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Upper Bounds for the Diameter and Height of Graphs of Convex Polyhedra*

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Abstract. Let $\Delta(d, n)$ be the maximum diameter of the graph of a d-dimensional polyhedron P with n-facets. It was conjectured by Hirsch in 1957 that $\Delta(d, n)$ depends linearly on n and d. However, all known upper bounds for $\Delta(d, n)$ were exponential in d. We prove a quasi-polynomial bound $\Delta(d, n) \leq n^{2\log d + 3}$.

Let P be a d-dimensional polyhedron with n facets, let φ be a linear objective function which is bounded on P and let v be a vertex of P. We prove that in the graph of P there exists a monotone path leading from v to a vertex with maximal φ-value whose length is at most $n^{2\sqrt{n}}$.

1. Introduction

Let P be a convex polyhedron. The *graph* of P denoted by $G(P)$ is an abstract graph whose vertices are the extreme points of P and two vertices u and v are adjacent if the interval $[v, u]$ is an extreme edge (=one-dimensional face) of P. The diameter of the graph of P is denoted by $\delta(P)$.

Let $\Delta(d, n)$ be the maximum diameter of the graphs of d-dimensional polyhedra P with n facets. (A facet is a $(d-1)$-dimensional face.) Thus, P is the set of solutions of n linear inequalities in d variables. It is an old standing problem to determine the behavior of the function $\Delta(d, n)$. The value of $\Delta(d, n)$ is a lower bound for the number of iterations needed for the simplex algorithm for linear programming with any pivot rule.

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In 1957 Hirsch conjectured [2] that $\Delta(d, n) \leq n - d$. Klee and Walkup [9] showed that the Hirsch conjecture is false for unbounded polyhedra. They found examples showing that, for $n \geq 2d$, $\Delta(d, n) \geq n - d + \lfloor d/5 \rfloor$. The Hirsch conjecture for convex polytopes (= bounded polyhedra) is still open. It is also open whether $\Delta(d, n)$ is bounded above by a linear function of n and d.

In view of its intrinsic interest and simplicity and its close connections with linear programming, the Hirsch conjecture drew substantial attention over the years. For a recent survey on the Hirsch conjecture and its relatives, see [7].

In 1967 Barnette proved [1], [4] that $\Delta(d, n) \leq n3^{d-3}$. Barnette's bound is linear in the number of facets but exponential in the dimension. An improved upper bound of a similar asymptotic behavior was found in 1970 by Larman who proved [10] that $\Delta(d, n) \leq n2^{d-3}$.

We give a quasi-polynomial upper bound for $\Delta(d, n)$. The existence of a polynomial or even a linear upper bound for $\Delta(d, n)$ is still open. In this paper $\log x$ stands for $\log_2 x$.

Theorem 1.

$$\Delta(d, n) \leq n^{2\log d + 3}. \tag{1}$$

Let P be d-dimensional polyhedron with n facets and let φ be a linear objective function on R^d which is bounded on P. A *φ-maximal vertex* of P is a vertex of P on which φ attains its maximum. A path on the graph of P is called *monotone* if φ is nondecreasing along it. For a vertex w of P let $h(w)$ be the minimum length of a monotone path in $G(P)$ from w to a φ-maximal vertex of P. (A length of a path in $G(P)$ is the number of its edges.) The *height* of P with respect to φ, denoted by $h_\varphi(P)$, is the maximum of $h(\omega)$ over all vertices w of P. (Note: our notion of height is different from the one in [4].) Let $H(d, n)$ denote the maximum value of $h_\varphi(P)$ for all d-dimensional polyhedra P with n facets, and all linear objective functions φ as described above. It is easy to see that $\Delta(d, n) \leq H(d, n)$.

Theorem 2.

$$H(d, n) \leq n^{2\sqrt{n}}. \tag{2}$$

A trivial upper bound on $h_\varphi(P)$ is the number vertices of P. This gives that $\log H(d, n) \leq c \log\binom{n}{d}$, for some positive constant c. The upper bound (2) gives $\log H(d, n) \leq \sqrt{n} \log n$. This is an improvement on the trivial bound when $n = o(d^2)$.

Let $\mathscr{P}[d, r]$ denote the class of simple *d-polytopes with the property that every* k-face has at most rk facets.

Theorem 3. *Let $r \geq 2$ be a fixed integer. The diameter of every polytope P in the class $\mathscr{P}[d, r]$ and the height of P with respect to every linear objective function are polynomial in d. The height and the diameter of every polytope in $\mathscr{P}[d, 2]$ is at most d.*

Theorem 3 applies to several classes of polyhedra which are of interest in linear programming theory. For example, it gives a polynomial upper bound on the height (and the diameter) of the feasible polyhedra of any generalized flow problem.

The proof of Theorem 1 is given in Section 2. The proofs of Theorems 2 and 3 are given in Section 3. In Section 4 we discuss a general combinatorial context for which our results apply. Section 5 contains speculations on the behavior of the function $\Delta(d, n)$ and on the relation with linear programming.

The results in Section 3 were obtained by the author in November 1990, and were the first subexponential bound for $\Delta(d, n)$. The better bounds of Section 2 were proved in March 1991. Several developments occurred since then. A substantial simplification of the proof of Theorem 1 was obtained by Kleitman (May 1991). His proof is given in a joint research announcement with the author [6]. The author observed (September 1991) how to modify the proofs of the quasi-polynomial bounds for $\Delta(d, n)$ to get similar bounds for $H(d, n)$. Finally, the author found (September 1991) a randomized simplex algorithm which takes an expected subexponential ($n^{3\sqrt{d}}$) number of arithmetic operations on every linear programming problem with d variables and n constraints.

2. Quasi-Polynomial Bounds for the Diameter

Let P be a d-dimensional polyhedron. For a face F of P let $N(F)$ be the set of facets of P with intersect F. Clearly, the number of facets of F itself is smaller than $N(F)$.

For two vertices u and w of P, define a *path of faces* between u and v as a sequence of the form $u = v_1, F_1, v_2, F_2 \ldots, v_t F_t, v_{t+1} = u$ where F_i is a face of P which contain the vertices v_i and v_{i+1}, $i = 1, 2, \ldots, t$. Such a path is called a *path of facets* if F_i is a facet of P for every i. A path of faces is *nested* if, for every i, $\dim F_i \geq \dim F_{i+1}$ and if $\dim F_i > \dim F_{i+1}$, then $F_i \supset F_j$ for every $j > i$.

For two vertices v and w of P let $I_P(u, v)$ denote a path of facets between u and v of minimal length. If $I_P(u, v)$ is of the form $u = v_1, F_1, v_2, F_2, \ldots, v_t F_t, v_{t+1} = u$, then it is easy to see that $N(F_i) \cap N(F_j) = \varnothing$ if $j \geq i + 3$. Since otherwise if $F \in N(F_i) \cap N(F_j)$ we could make a shortcut by replacing the facets F_l, $i < l < j$, by F. (This is the crucial observation in Barnette's proof [1], [4].)

Proof of Theorem 1. Let $g(d, n) = (6d)^{2 \log n} \leq n^{2 \log d + 3}$. Note that $g(d, n)$ is a monotone function of d and n, and that, for a fixed d, $g(d, n)$ is a convex function of n. We will prove by induction on d that $\Delta(d, n) \leq g(d, n)$. This is clear for $d = 1$. Assume that $\Delta(d', n) \leq g(d', n)$ for every $d' < d$ and every n.

Let P be a d-dimensional polyhedron with n facets. Call a face F of P *small* if $|N(F)| \leq n/2$, otherwise call F *big*. Note that for every two big faces there is a facet which intersects them both.

Let v and u be two vertices of P. We will find two nested paths of faces S_1 and S_2 of the form

S_1: $v = v_1, F_1, v_2, F_2, \ldots, F_s$ and
S_2: $u = u_1, G_1, u_2, G_2, \ldots, G_r$,

with the following properties:

(a) All faces F_i, $i \le s$, and G_i, $i \le t$, are small.
(b) There is a facet F which intersects both F_s and G_t.
(c) If F_i and F_{i+3} have the same dimension, then $N(F_i) \cap N(F_{i+3}) = \varnothing$, and the same property holds for the G_i's

Given S_1 and S_2, put $d_i = \dim F_i$, $d'_i = \dim G_i$, $n_i = |N(F_i)|$, and $n'_i = |N(G_i)|$. By the induction hypothesis, the distance between v and w in $G(P)$ is at most

$$\sum_{i=1}^{s} g(d_i, n_i) + \Delta(d-1, n-1) + \sum_{i=1}^{t} g(d'_i, n'_i).$$

Since $g(d, n)$ is a convex function of n this expression is at most $\Delta(d-1, n-1) + 6\sum_{i=1}^{d-1} g(d-i, n/2)$.

It follows that $\Delta(d, n) \le \Delta(d-1, n-1) + 6dg(d-1, n/2)$ and therefore $\Delta(d, n) \le 6d^2 g(d, n/2) = g(d, n)$.

We will now describe the construction of S_1 and S_2. This will be done in at most $(d-1)^2$ steps which correspond to pairs of positive integers (a, b), a, $b \le d-1$. In the (a, b)th step we construct two nested paths of faces R_1 and R_2:

R_1: $v = v_1, F_1, v_2, F_2, \ldots, F_s$ and
R_2: $u = u_1, G_1, u_2, G_2, \ldots, G_t$

with the properties:

(a′) All faces F_i, $i < s$, and G_i, $i < t$, are small. F_s and G_t are big. $\dim F_t = d - a$ and $\dim G_s = d - b$.
(c′) If F_i and F_{i+3} have the same dimension, then $N(F_i) \cap N(F_{i+3}) = \varnothing$, and the same holds for the G_i's.

To start the construction for $a = b = 1$ consider $I = I_p(v, u)$. If there is only one big facet F in I, break I into two parts S_1 and S_2 by deleting F. If there is no big facet in I, break I into S_1 and S_2 in an arbitrary way. Otherwise, let R_1 be the initial part of I which ends with the first big facet of I, and let R_2 be (in reverse order) the terminal part of I starting with the last big facet of I.

Assume that the sequences R_1 and R_2 are given. There is a facet F which intersect both F_s and G_t. (Since both are big faces.) Let x be a vertex in $F_s \cap F$ and let y be a vertex in $G_t \cap F$. Let $U = I_{F_s}(v_s, x)$ be a minimal path of facets of F_s between the vertices v_s and x and let $W = I_{G_t}(y, u_t)$ be a minimal path of facets of G_t between the vertices y and u_t.

If all faces in the paths U and W are small construct S_1 and S_2 by replacing in R_1 the face F_s by U and by replacing in R_2 the face G_t by W. Otherwise, assume that there is a big face in U and let H_j be the first big face in U. In this case, keep R_2 unchanged and replace in R_1 the face F_s by the initial part of U: $v_s = a_1, H_1, a_2, H_2, \ldots, a_j, H_j$. (Note that $\dim H_j = d - a - 1$.) It is immediate to check that these sequences satisfy the requirements for $(a+1, b)$.

Since every 1-face is small this process will terminate with sequences S_1 and S_2 as required. □

Remarks. 1. Let $f(d, n)$ be a monotone function of d and n which is a convex function of n for every fixed d. Suppose that $f(d, n)$ satisfies the relations $f(1, n) \geq 1$ and $f(d, n) \leq f(d-1, n-1) + \sum_{i=1}^{d-1} f(d-i, n/2)$. The proof of Theorem 1 actually shows that $\Delta(d, n) \leq f(d, n)$.

For positive real numbers a and b define

$$\bar{f}_{a,b}(d, n) = (ad/\log n)^{b \log n}, \tag{3}$$

$$f_{a,b}(d, n) = \max\{t \geq 1 : t \cdot \bar{f}_{a,b}(d, n/t)\}. \tag{4}$$

Note that $f_{a,b}(d, n)$ is a monotone function of d and n and is a convex function of n for every fixed d.

The behavior of the function $f_{a,b}(d, n)$ is demonstrated by the following relations. There are constants $c > 0$ and $C > 1$ which depend on a and b such that $f_{a,b}(d, n) \leq \min\{n^{b \log d + c}, C^d \cdot n\}$. On the other hand, when $\log \log n = o(\log d)$ then $f_{a,b}(d, n) \geq n^{b \log d(1+o(1))}$ and when $\log n \geq c'' \cdot d$ (for some constant c'' depending on a and b) then $f_{a,b}(d, n) \geq C^d \cdot n$.

For $b = 2$ and for a sufficiently large ($a = 20$ will do), the function $f(d, n) = f_{a,b}(d, n)$ satisfies the relation

$$f(d, n) \leq f(d-1, n) + \alpha f(d, \beta n), \tag{5}$$

where $\alpha = \sqrt{6}$ and $\beta = \sqrt{2}/2$. This relation (repeated twice) gives

$$f(d, n) \leq f(d-1, n) + 6 \sum_{i=0}^{d-1} f(d-i, n/2).$$

It is clear that $f(1, n) \geq 1$. Thus, the proof of Theorem 1 gives

$$\Delta(d, n) \leq f_{2,20}(d, n). \tag{6}$$

2. A substantial simplification of the proof giving a slightly better estimate, $\Delta(d, n) \leq n^{\log d + 1}$, was recently obtained by Kleitman. See [6]. Kleitman's proof gives the recurrence relation $\Delta(d, n) \leq \Delta(d-1, n-1) + 2\Delta(d, \lfloor n/2 \rfloor) + 2$.

3. Subexponential Bounds for the Height

We are given a d-dimensional polyhedron P and a linear objective function φ. For simplicity we assume that P is simple and that φ is not constant on any edge of P. (Recall that a d-polyhedron P is simple if every vertex of P belongs to exactly d facets.) There is no loss of generality in those assumptions. Let $r \geq 1$ be an integer. Given a vertex v of P consider the following algorithm GI(r) to reach the φ-maximal vertex. (GI stands for greatest improvement.)

GI(r): Start from a vertex v. If v is not the φ-maximal vertex of P replace v by the vertex with maximal value of φ among all vertices which belong to some r-face containing v, and repeat.

Note that GI(1) is just the simplex algorithm with Dantzig's original greatest improvement pivot rule.

Theorem 4. *If P is given by $n \leq \lceil kd/2 \rceil$ linear inequalities in d variables, then for $r = d - \lceil d/(k-1) \rceil$ the algorithm* GI(r) *terminates after at most $2k - 5$ steps.*

Lemma 5. *Let $F_1, F_2, \ldots, F_m$ be d-subsets of the set $\{1, 2, \ldots, n\}$, $n \leq \lceil kd/2 \rceil$. If $m \geq k - 1$, then there are i, j, $1 \leq i < j \leq m$, such that $|F_i \cap F_j| \geq \lceil d/(k-1) \rceil$.*

Proof. Otherwise

$$\left| \bigcup_{i=1}^{m} F_i \right| \geq (k-1)d - \binom{k}{2}(\lceil d/(k-1) \rceil - 1) > \lceil kd/2 \rceil \geq n. \qquad \square$$

Given a simple d-polyhedron P let $G_r(P)$ denote the graph whose vertices are the vertices of P and two vertices are adjacent if they are included in some r-face of P. (Thus $G_1(P) = G(P)$.)

Lemma 6. *Let P be a simple d-polyhedron with $n \leq \lceil kd/2 \rceil$ facets. Put $r = d - \lceil d/(k-1) \rceil$. Then $G_r(P)$ does not contain an independent set of $k - 1$ vertices.*

Proof. To each vertex v of P associate the set $S(v)$ of facets containing v. Two vertices w and u belongs to some r-face iff $|S(w) \cap S(u)| \geq d - r$. Let $r = d - \lceil d/(k-1) \rceil$. Since there are altogether at most $\lceil dk/2 \rceil$ facets, Lemma 5 follows from Lemma 6.

Proof of Theorem 4. Let φ be an objective function and let v be the φ-maximal vertex of P. Let w be a vertex of P, consider the sequence $w = w_0, w_1, \ldots, w_m = v$ where w_{i+1} is the vertex with maximum value of φ among all vertices which belong to some r-face containing w_i. If $m > 2k - 5$, then $w_0, w_2, w_4, \ldots, w_{2k-4}$ form an independent set of $k - 1$ vertices in $G_r(P)$. By Lemma 6 this is impossible and therefore $m \leq 2k - 5$. $\square$

Proof of Theorem 2. From Theorem 4 it follows that if $n \leq \lceil kd/2 \rceil$, then

$$H(d, n) \leq (2k-5)H(d - \lceil d/(k-1) \rceil, n - \lceil d/(k-1) \rceil). \tag{7}$$

Note that if $(n/d) \leq (k/2)$ for some integer $k \geq 2$, then

$$(n - \lceil d/(k-1) \rceil)/(d - \lceil d/(k-1) \rceil) \leq (k+1)/2.$$

Also note that $d - \lceil d/(k-1) \rceil = \lfloor d(k-2)/(k-1) \rfloor$.

By iterating (7) we obtain for every integer t that

$$H(d, n) \leq (2k-5) \cdot (2k-3) \cdot (2k-1) \cdots (2t-5)(2t-3)H(x, y),$$

where

$$x = d\left\lceil \frac{k-2}{k-1} \cdot \frac{k-1}{k} \cdots \frac{t-1}{t} \right\rceil \leq \left\lceil \frac{(k-2)d}{t} \right\rceil,$$

and $y \leq (t+2)x/2 \leq n$. Substituting $t = \sqrt{n}$ and using the trivial relation $H(d, n) \leq \binom{n}{d}$, we obtain that

$$H(d, n) \leq (2\sqrt{n})! \binom{n}{2\sqrt{n}} \leq n^{2\sqrt{n}}. \quad \square \tag{8}$$

Proof of Theorem 3. Let $h_r(d)$ be the maximum height of a polytope in $\mathscr{P}[d, r]$. It follows from Theorem 2 that $h_r(d) \leq (4r-5)h_r(d - \lceil d/(2r-1) \rceil)$. It follows that $h_r(d) \leq (4r-5)^{\log_{(2r-1)/(2r-2)} d} = d^{C(r)}$ where $C(r) = \log_{(2r-1)/(2r-2)}(4r-5)$. Note that $C(r) = O(r \log r)$. Let P be a polytope in $\mathscr{P}[d, 2]$. Let v be a vertex of P. There is a monotone path of length $h_2(d-1)$ from v to a vertex w with maximum φ value among all vertices which belong to a facet containing v. If w is not the φ-maximal vertex of P, then after moving along an improving edge from v we must reach the only vertex of P which belongs to no facet of P containing v, and this vertex must be optimal. $\square$

Remarks. 1. Let $w(k, d, n)$ denote the maximal cardinality of an independent set in $G_k(P)$ over all d-dimensional polyhedra with n facets. The value of $w(k, d, n)$ is closely related to the maximal possible number of binary vectors of length n, constant weight d, and Hamming distance at least $2k+2$ apart. See Chapter 17 of [5]. In particular, Lemma 5 is a weak form of Johnson's bound for constant weight codes.

2. Lemma 6 implies that if P is a d-polytope with at most $\lceil kd/2 \rceil$ facets, and $r = d - \lceil d/(k-1) \rceil$, then the diameter of $G_r(P)$ is at most $2k-4$. By a similar argument we can prove that the diameter of $G_r(P)$ is actually at most $k-1$. The value $2k-5$ in Theorem 4 can be reduced to $k-3$ by adding an improving move at the end of each step of algorithm GI(r).

3. In this remark we rely on the definitions in Section 1.5 of [3]. Let $\mathscr{G}\mathscr{F}$ denotes the class of polyhedra given as the set of generalized pseudoflows with a given excess function in some network (equipped with capacity and gain functions). It is easy to see that every face of a polyhedron in $\mathscr{G}\mathscr{F}$ is itself a polyhedron in $\mathscr{G}\mathscr{F}$. Another easy observation is that every polyhedron P in $\mathscr{G}\mathscr{F}$ can be defined over a network with the property that all vertex degrees are at least 3. If P is defined over such a network with v vertices and e edges, then $d = \dim P = e - v$ and the number of facets of P is at most $2e \geq 6(e-v) = 6d$. It follows that $P \in \mathscr{P}[d, 6]$ and that the height of P is at most $d^{\log_{(11/10)} 19} = d^{30.893\ldots}$.

4. General Combinatorial Setting

4.1. For the Diameter

Let K be a collection of d-sets. We call the elements of $\bigcup\{S: S \in K\}$ the *vertices* of K. Two sets in K are *adjacent* if their intersection is of cardinality $d-1$. K is *strongly connected* if between every two sets S and T in K there is a path of sets in K of the form $S = R_0, R_1, \ldots, R_t = T$ such that, for every i, R_i is adjacent to R_{i+1}. The distance between S and T is the minimum of t over all such paths, and the diameter of K is the maximal distance between two members of K. K is *ultraconnected* if for every S and T in K there is a path as above with the additional property that $(S \cap T) \subset R_i$ for every i.

Let $D(d, n)$ denote the maximal diameter of ultraconnected families of d-sets with n vertices.

Clearly, $D(d, n) \geq \Delta(d, n)$. See, e.g., [7]. (It is known that any upper bound on the diameter of simple d-polyhedra with n facets applies to general d-polyhedra with n facets. Next, to each vertex v of a simple d-polyhedron associate, as in the previous section, the set $S(v)$ of facets containing v. Two vertices v and w in P are adjacent in $G(P)$ iff $S(v)$ is adjacent to $S(u)$. The family $\{S(v)$: v is a vertex of $P\}$ is ultraconnected since for every two vertices u, v of P which belong to some face F of P there is a path in $G(P)$ connecting u and v which lies in F.)

Larman's proof applies in this generality (and beyond it, see [10]) and gives $D(d, n) \leq 2^{d-1}n$.

The proof of Section 2 applies to $D(d, n)$. In fact, this is the context for which the proof was found. To see this let K be an ultraconnected family of d-subsets of $[n] = \{1, 2, \ldots, n\}$. Let $\bar{K}$ be the simplicial complex spanned by K, i.e., the set of all subsets of sets in K. For $S \in \bar{K}$ define $N(S) = \{i \in [n]: \{i\} \cup S \in \bar{K}\}$. S is *small* if $|N(S)| \leq n/2$ or if $|S| = d - 1$. S is *big* otherwise. Now apply the proof of Section 2 word by word.

Kleitman's recent proof [6] can also be modified to apply to $D(d, n)$. To see this consider an ultraconnected family of d-subsets of $[m]$, $m \geq n$, and a subset V of $[m]$ of size n. Define the distance between two subsets of V *relative to* V to be minus one the minimal number of d-subsets of V in a path in K between S and T. (In other words, d-sets which contain vertices *not* in V are not counted.) The diameter of K *relative to* V is the maximum over all S and T of their distance relative to V. Let $\bar{D}(d, n)$ denote the maximal diameter of an ultraconnected family of sets relative to a set of n vertices. Kleitman's argument applies in this context and gives the recurrence $\bar{D}(d, n) \leq \bar{D}(d - 1, n - 1) + 2\bar{D}(d, n/2) + 2$.

Remarks. 1. All our arguments apply to the more general (graph-theoretic) context considered by Larman [10].

2. A family K of d-sets is ultraconnected iff the simplicial complex $\bar{K}$ is pure, and has the property that all links of faces of codimension one or more are connected. Note that this property is a topological property of $\bar{K}$.

4.2. For the Height

The proofs of the results on heights of polytopes apply directly to the following more general situation. Let $K = \{S_1, S_2, \ldots, S_t\}$ be a family of d-sets. For every U let $i(U)$ be the maximal index so that $U \subset S_{i(U)}$. Suppose that, for every U, if $j < i(U)$ and $U \subset S_j$, then there is $l > j$ so that $U \subset S_l$ and S_l is adjacent to S_j.

In other words, every terminal subfamily of K of the form $\{S_j, S_{j+1}, \ldots, S_t\}$ is ultraconnected. An "improving path" corresponds to a path of adjacent sets with increasing indices.

The required property for K is much stronger than ultraconnectivity. It is equivalent to the assertion that the simplicial complex spanned by K is shellable. Thus, the results on heights of polytopes apply to arbitrary shelling orders of the facets of a shellable complex.

5. Final Remarks

5.1. *What Is the Truth?*

The gap between the lower and upper bounds on $\Delta(d, n)$ is still substantial. The author's guess (which is as good as the reader's) is that the known upper bounds are asymptotically closer to the true value of $\Delta(d, n)$.

Conjecture 1. *For some positive reals a and b,*

$$D(d, n) \geq \Delta(d, n) \geq f_{a,b}(d, n). \tag{9}$$

(The function $f_{a,b}(d, n)$ is defined in Section 2.)

5.2. *Linear Programming*

There is a close relation between the complexity of edge-following algorithms for linear programming and the diameter problem for graphs of polytopes. Yet, good bounds for the diameter do not translate to quick pivot rules and, on the other hand, there are several transformations applicable to a linear program (such as LP duality) and it is possible that (allowing the use of these transformations) there is a worse-case polynomial variant of the simplex algorithm even if $\Delta(d, n)$ is not polynomial.

Klee and Minty [8] were the first to show that some variants of the simplex algorithm are exponential in the worse case. It would be of great interest to find a variant of the simplex algorithm for linear programming with subexponential worse-case behavior.

Consider the following algorithm RI(r).

RI(r): Start from a vertex v. If v is not the φ-maximal vertex of P choose a *random* r-face F containing v, replace v by the φ-maximal vertex in F and repeat.

Conjecture 2. *For every linear program with d variables and* $n \leq \lceil kd/2 \rceil$ *inequalities, the expected number of iterations of* RI($d - \lceil d/(k-1) \rceil$ *is bounded by* $O(kd \log n)$.

This conjecture is obtained by trying to estimate the number of iterations needed, in a similar way to the proof of Theorem 4, using a generous amount of unjustified probabilistic independence assumptions.

If true, Conjecture 2 would give a subexponential (randomized) variant for the simplex algorithm. Moreover, we may apply this algorithm recursively and simultaneously for faces and their duals. (One of these will have no more facets than twice the dimension.) This shows that an affirmative solution to Conjecture 2 would imply a quasi-polynomial combinatorial randomized algorithm for linear programming.

Late Remark. Recently the author [12] found a subexponential variant of the simplex algorithm. Recursive applications of algorithm RI(r) for $r = d - 1$ is subexponential but Conjecture 2 remains open.

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