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Upper Bounds for Fekete-Szego functions and the Second Hankel Determinant for a Class of Starlike functions.

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Abstract: *In this work,we provide upper estimate for Fekete-Szego functional and Second Hankel determinant for the class $S^*\{q\}$ consisting of functions analytic in the unit disk $U = \{z\in C : |z| < 1\}$ and normalized by $f(0) = f'(0) = 1 = 0$ and which satisfies the subordination condition*
$\frac{zf'(z)}{f(z)} \prec q(z), \quad z \in U.$ where $q(z) = \sqrt{1+z^2}+z$

Keywords:*Coefficient bounds,Fekete-Szego functional,Second Hankel Determinant and Subordination Keywords*

Introduction

Let A denote the class of functions $f(z)$ analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $S \subset A$ denote the class of analytic functions $f(z)$ in U which are univalent and has the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

Definition 1.1. *An analytic function $f(z)$ is said to be subordinate to the analytic function $g(z)$ and we write $f \prec g$ if there exist a function $\omega(z)$ analytic in U such that $\omega(0) = 1$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$. If $g(z)$ is univalent in U then $f \prec g \iff f(0) = g(0)$ and $f(|z| < 1) \subset g(|z| < 1)$*

A function $f(z)$ in S is said to be starlike in U if $f(z)$ maps U onto a starlike domain with respect to $\omega_0 = 0$. An analytic function $f(z)$ is starlike in U if and only if $Re\frac{zf'(z)}{f(z)} > 0$, $z \in U$.
The class of starlike functions in U is denoted as S^*. We define the class $S^*\{q\}$ to be the class of functions $f \in A$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z = q(z) \qquad z \in U$$

where q(0)=1,
Furthermore let Ω be the class of analytic functions of the form $\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + .\therefore$ in the unit disk U satisfying the condition $|\omega(z)| < 1$.

II. Preliminary Result

To establish our results we shall need the following lemmas:

Lemma1(Ali.R.M et al[5])

If $\omega \in \Omega$, then for any $t \in \mathbb{R}$

$$|\omega_2 - t\omega_1^2| \leq \left\{ \begin{array}{ccc} -t & if & t \leq -1 \\ 1 & if & -1 \leq t \leq 1 \\ t & if & t \geq 1 \end{array} \right\}$$

Lemma 2(Ali.R.M et al[5])

If $\omega \in \Omega$, for any complex number t $|\omega_2 - t\omega_1^2| \leq max\{1 : |t|\}$ The result is sharp for $\omega(z) = z^2$ or $\omega(z) = z$

III. Main Result

In This Research Work,The Following Are The Main Results.

Theorem 3.1. *Let $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$ and let $\sigma_1 = \frac{1}{4}, \sigma_2 \frac{5}{4}$ If $f(z)$ belong to $S^*(q)$ then for any real number λ*

$$|a_3 - \lambda a_2^2| \leq \left\{ \begin{array}{ccc} \frac{3}{4} - \lambda & if & \lambda \geq \sigma_1 \\ \frac{1}{2} & if & \sigma_1 \leq \lambda \leq \sigma_2 \\ -(\frac{3}{4} - \lambda) & if & \lambda \geq \sigma_2 \end{array} \right\}$$

Proof:

Let $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$

Also let $S^*(q)$ denote the class of functions $f(z)$ in the unit disk U normalized by $f(0) = f'(0) - 1 = 0$ satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1 + z^2} + z, \qquad z \in U$$

Whichimplies by definition of subordination that

$$\frac{z'f(z)}{f(z)} \prec \sqrt{1 + \omega^2(z)} + \omega(z), \qquad z \in U$$

And

$$zf'(z) - \omega(z)f(z) = f(z)\sqrt{1 + \omega^2(z)}$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$ for $|z| < 1$

$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots$

implies

$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots$

Hence,

$$zf'(z) = z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + \ldots$$
$$\omega(z)f(z) = [c_1z + c_2z^2 + c_3z^3 + \ldots][z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + \ldots]$$
$$= c_1z^2 + c_2z^3 + c_3z^4 + c_1a_2z^3 + c_2a_2z^4 + c_1a_3z^4 + \ldots$$
$$zf'(z) - \omega(z)f(z) = z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 - [c_1z^2 + c_2z^3 + c_3z^4 + c_1a_2z^3 + \ldots]$$

$$= z + (2a_2 - c_{1)}z^2 + (3a_3 - c_2 - c_1a_2)z^3 + (4a_4 - c_1a_3 - c_2a_2 - c_3)z^4 + \ldots \quad (1)$$
$$\omega(z) = c_1z + c_2z^2 + c_3z^3 + \ldots$$
$$\omega^2(z) = [c_1z + c_2z^2 + c_3z^3 + \ldots]^2$$
$$= c_1^2z^2 + c_1c_2z^3 + c_1c_3z^4 + c_1c_2z^3 + c_2^2z^4 + c_1c_3z^4 + \ldots$$
$$1 + \omega^2(z) = 1 + c_1^2z^2 + 2c_1c_2z^3 + 2c_1c_3z^4 + c_2^2z^4 + \ldots$$
$$\sqrt{1 + \omega^2(z)} = 1 + \frac{1}{2}[c_1^2z^2 + 2c_1c_2z^3 + 2c_1c_3z^4 + c_2^2z^4 + \ldots]$$
$$- \frac{1}{8}[c_1^2z^2 + 2c_1c_2z^3 + 2c_1c_3z^4 + c_2^2z^4 + \ldots]^2$$
$$= 1 + \frac{1}{2}c_1^2z^2 + c_1c_2z^3 + [c_1c_3 + \frac{1}{2}c_2^2 - \frac{1}{8}c_1^4]z^4 + \ldots$$
$$f(z)\sqrt{1 + \omega^2(z)} = [z + a_2z^2 + a_3z^3 + a_4z^4 + \ldots][1 + \frac{1}{2}c_1^2z^2 + c_1c_2z^3 + [c_1c_3 + \frac{1}{2}c_2^2 - \frac{1}{8}c_1^4]z^4 +$$

$$= z + a_2z^2 + (\frac{1}{2}c_1^2 + a_3)z^3 + (c_1c_2 + \frac{1}{2}c_1^2a_2 + a_4)z^4 + \ldots \quad (2)$$

On comparing the coefficient of equation **(1)** and **(2)** we have,

$$a_2 = c_1 \quad (i)$$
$$a_3 = \frac{c_2}{2} + \frac{3}{4}c_1^2 \quad (ii)$$
$$a_4 = \frac{c_3}{3} + \frac{5}{6}c_1c_2 + \frac{5}{12}c_1^3 \quad (iii)$$

Therefore from **(i)** and **(ii)** we have

$$a_3 - \lambda a_2^2 = \frac{1}{2}c_2 + \frac{3}{4}c_1^2 - \lambda c_1^2$$
$$= \frac{1}{2}c_2 - (\lambda - \frac{3}{4})c_1^2$$
$$= \frac{1}{2}[c_2 + (\frac{3 - 4\lambda}{2}c_1^2]$$
$$i.e$$
$$a_3 - \lambda a_2^2 = \frac{1}{2}[c_2 + vc_1^2]$$
$$where v = \frac{3 - 4\lambda}{2}$$
$$a_3 - \lambda a_2^2 = \frac{1}{2}[c_2 - (-v)c_1^2] \quad \ldots (iv)$$
$$= |\frac{1}{2}||c_2 - (-v)c_1^2|$$

Taking $t = -v$ in Lemma 1, we have

(a) For $t \leq -1$ i.e. $-v \leq -1$ implies $v \geq 1$.

Then we have

$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}[|-v|]$$
$$= \frac{1}{2}[v]$$
$$= \frac{1}{2}\left[\frac{3-4\lambda}{2}\right]$$

i.e.

$$|a_3 - \lambda a_2^2| \leq \frac{3}{4} - \lambda$$

and for the case of $v \geq 1$ gives

$\frac{3-4\lambda}{2} \geq 1$

and we have that

$\lambda \leq \frac{1}{4}$

i.e $\lambda \leq \sigma_1$ where $\sigma_1 = \frac{1}{4}$

Hence,

$$|a_3 - \lambda a_2^2| \leq \frac{3}{4} - \lambda$$

if $\lambda \leq \sigma_1$

(b). For the case when $-1 \leq t \leq 1$ in Lemma 1 , we have $-1 \leq -v \leq 1$ and

$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}(1)$$
$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}$$

And in this case $-1 \leq -v \leq 1$ implies $-1 \leq -(\frac{3-4\lambda}{2}) \leq 1$ which gives

$\sigma_1 \leq \lambda \leq \sigma_2$ where $\sigma_1 = \frac{1}{4}$ and $\sigma_2 = \frac{5}{4}$

For $t \geq 1$ i.e $-v \geq 1$ we have $v \leq -1$ and

$$|a_3 - \lambda a_2^2| \leq -\frac{1}{2}(\frac{3-4\lambda}{2})]$$
$$|a_3 - \lambda a_2^2| \leq \lambda - \frac{3}{4}$$

For this case $v \leq -1$ gives $\frac{3-4\lambda}{2} \leq -1$

which gives $\lambda \geq \sigma_2$ when $\sigma_2 = \frac{5}{4}$

Hence,

$$|a_3 - \lambda a_2^2| \leq -(\frac{3}{4} - \lambda)$$

if $\lambda \geq \sigma_2$

And This Completes The Proof Of The Theorem.

Theorem

3.2. *Let* $\omega(z) = \sum_{k=1}^{\infty} c_k z^k$

If $f(z)$ *belong to* S^*q *then the for any complex number* λ

$|a_3 - \lambda a_2^2| \leq \frac{1}{2}\max\{1 : |\frac{3-4\lambda}{2}|\}$

Proof:

From theorem 3.1

$$a_2 = c_1$$

$$a_3 = \frac{1}{2}c_2 + \frac{3}{4}c_1^2$$

Therefore,

$$|a_3 - \lambda a_2^2| = \left|\frac{1}{2}[c_2 + \left(\frac{3-4\lambda}{2}\right) c_1^2]\right|$$

$$= \frac{1}{2}\left|[c_2 - (-v)c_1^2]\right|$$

Where $v = \frac{3-4\lambda}{2}$

Applying Lemma 2 and taking $t = -v$ we have that

$$|\omega_2 - (-v)\omega_1^2| \leq \max\{1 : |-v|\}$$

$$|\omega_2 - (-v)\omega_1^2| \leq max\{1 : |v|\}$$

Hence we obtain

$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}\max\{1 : |-v|\}$$

$$= \frac{1}{2}\max\{1 : |v|\}$$

$$= \frac{1}{2}\max\{1 : |\frac{3-4\lambda}{2}|\}$$

i.e.

$$|a_3 - (\lambda)a_2^2| \leq \frac{1}{2}\max\{1 : |\frac{3-4\lambda}{2}|\}$$

And This Completes The Proof Of The Theorem.

Theorem

3.3. *Let* $f(z) \in S^*(q)$ *then* $H_2(2) = |a_2a_4 - a_3^2| \leq \frac{39}{48}$

Proof:

From Theorem 3.1

$$\begin{aligned}
a_2 &= c_1 \\
a_3 &= \frac{1}{2}c_2 + \frac{3}{4}c1^2 \\
a_4 &= \frac{1}{3}c_3 + \frac{5}{6}c_1c_2 + \frac{5}{12}c_1^3 \\
a_2a_4 - a_3^2 &= c_1\left(\frac{1}{3}c_3 + \frac{5}{6}c_1c_2 + \frac{5}{12}c_1^3\right) - \left(\frac{1}{2}c_2 + \frac{3}{4}c_1^2\right)^2 \\
&= \frac{1}{3}c_3c_1 + \frac{5}{6}c_2c_2 + \frac{5}{12}c_1^4 - \frac{1}{4}c_2^2 - \frac{6}{8}c_1^2c_2 - \frac{9}{16}c_1^4 \\
&= \frac{c_1^2}{12}\left(c_2 - \frac{7}{4}c_1^2\right) + \frac{c_1c_3}{3} - \frac{c_2^2}{4} \\
H_2(2) &= |a_2a_4 - a_3^2| \\
&= \left|\frac{c_1^2}{12}\left(c_2 - \frac{7}{4}c_1^2\right) + \frac{c_1c_3}{3} - \frac{c_2^2}{4}\right| \\
&\leq \frac{|c_1^2|}{12}|c_2 - \frac{7}{4}c_1^2| + \frac{|c_1||c_3|}{3} + \frac{|c_2|^2}{4} \\
&\leq \frac{|c_1^2|}{12}\left(|c_2| + \frac{7}{4}|c_1|^2\right) + \frac{|c_1||c_3|}{3} + \frac{|c_2|^2}{4} \\
&\leq \frac{1}{12}\left(1 + \frac{7}{4}\right) + \frac{1}{3} + \frac{1}{4} \\
&= \frac{39}{48}
\end{aligned}$$

Therefore, $H_2(2) \leq \frac{39}{48}$

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