Discrete Comput Geom 1:219-227 (1986)

Discrete & Computational Geometry
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Upper Bounds for Configurations and Polytopes in R^d

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Communicated by Branko Grünbaum

Abstract. We give a new upper bound on $n^{d(d+1)^n}$ on the number of realizable order types of simple configurations of n points in R^d, and of $n^{2d^2 n}$ on the number of realizable combinatorial types of simple configurations. It follows as a corollary of the first result that there are no more than $n^{d(d+1)n}$ combinatorially distinct labeled simplicial polytopes in R^d with n vertices, which improves the best previous upper bound of $n^{cn^{d/2}}$.

1. Introduction

We consider simple numbered configurations of points in R^d, i.e., labeled sets $\{P_1, \ldots, P_n\} \subset R^d$, with $n > d$, and no hyperplane containing more than d of the P_i. There are several natural equivalence relations on such configurations, one being oriented matroid equivalence [2], [5], also known as chirotope equivalence [3], semispace equivalence [8], or order equivalence [7]; another being what we have called combinatorial equivalence in the case $d = 2$ [6], [8], but which extends easily to the case $d > 2$ (see below). The purpose of this paper is to give upper bounds on the number of equivalence classes of simple numbered configurations in each of these two equivalence relations.

It follows from the results of [7] that since the order type of a configuration S is determined by its λ-function (which assigns to each ordered $d+1$-tuple the number of points of S lying on the positive side of the oriented hyperplane spanned by the $d+1$-tuple), the number of order types is bounded above by roughly n^{n^d} (this is the so-called "information-theoretic bound"). But the λ-

* Supported in part by NSF Grant DMS-8501492 and PSC-CUNY Grant 665258.

† Supported in part by NSF Grant DMS-8501947.

matrix also classifies the order type of a *generalized* configuration of n points (in which the points are connected by an arrangement of pseudohyperplanes); hence we are clearly overcounting the number of *genuine* configurations. The question is: by how much? The surprising answer is: by a great deal. More precisely,

Theorem 1. *Let $f(n, d)$ be the number of distinct order types of simple numbered configurations of n points in R^d. Then*

$$f(n, d) \leq n^{d(d+1)n}.$$

The key step in the proof of Theorem 1 is the following result of J. Milnor [11, Theorem 3] (which has been used in a similar way in [14], among other places):

> If a set $X \subset R^m$ is defined by polynomial inequalities of the form
>
> $$f_1 \geq 0, \quad \ldots, \quad f_p \geq 0$$
>
> with total degree $d = \deg(f_1) + \cdots + \deg(f_p)$, then
>
> $$\operatorname{rank} H^*X \leq \tfrac{1}{2}(2+d)(1+d)^{m-1}.$$

Here, H^*X is the direct sum of the (Čech) cohomology groups of the semialgebraic set X; hence rank H^*X represents the sum of the Betti numbers of X. In particular, since rank H^0X is the number of connected components of X, we have:

> The number of connected components of the set X defined as above is at most $\frac{1}{2}(2+d)(1+d)^{m-1}$.

It is in this form that we shall use Milnor's theorem.

In Section 3 we apply Theorem 1 to the vertex sets of simplicial polytopes and derive a new bound on the number of combinatorial equivalence classes of labeled simplicial polytopes.

The problem of counting polytopes, even simplicial polytopes, has a long and venerable history [9, Section 13.6]. While significant progress has been made when the number of vertices is not too much larger than the dimension [9], little is known above dimension 3 in the general case. As for bounds, the best upper bound known until now for simplicial polytopes was apparently the one easily derivable from the (asymptotic) Upper Bound Theorem [10]: Each simplicial polytope has $\leq n^{[d/2]}$ facets, and a (legitimate) choice of facets, i.e., of sets of d vertices, determines the whole combinatorial structure of the polytope. Hence the number of combinatorial types is bounded above by

$$n^{[d/2]}\binom{\binom{n}{d}}{n^{[d/2]}} \leq n^{[d/2]}(en^{[(d+1)/2]})^{n^{[d/2]}} \approx n^{c(d)n^{[d/2]}}.$$

However, as a corollary of Theorem 1 we get

Theorem 2. *Let $g(n, d)$ be the number of combinatorially distinct labeled simplicial polytopes with n vertices in R^d. Then*

$$g(n, d) \leq n^{d(d+1)n}.$$

In Section 4 we extend the concept of "combinatorial equivalence" of n-point configurations in R^d from the case $d=2$, treated in [6] and [8], to the case $d>2$. Just as in the plane, combinatorial equivalence in R^d is a finer relation than order equivalence, and we prove

Theorem 3. *Let $h(n, d)$ be the number of distinct combinatorial equivalence classes of numbered configurations of n points in general position in R^d. Then*

$$h(n, d) \leq n^{2d^2 n}.$$

We relate this bound to the one which follows from Stanley's [13] and Edelman and Greene's [4] enumeration of maximal chains in the weak Bruhat order of S_n for the case $d=2$.

Finally, in Section 5, we discuss the question of lower bounds, as well as some consequences of our results for geometric sorting and for the isotopy problem for configurations.

We wish to thank Herbert Edelsbrunner and Emo Welzl for bringing Milnor's paper [11] to our attention, and Noga Alon for several stimulating conversations.

2. An Upper Bound on Order Types

Lemma 1. *Suppose $P \in R[X_1, \ldots, X_k]$, and $V = \{(x) \mid (x) \in R^k,\ P(x) = 0\}$. Let $U_1, \ldots, U_m$ be distinct connected components of $R^k \setminus V$. Then there is an $\varepsilon > 0$ such that the set $W = \{(x) \mid |P(x)| \geq \varepsilon\}$ has at least m connected components.*

Proof. Choose $(x^i) \in U_1$ for $1 \leq i \leq m$, and let

$$\varepsilon = \min_{1 \leq i \leq m} |P(x^i)|, \qquad W = \{(x) \mid |P(x)| \geq \varepsilon\}.$$

Then $(x^i) \in W$ for $i = 1, \ldots, m$, and clearly (x^i), (x^j) belong to different components of W for $i \neq j$, since any arc connecting them must cross V. The conclusion follows. $\square$

Lemma 2. *Let $P \in R[X_1, \ldots, X_k]$, $\deg(P) = r$, and $U = \{(x) \mid (x) \in R^m,\ P(x) \neq 0\}$. Then U has at most*

$$(2+r)(1+r)^{k-1}$$

connected components.

Proof. Let ε, W be as in Lemma 1. By that lemma, the number of connected components of U is bounded above by the number of connected components of W. Let

$$W_1 \text{ (resp. } W_2) = \{(x) \mid P(x) \geq \varepsilon\} \text{ (resp. } \{(x) \mid P(x) \leq -\varepsilon\}).$$

Then $W = W_1 \cup W_2$ and the result follows by applying Theorem 3 of [11] to each of W_1, W_2. □

Proof of Theorem 1. To each numbered configuration $S = \{(x^1), \ldots, (x^n)\}$ of points in R^d we associate a point $(x) \in R^{dn}$. The *order type* of S can then be viewed as a mapping

$$\omega: R^{dn} \longrightarrow \{-1, 0, 1\}^{\binom{n}{d+1}},$$

with ω defined by

$$\omega((x^1), \ldots, (x^n)) = \left(\operatorname{sgn} \det \begin{pmatrix} 1 & x_1^{i(0)} & \cdots & x_d^{i(0)} \\ \vdots & \vdots & & \vdots \\ 1 & x_1^{i(d)} & \cdots & x_d^{i(d)} \end{pmatrix} \right)_{1 \leq i(0) < \cdots < i(d) \leq n}$$

(see [7] for details). To say that S is simple means that

$$\omega(S) \in \{-1, 1\}^{\binom{n}{d+1}},$$

i.e., that none of the determinants above vanishes at the point corresponding to S.

Each of these determinants is a polynomial of degree d in the dn variables $X_1^1, \ldots, X_d^n$, so if we multiply them we get a single polynomial

$$P(X_1^1, \ldots, X_d^n) \qquad \text{of degree } d\binom{n}{d+1},$$

whose zero locus V corresponds precisely to the set of nonsimple configurations. Let U be the complement of V. Then a connected component of U is a full isotopy class of simple configurations, i.e., a maximal set such that any two can be deformed, one into the other, by a continuous family of configurations all having the same order type. In particular, *the number of distinct simple order types is bounded above by the number of connected components of U.* It follows from Lemma 2 that

$$f(n, d) \leq \left(2 + d\binom{n}{d+1}\right)\left(1 + d\binom{n}{d+1}\right)^{dn-1} \leq n^{d(d+1)n}. \qquad \square$$

3. An Upper Bound on Simplicial Polytopes

Proof of Theorem 2. Consider a simplicial polytope with n labeled vertices in R^d. Note first that any such polytope can be "jiggled" slightly so that its vertices form a simple configuration. Now two combinatorially equivalent polytopes may have inequivalent vertex sets (one can jiggle the vertices of a regular octahedron in two different ways to get inequivalent simple configurations, for example), but it is clear that *two inequivalent polytopes cannot give rise to two vertex sets of the same order type,* since—as is shown in [7, Theorem 1.8]—the order type of a configuration determines its intersections with the supporting hyperplanes of its convex hull, and—in the case of polytopes—these determine the combinatorial type completely.

It follows that the bound of Theorem 1 applies equally well to combinatorial equivalence classes of polytopes, and so we have

$$g(n, d) \leq n^{d(d+1)n}. \qquad \square$$

4. An Upper Bound on Combinatorial Types of Configurations

In [6] and [8] we have introduced the concept of "combinatorial equivalence" of numbered planar configurations of points. In brief, to each such configuration S we associate the circular sequence of permutations of $1, \ldots, n$ obtained by projecting the points of S orthogonally onto a line which rotates counterclockwise around a fixed point. The resulting "allowable sequences" of permutations provide a somewhat finer classification of planar configurations than does "order type," and allow one to examine many geometric properties of a configuration in purely combinatorial terms.

In [8, Corollary 1.14] we prove that the combinatorial type of a configuration is determined (up to a reversal of its allowable sequence, which corresponds to a reflection in a line) by its associated *set* of permutations. This suggests the following definition of combinatorial equivalence in higher dimensions:

Definition 1. Let $S = \{(x^1), \ldots, (x^n)\}$ be a numbered configuration of n points in R^d. Let L be a directed line in R^d passing through O such that the points of S have distinct images under the orthogonal projection

$$p_L: R^d \to L,$$

and let π_L be the associated element of the symmetric group S_n (induced by the direction on L). The set $\Pi(S) \subset S_n$ consisting of all the permutations π_L obtained in this way we will call the *permutation set* of S. S and T will be called *combinatorially equivalent* if $\Pi(S) = \Pi(T)$.

Remark 1. Just as in the plane the members of the permutation set of a configuration S fall—in a natural way—into a circular sequence, so too, in R^d, they form a complex on the unit sphere S^{d-1}, whose points correspond to the directed lines L in Definition 1. The structure of this spherical complex will be explored elsewhere. It is easy to see, however, that for each connecting line $\overline{(x^i)(x^j)}$ determined by a pair of points of S there is a great $(d-2)$-sphere Σ_{ij} lying on S^{d-1}, on each side of which every corresponding directed line L induces a permutation in which the indices i, j appear in the same order. These Σ_{ij}'s cut S^{d-1} up into $O(n^{2d})$ cells, the maximum being obtained if and only if the points of S are in *general position*, in the following sense: For generic S, each cell is bounded by at least d of the Σ_{ij}'s. As S moves to a special position in which one of these cells vanishes, then d of the Σ_{ij}'s must meet, and—conversely—if d of them meet, the corresponding cell in S^{d-1} will be empty. Thus the condition that the points of S lie in general position amounts to saying that *no d among the Σ_{ij}'s should meet, unless they do so generically.* More precisely, if K_n is the complete graph on the set $\{1, \ldots, n\}$, we must have that *for any choice of d pairs $\{i, j\}$ which induces an acyclic subgraph of K_n, the great spheres Σ_{ij} have no point in common.* (Depending on how the indices in the d pairs $\{i, j\}$ are related, this condition has various geometric interpretations. For example, for $d = 2$ it amounts to saying that no three points of S are collinear and that no two lines are parallel. For $d = 3$ it says that no four points are coplanar, no line is parallel to a plane, and no three lines have a common perpendicular.)

Proof of Theorem 3. Just as in the proof of Theorem 1, since combinatorial isotopy equivalence implies combinatorial equivalence, it is sufficient to give an upper bound on the number of combinatorial isotopy classes.

It follows from Definition 1 and Remark 1 that two configurations, S and T, are combinatorially isotopic, i.e., each can be deformed to the other without leaving its combinatorial equivalence class, if and only if S can be deformed to T (say) without any d Σ_{ij}'s which formed a cell in S collapsing, i.e., without any d connecting lines of S which had no common perpendicular acquiring one. The condition that d connecting lines in R^d,

$$\overline{(x^{i_1}), x^{(j_1)}}, \quad \ldots, \quad \overline{(x^{i_d}), (x^{j_d})},$$

have a common perpendicular amounts to saying that the vectors along them are linearly dependent, i.e., that the determinant

$$\det\begin{pmatrix} x_1^{j_1} - x_1^{i_1} & \cdots & x_d^{j_1} - x_d^{i_1} \\ \vdots & & \\ x_1^{j_d} - x_1^{i_d} & \cdots & x_d^{j_d} - x_d^{i_d} \end{pmatrix}$$

vanishes, and this condition is given by a polynomial of degree d. Hence the number of isotopy classes is precisely the number of cells into which R^{dn} is cut by the zero loci of these

$$\leq \binom{n}{2}^d$$

polynomials in the indeterminates $X_1^1, \ldots, X_d^n$. As in Theorem 1, if we let

$$Q(X_1^1, \ldots, X_d^n)$$

be their product, then

$$\deg(Q) \le \frac{dn^{2d}}{2^d},$$

and—by Lemma 2—we have

$$h(n, d) \le \left(2 + \frac{dn^{2d}}{2^d}\right)\left(1 + \frac{dn^{2d}}{2^d}\right)^{dn-1} \le n^{2d^2 n}. \qquad \square$$

Edelman and Greene [4] and Stanley [13] have shown that the number of simple allowable sequences on n indices which contain the permutation $12 \cdots n$ is precisely

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2} \cdots (2n-3)^1}.$$

Since each such allowable sequence has $n(n-1)$ terms, the permutation $12 \cdots n$ appears in $1/(n-2)!$ of the total number of allowable sequences, and it follows that there are precisely

$$\frac{(n-2)!\binom{n}{2}!}{1^{n-1}3^{n-2} \cdots (2n-3)^1}$$

of these. A comparison of this constant (whose logarithm is asymptotic to $cn^2 \log n$) with the result of Theorem 3 in the case $d = 2$ shows immediately that most (in a very strong sense) allowable sequences are not geometrically realizable.

5. Remarks

(i) The most naive way of counting configurations yields a lower bound on the number of realizable simple order types which is surprisingly close to our upper bound, in fact agreeing with it in the highest order term in the exponent. A simple configuration of n points in R^d determines $\binom{n}{d}$ hyperplanes, and these in turn determine

$$\binom{\binom{n}{d}}{d}+\binom{\binom{n}{d}}{d-1}+\cdots+\binom{\binom{n}{d}}{0}-n\binom{\binom{n-1}{d-1}-1}{d}$$

cells [15, p. 65]. Any n-point configuration can be extended to an $(n+1)$-point configuration by placing a new point in any one of these cells. (Of course we are undercounting, since different realizations of the same order type may have noncorresponding cells.) This gives a lower bound of roughly

$$\frac{(n!)^{d^2}}{(d!)^{(d+1)n}}$$

on the number of realizable order types, and by Stirling's formula this comes down to

$$n^{d^2n+O(n/\log n)}.$$

This shows that our upper bound in Theorem 1 is quite close to the truth, at least asymptotically.

It also shows that the isotopy classes that make up an order type are not too numerous, in the scheme of things. (The conjectured result, of course, is that each order type contains only *one* isotopy class.)

(ii) An argument of N. Alon [1] shows that there are at least n^{cdn} labeled simplicial polytopes with n vertices in R^d. (Even a restricted class of polytopes has been shown to have at least n^{cn} members: in [12] Shemer proves that the number of *neighborly* polytopes with n vertices in R^d is asymptotically bounded below by $n^{n/2}$.) Thus the gap between the lower and upper bounds for simplicial polytopes is no longer impossibly wide.

(iii) Theorem 1 points up the need for a new way of encoding the order type of a configuraton of points. In [7] we have suggested several possible applications of such an encoding, which we call *geometric sorting*—to pattern recognition, to stereochemistry, and to cluster analysis. It is important, in these applications, to find an *efficient* way of encoding the order type of a configuration, i.e., the orientations of all the $(d+1)$-tuples in it. It is now clear that the λ-function, although it is the most efficient way known at present, becomes less and less efficient as the dimension goes up; already in dimension 2 there is a significant gap between the number of bits needed for a λ-matrix ($n^2 \log n$) and the logarithm of the number of objects we are using it to encode ($6n \log n$). Since λ-matrices also encode generalized configurations, and—as we have shown in [7]—there are at least $\exp(cn^2)$ of these in the plane, this discrepancy is not unexpected. But if we are interested in ordinary point configurations, as in most applications, there should be a more compact way of encoding them, one which takes at most $d(d+1)n \log n$ bits, and which—hopefully—can be accomplished in close to linear time.

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Received March 12, 1985.

Note added in proof. Since the original submission of this paper, there have been two improvements in the upper bound for the number of realizable order types, hence also for the number of labeled polytopes. N. Alon (The number of polytopes, configurations, and real matroids, *Mathematika*, to appear) has been able to reduce the upper bound given in Theorems 1 and 2 above to

$$\left(\frac{n}{d}\right)^{d^2 n\left(1+O\left(\frac{\log\log(n/d)}{\log(n/d)}\right)\right)} \qquad \text{for} \quad \frac{n}{d}\to\infty,$$

and to remove the restriction that the configuration be simple (hence also that the polytope be simplicial). More recently, using a result of H. E. Warren (Lower bounds for approximation by nonlinear manifolds, *Trans. Amer. Math. Soc.* **133** (1968), 167–178) in place of the Milnor theorem, we have been able to show that the bound can be further reduced to

$$\left(\frac{n}{d}\right)^{d^2 n\left(1+O\left(\frac{1}{\log(n/d)}\right)\right)} \qquad \text{for} \quad \frac{n}{d}\to\infty,$$

again for arbitrary configurations and polytopes.