# UPPER BOUNDS FOR THE STANLEY DEPTH 

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#### Abstract

Let $I \subset J$ be monomial ideals of a polynomial algebra $S$ over a field. Then the Stanley depth of $J / I$ is smaller or equal with the Stanley depth of $\sqrt{J} / \sqrt{I}$. We give also an upper bound for the Stanley depth of the intersection of two primary monomial ideals $Q, Q^{\prime}$, which is reached if $Q, Q^{\prime}$ are irreducible, $\operatorname{ht}\left(Q+Q^{\prime}\right)$ is odd and $\sqrt{Q}, \sqrt{Q^{\prime}}$ have no common variable.


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## 1. Introduction

Let $S=K\left[x_{1}, \ldots, x_{n}\right]$ be a polynomial ring over a field $K$ and $M$ be a finitely generated $\mathbb{Z}^{n}$-graded $S$-module. For a homogenous element $w \in M$ and a subset $Z \subset$ $\left\{x_{1}, \ldots, x_{n}\right\}, w K[Z]$ denotes the $K$-subspace of $M$ generated by all homogeneous elements of the form $w v$, where $v$ is a monomial in $K[Z]$. The linear $K$-subspace $w K[Z]$ is called a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where $|Z|$ denotes the number of indeterminates in $Z$. A Stanley decomposition of $M$ is a presentation of the $K$-vector space $M$ as a finite direct sum of Stanley spaces

$$
\mathcal{D}: M=\bigoplus_{i=1}^{s} u_{i} K\left[Z_{i}\right]
$$

and the Stanley depth of a decomposition $\mathcal{D}$ is sdepth $\mathcal{D}=\min \left\{\left|Z_{i}\right|, i=1, \ldots, s\right\}$. The Stanley depth of $M$ is

$$
\operatorname{sdepth}_{S}(M)=\max \{\operatorname{sdepth} \mathcal{D}: \mathcal{D} \text { is a Stanley decomposition of } M\}
$$

(sometimes we write $\operatorname{sdepth}(M)$ if no confusion is possible). Several properties of the Stanley depth are given in [4], [9], [15], [16] and [17]. Stanley conjectured in [19] that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ (for terminology of commutative algebra we refer to [3]). This conjecture has been proved in several special cases, for example see ([1], [7], [12], [13], [14]) but it is still open in general.

Let $I \subset S$ be a monomial ideal. It is well known that depth $S / I \leq \operatorname{depth} S / \sqrt{I}$ (see the proof of [8, Theorem 2.6]) and equivalently depth $I \leq \operatorname{depth} \sqrt{I}$. The first inequality holds also for sdepth, that is sdepth $S / I \leq \operatorname{sdepth} S / \sqrt{I}$ (see [2, Theorem 1]). Moreover if $I \subset J$ are monomial ideals of $S$ then our Theorem [2.1 says that

[^0]sdepth $J / I \leq \operatorname{sdepth} \sqrt{J} / \sqrt{I}$. In particular if $J / I$ is a kind of "Stanley-CohenMacaulay" module, that is sdepth $J / I=\operatorname{dim} J / I$ then $\sqrt{J} / \sqrt{I}$ is too. The idea of the proof is inspired by [14, Lemma 2.1].

Next we give upper bounds for the intersection $I$ of two monomial primary ideals $Q, Q^{\prime}$ (see Theorem 2.19). It is enough to find upper bounds for the intersection $J$ of two monomial prime ideals by Theorem 2.1. When these prime ideals have no common variables, this is done in Theorem 2.8, which says sdepth $J \leq \frac{n+2}{2}$, using an idea of [10, Lemma 2.2] and the algorithm for computation of the Stanley depth given in [9]. For the general case we need to show that $\operatorname{sdepth}_{S\left[x_{n+1}\right]}\left(I, x_{n+1}\right) \leq$ $\operatorname{sdepth}_{S} I+1$, which is stated by the elementary Lemma 2.11. Using the lower bound given in [14, Lemma 4.1] we noticed that our upper bound is reached when $Q, Q^{\prime}$ are irreducible ideals, $\operatorname{ht}\left(Q+Q^{\prime}\right)$ is odd and $\sqrt{Q}, \sqrt{Q^{\prime}}$ have no common variables. In general our upper bound is big as shows Lemma 2.15 and several examples.

## 2. UPPER BOUNDS FOR THE STANLEY DEPTH

Let $I$ and $J$ be two monomial ideals of $S$ such that $I \subset J$ and $\sqrt{I}$ and $\sqrt{J}$ be the radical ideals of $I$ and $J$ respectively. Let $G(I)$ be the minimal system of monomial generators of $I$. Then
Theorem 2.1. $\operatorname{sdepth}(J / I) \leq \operatorname{sdepth}(\sqrt{J} / \sqrt{I})$.
Proof. The ideals $I$ and $J$ have an irredundant monomial decomposition $J=\bigcap_{i=1}^{q} Q_{i}$ and $I=\bigcap_{j=1}^{r} Q_{j}^{\prime}$, where $Q_{i}{ }^{\prime} s$ and $Q_{j}^{\prime}$ 's are primary monomial ideals (if $J=S$ then $\left(Q_{i}\right)_{i}$ is the empty set of primary ideals). Let $T=K\left[y_{1}, \ldots, y_{n}\right]$ and $a_{i} \in \mathbb{N}$ be the maximum power of $x_{i}$ which appear in $\bigcup_{i=1}^{q} G\left(Q_{i}\right) \cup \bigcup_{j=1}^{r} G\left(Q_{j}\right)$, if no power of $x_{i}$ is there then take $a_{i}=0$. Set $a=\max _{i}\left(a_{i}\right)$. Let $P_{i}=\left(y_{c}: x_{c} \in \sqrt{Q_{i}}\right)$ and $P_{j}^{\prime}=$ $\left(y_{d}: x_{d} \in \sqrt{Q_{j}^{\prime}}\right)$. These $P_{i}^{\prime} s$ and $P^{\prime}{ }_{j} s$ are uniquely determined by $I$ and $J$. Let $\varphi: T \longrightarrow S$ be the $K$-morphism given by $y_{i} \longrightarrow x_{i}^{a}$. Let $\mathcal{D}: J / I=\bigoplus_{i=1}^{s} u_{i} K\left[Z_{i}\right]$ be a Stanley decomposition of $J / I$ such that $\operatorname{sdepth}(\mathcal{D})=\operatorname{sdepth}(J / I)$. Then we have

$$
L:=\varphi^{-1}(J) / \varphi^{-1}(I)=\cap_{i=1}^{q} P_{i} / \cap_{j=1}^{r} P_{j}^{\prime},
$$

and $\varphi$ defines an injection $\tilde{\varphi}: L \longrightarrow J / I$. Thus

$$
L=\tilde{\varphi}^{-1}\left(\bigoplus_{i=1}^{s} u_{i} K\left[Z_{i}\right]\right)=\bigoplus_{i=1}^{s} \tilde{\varphi}^{-1}\left(u_{i} K\left[Z_{i}\right]\right)
$$

Suppose that $\tilde{\varphi}^{-1}\left(u_{j} K[Z j]\right) \neq 0$, then $u_{j}=x_{l_{1}}^{b_{1}} \ldots x_{l_{m}}^{b_{l_{m}}}, b_{l_{k}} \in \mathbb{Z}_{+}$is such that if $x_{l_{k}} \notin Z_{j}$ then $a \mid b_{l_{k}}$ where $1 \leq l_{k} \leq n$, let us say $b_{l_{k}}=a c_{l_{k}}$ for some $c_{l_{k}} \in \mathbb{Z}_{+}$. Denote $c_{l_{k}}=\left\lceil\frac{b_{l_{k}}}{a}\right\rceil$ when $x_{l_{k}} \in Z_{j}$. We get

$$
\tilde{\varphi}^{-1}\left(u_{j} k\left[Z_{j}\right]\right)=y_{l_{1}}^{c_{l_{1}}} \ldots y_{l_{m}}^{c_{l_{m}}} k\left[V_{j}\right]
$$

where $V_{j}=\left\{y_{l}: 1 \leq l \leq n, x_{l} \in Z_{j}\right\}$. Thus $\tilde{\varphi}^{-1}\left(u_{j} k\left[Z_{j}\right]\right)$ is a Stanley space of $L$ and so $\mathcal{D}$ induces a Stanley decomposition $\mathcal{D}^{\prime}$ of $L$ such that

$$
\operatorname{sdepth}(J / I)=\operatorname{sdepth}(\mathcal{D}) \leq \operatorname{sdepth}\left(\mathcal{D}^{\prime}\right) \leq \operatorname{sdepth}(L)
$$

Now let us define an isomorphism $\psi: T \longrightarrow S$ by $\psi\left(y_{i}\right)=x_{i}$. Under this isomorphism, we have

$$
L \cong \cap_{i=1}^{q} \sqrt{Q_{i}} / \cap_{j=1}^{r} \sqrt{Q_{j}^{\prime}} .
$$

Thus

$$
\operatorname{sdepth}(L)=\operatorname{sdepth}\left(\cap_{i=1}^{q} \sqrt{Q_{i}} / \cap_{j=1}^{r} \sqrt{Q_{j}^{\prime}}\right)=\operatorname{sdepth}(\sqrt{J} / \sqrt{I})
$$

Hence

$$
\operatorname{sdepth}(J / I) \leq \operatorname{sdepth}(\sqrt{J} / \sqrt{I})
$$

Corollary 2.2. Let $I \subset S$ be a monomial ideal and $\sqrt{I}$ be its radical. Then $\operatorname{sdepth}(S / I) \leq \operatorname{sdepth}(S / \sqrt{I})$ and $\operatorname{sdepth}(I) \leq \operatorname{sdepth}(\sqrt{I})$.

First part of the above corollary is already done in [2, Theorem 1].
Corollary 2.3. Let $I$ and $J$ be two monomial ideals of $S$ such that $I \subset J$ and $\sqrt{I}$ and $\sqrt{J}$ be the radical ideals of $I$ and $J$ respectively. If $\operatorname{sdepth}(J / I)=\operatorname{dim}(J / I)$. Then $\operatorname{sdepth}(\sqrt{J} / \sqrt{I})=\operatorname{dim}(\sqrt{J} / \sqrt{I})$.

Proof. Given that $\operatorname{dim}(J / I)=\operatorname{sdepth}(J / I)$, and from Theorem 2.1, we have

$$
\operatorname{dim}(J / I)=\operatorname{sdepth}(J / I) \leq \operatorname{sdepth}(\sqrt{J} / \sqrt{I}) \leq \operatorname{dim}(\sqrt{J} / \sqrt{I})
$$

by [17]. Also since $\operatorname{dim}(\sqrt{J} / \sqrt{I})=\operatorname{dim}(S / \sqrt{I}: \sqrt{J}) \leq \operatorname{dim}(S / \sqrt{I: J})=\operatorname{dim}(S / I$ : $J)=\operatorname{dim}(J / I)$, we get $\operatorname{sdepth}(\sqrt{J} / \sqrt{I})=\operatorname{dim}(\sqrt{J} / \sqrt{I})$.

The inequality given by Theorem 2.1 can be strict as shows the following example.
Example 2.4. Let $I \subset K\left[x_{1}, x_{2}\right]$ and $I=\left(x_{1}^{2}, x_{1} x_{2}\right)$. Since $I=x_{1}^{2} K\left[x_{1}, x_{2}\right] \oplus$ $x_{1} x_{2} K\left[x_{2}\right]$, we see that $\operatorname{sdepth}(I)=1$. Since $\sqrt{I}=\left(x_{1}\right)$, we have $\operatorname{sdepth}(\sqrt{I})=2$.

A lower bound for Stanley depth of intersection of two monomial primary ideals is discussed in [14]. We give an upper bound for the Stanley depth of intersection of two monomial primary ideals. Let $Q$ and $Q^{\prime}$ be any two monomial primary ideal of $S$, then after renumbering the variables we can always assume that $\sqrt{Q}=\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{r+1}, \ldots, x_{p}\right)$, where $0 \leq r \leq t \leq p \leq n$. We start with the case $n=p$. A special case is given by the following:

Lemma 2.5. Let $Q$ and $Q^{\prime}$ be two monomial primary ideals with $\sqrt{Q}=\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{1}, \ldots, x_{n}\right)$. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq n-\left\lfloor\frac{t}{2}\right\rfloor
$$

Proof. $\sqrt{Q} \subseteq \sqrt{Q^{\prime}}$ implies that $\sqrt{Q} \cap \sqrt{Q^{\prime}}=\sqrt{Q}$ by [5, Theorem 1.3] $\operatorname{sdepth}(\sqrt{Q})=$ $n-\left\lfloor\frac{t}{2}\right\rfloor$. So by Corollary 2.2 it follows that $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq n-\left\lfloor\frac{t}{2}\right\rfloor$.

To find an upper bound for Stanley depth of $Q \cap Q^{\prime}$ it is necessary to find an upper bound for $\sqrt{Q} \cap \sqrt{Q^{\prime}}$. We consider two cases
$\operatorname{Case}(1): \sqrt{Q}=\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{t+1}, \ldots, x_{n}\right)$.

Case(2): $\sqrt{Q}=\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{r+1}, \ldots, x_{n}\right)$ where $1<r<t<n$.
First we consider Case(1) where the proof idea comes from 10. We recall the method of Herzog et al. [9] for computing the Stanley depth of a squarefree monomial ideal $I$ using posets. Let $G(I)=\left\{v_{1}, \ldots, v_{m}\right\}$ be the set of minimal monomial generators of $I$. The characteristic poset of $I$ with respect to $g=(1, \ldots, 1)$ (see [9]), denoted by $\mathcal{P}_{I}^{(1, \ldots, 1)}$ is in fact the set

$$
\mathcal{P}_{I}^{(1, \ldots, 1)}=\left\{C \subset[n] \mid C \text { contains supp }\left(v_{i}\right) \text { for some } i\right\},
$$

where $\operatorname{supp}\left(v_{i}\right)=\left\{j: x_{j} \mid v_{i}\right\} \subseteq[n]:=\{1, \ldots, n\}$. For every $A, B \in \mathcal{P}_{I}^{(1, \ldots, 1)}$ with $A \subseteq B$, define the interval $[A, B]$ to be $\left\{C \in \mathcal{P}_{I}^{(1, \ldots, 1)}: A \subseteq B \subseteq C\right\}$. Let $\mathcal{P}$ : $\mathcal{P}_{I}^{(1, \ldots, 1)}=\cup_{i=1}^{r}\left[C_{i}, D_{i}\right]$ be a partition of $\mathcal{P}_{I}^{(1, \ldots, 1)}$, and for each $i$, let $c(i) \in\{0,1\}^{n}$ be the tuple such that $\operatorname{supp}\left(x^{c(i)}\right)=C_{i}$. Then there is a Stanley decomposition $\mathcal{D}(\mathcal{P})$ of $I$

$$
\mathcal{D}(\mathcal{P}): I=\bigoplus_{i=1}^{s} x^{c(i)} K\left[\left\{x_{k} \mid k \in D_{i}\right\}\right] .
$$

Clearly sdepth $\mathcal{D}(\mathcal{P})=\min \left\{\left|D_{1}\right|, \ldots,\left|D_{s}\right|\right\}$. It is shown in [9] that

$$
\operatorname{sdepth}(I)=\max \left\{\operatorname{sdepth} \mathcal{D}(P) \mid \mathcal{P} \text { is a partition of } \mathcal{P}_{I}^{(1, \ldots, 1)}\right\}
$$

An easy case which is enough when $n \leq 3$ is given by the following:
Lemma 2.6. Let $Q$ and $Q^{\prime}$ be two monomial primary ideals with $\sqrt{Q}=\left(x_{1}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{2}, \ldots, x_{n}\right)$. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq 1+\left\lceil\frac{n-1}{2}\right\rceil
$$

Proof. Let $I$ be a monomial ideal and $v=G C D(u \mid u \in G(I))$. Then $I=v I^{\prime}$ where $I^{\prime}=(I: v)$. By [6, Proposition 1.3(2)] $\operatorname{sdepth}(I)=\operatorname{sdepth}\left(I^{\prime}\right)$. Since in our case $v=x_{1}$ then we have $\operatorname{sdepth}\left(\sqrt{Q} \cap \sqrt{Q^{\prime}}\right)=\operatorname{sdepth}\left(\left(\sqrt{Q} \cap \sqrt{Q^{\prime}}\right): x_{1}\right)=$ $\operatorname{sdepth}\left(\sqrt{Q^{\prime}}\right)=n-(n-1)+\left\lceil\frac{n-1}{2}\right\rceil=1+\left\lceil\frac{n-1}{2}\right\rceil$ by [5, Theorem 1.3]. Hence from Corollary 2.2 the result follows.

Remark 2.7. If in Lemma $2.6 Q$ and $Q^{\prime}$ are irreducible monomial ideals then $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=1+\left\lceil\frac{n-1}{2}\right\rceil$ by [6, Theorem 1.3(2)]. Note that $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=\frac{n}{2}+1$ if $n$ is even.

Theorem 2.8. Let $Q$ and $Q^{\prime}$ be two primary ideals with $\sqrt{Q}=\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{t+1}, \ldots, x_{n}\right)$, where $t \geq 2$ and $n \geq 4$. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \frac{n+2}{2}
$$

Proof. Case $t=2, n=4$. Applying Lemma 2.6 it is enough to consider the case $\sqrt{Q}=\left(x_{1}, x_{2}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{3}, x_{4}\right)$. We have

$$
\begin{aligned}
\sqrt{Q} \cap \sqrt{Q^{\prime}}=x_{1} x_{3} K\left[x_{1}, x_{3}, x_{4}\right] \oplus & x_{1} x_{4} K\left[x_{1}, x_{2}, x_{4}\right] \oplus x_{2} x_{3} K\left[x_{1}, x_{2}, x_{3}\right] \\
& \oplus x_{2} x_{4} K\left[x_{2}, x_{3}, x_{4}\right] \oplus x_{1} x_{2} x_{3} x_{4} K\left[x_{1}, x_{2}, x_{3}, x_{4}\right] .
\end{aligned}
$$

The above Stanley decomposition shows that $\operatorname{sdepth}\left(\sqrt{Q} \cap \sqrt{Q^{\prime}}\right)=3$ because $\sqrt{Q} \cap \sqrt{Q^{\prime}}$ is not principal. Then by Corollary $2.2 \operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq 3$, which is enough. For the remaining cases we proceed as follows:

Note that $\sqrt{Q} \cap \sqrt{Q^{\prime}}$ is a squarefree monomial ideal generated in momomials of degree 2. Let $k:=\operatorname{sdepth}\left(\sqrt{Q} \cap \sqrt{Q^{\prime}}\right)$. The poset $P_{\sqrt{Q} \cap \sqrt{Q^{\prime}}}$ has a partition $\mathcal{P}$ : $P_{\sqrt{Q} \cap \sqrt{Q^{\prime}}}=\bigcup_{i=1}^{s}\left[C_{i}, D_{i}\right]$, satisfying $\operatorname{sdepth}(\mathcal{D}(\mathcal{P}))=k$ where $\mathcal{D}(\mathcal{P})$ is the Stanley decomposition of $\sqrt{Q} \cap \sqrt{Q^{\prime}}$ with respect to partition $\mathcal{P}$. For each interval [ $\left.C_{i}, D_{i}\right]$ in $\mathcal{P}$ with $\left|C_{i}\right|=2$ we have $\left|D_{i}\right| \geq k$. There are $\left|D_{i}\right|-\left|C_{i}\right|$ subsets of cardinality 3 in this interval. These intervals are disjoint.
Case $t=2, n \geq 5$
Since the number of subsets of cardinality 3 from that intervals $\left[C_{i}, D_{i}\right]$ with $\left|C_{i}\right|=2$ is at least

$$
a:=\left[\binom{n}{2}-\binom{n-2}{2}-1\right](k-2)
$$

it follows that

$$
a \leq\binom{ n}{3}-\binom{n-2}{3}
$$

We get

$$
k \leq \frac{\binom{n}{3}-\binom{n-2}{3}}{\binom{n}{2}-\binom{n-2}{2}-1}+2=\frac{n+2}{2}
$$

Thus we have

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \operatorname{sdepth}\left(\sqrt{Q} \cap \sqrt{Q^{\prime}}\right) \leq \frac{n+2}{2}
$$

Case $t \geq 3, n \geq 6$.
Now the number of subsets of cardinality 3 from the intervals $\left[C_{i}, D_{i}\right]$ with $\left|C_{i}\right|=2$ is at least

$$
b:=\left[\binom{n}{2}-\binom{t}{2}-\binom{n-t}{2}\right](k-2)
$$

and it follows

$$
b \leq\binom{ n}{3}-\binom{t}{3}-\binom{n-t}{3}
$$

We get

$$
k \leq \frac{\binom{n}{3}-\binom{t}{3}-\binom{n-t}{3}}{\binom{n}{2}-\binom{t}{2}-\binom{n-t}{2}}+2=\frac{n+2}{2}
$$

Thus we have

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \operatorname{sdepth}\left(\sqrt{Q} \cap \sqrt{Q^{\prime}}\right) \leq \frac{n+2}{2}
$$

Corollary 2.9. Let $Q$ and $Q^{\prime}$ be two irreducible monomial ideals such that $\sqrt{Q}=$ $\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{t+1}, \ldots, x_{n}\right)$. Suppose that $n$ is odd. Then $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=$ $\left\lceil\frac{n}{2}\right\rceil$.

Proof. Using [14, Lemma 4.1], we have, $\frac{n}{2} \leq \operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \frac{n}{2}+1$ and so we get $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=\left\lceil\frac{n}{2}\right\rceil$.

Corollary 2.10. Let $Q$ and $Q^{\prime}$ be two irreducible monomial ideals such that $\sqrt{Q}=$ $\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{t+1}, \ldots, x_{n}\right)$. Suppose that $n$ is even. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)= \begin{cases}\frac{n}{2}+1, & \text { if } t \text { is odd } \\ \frac{n}{2} \text { or } \frac{n}{2}+1, & \text { if } t \text { is even }\end{cases}
$$

Proof. Using again [14, Lemma 4.1], we have, $\left\lceil\frac{t}{2}\right\rceil+\left\lceil\frac{n-t}{2}\right\rceil \leq \operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \frac{n}{2}+1$. Case $t$ is odd,
If $t$ is odd then $\left\lceil\frac{t}{2}\right\rceil+\left\lceil\frac{n-t}{2}\right\rceil=\frac{n}{2}+1$ and so we get $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=\frac{n}{2}+1$ in this case.
Case $t$ is even,
If $t$ is even then $\left\lceil\frac{t}{2}\right\rceil+\left\lceil\frac{n-t}{2}\right\rceil=\frac{n}{2}$ and we get $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=\frac{n}{2}$ or $\frac{n}{2}+1$.
We need next the following elementary lemma:
Lemma 2.11. Let I be a monomial ideal of $S$, and let $I^{\prime}=\left(I, x_{n+1}\right)$ be the monomial ideal of $S^{\prime}=S\left[x_{n+1}\right]$. Then

$$
\operatorname{sdepth}_{S}(I) \leq \operatorname{sdepth}_{S^{\prime}}\left(I^{\prime}\right) \leq \operatorname{sdepth}_{S}(I)+1
$$

Proof. Since $I^{\prime}=\left(I, x_{n+1}\right)$, so $I^{\prime} \cap S=I$. Now let $\mathcal{D}: I^{\prime}=\bigoplus_{i=1}^{r} u_{i} K\left[Z_{i}\right]$ be the Stanley decomposition of $I^{\prime}$, with $\operatorname{sdepth}(\mathcal{D})=\operatorname{sdepth}_{S^{\prime}}\left(I^{\prime}\right)$. Since

$$
\begin{aligned}
I=I^{\prime} \cap S & =\bigoplus_{i=1}^{r} u_{i} K\left[Z_{i}\right] \cap S \\
& =\bigoplus_{x_{n+1} \notin \operatorname{supp}\left(u_{i}\right)} u_{i} K\left[Z_{i} \backslash\left\{x_{n+1}\right\}\right]
\end{aligned}
$$

we conclude that

$$
\operatorname{sdepth}_{S^{\prime}}\left(I^{\prime}\right) \leq \operatorname{sdepth}_{S}(I)+1
$$

For the other inequality note that a Stanley decomposition $\mathcal{D}: I=\bigoplus_{i} u_{i} K\left[Z_{i}\right]$ with sdepth $\mathcal{D}=\operatorname{sdepth}_{S}(I)$ induces a Stanley decomposition

$$
\mathcal{D}^{\prime}: I^{\prime}=\bigoplus_{i} u_{i} K\left[Z_{i}\right] \bigoplus x_{n+1} S^{\prime}
$$

with sdepth $\mathcal{D}^{\prime}=\operatorname{sdepth}(I)$.

Example 2.12. If $I$ is any monomial complete intersection ideal of $S$ with $|G(I)|=$ $m$ then $\operatorname{sdepth}_{S}(I)=n-\left\lfloor\frac{m}{2}\right\rfloor$ by [18]. Note that $I^{\prime}=\left(I, x_{n+1}\right)$ is again a monomial complete intersection ideal of $S^{\prime}$, so we have $\operatorname{sdepth}_{S^{\prime}}\left(I^{\prime}\right)=n+1-\left\lfloor\frac{m+1}{2}\right\rfloor$. Now if $m$ is odd then $\operatorname{sdepth}_{S^{\prime}}\left(I^{\prime}\right)=\operatorname{sdepth}_{S}(I)$ and $\operatorname{sdepth}_{S^{\prime}}\left(I^{\prime}\right)=\operatorname{sdepth}_{S}(I)+1$ if $m$ is even.

Proposition 2.13. Let $Q$ and $Q^{\prime}$ be two primary ideals with $\sqrt{Q}=\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{r+1}, \ldots, x_{n}\right)$, where $1<r \leq t<n, n \geq 4$. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \frac{n+t-r+2}{2}
$$

Proof. Let $S^{\prime}=K\left[x_{1}, \ldots, x_{r}, x_{t+1}, \ldots, x_{n}\right]$ and $q=Q \cap S^{\prime}, q^{\prime}=Q^{\prime} \cap S^{\prime}$. Then

$$
\operatorname{sdepth}\left(q \cap q^{\prime}\right) \leq \frac{(n-t+r)+2}{2} \quad \text { by Theorem 2.8. }
$$

But $\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \operatorname{sdepth}\left(q \cap q^{\prime}\right)+t-r$ applying Lemma 2.11 by recurrence and the inequality follows.

The bound given by Proposition 2.13 sounds reasonable for $t=r$, otherwise seems to be too big as shows our Example 2.14 and Lemma 2.15.

Example 2.14. Let $\mathfrak{m}=\left(x_{1}, \ldots, x_{n}\right)$ be the maximal ideal of $S$. Then $\operatorname{sdepth}(\mathfrak{m})=$ $\left\lceil\frac{n}{2}\right\rceil$. Now let $S^{\prime}=S\left[x_{n+1}, \ldots, x_{n+r}\right]$ and $\mathfrak{m}^{\prime}=\left(\mathfrak{m}, x_{n+1}, \ldots, x_{n+r}\right)$. We see that $\operatorname{sdepth}_{S^{\prime}}\left(\mathfrak{m}^{\prime}\right)=\left\lceil\frac{n+r}{2}\right\rceil$, which is much smaller than $\operatorname{sdepth}_{S} \mathfrak{m}+r$.

Lemma 2.15. Let $Q$ and $Q^{\prime}$ be two primary monomial ideals with $\sqrt{Q}=\left(x_{1}, \ldots, x_{n-1}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{2}, \ldots, x_{n}\right)$. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq n-\left\lfloor\frac{n-1}{2}\right\rfloor
$$

Proof. Since $\sqrt{Q} \cap \sqrt{Q^{\prime}}=\left(x_{1} x_{n}, x_{2}, \ldots, x_{n-1}\right)$ is a complete intersection ideal we have $\operatorname{sdepth}\left(\sqrt{Q} \cap \sqrt{Q^{\prime}}\right)=n-\left\lfloor\frac{n-1}{2}\right\rfloor$ by [18, Theorem 2.4]. So by Corollary [2.2 we have that

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq n-\left\lfloor\frac{n-1}{2}\right\rfloor
$$

Another possible bound is given below.
Proposition 2.16. Let $Q$ and $Q^{\prime}$ be two monomial primary ideals with $\sqrt{Q}=$ $\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{r+1}, \ldots, x_{n}\right)$ where $1<r \leq t<n$. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \min \left\{n-\left\lfloor\frac{t}{2}\right\rfloor, n-\left\lfloor\frac{n-t}{2}\right\rfloor\right\}
$$

Proof. Let $I^{\prime}=\left(x_{1}, \ldots, x_{t}\right) \cap\left(x_{r}, \ldots, x_{n}\right)$ be an ideal of $S$. Let $\varphi: S \longrightarrow S^{\prime}$ where $S^{\prime}=K\left[x_{1}, \ldots, x_{n-1}\right]$ be the homomorphism given by $\varphi\left(x_{i}\right)=x_{i}$ for $i \leq n-1$ and $\varphi\left(x_{n}\right)=1$. We see that $I=\varphi\left(I^{\prime}\right)=\left(x_{1}, \ldots, x_{t}\right)$ where $I \subset S^{\prime}$. Then by [5, Lemma 2.2], we have

$$
\operatorname{sdepth}_{S}\left(I^{\prime}\right) \leq \operatorname{sdepth}_{S^{\prime}}(I)+1
$$

Since

$$
\operatorname{sdepth}_{S^{\prime}}(I)=(n-1)-t+\left\lceil\frac{t}{2}\right\rceil
$$

by [5, Theorem 1.3] we have

$$
\operatorname{sdepth}_{S}\left(I^{\prime}\right) \leq(n-1)-t+\left\lceil\frac{t}{2}\right\rceil+1=n-\left\lfloor\frac{t}{2}\right\rfloor
$$

By Corollary 2.2 we have

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \operatorname{sdepth}_{S}\left(I^{\prime}\right) \leq n-\left\lfloor\frac{t}{2}\right\rfloor
$$

and similarly

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq n-\left\lfloor\frac{n-t}{2}\right\rfloor
$$

Example 2.17. Let $I=\left(x_{1}, \ldots, x_{6}\right) \cap\left(x_{3}, \ldots, x_{8}\right), n=8$. Then by Proposition 2.16, we have

$$
\operatorname{sdepth}(I) \leq 8-\left\lfloor\frac{6}{2}\right\rfloor=5
$$

But using Proposition 2.13, we have sdepth $(I) \leq 7$, which shows that Proposition 2.16] gives better bound than Proposition [2.13] in this case.

Example 2.18. Let $I=\left(x_{1}, \ldots, x_{5}\right) \cap\left(x_{5}, \ldots, x_{9}\right), n=9$. Then by Proposition 2.16, we have $\operatorname{sdepth}(I) \leq 7$. But using Proposition [2.13, we have $\operatorname{sdepth}(I) \leq 6$.

These examples show somehow that the upper bound given by Proposition 2.13 is good if less number of variables are in common. The upper bound of Proposition 2.16 seems to be better if we have large number of variables in common.

Theorem 2.19. Let $Q$ and $Q^{\prime}$ be two monomial primary ideals with $\sqrt{Q}=\left(x_{1}, \ldots, x_{t}\right)$ and $\sqrt{Q^{\prime}}=\left(x_{r+1}, \ldots, x_{p}\right)$, where $1<r \leq t<p \leq n, n \geq 4$. Then

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right) \leq \min \left\{\frac{2 n+t-p-r+2}{2}, n-\left\lfloor\frac{t}{2}\right\rfloor, n-\left\lfloor\frac{p-t}{2}\right\rfloor\right\}
$$

The inequality becomes equality if $t=r, n$ is odd and $Q, Q^{\prime}$ are irreducible.
Proof. Let $S^{\prime}=K\left[x_{1}, \ldots, x_{p}\right]$ and $q=Q \cap S^{\prime}, q^{\prime}=Q^{\prime} \cap S^{\prime}$. Then

$$
\operatorname{sdepth}\left(q \cap q^{\prime}\right) \leq \frac{p-t+r+2}{2} \quad \text { by Proposition } 2.13
$$

and

$$
\operatorname{sdepth}\left(q \cap q^{\prime}\right) \leq \min \left\{p-\left\lfloor\frac{t}{2}\right\rfloor, p-\left\lfloor\frac{p-t}{2}\right\rfloor\right\} \quad \text { by Proposition 2.16. }
$$

Thus

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=\operatorname{sdepth}\left(q \cap q^{\prime}\right)+n-p \leq \frac{2 n+t-p-r+2}{2}
$$

and

$$
\operatorname{sdepth}\left(Q \cap Q^{\prime}\right)=\operatorname{sdepth}\left(q \cap q^{\prime}\right)+n-p \leq \min \left\{n-\left\lfloor\frac{t}{2}\right\rfloor, n-\left\lfloor\frac{p-t}{2}\right\rfloor\right\}
$$

by [9, Lemma 3.6]. For the second statement apply Corollary 2.9 and [9, Lemma 3.6].

Example 2.20. Let $I=\left(x_{1}^{a_{1}}, x_{2}^{a_{2}}\right) \cap\left(x_{2}^{a_{2}}, x_{3}^{a_{3}}, \ldots, x_{n}^{a_{n}}\right)$, where $n=2 k$ and $k \geq 2$. Then $\operatorname{sdepth}(I)=k+1$. Indeed, from Proposition 2.13 we have

$$
\operatorname{sdepth}(I) \leq\left\lfloor\frac{2 k+3}{2}\right\rfloor=k+1
$$

Also we know from [11] that

$$
\operatorname{sdepth}(I) \geq n-\left\lfloor\frac{|G(I)|}{2}\right\rfloor=2 k-\left\lfloor\frac{2 k-1}{2}\right\rfloor=2 k-k+1=k+1,
$$

which is enough.

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