\leftrightarrow x\}-\|x\|$, the outwards "radius" of C_x;
$b(x_i)=\mathbf{1}_{\{x_i\not\in\bigcup_{j<i}A_j\}}$;
$R(x)=\max\{\|y-x\|\,:\,y\in C_x\}$, the "radius" of C_x. Notice here that $R(x)\geq r(x)$, and that the distribution of this quantity is actually independent of x.

By construction, if x,y are in the same cluster of A we have that $m_A(x)=m_A(y)$. Moreover, if $b(x)=1$ then x minimizes $\|y\|$ among all $y\in A_x$; in particular, vertices x with $b(x)=1$ maximize the ratio $r_A(x)/\|x\|$. We thus have

$$\begin{aligned}\sum_{x\in\Delta_R}\frac{r_A(x)^2}{\|x\|^2}&\leq\sum_{x\in\Delta_R}b(x)n_A(x)\frac{r_A(x)^2}{\|x\|^2}\\&\leq\sum_{x\in\Delta_R}b(x)N(x)\frac{r(x)^2}{\|x\|^2}\leq\sum_{x\in\Delta_R}N(x)\frac{R(x)^2}{\|x\|^2}.\end{aligned}$$

□

A.2. Some technical estimates.

Lemma 2. *For all $k \geq 1$ and $\alpha > 1$,*

$$\sum_{\ell=1}^{k-1} \frac{1}{(k-\ell)^\alpha \ell^\alpha} \leq \frac{2^{\alpha+1}\zeta(\alpha)}{k^\alpha}.$$

Proof. Since $\alpha > 1$,

$$\sum_{\ell=1}^{k-1} \frac{1}{(k-\ell)^\alpha \ell^\alpha} \leq 2 \sum_{\ell=1}^{(k-1)/2} \frac{1}{(k-\ell)^\alpha \ell^\alpha} \leq 2 \sum_{\ell=1}^{(k-1)/2} \frac{1}{(k/2)^\alpha \ell^\alpha} \leq 2^{1+\alpha} k^{-\alpha} \zeta(\alpha).$$

□

The next result is classical (see, e.g. [19]); we include its short proof for the convenience of the reader.

Lemma 3. *Let $(X_k)_{k\geq 1}$ be independent random variables such that $0 \leq X_k \leq 1$. Define $S_n = \sum_{k=1}^n X_k$, $\mu \geq \mathbb{E}[S_n]$ and $p = \mathbb{E}[S_n]/n$. Then, for all $\epsilon > 0$,*

$$\mathbb{P}(S_n \geq (1+\epsilon)\mu) \leq e^{-((1+\epsilon)\log(1+\epsilon)-\epsilon)\mu}.$$

Proof. The proof uses a control of the exponential moments of S_n and the Markov inequality. Let $h > 0$ and recall that for $t \in [0,1]$ we have $e^{ht} \leq 1-t+te^h$. Using the inequality between arithmetic and geometric means,

$$\begin{aligned}
\mathbb{E}\big[e^{hS_n}\big] = \prod_{k=1}^n \mathbb{E}\big[e^{hX_k}\big] &\leq \prod_{k=1}^n \big(1-\mathbb{E}[X_k]+\mathbb{E}[X_k]e^h\big) \\
&\leq \Big(\frac{1}{n}\sum_{k=1}^n \big(1-\mathbb{E}[X_k]+\mathbb{E}[X_k]e^h\big)\Big)^n \\
&\leq \Big(1-\frac{\mathbb{E}[S_n]}{n}+e^h\frac{\mathbb{E}[S_n]}{n}\Big)^n \\
&\leq \big(1-p+pe^h\big)^n.
\end{aligned}$$

Now, we can apply Markov's inequality to obtain, for all $h > 0$,

$$\mathbb{P}(S_n \geq m) \leq e^{-hm}\mathbb{E}\big[e^{hS_n}\big] \leq e^{-hm}\big(1-p+pe^h\big)^n.$$

Setting $m = (1+\epsilon)\mu$ and choosing h such that $e^h = (1+\epsilon)$ yields

$$\begin{aligned}
\mathbb{P}(S_n \geq (1+\epsilon)\mu) &\leq (1+\epsilon)^{-(1+\epsilon)\mu}\,(1-p+p(1+\epsilon))^n \leq e^{-(1+\epsilon)\log(1+\epsilon)\mu}e^{\epsilon pn} \\
&\leq e^{-(1+\epsilon)\log(1+\epsilon)\mu}e^{\epsilon\mu}.
\end{aligned}$$

□

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