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UPPER BOUNDS FOR THE INDEX OF MINIMAL SURFACES

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Introduction. A minimal submanifold in a Riemannian manifold is a critical point of the volume functional. Therefore, problems on the index arise naturally. Here the index in the sense of Morse is defined to be the number of negative eigenvalues of the Jacobi operator corresponding to the second variation.

In this paper we obtain upper bounds for the index of a compact domain with boundary on a minimal surface in an Hadamard manifold or a space form, where an Hadamard manifold is a simply connected complete Riemannian manifold of nonpositive curvature. In [2] Berard and Besson obtained an upper bound for the index of a compact domain with boundary on a minimal submanifold of dimension greater than 2 in an Hadamard manifold. We note that their method does not apply in dimension 2. First we have the following:

THEOREM 1. *Let $f: M \to N$ be a minimal immersion of a 2-dimensional manifold M into an n-dimensional Hadamard manifold N, and let D be a simply connected compact domain on M with piecewise smooth boundary ∂D. Then we have*

$$\mathrm{Index}(D) \leqq \frac{(n-2)e}{4\pi^2} \mathrm{area}(D) \int_D |A|^4 dM ,$$

where A is the second fundamental form of f and dM is the area element of M induced by f.

We also show the following:

REMARK 1. Let $f: M \to N$ be a minimal immersion of a 2-dimensional manifold M into an n-dimensional Hadamard manifold N, and let D be a compact domain on M with piecewise smooth boundary ∂D. Then we have

$$\mathrm{Index}(D) \leqq \frac{c(n-2)e}{\pi^2} \mathrm{area}(D) \int_D |A|^4 dM ,$$

where A is the second fundamental form of f, dM is the area element of M induced by f and $c = 8(\sqrt{57}-3)^2/(9-\sqrt{57})^2(\sqrt{57}-7)$ $(=143.22\cdots)$.

Let $f: M \to N$ be an immersion of a manifold M into a Riemannian manifold N. Then a point p on M is said to be a geodesic point if the second fundamental form of f vanishes at p. We denote by $N^n(a)$ the n-dimensional simply connected space form of constant curvature a. Then we obtain:

THEOREM 2. *Let $f: M \to N^n(a)$ be a minimal immersion of a 2-dimensional manifold M into $N^n(a)$ with $a \leqq 0$, and let D be a simply connected compact domain on M with piecewise smooth boundary ∂D and without geodesic points. Then*

$$\mathrm{Index}(D) \leqq \frac{9(n-2)e}{16\pi^2} \int_D |A|^\alpha dM \int_D |A|^{4-\alpha} dM$$

for all $\alpha \in [0.2/3]$, where A is the second fundamental form of f and dM is the area element of M induced by f.

In Section 3, we use this theorem to estimate the index of a domain of infinite area.

In the above results, the ambient spaces are all nonpositively curved. However, we do not assume $a \leqq 0$ in the following theorem. Let $f: M \to N$ be an immersion of a manifold M into a Riemannian manifold N, and let A be the second fundamental form of f. Then a point p on M is said to be an isotropic point if the length of $A(X, X)$ is constant for any unit vector X at $f(p)$ tangent to $f(M)$. For a function F, we denote $\max\{F, 0\}$ by F^+.

THEOREM 3. *Let $f: M \to N^n(a)$ be a minimal immersion of a 2-dimensional manifold M into $N^n(a)$, and let D be a simply connected compact domain on M with piecewise smooth boundary ∂D and without isotropic points. Then*

$$\mathrm{Index}(D) \leqq \frac{9(n-2)e}{4\pi^2} \int_D \{(a-K)^2 - {}^\perp K\}^{1/4} dM \int_D \frac{((2a-K)^+)^2}{\{(a-K)^2 - {}^\perp K\}^{1/4}} dM,$$

where K, ${}^\perp K$ and dM denote the Gaussian curvature, the normal scalar curvature and the area element of M induced by f, respectively.

See [10] for the definition of the normal scalar curvature. We note that $\{(a-K)^2 - {}^\perp K\}$ is positive on D because D contains no isotropic points.

As applications of these results, we give sufficient conditions for the stability (cf. Corollaries 1, 2 and 3).

REMARK 2. (i) Tysk [20], Cheng and Tysk [6] obtained upper bounds for the index of a complete minimal surface in the Euclidean space.

(ii) In the previous paper [17], we discussed the index of a surface with constant mean curvature.

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1. Preliminaries. Let $f: M \to N$ be a minimal immersion of an m-dimensional manifold M into a Riemannian manifold N, and let D be a compact domain on M with piecewise smooth boundary ∂D. We denote by $W(D)$ the space of smooth vector fields normal to $f(M)$ *on* D which vanish on ∂D. For $V \in W(D)$ we consider a smooth one-parameter family of immersions $\{f_t; 0 \leqq t \leqq 1\}$ of D into N such that $f_0 = f$, $f_t|_{\partial D} = f|_{\partial D}$ and $(d/dt)f_t|_{t=0} = V$. The second variation $I(V, V)$ of the volume functional of D for the variational vector field V is defined by $I(V, V) = (d^2/dt^2)\mathrm{Vol}(D, t)|_{t=0}$, where $\mathrm{Vol}(D, t)$ is the volume of D with respect to the metric induced by f_t. Let ${}^\perp\nabla$ and ${}^\perp\Delta$ be the normal connection and the Laplacian of the normal bundle $T^\perp M$ of M induced by f, respectively. Let ${}^N R$ and A denote the curvature tensor of N and the second fundamental form of f, respectively. We define two smooth sections R and $\tilde{A}$ of $\mathrm{End}(T^\perp M)$ by $\langle Rv, w\rangle = \sum_{i=1}^m \langle {}^N R(v, f_* e_i) f_* e_i, w\rangle$ and $\langle \tilde{A} v, w\rangle = \langle A^v, A^w\rangle$ for $v, w \in T_p^\perp M$, where $\{e_1, \cdots, e_m\}$ is an orthonormal basis for the tangent space of M at p with respect to the induced metric ds^2. We denote by dM the volume element of (M, ds^2). Then by the second variational formula for minimal submanifolds (see [18]), we have

$$(1.1) \qquad I(V, V) = \int_D (|{}^\perp\nabla V|^2 - \langle RV, V\rangle - |A^V|^2) dM = \int_D \langle -({}^\perp\Delta + R + \tilde{A})V, V\rangle dM$$

for $V \in W(D)$. The index of D is defined to be the number of negative eigenvalues, counted with multiplicities, of the eigenvalue problem

$$(1.2) \qquad ({}^\perp\Delta + R + \tilde{A})V + \lambda V = 0 \qquad \text{for} \quad V \in W(D)\,.$$

The domain D is stable if all the eigenvalues of (1.2) are positive, and D is unstable if (1.2) has a negative eigenvalue. The index of D thus measures how far D is from being stable.

2. Proof of Theorem 1.

PROOF OF THEOREM 1. Let $W(D)$, $I(\,,\,)$, ${}^\perp\nabla$, ${}^\perp\Delta$ and ds^2 be as in Section 1. Since N has nonpositive curvature, we have by (1.1)

$$(2.1) \qquad I(V, V) \geqq \int_D (|{}^\perp\nabla V|^2 - |A|^2 |V|^2) dM$$

for $V \in W(D)$. Let μ_i and V_i be the i-th eigenvalue and the i-th eigenvector field of the eigenvalue problem

$${}^\perp\Delta V + \mu V = 0 \qquad \text{for} \quad V \in W(D)\,,$$

respectively. It is easy to see that $\mu_i > 0$. Set

$$W_i(D) = \left\{ V \in W(D); \int_D \langle V, V_j\rangle dM = 0 \text{ for } 1 \leqq j \leqq i-1 \right\}.$$

We note that (M, ds^2) has nonpositive curvature because $f(M)$ is a minimal surface in an Hadamard manifold. Let ∇ denote the Riemannian connection of (M, ds^2). For $V \in W_i(D)$ we get

$$\begin{aligned}(2.2)\qquad \int_D |A|^2|V|^2 dM &\leqq \left(\int_D |A|^4 dM\right)^{1/2}\left(\int_D |V|^4 dM\right)^{1/2}\\ &\leqq 2^{-1}\pi^{-1/2}\left(\int_D |A|^4 dM\right)^{1/2}\int_D |\nabla(|V|^2)| dM\\ &\leqq \pi^{-1/2}\left(\int_D |A|^4 dM\right)^{1/2}\int_D |V||{}^\perp\nabla V| dM\\ &\leqq \pi^{-1/2}\left(\int_D |A|^4 dM\right)^{1/2}\left(\int_D |V|^2 dM\right)^{1/2}\left(\int_D |{}^\perp\nabla V|^2 dM\right)^{1/2}\\ &\leqq \pi^{-1/2}\mu_i^{-1/2}\left(\int_D |A|^4 dM\right)^{1/2}\int_D |{}^\perp\nabla V|^2 dM\end{aligned}$$

(cf. [9, p. 69]), where for the second inequality we use the Sobolev inequality on the simply connected nonpositively curved domain (D, ds^2) (see [4]). Therefore if

$$\mu_i \geqq \frac{1}{\pi}\int_D |A|^4 dM,$$

then by (2.1) and (2.2), we have $I(V, V) \geqq 0$ for any $V \in W_i(D)$. From this fact we find that

$$\text{(2.3)}\qquad \mathrm{Index}(D) \leqq \mathrm{Card}\left\{i;\ \mu_i < \frac{1}{\pi}\int_D |A|^4 dM\right\}.$$

Let $F(D)$ be the space of smooth functions on D which vanish on ∂D. Let λ_i be the i-th eigenvalue of the eigenvalue problem

$$\Delta\psi + \lambda\psi = 0 \qquad \text{for} \quad \psi \in F(D),$$

where Δ is the Laplacian of (M, ds^2). Then

$$\text{(2.4)}\qquad \sum_{i=1}^{\infty} \exp(-\mu_i t) \leqq (n-2)\sum_{i=1}^{\infty} \exp(-\lambda_i t)$$

for $t > 0$ (see [21, Theorem 2.1] and [8]). Since $f(D)$ is on a minimal surface in an Hadamard manifold, we have

$$\text{(2.5)}\qquad \sum_{i=1}^{\infty} \exp(-\lambda_i t) \leqq \frac{1}{4\pi t}\,\mathrm{area}(D)$$

for $t > 0$ (see [5] and [11]). Using (2.3), (2.4) and (2.5) we see that

$$\text{(2.6)}\quad \mathrm{Index}(D)\exp\left(-\frac{t}{\pi}\int_D |A|^4 dM\right) \leqq \sum_{i \in I} \exp(-\mu_i t) \leqq \sum_{i=1}^{\infty} \exp(-\mu_i t) \leqq \frac{n-2}{4\pi t}\,\mathrm{area}(D)$$

for $t>0$, where $I=\{i;\ \mu_i<(1/\pi)\int_D |A|^4 dM\}$. Thus we have

$$\mathrm{Index}(D)\leqq\frac{n-2}{4\pi}\,\mathrm{area}(D)\inf_{t>0}\frac{1}{t}\exp\left(\frac{t}{\pi}\int_D |A|^4 dM\right)=\frac{(n-2)e}{4\pi^2}\,\mathrm{area}(D)\int_D |A|^4 dM\,.$$

q.e.d.

Next we show the fact in Remark 1. The proof is the same as that of Theorem 1, except that we use in (2.2) the Sobolev inequality on a compact domain with boundary on a minimal surface in an Hadamard manifold (see [19, p. 324] and [9]).

COROLLARY 1. *Let $f: M\to N$ be a minimal immersion of a 2-dimensional manifold M into an Hadamard manifold N whose sectional curvature is not greater than $a<0$. Let D be a simply connected compact domain on M with piecewise smooth boundary ∂D. Then D is stable if*

$$\int_D |A|^4 dM<\frac{\pi|a|}{4},$$

where A is the second fundamental form of f and dM is the area element of M induced by f.

PROOF. Let $I(\,,)$, ds^2, ∇, $F(D)$ and λ_i be as in the proof of Theorem 1. Let ν be a unit vector field normal to $f(M)$ on D. Then as in (2.1) we have

$$I(\psi\nu,\psi\nu)\geqq\int_D (|\nabla\psi|^2-|A|^2\psi^2)dM \tag{2.7}$$

for $\psi\in F(D)$. As in (2.2) we have

$$\int_D |A|^2\psi^2 dM\leqq\pi^{-1/2}\lambda_1^{-1/2}\left(\int_D |A|^4 dM\right)^{1/2}\int_D |\nabla\psi|^2 dM \tag{2.8}$$

for $\psi\in F(D)$. We note that the Gaussian curvature of (M, ds^2) is not greater than $a<0$ because of the hypothesis. Combining Theorem 4.4 in [14] with Lemma 2 in [13], we have $\lambda_1\geqq|a|/4$. Hence by (2.7) and (2.8), we have $I(\psi\nu,\psi\nu)>0$ for any ν and any $\psi\in F(D)$ which is not identically zero, under the hypothesis of the corollary. q.e.d.

COROLLARY 2. *Let $f: M\to N$ be a minimal immersion of a 2-dimensional manifold M into an Hadamard manifold N whose sectional curvature is not greter than $a<0$. Let D be a doubly connected compact domain on M with piecewise smooth boundary ∂D. Then D is stable if*

$$\int_D |A|^4 dM<\frac{\pi|a|}{16c},$$

where A is the second fundamental form of f, dM is the area element of M induced by f and c is as in Remrark 1.

PROOF. The proof is the same as that of Corollary 1, except that we use in (2.8) the Sobolev inequality in [19]. q.e.d.

3. Proof of Theorem 2. For the proof of Theorem 2 we need the following lemma (cf. [1, Proposition 2.2]):

LEMMA. *Let $f: M \to N^n(a)$ be a minimal immersion of a 2-dimensional manifold M into $N^n(a)$ with $a \leqq 0$. Let A and ds^2 denote the second fundamental form and the metric on M induced by f, respectively. Set $ds_\alpha^2 = |A|^\alpha ds^2$. Then the Gaussian curvature K_α of (M, ds_α^2) satisfies $K_\alpha \leqq 0$ for $\alpha \in [0, 2/3]$ except at geodesic points.*

PROOF. Suppose that p is a non-geodesic point on M. Let K, ∇ and Δ denote the Gaussian curvature, the Riemannian connection and the Laplacian of (M, ds^2), respectively. Then we have

$$(3.1)\qquad \begin{aligned} K_\alpha &= \frac{K}{|A|^\alpha} - \frac{1}{2|A|^\alpha}\Delta \log(|A|^\alpha) = \frac{2a-|A|^2}{2|A|^\alpha} - \frac{\alpha}{4|A|^\alpha}\Delta \log(|A|^2) \\ &= \frac{2a-|A|^2}{2|A|^\alpha} - \frac{\alpha}{4|A|^\alpha}\left\{\frac{\Delta(|A|^2)}{|A|^2} - \frac{|\nabla(|A|^2)|^2}{|A|^4}\right\} = \frac{2a-|A|^2}{2|A|^\alpha} \\ &\quad - \frac{\alpha}{2|A|^{2+\alpha}}\langle A, \Delta A\rangle + \frac{\alpha}{|A|^{4+\alpha}}\left\{-\frac{1}{2}|A|^2|\nabla A|^2 + \left|\frac{1}{2}\nabla(|A|^2)\right|^2\right\}, \end{aligned}$$

where we use the Gauss equation for the second equality. In [15] we showed that

$$(3.2)\qquad -\langle A, \Delta A\rangle \leqq \frac{3}{2}|A|^4 - 2a|A|^2$$

(cf. [7] and [18]). Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for the tangent space of $N^n(a)$ at $f(p)$ such that e_1, e_2 are tangent to $f(M)$. Let h_{ij}^β and h_{ijk}^β be the components of A and ∇A with respect to the basis, respectively. Here and in what follows, we use the following convention on the ranges indices: $1 \leqq i, j, k \leqq 2$, $3 \leqq \beta \leqq n$. We may choose $\{e_1, \cdots, e_n\}$ so that

$$(h_{ij}^3) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h_{ij}^5) = \cdots = (h_{ij}^n) = (0)$$

for some λ and μ. We note that the components h_{ijk}^β are symmetric in i, j, k, and satisfy $h_{11i}^\beta + h_{22i}^\beta = 0$. Thus we have

$$(3.3)\qquad \begin{aligned} -\frac{1}{2}|A|^2|\nabla A|^2 + \left|\frac{1}{2}\nabla(|A|^2)\right|^2 &= -\frac{1}{2}\sum_{i,j,\beta}(h_{ij}^\beta)^2 \sum_{i,j,k,\beta}(h_{ijk}^\beta)^2 + \sum_k\left(\sum_{i,j,\beta} h_{ij}^\beta h_{ijk}^\beta\right)^2 \\ &\leqq -4(\lambda^2+\mu^2)\{(h_{111}^3)^2 + (h_{112}^3)^2 + (h_{111}^4)^2 + (h_{112}^4)^2\} + 4(\lambda h_{111}^3 + \mu h_{112}^4)^2 \\ &\quad + 4(\lambda h_{112}^3 - \mu h_{111}^4)^2 = -4(\lambda h_{112}^4 - \mu h_{111}^3)^2 - 4(\lambda h_{111}^4 + \mu h_{112}^3)^2 \leqq 0\,. \end{aligned}$$

Using (3.1), (3.2) and (3.3) we see that for $\alpha\in[0, 2/3]$

$$K_\alpha\leqq\frac{4a(1-\alpha)+(3\alpha-2)|A|^2}{4|A|^\alpha}\leqq 0\,. \qquad \text{q.e.d.}$$

PROOF OF THEOREM 2. Let $W(D)$, $I(\,,)$, $T^\perp M$, ${}^\perp\nabla$ and ds^2 be as in Section 1. As in (2.1) we have

$$I(V, V)\geqq\int_D (|{}^\perp\nabla V|^2-|A|^2|V|^2)dM \tag{3.4}$$

for $V\in W(D)$. Set $ds_\alpha^2=|A|^\alpha ds^2$ for $\alpha\in[0, 2/3]$, which is nondegenerate on D because D contains no geodesic points. Let ${}^\perp\nabla_\alpha$ and dM_α denote the connection of $T^\perp M$ with respect to ds_α^2 and the area element of (M, ds_α^2), respectively. We see that

$$|{}^\perp\nabla_\alpha V|^2=\frac{|{}^\perp\nabla V|^2}{|A|^\alpha} \quad\text{and}\quad dM_\alpha=|A|^\alpha dM\,, \tag{3.5}$$

where V is a smooth vector field normal to $f(M)$ on D. By (3.4) and (3.5) we have for $V\in W(D)$

$$I(V, V)\geqq\int_D (|{}^\perp\nabla_\alpha V|^2-|A|^{2-\alpha}|V|^2)dM_\alpha\,. \tag{3.6}$$

Let μ_i^α and V_i^α be the i-th eigenvalue and the i-th eigenvector field of the eigenvalue problem

$$^\perp\Delta_\alpha V+\mu V=0 \qquad\text{for}\quad V\in W(D)\,,$$

respectively, where ${}^\perp\Delta_\alpha$ is the Laplacian of $T^\perp M$ with respect to ds_α^2. Set

$$W_i^\alpha(D)=\left\{V\in W(D);\ \int_D \langle V, V_j^\alpha\rangle dM_\alpha=0 \text{ for } 1\leqq j\leqq i-1\right\}.$$

We note that (D, ds_α^2) has nonpositive curvature by the lemma. As in (2.2) for $V\in W_i^\alpha(D)$ we have

$$\int_D |A|^{2-\alpha}|V|^2dM_\alpha\leqq\pi^{-1/2}(\mu_i^\alpha)^{-1/2}\left(\int_D |A|^{4-2\alpha}dM_\alpha\right)^{1/2}\int_D |{}^\perp\nabla_\alpha V|^2dM_\alpha\,, \tag{3.7}$$

where we use the Sobolev inequality on the simply connected nonpositively curved domain (D, ds_α^2) (see [4]). Therefore, as in (2.3) we have

$$\mathrm{Index}(D)\leqq\mathrm{Card}\left\{i;\ \mu_i^\alpha<\frac{1}{\pi}\int_D |A|^{4-2\alpha}dM_\alpha\right\}. \tag{3.8}$$

Let $F(D)$ be the space of smooth functions on D which vanish on ∂D. Let λ_i^α be the i-th eigenvalue of the eigenvalue problem

$$\Delta_\alpha\psi+\lambda\psi=0 \qquad \text{for} \quad \psi\in F(D),$$

where Δ_α is the Laplacian of (M, ds_α^2). Then

$$\sum_{i=1}^{\infty}\exp(-\mu_i^\alpha t)\leqq(n-2)\sum_{i=1}^{\infty}\exp(-\lambda_i^\alpha t) \tag{3.9}$$

for $t>0$ (see [21, Theorem 2.1]). Since (D, ds_α^2) is simply connected and nonpositively curved, we have

$$\sum_{i=1}^{\infty}\exp(-\lambda_i^\alpha t)\leqq\frac{9}{16\pi t}\int_D dM_\alpha \tag{3.10}$$

for $t>0$ (see the proof of Theorem 2 in [4]). By (3.8), (3.9) and (3.10) we find as in (2.6)

$$\mathrm{Index}(D)\exp\left(-\frac{t}{\pi}\int_D |A|^{4-2\alpha}dM_\alpha\right)\leqq\frac{9(n-2)}{16\pi t}\int_D dM_\alpha$$

for $t>0$. Thus we have

$$\begin{aligned}\mathrm{Index}(D)&\leqq\frac{9(n-2)}{16\pi}\int_D dM_\alpha\inf_{t>0}\frac{1}{t}\exp\left(\frac{t}{\pi}\int_D |A|^{4-2\alpha}dM_\alpha\right)\\&=\frac{9(n-2)e}{16\pi^2}\int_D dM_\alpha\int_D |A|^{4-2\alpha}dM_\alpha=\frac{9(n-2)e}{16\pi^2}\int_D |A|^\alpha dM\int_D |A|^{4-\alpha}dM.\end{aligned}$$

q.e.d.

COROLLARY 3. *Let $f: M\to N^n(a)$ be a minimal immersion of a 2-dimensional manifold M into $N^n(a)$ with $a\leqq 0$, and let D be a simply connected compact domain on M with piecewise smooth boundary ∂D and without geodesic points. Then D is stable if*

$$\int_D |A|^\alpha dM\int_D |A|^{4-\alpha}dM<j_0^2\pi^2$$

for some $\alpha\in[0, 2/3]$, where A is the second fundamental form of f, dM is the area element of M induced by f and j_0 $(=2.40483\cdots)$ is the smallest zero of the Bessel function of order zero.

PROOF. Let $I(\ ,\ )$, ds_α^2, dM_α, $F(D)$ and λ_i^α be as in the proof of Theorem 2. Let ∇_α denote the Riemannian connection of (M, ds_α^2), and let ν be a unit vector field normal to $f(M)$ on D. Then as in (3.6) we have

$$I(\psi\nu, \psi\nu)\geqq\int_D (|\nabla_\alpha\psi|^2-|A|^{2-\alpha}\psi^2)dM_\alpha \tag{3.11}$$

for $\psi\in F(D)$. As in (3.7) we get

$$(3.12)\qquad \int_D |A|^{2-\alpha}\psi^2 dM_\alpha \leqq \pi^{-1/2}(\lambda_1^\alpha)^{-1/2}\left(\int_D |A|^{4-2\alpha} dM_\alpha\right)^{1/2}\int_D |\nabla_\alpha \psi|^2 dM_\alpha$$

for $\psi \in F(D)$. Since (D, ds_α^2) is simply connected and nonpositively curved, we have

$$(3.13)\qquad \lambda_1^\alpha \geqq j_0^2 \pi\left(\int_D dM_\alpha\right)^{-1}$$

(see [1, Proposition 3.3]). By (3.11), (3.12) and (3.13), we have $I(\psi v, \psi v) > 0$ for any v and any $\psi \in F(D)$ which is not identically zero, under the hypothesis of the corollary.

q.e.d.

Now we recall Mori's examples of complete minimal surfaces in the 3-dimensional hyperbolic space H^3 of constant curvature -1 (see [12]). Let L^4 be the space of 4-tuples $x = (x_1, x_2, x_3, x_4)$ with the Lorentzian metric $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ for $x, y \in L^4$. We regard H^3 as a hypersurface $\{x \in L^4; \langle x, x\rangle = -1, x_1 \geqq 1\}$ in L^4. For each $a > 1/2$, we define an immersion f_a: $M = \mathbf{R} \times S^1 \to H^3$ by

$$f_a(s, \theta) = {}^t\begin{pmatrix} A(a, s)\cosh\phi_a(s) \\ A(a, s)\sinh\phi_a(s) \\ B(a, s)\cos\theta \\ B(a, s)\sin\theta \end{pmatrix}$$

for $s \in (-\infty, \infty)$, $\theta \in [0, 2\pi]$, where

$$A(a, s) = \left(a\cosh(2s) + \frac{1}{2}\right)^{1/2}, \qquad B(a, s) = \left(a\cosh(2s) - \frac{1}{2}\right)^{1/2}$$

and

$$\phi_a(s) = \left(a^2 - \frac{1}{4}\right)^{1/2}\int_0^s \frac{dt}{\{A(a, t)\}^2 B(a, t)}.$$

Then f_a is a minimal immersion and M is complete in the induced metric. Let A_a and dM_a denote the second fundamental form and the area element of M induced by f_a, respectively. Then

$$|A_a|^2 = \frac{2(a^2 - 1/4)}{\{B(a, s)\}^4} > 0, \qquad dM_a = B(a, s)ds d\theta.$$

For $\alpha \in (1/2, 7/2)$ we have

$$\int_M |A_a|^\alpha dM_a \int_M |A_a|^{4-\alpha} dM_a = 64\pi^2\left(a^2 - \frac{1}{4}\right)^2 \int_0^\infty \frac{ds}{\{B(a, s)\}^{2\alpha-1}} \int_0^\infty \frac{ds}{\{B(a, s)\}^{7-2\alpha}} < \infty.$$

Therefore, we can estimate the index of a simply connected domain of infinite area on

M by applying Theorem 2 to $\alpha \in (1/2, 2/3]$. Similarly, we may use Corollaries 1, 2 and 3 to estimate the stability of a domain of infinite area (cf. [16]).

REMARK 3. (i) The surface M is stable with respect to f_a if $a \geqq 17/2$ (see [12, Theorem 2]), and M is unstable with respect to f_a if $1/2 < a < c_0$ ($\simeq 0.69$) (see [3, p. 708]).

(ii) There are classical examples of complete minimal surfaces in the 3-dimensional Euclidean space. But the author could not find among them a suitable one to which Theorem 2 is applied as above.

4. Proof of Theorem 3.

PROOF OF THEOREM 3. Let $W(D)$, $I(\,,)$, $T^{\perp}M$, ${}^{\perp}\nabla$ and ds^2 be as in Section 1. Then we have by (1.1)

$$(4.1)\quad I(V, V) \geqq \int_D \{|{}^{\perp}\nabla V|^2 - (2a + |A|^2)|V|^2\} dM = \int_D \{|{}^{\perp}\nabla V|^2 - 2(2a - K)|V|^2\} dM$$

for $V \in W(D)$, where we use the Gauss equation. We consider a flat metric $d\tilde{s}^2 = \{(a-K)^2 - {}^{\perp}K\}^{1/4} ds^2$ on M (see [10, p. 207]), which is nondegenerate on D because D contains no isotropic points. Let ${}^{\perp}\tilde{\nabla}$ and $d\tilde{M}$ denote the connection of $T^{\perp}M$ with respect to $d\tilde{s}^2$ and the area element of $(M, d\tilde{s}^2)$, respectively. We see that

$$(4.2)\qquad |{}^{\perp}\tilde{\nabla} V|^2 = \frac{|{}^{\perp}\nabla V|^2}{\{(a-K)^2 - {}^{\perp}K\}^{1/4}}$$

for a smooth vector field V normal to $f(M)$ on D, and

$$(4.3)\qquad d\tilde{M} = \{(a-K)^2 - {}^{\perp}K\}^{1/4} dM .$$

By (4.1), (4.2) and (4.3) we have for $V \in W(D)$

$$I(V, V) \geqq \int_D \left[|{}^{\perp}\tilde{\nabla} V|^2 - \frac{2(2a-K)}{\{(a-K)^2 - {}^{\perp}K\}^{1/4}} |V|^2 \right] d\tilde{M} \geqq \int_D (|{}^{\perp}\tilde{\nabla} V|^2 - 2u|V|^2) d\tilde{M} ,$$

where

$$u = \frac{(2a-K)^+}{\{(a-K)^2 - {}^{\perp}K\}^{1/4}} .$$

The rest of the proof is identical to those of Theorems 1 and 2. q.e.d.

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