UPPER BOUNDS ON THE SPLITTING OF THE EIGENVALUES

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ABSTRACT OF DISSERTATION

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The Graduate School
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A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Phuoc L. Ho
Lexington, Kentucky

Director: Dr. Peter Hislop, Professor of Mathematics
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We establish the upper bounds for the difference between the first two eigenvalues of the relative and absolute eigenvalue problems. Relative and absolute boundary conditions are generalization of Dirichlet and Neumann boundary conditions on functions to differential forms respectively. The domains are taken to be a family of symmetric regions in $\mathbb{R}^n$ consisting of two cavities joined by a straight thin tube. Our operators are Hodge Laplacian operators acting on k-forms given by the formula $\Delta^{(k)} = d\delta + \delta d$, where d and δ are the exterior derivatives and the codifferentials respectively. A result has been established on Dirichlet case (0-forms) by Brown, Hislop, and Martinez [2]. We use the same techniques to generalize the results on exponential decay of eigenforms, singular perturbation on domains [1], and matrix representation of the Hodge Laplacian restricted to a suitable subspace [2]. From matrix representation, we are able to find exponentially small upper bounds for the difference between the first two eigenvalues.

KEYWORDS: gap estimate, Hodge Laplacian, Sobolev space, deRham cohomology, relative eigenvalues

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Chapter 1 Introduction

The topic of this dissertation originated with the problem of estimating the difference between the first two eigenvalues of an elliptic operator on a bounded domain with Dirichlet boundary conditions. This difference is called the splitting of the first two eigenvalues. To be more specific, one can take the self-adjoint Schödinger operator $L = -\Delta + V$ on some smooth domain $\Omega \subset \mathbb{R}^n$ and search for lower and upper bounds of the splitting of the first two eigenvalues. For our work, we are only interested in the upper bounds. One can impose more conditions on the potential V and the domain Ω. For instant, let V be the double-well potential on $\Omega = \mathbb{R}^n$. Then the splitting is roughly less than $ce^{-\rho(x_a,x_b)}$, where ρ is the geodesic distance in the Agmon metric between the two nondegenerate minima located at x_a and x_b [11, Theorem 12.3]. The exponential factor in the upper bound of this splitting is the tunneling phenomenon originated from V [2]. There is another type of tunneling phenomenon that due to the geometry of Ω. That is, we set $V = 0$ and take $\Omega \subset \mathbb{R}^n$ to be a symmetric region consisting of two cavities connected by a straight thin tube of radius ε and length L. Then the splitting is less than $c\varepsilon^{n-2}e^{-\gamma(\varepsilon)L}$ for ε sufficiently small. Here $\gamma(\varepsilon) \approx \alpha/\varepsilon$, α^2 is the first Dirichlet eigenvalue on the unit ball in $\mathbb{R}^{n-1}$ [2]. Generally speaking, the straight thin tube plays the role of the potential V.

In this dissertation, we will generalize the latter result to the Hodge Laplacian operator acting on differential forms with relative boundary conditions. That is, we obtain similar upper bounds for the splitting of the first two relative eigenvalues. By duality of the Hodge star operator, we also obtain upper bounds for the splitting of the first two absolute eigenvalues. Relative and absolute boundary conditions are generalization of Dirichlet and Neumann conditions respectively. More specifically, let $M \subset \mathbb{R}^n$ be a compact symmetric region consisting of two cavities connected by a straight thin tube of radius ε and length L. Let $\Delta_M^{(k)}$ be the Hodge Laplacian acting on k-forms given by the formula $\Delta_M^{(k)} = d\delta + \delta d$, where d and δ are the exterior derivative and the codifferential respectively. For ω in the space of differential k-forms, define the relative eigenvalue problem [4]

$$\begin{cases} \Delta_M^{(k)}\omega = \lambda\omega & \text{on } M \\ j^*\omega = j^*\delta\omega = 0 & \text{on } \partial M \end{cases}$$

where j^* is the pullback induced by the inclusion map $j : \partial M \to M$. When ω is a 0-form, the Hodge Laplacian reduced to the usual Laplacian $-\Delta$ on functions. The relative boundary conditions $j^*\omega = j^*\delta\omega = 0$ reduced to the Dirichlet boundary condition. Similarly, the absolute boundary conditions $j^*i_\nu\omega = j^*i_\nu d\omega = 0$ reduced to Neumann boundary condition. Here ν is the inward unit normal field on ∂M, and i_ν is the interior multiplication. Our main objective is to prove the upper bounds of the splitting of relative eigenvalues, from which the Hodge duality gives the upper bounds on the splitting of absolute eigenvalues. These upper bounds constitute the main result of this dissertation.

We give a brief outline of the dissertation's content. In chapter 2, we give the necessary background material on tangent spaces, differential forms, and operators acting on differential forms. The domains will always be compact connected subsets in $\mathbb{R}^n$. Chapter 3 presents the Sobolev theory of differential forms. We will define Sobolev spaces of differential k-forms. The Sobolev spaces of 0-forms coincide with the Sobolev spaces of functions on M. We also state several important theorems that are necessary for our work such as Stokes' theorem, trace theorem, and Sobolev embedding theorem. Chapter 4, 5, and 6 are the main work of this dissertation. We first prove that eigenforms decay exponentially inside the tube. Using the stability of eigenvalues in Section 6.2, we calculate the matrix representation for the Hodge Laplacian restricted to a suitable two dimensional subspace. From there we estimate the upper bounds of the splitting of the first two relative eigenvalues. In Chapter 7, we give brief discussions on the boundary of the cavity, the generic cavities with simple multiplicity, and the first relative eigenvalue having multiplicity $m > 1$. Finally, the Appendix gives calculations and formulas that are needed in the main work.

Chapter 2 Background

In this chapter, we give a brief survey of differential forms on compact connected sets $M \subset \mathbb{R}^n$. The space of all differential forms of order k on M is denoted by $C^\infty\Omega^k(M)$. We will provide the definitions of operators on $C^\infty\Omega^k(M)$ that are necessary for our work in the later chapters. We use Morita [6] for our main reference. For a detail presentation of of differential forms on a manifold with boundary, see Schwarz [7].

2.1 Tangent vectors and vector fields

Let M be a compact connected set in $\mathbb{R}^n$. We define the tangent space at a point $p \in M$. A function $f : M \to \mathbb{R}$ is smooth if there exists a smooth function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that $f = \tilde{f}|_M$. Let $C^\infty(M)$ denote the space of all smooth function on M. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a tangent vector to M at p if

$$v(fg) = f(p)v(g) + v(f)g(p)$$

for all $f, g \in C^\infty(M)$. The tangent space T_pM of M at p is the vector space of all tangent vectors at p. The tangent space T_pM at $p \in M$ is an n-dimensional vector space [6, Theorem 1.33]. Let $\{\partial_{x_i}\}_{i=1}^n$ be the standard basis for T_pM. Hence, an arbitrary vector $X_p \in T_pM$ can be written uniquely as $X_p = a_1\partial_{x_1} + \cdots + a_n\partial_{x_n}$, where $X_p : C^\infty(M) \to \mathbb{R}$ is a map defined by $X_p(f) = a_1\partial_{x_1}f(p) + \cdots + a_n\partial_{x_n}f(p)$. The tangent bundle TM of M is defined as the union $\bigcup_{p\in M} T_pM$ of all tangent spaces.

Define a smooth vector field X on M to be a map $X : M \to TM$ such that $X(p) := X_p \in T_pM$ is smooth with respect to p. That is, we required the functions $a_i(p)$ to be smooth, where $X_p = \sum_{i=1}^n a_i(p)\partial_{x_i}$. Let $\Gamma(TM)$ denote the space of all smooth vector fields on M. A smooth vector field X acts on $f \in C^\infty(M)$ by putting $(Xf)(p) = X_p(f)$ for $p \in M$. So we get a function $Xf \in C^\infty(M)$. Define the bracket vector field $[X, Y]$ to be $[X, Y]f = X(Yf) - Y(Xf)$ for any two smooth vector fields X and Y. We use these definitions in Section 2.3.

2.2 Differential forms

Let $(\Lambda_n^*, +, \wedge)$ denote the algebra generated by $dx_1, ..., dx_n$ over $\mathbb{R}$ with unity 1 that satisfies $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for all i, j. Here $\wedge$ is the product of this algebra, and dx_i is the dual of ∂_{x_i} for each $i = 1, ..., n$. Let Λ_n^k be the linear vector space generated by the bases $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ of degree k in Λ_n^*, $1 \leq i_1 < \cdots < i_k \leq n$. A k-form on M is a linear combination

$$\omega = \sum_{i_1 < \cdots < i_i} f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \ ,$$

where $f_{i_1 \cdots i_k}$'s are functions on M. We denote the space of all k-forms on M by $\Omega^k(M)$. A k-form $\omega \in \Omega^k(M)$ is called smooth (differentiable) if $f_{i_1 \cdots i_k} \in C^\infty(M)$ for all indexes $i_1 \cdots i_k$. Let $C^\infty \Omega^k(M)$ denote the space of all smooth k-forms on M.

Each k-form ω is a multilinear alternating map $T_pM \times \cdots \times T_pM \to \mathbb{R}$ on the k-fold product of tangent spaces for $p \in M$. The map is defined on the basis elements by

$$dx_1 \wedge \cdots \wedge dx_k(X_1, ..., X_k) = \frac{1}{k} \det(dx_i(X_j)) \ ,$$

where $X_1, ..., X_k \in T_pM$ and $dx_i(\partial_{x_j}) = \delta_{ij}$. If ω is smooth, then putting all p together induces a multilinear alternating map $\Gamma(TM) \times \cdots \times \Gamma(TM) \to C^\infty(M)$ on the k-fold product of tangent bundles.

Now, let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$. For simplicity, we write $\omega = \sum_I f_I dx_I$ and $\eta = \sum_J g_J dx_J$ for some multi-index sets I and J with $|I| = k$ and $|J| = l$. The wedge product $\omega \wedge \eta \in \Omega^{k+l}(M)$ is defined by

$$\omega \wedge \eta = \sum_{I,J} f_I g_J dx_I \wedge dx_J \ . \tag{2.1}$$

Observe that $\omega \wedge \eta = 0$ if $k + l > n$.

2.3 Operators and maps on $\Omega^k(M)$

Let $\omega = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_I f_I dx_I$ be a k-form on M. The exterior derivative $d : C^\infty\Omega^k(M) \to C^\infty\Omega^{k+1}(M)$ is defined as

$$d\omega = \sum_I \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I \ . \tag{2.2}$$

The Hodge star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$, $0 \le k \le n$, is defined by

$$*\omega = \sum_{j_1<\cdots<j_{n-k}} \operatorname{sgn}(I, J) f_I dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}} \ , \tag{2.3}$$

where $J = \{j_1, ..., j_{n-k}\}$ is the complement of I in $\{1, ..., n\}$ and $\operatorname{sgn}(I, J)$ is the sign of the permutation $(i_1, ..., i_k, j_1, ..., j_{n-k})$. We define the codifferential $\delta : C^\infty\Omega^k(M) \to C^\infty\Omega^{k-1}(M)$ by the formula $\delta = (-1)^{nk+n+1} * d*$. With some calculation, one obtains an explicit formula for δ:

$$\delta\omega = \sum_I \sum_{s=1}^{k} (-1)^s \frac{\partial f_I}{\partial x_{i_s}} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_s}} \wedge \cdots \wedge dx_{i_k} \ , \tag{2.4}$$

where $\widehat{dx_{i_s}}$ indicates that the factor dx_{i_s} is deleted from the basis. We show $d^2 = 0$ and $\delta^2 = 0$. Computing $d^2\omega$,

$$dd\omega = \sum_I \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I \ .$$

Since $dx_i \wedge dx_j = -dx_j \wedge dx_i$, we have $dx_i \wedge dx_i = 0$. Hence

$$dd\omega = \sum_I \sum_{i<j} \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I + \sum_I \sum_{j<i} \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I \ .$$

The two sums on the right hand side are identical (since f_I is smooth) except $dx_j \wedge dx_i$ in the first and $dx_i \wedge dx_j$ in the second. So they cancel out. Next, using the definition $\delta = (-1)^{nk+k+1} * d*$ and the fact that $** = (-1)^{k(n-k)}$, we get $\delta^2 = (-1)^{nk+k+1} * d^2 *$. Thus $\delta^2 = 0$.

The interior product $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined by $i_X\omega(X_1, ..., X_{k-1}) = \omega(X, X_1, ..., X_{k-1})$, where $X, X_1, ..., X_{k-1}$ are vector fields on M.

A tangent vector $X_p \in T_p(\partial M)$ is a linear map $X_p : C^\infty(\partial M) \to \mathbb{R}$ satisfying $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(\partial M)$. Let $j : \partial M \to M$ be the inclusion map. Define the differential map $j_* : T_p(\partial M) \to T_pM$ by

$$j_* X_p(f) = X_p(f \circ j) \ ,$$

where $X_p \in T_p(\partial M)$ and $f \in C^\infty(M)$. Define the pullback map $j^* : C^\infty\Omega^k(M) \to C^\infty\Omega^k(\partial M)$ by

$$j^*\omega(X_1, ..., X_k) = \omega(j_*X_1, ..., j_*X_k)$$

for all k-forms $\omega \in C^\infty\Omega^k(M)$, $k > 0$, and $X_1, ..., X_k \in T_p(\partial M)$. Define $j^*\omega = \omega \circ j$ for $\omega \in C^\infty\Omega^0(M)$.

For example, let $T(1) = D \times [-L, L]$ be a tube in $\mathbb{R}^3$, where D is a unit disk. In cylindrical coordinates, let $\omega = f dt$ be a smooth 1-form on $T(1)$. Then $j^*\omega = (f \circ j)d(t \circ j)$ and $j^*\delta\omega = j^*(-\partial_t f) = -\partial_t f \circ j$, where $j : \partial T(1) \to T(1)$ is the inclusion map. If ω satisfies the relative boundary conditions $j^*\omega = j^*\delta\omega = 0$, then we have $f|_{\partial D\times[-L,L]} = 0$ and $\partial_t f|_{\partial T(1)} = 0$.

Finally, we would like to define the covariant derivative ∇_X on k-forms for $X \in \Gamma(TM)$. A connection on M is a map $\nabla_X : \Gamma(TM) \to \Gamma(TM)$ satisfying:

(i) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$ for $a, b \in \mathbb{R}$,

(ii) $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for $f, g \in C^\infty(M)$,

(iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$ for $f \in C^\infty(M)$.

Moreover, we define $\nabla_X f = Xf$ for $f \in C^\infty(M)$. We assume our connection satisfies $\nabla_X Y - \nabla_Y X = [X, Y]$ and $\nabla_X d(Y, Z) = d(\nabla_X Y, Z) + d(Y, \nabla_X Z)$, where d is some metric on M. For $X_i = \partial_i$, we define the Christoffel symbols Γ^k_{ij} associate with this connection by $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$. Observe that $\Gamma^k_{ij} = 0$ for the Euclidean metric $ds^2 = dx_1^2 + \cdots + dx_n^2$. We want to transfer $\nabla_{X_i} := \nabla_i$ to k-forms. We define $\nabla_i dx_j = -\sum_k \Gamma^j_{ik} dx_k$ for 1-form dx_j. We extend this definition to k-forms by requiring $\nabla_i(\omega \wedge \eta) = \nabla_i\omega \wedge \eta + \omega \wedge \nabla_i\eta$. For example, let $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ be a k-form. Then we have

$$\nabla_i \omega = \partial_i f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \sum_{s=1}^{k} f dx_{i_1} \wedge \cdots \wedge \nabla_i dx_{i_s} \wedge \cdots \wedge dx_{i_k} \ .$$

By linearity, we have defined $\nabla_X : C^\infty\Omega^k(M) \to C^\infty\Omega^k(M)$ for any arbitrary smooth k-form. The operator ∇_X is called the covariant derivative of differential forms on M. We use the covariant derivative to define Sobolev spaces in the next chapter.

2.4 Integration of n-forms

Let M be a compact connected subset in $\mathbb{R}^n$. Let $\omega \in \Omega^n(M)$ be an n-form. Then ω can be written as $\omega = f dx_1 \wedge \cdots \wedge dx_n = f\mu$; here μ is called the volume element on M. We define the integral of ω on M to be

$$\int_M \omega = \int_M f dV \ ,$$

where $dV = dx_1 \cdots dx_n$ is the standard Lebesgue measure. Note that the integral on the right hand side may not exist. For $\omega, \eta \in \Omega^k(M)$, we define the L^2-inner product

$$(\omega, \eta)_{L^2} = \int_M \omega \wedge *\eta \ , \tag{2.5}$$

so that the norm is

$$\|\omega\|_{L^2}^2 = \int_M \omega \wedge *\omega \ .$$

We show that the above L^2-inner product is is symmetric. Since the wedge product is linear, we may assume $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k} = f dx_I$. Then η must have the same basis as ω, otherwise $\omega \wedge *\eta = 0$ because the basis of $*\eta$ contains some factor dx_{i_l} belong to the basis of ω. That is, $\eta = g dx_I$. Hence

$$\omega \wedge *\eta = \mathrm{sgn}(I, J)\omega \wedge g dx_J = f g \mu \ ,$$

where $\mathrm{sgn}(I, J)$ is defined in Section 2.3. Similarly,

$$\eta \wedge *\omega = \mathrm{sgn}(I, J)\eta \wedge f dx_J = f g \mu \ .$$

Therefore, $(\omega,\eta)_{L^2} = (\eta,\omega)_{L^2}$. It also follows that $\|\omega\|^2_{L^2} = \int_M f^2\mu$.

We comment that this is a real inner product. One can define the complex inner product by integrating $\omega \wedge * \bar{\eta}$ over M, see Chapter 6. A k-form is said to be measurable if all its coefficients are measurable functions on M. We say a k-form ω is square integrable if it is measurable, and $\|\omega\|_{L^2}$ exists and finite. Denote $L^2\Omega^k(M)$ the real Hilbert space of all square integrable k-forms on M.

With this definition, we can define the pointwise inner product of k-forms on M. For $\omega, \eta \in L^2\Omega^k(M)$, define their pointwise inner product $\langle \omega, \eta \rangle$ to be the function on M so that

$$\langle \omega, \eta \rangle \mu := \omega \wedge * \eta \ . \tag{2.6}$$

The pointwise inner product $\langle \omega, \eta \rangle$ can be defined explicitly for two k-forms ω and η. However, it is enough for us to draw conclusions from this implicit definition. Since as computed above, $\langle f dx_I, g dx_I \rangle = f(x)g(x)$ for $x \in M$, and $\langle f dx_I, g dx_J \rangle = 0$ for $I \neq J$, where I and J are written in an increasing order of indexes and $|I| = |J|$. We can extend by linearity to get the pointwise inner product on arbitrary k-forms.

For example, let $\omega = a dx_1 \wedge dx_2 + b dx_2 \wedge dx_3$ and $\eta = c dx_2 \wedge dx_3 + e dx_1 \wedge dx_3$ be two forms on a compact set M in $\mathbb{R}^3$. Then $\langle \omega, \eta \rangle = bc$. Furthermore, the following properties hold for pointwise inner product:

(i) $\langle a\omega + b\omega', \eta \rangle = a\langle \omega, \eta \rangle + b\langle \omega', \eta \rangle$,

(ii) $\langle \omega, \eta \rangle = \langle \eta, \omega \rangle$.

Property (ii) follows from the symmetry of the L^2-inner product.

Chapter 3 Sobolev Theory

In this chapter, we give the definition of Sobolev spaces of k-forms on a compact connected subset in $\mathbb{R}^n$. We also give Stokes' and Green's Theorems for k-forms. We use Taylor [12] as our main reference.

3.1 Sobolev Spaces

Let M be a compact connected subset of $\mathbb{R}^n$. Recall that $L^2\Omega^k(M)$ is the space of square integrable k-forms on M. In general, we define the L^p-norm on $\Omega^k(M)$ for $p \in [1, \infty)$ by

$$\|\omega\|_{L^p}^p = \int_M |\omega|^p \mu \ ,$$

where $\omega \in \Omega^k(M)$ and $|\omega| = \langle \omega, \omega \rangle^{1/2}$ is the pointwise inner product defined in Section 2.4. Let $L^p\Omega^k(M)$ denote the space of all measurable k-forms ω such that $\|\omega\|_{L^p}$ exists and is finite.

Let us define the weak derivative. We say that a function $f \in L^p(M)$ has a weak derivative with respect to x_j if there exists $g \in L^p(M)$ such that

$$\int_M f \frac{\partial \phi}{\partial x_j} dV = -\int_M g\phi dV \ ,$$

where ϕ is any C^∞ function with compact support in the interior of M. Here g is called the weak L^p-derivative of f with respect to x_j, written $\partial_{x_j} f = g$. Similarly, we can define higher order of weak L^p-derivatives.

Now, let $X = \partial_{x_1} + \cdots + \partial_{x_n}$ be a smooth vector field. For a nonnegative integer m, define the Sobolev space $W^{m,p}\Omega^k(M)$ to be the space of all $\omega \in L^p\Omega^k(M)$ such that $\nabla_X^l \omega \in L^p\Omega^k(M)$ for all $l = 0, ..., m$. Here the derivatives are the covariant derivative defined in Section 2.3 and are taken in the sense of weak derivatives. The

Sobolev $W^{m,p}$-norm is defined as

$$\|\omega\|_{W^{m,p}}^p = \sum_{l=0}^{m} \|\nabla_X^l \omega\|_{L^p}^p \tag{3.1}$$

for all $\omega \in W^{m,p}\Omega^k(M)$. We write $H^m\Omega^k(M)$ for $W^{m,2}\Omega^k(M)$.

Remark. One can replace the covariant derivative ∇ by all differential operators P acting on forms of orders $\leq m$ with coefficients in $C^\infty(M)$. Also, one can replace the $W^{m,p}$-norm by any equivalent norms. For instant, one can show that the H^1-norm $\|\omega\|_{H^1}^2$ is equivalent to $\|d\omega\|_{L^2} + \|\delta\omega\|_{L^2} + \|\omega\|_{L^2}$.

We want to extend the operators in Section 2.3 to Sobolev spaces. Let $\omega = f dx_I \in L^p\Omega^k(M)$. Recall that $d\omega = \sum_{i=1}^n \partial_{x_i} f dx_i \wedge dx_I$. The exterior derivative d can be extended to Sobolev spaces by taking $\partial_{x_i} f$ to be the weak derivatives on M. The extension of d is also denoted by $d : W^{m,p}\Omega^k(M) \to W^{m-1,p}\Omega^{k+1}(M)$. Hence, we have the codifferential operator $\delta : W^{m,p}\Omega^k(M) \to W^{m-1,p}\Omega^{k-1}(M)$.

Next, we state a few basic results in the theory of differential forms. A point $p \in \partial M$ is called a corner if there is a neighborhood U of p in M and a diffeomorphism of U onto a neighborhood V of 0 in $K = \{x \in \mathbb{R}^n : x_j \geq 0,\ j = 1, ..., d\}$ for some $d \in \{1, ..., n\}$. For example, a closed rectangular box in $\mathbb{R}^3$ has boundary with corners. The generalized Stokes formula [12, Proposition 13.4], [7, Proposition 2.1.1]

Theorem 3.1.1 (Stokes' Theorem) *Let M be a compact connected subset in $\mathbb{R}^n$ with boundary ∂M (possibly with corners). Then*

$$\int_M d\omega = \int_{\partial M} j^*\omega$$

for all $\omega \in W^{1,1}\Omega^{n-1}(M)$ and j^* is defined in Section 2.3.

This theorem is a generalization of the classical Stokes' theorem to Sobolev spaces. We next state Holder Inequality and Green's formula.

Theorem 3.1.2 (Holder inequality) *Let $\omega \in L^p\Omega^k(M)$ and $\eta \in L^q\Omega^l(M)$. Then $\omega \wedge \eta \in L^1\Omega^{k+l}(M)$, and*

$$\|\omega \wedge \eta\|_{L^1} \leq \|\omega\|_{L^p}\|\eta\|_{L^q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $p > 1$.

Theorem 3.1.3 (Green's Formula) *Let $M \subset \mathbb{R}^n$ be a compact connected set. Let $\omega \in W^{1,p}\Omega^{k-1}(M)$ and $\eta \in W^{1,q}\Omega^k(M)$ be differential forms on M with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$(d\omega, \eta) = (\omega, \delta\eta) + \int_{\partial M} j^*\omega \wedge j^*(*\eta) \ .$$

Proof. Let $\chi := \omega \wedge *\eta$ be an $(n-1)$-form, where ω, η are as described in the theorem. Since $\eta \in W^{1,q}\Omega^k(M)$, we have $*\eta \in W^{1,q}\Omega^{n-k}(M)$. By Holder inequality, $\chi \in W^{1,1}\Omega^{n-1}(M)$. Further, $d\chi = d\omega \wedge *\eta + (-1)^{k-1}\omega \wedge d(*\eta) = d\omega \wedge *\eta - \omega \wedge *\delta\eta$. Applying Stokes' theorem, we have

$$\int_M d\omega \wedge *\eta - \int_M \omega \wedge \delta\eta = \int_{\partial M} j^*(\omega \wedge *\eta) \ .$$

Thus, the theorem follows. □

3.2 The Hodge Laplacian

We give a brief discussion on self-adjointness of the Hodge Laplacian. We define the Hodge Laplacian $\Delta_M^{(k)} : C^\infty\Omega^k(M) \to C^\infty\Omega^k(M)$ by $\Delta_M^{(k)} = d\delta + \delta d$. Extend $\Delta_M^{(k)}$ by weak derivatives to $\Delta_M^{(k)} : W^{m,p}\Omega^k(M) \to W^{m-2,p}\Omega^k(M)$. Let $D(\Delta_M^{(k)}) = \{\omega \in H^2\Omega^k(M) : j^*\omega = j^*\delta\omega = 0\}$ be the natural domain of $\Delta_M^{(k)}$. Since $C_0^\infty\Omega^k(M\backslash\partial M)$ is dense in $L^2\Omega^k(M)$ and $C_0^\infty\Omega^k(M\backslash\partial M) \subset D(\Delta_M^{(k)})$, $D(\Delta_M^{(k)})$ is dense in $L^2\Omega^k(M)$.

We show that $\Delta_M^{(k)}$ with domain $D(\Delta_M^{(k)})$ is a closed operator. Let $\omega_n \in D(\Delta_M^{(k)})$ such that ω_n converges in L^2-norm to $\omega \in L^2\Omega^k(M)$ and $\Delta_M^{(k)}\omega_n$ converges in L^2-norm to $\eta \in L^2\Omega^k(M)$. Since $H^1\Omega^k(M)$ is complete for all $0 \leq k \leq n$, it follows that ω_n

and $\delta\omega_n$ converges in H^1-norm to ω and $\delta\omega$ respectively, where $\delta\omega \in H^1\Omega^{k-1}(M)$. Moreover for $\alpha \in C_0^\infty\Omega^k(M\backslash\partial M)$, it follows that

$$(\omega, \Delta_M^{(k)}\alpha)_{L^2} = \lim_{n\to\infty}(\omega_n, \Delta_M^{(k)}\alpha)_{L^2} = \lim_{n\to\infty}(\Delta_M^{(k)}\omega_n, \alpha)_{L^2} = (\eta, \alpha)_{L^2} \ .$$

Hence, $\omega \in H^2\Omega^k(M)$ with weak derivative $\Delta_M^{(k)}\omega = \eta$. Next, observe that j^* maps $H^1\Omega^k(M)$ continuously to $L^2\Omega^k(\partial M)$, see discussion on the trace theorem [Proposition 3.3.1]. So

$$\|j^*\omega\|_{L^2\Omega^k(\partial M)} = \|j^*(\omega_n - \omega)\|_{L^2\Omega^k(\partial M)} \le C\|\omega_n - \omega\|_{H^1\Omega^k(M)} \to 0$$

as $n \to \infty$. Thus, $j^*\omega = 0$. Similarly, the map $j^*\delta : H^2\Omega^k(M) \to L^2\Omega^{k-1}(\partial M)$ is continuous. We have

$$\|j^*\delta\omega\|_{L^2\Omega^{k-1}(\partial M)} = \|j^*\delta(\omega_n - \omega)\|_{L^2\Omega^{k-1}(\partial M)} \le C\|\delta(\omega_n - \omega)\|_{H^1\Omega^{k-1}(M)} \to 0 \ .$$

Therefore, $j^*\delta\omega = 0$ and ω belongs to $D(\Delta_M^{(k)})$. We have shown that $\Delta_M^{(k)}$ is a closed densely defined operator on $D(\Delta_M^{(k)})$.

Now, let A be a closed densely defined operator. The spectrum $\sigma(A)$ of A is the set of all points $z \in \mathbb{C}$ such that $z - A$ does not have a bounded inverse. The resolvent set $\rho(A)$ of A is the set of all points $z \in \mathbb{C}$ such that $z - A$ is invertible. For $z \in \rho(A)$, the inverse of $z - A$ is called the resolvent of A at z; the resolvent of A is written as $R_A(z) = (z - A)^{-1}$.

We return to our discussion on the Hodge Laplacian. Let us define a bilinear form $\mathcal{D} : H^1\Omega^k(M) \times H^1\Omega^k(M) \to \mathbb{R}$,

$$\mathcal{D}(\omega, \eta) = (d\omega, d\eta) + (\delta\omega, \delta\eta).$$

Here $\mathcal{D}$ is called the Dirichlet integral.

Corollary 3.2.1 (Corollary to Green's formula) *For all $\omega \in H^2\Omega^k(M)$ and $\eta \in H^1\Omega^k(M)$,*

$$\mathcal{D}(\omega, \eta) = (\Delta_M^{(k)}\omega, \eta) + \int_{\partial M} j^*\eta \wedge j^*(*d\omega) - \int_{\partial M} j^*\delta\omega \wedge j^*(*\eta) \ .$$

Theorem 3.2.2 (Gaffney's inequality) *Let $M \subset \mathbb{R}^n$ be a compact connected set, and let $\omega \in H^1\Omega^k(M)$ with $j^*\omega = 0$. Then*

$$\|\omega\|_{H^1}^2 \le C(\mathcal{D}(\omega,\omega) + \|\omega\|_{L^2}^2)$$

for some finite constant $C > 0$.

By corollary to Green's formula, $\Delta_M^{(k)}$ is symmetric on $D(\Delta_M^{(k)})$; that is $(\Delta\omega, \eta) = (\omega, \Delta\eta)$ for all $\omega, \eta \in D(\Delta_M^{(k)})$. We state the basic criterion for self-adjointness [19, Theorem VIII.3]

Theorem 3.2.3 (Reed and Simon) *Let A be a symmetric operator with domain $D(A)$ on a Hilbert space $\mathcal{H}$. The following statements are equivalent*

(a) A is self-adjoint on $D(A)$.

(b) A is closed and $\ker(A^* \pm i) = \{0\}$.

(c) The range of $A \pm i$ on $D(A)$ is equal to $\mathcal{H}$.

We want to show that the range of $\Delta_M^{(k)} \pm i$ is equal to the complex Hilbert space $L^2\Omega^k(M)$. Observe that $(\Delta_M^{(k)}\omega, \omega) = \|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 \ge 0$ for all $\omega \in D(\Delta_M^{(k)})$. Hence $\Delta_M^{(k)}$ is a positive operator, and $\pm i$ cannot be an eigenvalue of $\Delta_M^{(k)}$. Otherwise we have $\omega \in L^2\Omega^k(M)$ so that $(\Delta_M^{(k)}\omega, \omega) = (\pm i\omega, \omega) = \pm i\|\omega\|_{L^2}^2$, which contradicts the positivity of $\Delta_M^{(k)}$. So $\pm i$ is in the resolvent set of $\Delta_M^{(k)}$ and $\Delta_M^{(k)} \pm i$ has bounded inverse. This implies the range of $\Delta_M^{(k)} \pm i$ is equal to $L^2\Omega^k(M)$. Therefore the Hodge Laplacian $\Delta_M^{(k)}$ with domain $D(\Delta_M^{(k)})$ is self-adjoint.

3.3 Fractional Sobolev Spaces

We need complex interpolation to define the fractional Sobolev spaces. Fractional Sobolev spaces are needed for the trace theorem and Sobolev embedding theorem. First, we recall the complex interpolation method. Let E and F be Banach spaces. Suppose they both continuously inject into V, a locally convex topological vector space. Let $G = \{e + f : e \in E, f \in F \text{ and } E, F \subset V\}$. The set G is a Banach space with norm

$$\|g\|_G = \inf\{\|e\|_E + \|f\|_F : g = e + f \in V,\ e \in E,\ f \in F\}.$$

Let $S = \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$ be a vertical strip in the complex plane. Define $\mathcal{H}_{E,F}(S)$ the set of all bounded continuous functions u in $\overline{S}$ with values in G and holomorphic in S such that $\|u(ib)\|_E$ and $\|u(1 + ib)\|_F$ are bounded for each $b \in \mathbb{R}$. For $\theta \in [0, 1]$, define the interpolation space $[E, F]_\theta$ by

$$[E, F]_\theta = \{u(\theta) : u \in \mathcal{H}_{E,F}(S)\}.$$

We now show how to use this technique to define fractional order Sobolev spaces of functions. Let M be a compact connected set in $\mathbb{R}^n$ with smooth boundary ∂M. We recall the definition of the 0-form Sobolev spaces for nonnegative integers m is

$$H^m(M) = \{u \in L^2(M) : D^\alpha u \in L^2(M)\}\ ,$$

for all $|\alpha| \leq m$. Here the covariant derivative ∇_i reduces to the gradient D_i. For any real number $s \geq 0$, define

$$H^s(M) = [L^2(M), H^m(M)]_\theta$$

where $m \geq s$ and $s = \theta m$. The definition is independent of the choice of m satisfying this condition [12, Chapter 4].

If $s = m$ is an integer, then we see that $H^s(M) = [L^2(M), H^m(M)]_1$. Let $u(1) \in H^s(M)$, then by definition, $\|u(1)\|_{H^m}$ is bounded. Thus $u(1) \in H^m(M)$. Now let $f \in H^m(M)$. We define $u(z) = af$, where $z = a + ib \in \overline{S}$. It follows that $u \in \mathcal{H}_{L^2,H^m}(S)$, and hence $u(1) = f \in H^s(M)$. Therefore, we have $H^s(M) = H^m(M)$ when s is a nonnegative integer.

Example. Let $I = [0, 1]$ be an closed interval in $\mathbb{R}$. We show that $H^1[0, 1]$ is a proper subspace of $H^{1/2}[0, 1] = [L^2[0, 1], H^1[0, 1]]_{1/2}$. Define $u : \overline{S} \to L^2[0, 1]$, $u(z) = x^a$ for

$z = a + ib \in \overline{S}$. We see that u is bounded continuous on $\overline{S}$ and holomorphic on S. Further, $\|u(ib)\|_{L^2} = \int_0^1 dx = 1$, and $\|u(1+ib)\|_{H^1} = \int_0^1 (x^2+1)dx = 4/3$. Hence $u \in \mathcal{H}_{L^2,H^1}(S)$, and $u(1/2) \in H^{1/2}[0,1]$. However $u(1/2) = x^{1/2}$ does not belong to $H^1[0,1]$.

Proposition 3.3.1 (Trace theorem for functions) *Let T be the trace map, that is, $Tu = u|_{\partial M}$. Then for $s > 1/2$, T extends uniquely to a continuous map $T : H^s(M) \to H^{s-1/2}(\partial M)$.*

We define the fractional Sobolev spaces of k-forms by

$$H^s\Omega^k(M) = [L^2\Omega^k(M), H^m\Omega^k(M)]_\theta \tag{3.2}$$

for any real $s \geq 0$, where $m \geq s$ and $s = \theta m$. It is not hard to show that (3.2) is equivalent to the definition $H^s\Omega^k(M) = \{\omega = \sum_I f_I dx_I \in L^2\Omega^k(M) : f_I \in H^s(M)\}$.

From the equivalent definition, the results on $H^s(M)$ can be translated to $H^s\Omega^k(M)$. We define the trace operator T on $H^s\Omega^k(M)$ for $s > 1/2$ as follows. Suppose $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in H^s\Omega^k(M)$. Then

$$T\omega = \omega|_{\partial M} = f|_{\partial M} dx_{i_1}|_{\partial M} \wedge \cdots \wedge dx_{i_k}|_{\partial M} \ .$$

We extend the definition to an arbitrary k-form ω by linearity. The space of all $\omega|_{\partial M}$ is denoted by $H^{s-1/2}\Omega^k(M)|_{\partial M}$. Hence, the trace theorem on functions can be generalized to forms.

Next, let $j : \partial M \to M$ be the inclusion map. Consider the pullback $j^* : C^\infty\Omega^k(M) \to C^\infty\Omega^k(\partial M)$. Extend j^* to be another version of the trace operator acting on $H^s\Omega^k(M)$ by

$$j^*\omega = f|_{\partial M} d(x_{i_1} \circ j) \wedge \cdots \wedge d(x_{i_k} \circ j) \ .$$

It follows that $j^*\omega \in H^{s-1/2}\Omega^k(\partial M)$ for $s > 1/2$. Note that $dx_i|_{\partial M}$ is a covector field on M taking values on the boundary ∂M, whereas $d(x_i \circ j)$ is a covector field on the boundary ∂M. Similarly, we can extend the Sobolev embedding theorem on functions [13, Proposition 6.4] to differential forms.

Theorem 3.3.2 (Sobolev embedding) *Let M be a compact connected subset in $\mathbb{R}^n$ (possibly with non-empty smooth boundary). The embedding*

$$H^s\Omega^k(M) \hookrightarrow L^{2n/(n-2s)}\Omega^k(M)$$

is continuous for all real $s \in [0, n/2)$.

Combining the Sobolev embedding and trace theorems, we obtain the boundary trace embedding theorem by replacing the trace T by j^*. See [18, Theorem 1.5.1.3] for the definition of fractional Sobolev spaces and the trace theorem on Lipschitz domains.

Theorem 3.3.3 (Boundary trace) *Let $M \subset \mathbb{R}^n$ be a compact region with piecewise smooth boundary ∂M. Then there is a continuous embedding $H^1\Omega^k(M) \hookrightarrow L^{\frac{2(n-1)}{(n-2)}}\Omega^k(\partial M)$.*

More generally, we have a continuous embedding $W^{1,p}\Omega^k(M) \hookrightarrow L^{\frac{(n-1)p}{n-p}}\Omega^k(\partial M)$ for $p \in [1, n)$ [17, Theorem 7.43]. However, we only need the trace embedding on $H^1\Omega^k(M)$.

Finally, we want to define Sobolev spaces of negative orders. For m a positive integer, let $H^m_0\Omega^k(M) = \{\omega \in H^1\Omega^k(M) : \omega|_{\partial M} = 0\}$. We define $H^{-m}\Omega^k(M)$ to be the dual of $H^m_0\Omega^k(M)$. That is, $H^{-m}\Omega^k(M)$ is the space of all continuous linear functionals on $H^m_0\Omega^k(M)$.

3.4 Regularity of eigenforms

In this section, we give a brief discussion on the regularity of eigenforms. We use Taylor [12] for our main reference. Let $M \subset \mathbb{R}^n$ be a compact set with smooth boundary ∂M. We have the following proposition [12, Proposition 9.7]

Proposition 3.4.1 *Let $\eta \in H^j\Omega^k(M)$ for some $j \geq 1$. If $\omega \in H^{j+1}\Omega^k(M)$ satisfies $\Delta_M^{(k)}\omega = \eta$ on M and $j^*\omega = j^*\delta\omega = 0$ on ∂M, then ω belongs to $H^{j+2}\Omega^k(M)$.*

We state a corollary that is needed for Section 4.3 in the next chapter.

Corollary 3.4.2 *The eigenforms of $\Delta_M^{(k)}$ belong to $C^\infty\Omega^k(M)$.*

Remark. If the boundary of M is not smooth, the eigenforms may have singular behavior near the irregular points of ∂M. However when M is a tube, we can separate variables [Chapter 4] to see that the eigenforms belong to $C^\infty\Omega^k(M)$.

Relative harmonic spaces

We define the relative harmonic space $\mathcal{H}_R^k(M)$ by

$$\mathcal{H}_R^k(M) = \{\omega \in H^1\Omega^k(M) : d\omega = \delta\omega = 0 \text{ and } j^*\omega = 0\} . \tag{3.3}$$

Since M is compact, the space $\mathcal{H}_R^k(M)$ is a finite dimensional subspace of $C^\infty\Omega^k(M)$ [7, Theorem 2.2.2].

We want to compute the relative harmonic spaces for some class of domains in $\mathbb{R}^n$. Let $B \in \mathbb{R}^n$ be an n-dimensional closed unit ball centered at the origin. In order to compute $\mathcal{H}_R^k(B)$, we relate it to cohomology spaces. Define the relative cohomology space $H^k(B, \partial B)$ of B to be the quotient of $\{\omega \in C^\infty\Omega^k(B) : d\omega = 0, j^*\omega = 0\}$ over $d\{\omega \in C^\infty\Omega^{k-1}(B) : j^*\omega = 0\}$, where $C^\infty\Omega^k(B)$ is the space of smooth k-forms on B. It follows that $\mathcal{H}_R^k(B)$ is isomorphic to $H^k(B, \partial B)$ [12, Proposition 9.9]. We state a proposition (see Taylor [12, Exer 4]).

Proposition 3.4.3 *Let $B \subset \mathbb{R}^n$ be an n-dimensional closed unit ball. Then*

$$\mathcal{H}_R^k(B) = \begin{cases} 0 & 0 \leq k \leq n-1 \\ \mathbb{R} & k = n \end{cases}$$

Since $\mathcal{H}_R^k(B) \cong H^k(B, \partial B)$, the proof of the proposition will follows if we know $H^k(B, \partial B)$. To compute $H^k(B, \partial B)$, one can use the proof of the Poincaré lemma to show directly that the deRham cohomology $H^k(B)$ is zero for $1 \leq k \leq n$. Here the deRham cohomology is defined as

$$H^k(B) = \frac{\ker[d : C^\infty\Omega^k(M) \to C^\infty\Omega^{k+1}(M)]}{\operatorname{im}[d : C^\infty\Omega^{k-1}(M) \to C^\infty\Omega^k(M)]} \,.$$

Furthermore, observe that the zero dimensional cohomology $H^0(B) = \mathbb{R}$ because M is connected. Hence, the proposition follows from the fact that $H^k(B) \cong H^{n-k}(B, \partial B)$.

Note that $\mathcal{H}_R^n(B)$ can be compute directly. We give an example for a 2-dimensional ball B^2 of radius 1. Let $\omega = f\, r d\theta \wedge dr$ be a two form in $H^1\Omega^2(B^2)$. Then $d\omega = 0$ and $j^*\omega = f(1, \theta) d(\theta \circ j) \wedge d(r \circ j)$. Since the boundary is given by $r = 1$, we have $d(r \circ j) = 0$. From definition 3.3, we need $\delta\omega = 0$. It follows form the definition of δ that $\delta\omega(-1)^{nk+n+1} * d * \omega = (-1)^{nk+n+1} * df = 0$, so we must have $df = 0$. It follows that $\partial_{x_i} f = 0$ for all $i = 1, ..., n$. Hence, f is a constant function on B^2. Since B^2 is connected, we have $\mathcal{H}_R^2(B^2) \cong \mathbb{R}$.

Now, let M be a compact set in $\mathbb{R}^n$ which has the same homotopy type as B. Then by homotopy invariance [6, Corollary 3.16], we have $H^k(M) \cong H^k(B)$. This implies $\mathcal{H}_R^k(M)$ is isomorphic to $\mathcal{H}_R^k(B)$. In Chapter 4, we choose the cavity $\mathcal{C}$ to be a compact set in $\mathbb{R}^n$ that has the same homotopy type as B. So there are no relative harmonic k-forms on $\mathcal{C}$ for all $k < n$. Thus, the first eigenvalue $\lambda_1^{(k)}(\mathcal{C})$ of the relative eigenvalue problem $\Delta_{\mathcal{C}}^{(k)}\omega = \lambda\omega$ on $\mathcal{C}$, $j^*\omega = j^*\delta\omega = 0$ on $\partial\mathcal{C}$ is positive for $k < n$.

Finally, we can pass all the results from relative harmonic space $\mathcal{H}_R^k(M)$ to the absolute harmonic space $\mathcal{H}_A^k(M)$. That is, define $\mathcal{H}_A^k(M)$ by

$$\mathcal{H}_A^k(M) = \{\omega \in H^1\Omega^k(M) : d\omega = \delta\omega = 0 \text{ and } j^* i_\nu \omega = 0\} \,. \tag{3.4}$$

By duality of the Hodge star operator and the fact that $\Delta_M^{(k)} * = * \Delta_M^{(n-k)}$, we have [12, Proposition 9.12]

Proposition 3.4.4 *If $M \subset \mathbb{R}^n$ is a compact set with nonempty interior and smooth boundary, then*

$$* : \mathcal{H}_R^k(M) \to \mathcal{H}_A^{n-k}(M)$$

is an isomorphism.

Chapter 4 Poincaré Inequality and Exponential Decay of Eigenforms

4.1 Introduction

We give the basic definitions and state our main results. Let M be a compact connected subset in $\mathbb{R}^n$, $n \geq 3$. Recall that the Hodge Laplacian on M is defined by $\Delta_M^{(k)} = (d\delta + \delta d)$, $k = 0, 1, ...n$. Here d and δ are the exterior derivative and the codifferential respectively. We refer the readers to Sections 2.3 and 3.1 for the definition of operators acting on k-forms. Consider the relative eigenvalue problem for k-forms

$$\begin{cases} \Delta_M^{(k)}\omega = \lambda\omega \\ j^*\omega = j^*\delta\omega = 0 \end{cases}$$

where $j : \partial M \hookrightarrow M$ is the inclusion map, and j^* is the pullback induced by j. When $k = 0$, the relative boundary conditions become $\omega|_{\partial M} = 0$, ω a function on M. Thus for $k = 0$, the above relative eigenvalue problem reduced to a Dirichlet boundary problem. Similarly, the absolute eigenvalue problem of k-forms defined as follows. Let ν be the inward unit normal vector at each point on the boundary ∂M. Define

$$\begin{cases} \Delta_M^{(k)}\omega = \mu\omega \\ j^*i_\nu\omega = j^*i_\nu d\omega = 0 \end{cases}$$

where i_ν is the interior product acting on k-forms. When $k = 0$, the absolute eigenvalue problem reduced to a Neumann boundary problem. Henceforth, relative and absolute eigenvalue problems are generalization of Dirichlet and Neumann boundary problems respectively.

To state our main theorem, we need to define a family of domains. First, let $\mathcal{C}$ be a compact region in $\mathbb{R}^n$ with nonempty interior and smooth boundary $\partial\mathcal{C}$. We call such region $\mathcal{C}$ a cavity. We make the following assumptions on the cavity $\mathcal{C}$:

Assumption 1. $\mathcal{C}$ is homotopy equivalent to a closed ball.

Assumption 2. The relative eigenvalue $\lambda_1^{(k)}(\mathcal{C})$ is nondegenerate.

Assumption 1 gives the following implications. As mentioned in Section 3.4, the first relative eigenvalue $\lambda_1^{(k)}(\mathcal{C})$ on $\mathcal{C}$ is positive for $k < n$ (by homotopy invariance and Proposition 3.4.3). Next, $\mathcal{C}$ is simply-connected. Topologically, the cavity $\mathcal{C}$ has no 'hole' in it. Hence, we may assume $\mathcal{C}$ to be a compact simply-connected region in $\mathbb{R}^n$ with nonempty interior.

Assumption 2 is needed in order to obtain a 2×2 matrix representation of the Hodge Laplacian restricted to a suitable 2-dimensional basis in Section 5.2. To gain some insight of such a cavity, we take $\mathcal{C}$ to be a cube (non-smooth boundary) in $\mathbb{R}^3$. Then the first relative eigenvalue has multiplicity 3. If $\mathcal{C}$ is a rectangular box with square base in $\mathbb{R}^3$, then the first relative eigenvalue has multiplicity 2. If $\mathcal{C}$ is box with all sides not equal, then the first relative eigenvalue has multiplicity 1; in this case, the eigenvalue is said to be simple or nondegenerate. This can be generalize to n-dimensional rectangular box. Similarly, if we take $\mathcal{C}$ to be a ball in $\mathbb{R}^3$, we will see that the multiplicity of the first eigenvalue is at least 2. We observed that the multiplicity of the first eigenvalue depends on the symmetry of $\mathcal{C}$; and a cavity that satisfies Assumption 2 (for $k > 0$) cannot be 'too' symmetric. See Section 7.2 for calculation and further discussion.

Returning to our domains, let $R : \mathbb{R}^n \to \mathbb{R}^n$, $R(x,t) = (x,-t)$ for all $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, be the reflection operator. We choose coordinates so $(0,-L/2) \in \partial\mathcal{C}$ such that $\mathcal{C} \cap (\mathbb{R}^{n-1} \times [0,\infty)) \subset \partial\mathcal{C}$. This can be done by rotating the cavity $\mathcal{C}$. Define $\tilde{T}(\varepsilon) := B^{n-1}(0,\varepsilon) \times [-L/2 - a, L/2]$ for some small $a > 0$, where a is chosen so that the line segments $\{x'\} \times [-a, L]$ intersect $\partial\mathcal{C}$ exactly once for all $x' \in B^{n-1}$. Let $\hat{T}(\varepsilon) = \overline{\mathcal{C}^c} \cap \tilde{T}(\varepsilon)$, and let $M_1(\varepsilon)$ be the union $\mathcal{C} \cup \hat{T}(\varepsilon)$. Define $M(\varepsilon) := M_1(\varepsilon) \cup RM_1(\varepsilon)$. By construction, $M(\varepsilon)$ is a region that consists of two cavities joined by a straight thin tube centered on the t-axis with length $L + 2a$ $(L > 0)$ and cross sectional diameter 2ε satisfying $RM(\varepsilon) = M(\varepsilon)$. See Figure 4.1.

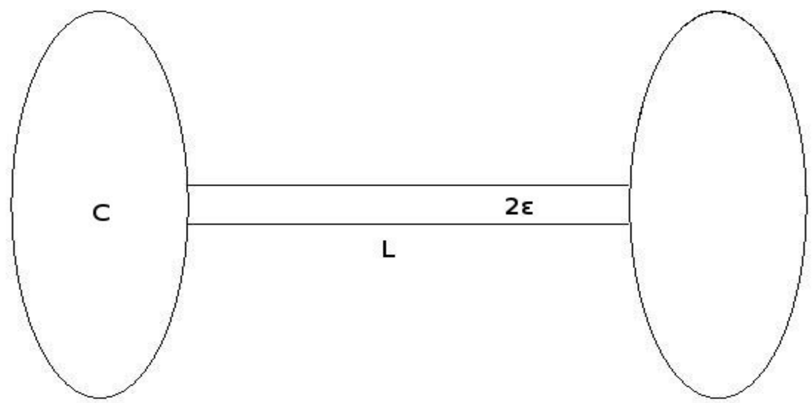

Figure 4.1: 2D cross-section of $M(\varepsilon)$ along t-axis.

We now discuss the splitting of the relative eigenvalues. Let ω be an eigenform corresponding to the simple relative eigenvalue $\lambda_1^{(k)}(\mathcal{C})$. Then $\omega \circ R$ is an eigenform on $R\mathcal{C}$ with the same relative eigenvalue $\lambda_1^{(k)}(\mathcal{C})$. Furthermore, ω and $\omega \circ R$ are linearly independent. Hence, the first relative eigenvalue on $\mathcal{C} \cup R\mathcal{C}$ is doubly degenerate. If we attach a thin tube between $\mathcal{C}$ and $R\mathcal{C}$, the first eigenvalue may split. We state the upper bounds of this splitting as our main theorem. Let $\lambda_1^{(k)}(M(\varepsilon))$ and $\lambda_2^{(k)}(M(\varepsilon))$ denote the first and second relative eigenvalues on $M(\varepsilon)$ respectively. For ε sufficiently small, Corollary 6.2.4 implies $\lambda_1^{(k)}(M(\varepsilon))$ is positive for $k < n$. Similarly, $\mu_1^{(k)}(M(\varepsilon))$ and $\mu_2^{(k)}(M(\varepsilon))$ denote the first and second absolute eigenvalues on $M(\varepsilon)$ respectively.

Theorem 4.1.1 *Let $M(\varepsilon)$ be a symmetric region as described above with the Assumptions 1 and 2. Then for $k \neq n-1, n$, and for all ε sufficiently small and any $d \in (0,1)$, there exists a constant $c > 0$ depending only on d and n such that*

$$0 \leq \lambda_2^{(k)}(M(\varepsilon)) - \lambda_1^{(k)}(M(\varepsilon)) \leq c\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon}.$$

Hodge star duality gives an immediate corollary.

Corollary 4.1.2 *Let $M(\varepsilon)$ be as described in Theorem 4.1.1. Then for $k \neq 0, 1$, and for all ε sufficiently small and any $d \in (0,1)$, there exists a constant $c > 0$ depending only on d and n such that*

$$0 \leq \mu_2^{(k)}(M(\varepsilon)) - \mu_1^{(k)}(M(\varepsilon)) \leq c\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon}.$$

We will restate these results in Section 5.3 (see Theorem 5.3.1). In Section 5.4, we sharpen these results with a prefactor of ε^{n-2}. So we have the same upper bounds as in Brown-Hislop-Martinez [2] for 0-forms.

We study some basic facts about the relative k-eigenvalues on a manifold M with boundaries. Denote $\lambda_1^{(k)}(M)$ the first positive eigenvalue of $\Delta_M^{(k)}$ with relative boundary conditions on M. We then have the min-max principle [3]

$$\lambda_1^{(k)}(M) = \inf\{\mathcal{R}(\omega) : \omega \neq 0, j^*\omega = 0, \omega \in \mathcal{H}_R^k(M)^{\perp}\} \tag{4.1}$$

where

$$\mathcal{R}(\omega) = \frac{\int_M |d\omega|^2 + |\delta\omega|^2}{\int_M |\omega|^2}$$

is the Rayleigh quotient, and $\mathcal{H}_R^k(M)$ is the space of relative harmonic k-forms on M [see definition (3.3)].

We state a couple useful results on the lower bound of $\lambda_1^{(k)}(M)$. A subset $M \subset \mathbb{R}^n$ is convex if for all $x, y \in M$, the line segment from x to y is contained in M. Let us drop the 'M' in our notation and write $\lambda_1^{(k)}$ and $\mu_1^{(k)}$ for the first relative and absolute eigenvalues on M respectively. A special case of Guerini-Savo result [4, Theorem 2.6]

Theorem 4.1.3 (Guerini-Savo [4]) *For M a convex compact set homotopy equivalent to a closed unit ball in $\mathbb{R}^n$, the sequence $\{\mu_1^{(k)}\}_{k=1}^n$ is nondecreasing with respect to the degree k:*

$$0 < \mu_2^{(0)} = \mu_1^{(1)} \leq \mu_1^{(2)} \leq \cdots \leq \mu_1^{(n)}.$$

By Proposition 3.4.4, 3.4.3, and homotopy invariance, we have $\mu_1^{(k)} > 0$ for all $k \geq 1$ and $\mu_1^{(0)} = 0$. The proof of Theorem 4.1.3 (without the homotopy assumption) can be found in [4, Theorem 2.6]. From this theorem, we see that a lower bound for $\mu_2^{(0)}$ will be a lower bound for $\mu_1^{(k)}$, $k \geq 1$.

For a lower bound of $\mu_2^{(0)}$, we state a special case of the Payne-Weinberger inequality [14, Equation 4.12] on convex domains. Let $\mathbb{R}_i^{n-1}$ be the coordinate plane $\mathbb{R}_1 \times \cdots \times \mathbb{R}_{i-1} \times \mathbb{R}_{i+1} \times \cdots \times \mathbb{R}_n$, where $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ is a point in $\mathbb{R}_i^{n-1}$ and $1 \leq i \leq n$. A region $M \subset \mathbb{R}^n$ is symmetric with respect to the coordinate plane $\mathbb{R}_i^{n-1}$ if $(x_1, ..., x_i, ..., x_n) \in M$ implies $(x_1, ..., -x_i, ..., x_n) \in M$. Since M is compact, the intersection of M with lines parallel to the x_i-axis is a set of line segments parallel to the x_i-axis. Let L_{x_i} be the maximum length of line segments in this set.

Theorem 4.1.4 (Payne-Weinberger [14]) *Let $M \subset \mathbb{R}^n$ be a symmetric region with respect to all n coordinate planes. Then*

$$\mu_2^{(0)} \geq \pi^2/L^2 \ ,$$

where $L = \sup\{L_{x_i}\}$.

Now, let $B^n := B^n(0, \varepsilon)$ be a ball in $\mathbb{R}^n$ centered at the origin with radius ε. Observe that B^n satisfies the assumptions in Theorem 4.1.3 and Theorem 4.1.4. Thus, we have $\mu_2^{(0)} \geq \pi^2/(2\varepsilon)^2$ and $\mu_1^{(k)}(B^n) \geq \pi^2/(2\varepsilon)^2$ for $k = 1, ...n$. Furthermore, observe that duality of the Hodge star operator implies $\mu_1^{(k)} = \lambda_1^{(n-k)}$. Hence, the reverse inequality holds for $\lambda_1^{(k)}$, $k = 0, ..., n-1$. That is,

$$\mu_2^{(0)} = \lambda_2^{(n)} = \lambda_1^{(n-1)} \leq \cdots \leq \lambda_1^{(0)}.$$

So $\lambda_1^{(k)}(B^n) \geq \pi^2/(2\varepsilon)^2$ for $k = 0, ..., n-1$. We state this result as a corollary.

Corollary 4.1.5 *Let B^n be a ball in $\mathbb{R}^n$ with radius ε, and let $\lambda_1^{(k)}(B^n)$ denote the first relative eigenvalue on B^n. Then*

$$\lambda_1^{(k)}(B^n) \geq \pi^2/(2\varepsilon)^2$$

for all $0 \leq k < n$.

The case of $k = n$ is excluded due to Proposition 3.4.3.

4.2 Poincaré inequality on Straight Tubes

In this section, we use the results in Section 4.1 to prove the following lemma. Let $B^{n-1}(0,\varepsilon) \subset \mathbb{R}^{n-1}$ be a closed ball centered at the origin with radius ε, and let $T(\varepsilon) = B^{n-1}(0,\varepsilon) \times [-L/2, L/2]$. For simplicity, we drop the volume form μ from the integrals [see Section 2.4].

Lemma 4.2.1 *Let $T(\varepsilon)$ be as described above. Then*

$$\int_{T(\varepsilon)} |\omega|^2 \leq \varepsilon^2 \int_{T(\varepsilon)} |d\omega|^2 + |\delta\omega|^2$$

for all $\omega \in H^1\Omega^k(M)$ satisfying the conditions $j^*\omega = 0$, $k < n-1$.

We observe that for $k < n-1$, the relative harmonic space $\mathcal{H}_R^k(B)$ is zero by Proposition 3.4.3. So if α_i is not identically zero, α_i is a test form because $j^*\omega = 0$ implies $j_B^*\alpha_i = 0$, $i = 1, 2$. Hence, α_i can be used in the Rayleigh quotient (4.1). We need some preliminary calculations before giving the proof of the lemma.

Consider an orthonormal coframe $\{f_1 d\theta_1, ..., f_{n-2} d\theta_{n-2}, dr, dt\}$ on $T(\varepsilon)$. Let $\omega = \alpha_1 + \alpha_2 \wedge dt$ be a k-form with $j^*\omega = 0$, where j^* is the pull-back induced by the inclusion map $j : \partial T(\varepsilon) \to T(\varepsilon)$. Define $Z = \partial B \times [-L/2, L/2]$, and let j_Z^* be the restriction of j^* to Z. Let f be a 0-form on $T(\varepsilon)$ with $j^*f = f \circ j = f|_{\partial T(\varepsilon)} = 0$. In particular, $j_Z^* f = 0$. It follows that $j_Z^*(\partial_t f) = 0$. To see this, let $p \in Z$, then

$$\frac{\partial f}{\partial t}(p) = \lim_{h \to 0} \frac{f(p + he_t) - f(p)}{h} = 0 \ ,$$

where e_t is the unit vector in cylindrical coordinates parallel to the t-axis. Similarly, we show that $j_Z^*(\partial_t \alpha_2) = 0$. Since $j^*\omega = 0$, we have $j^*\alpha_1 = 0$ and $j^*(\alpha_2 \wedge dt) = 0$. Observe that $j^*(\alpha_2 \wedge dt) = j^*\alpha_2 \wedge d(t \circ j) = 0$ and $d(t \circ j) \neq 0$ on Z. So $j^*\alpha_2 = 0$

on Z, or $j_Z^*\alpha_2 = 0$. Now, let $\alpha_2 = f dx_J$ be a $(k-1)$-form on B. If dr is part of the wedge product dx_J, then $j_Z^*(\partial_t \alpha_2) = 0$. Otherwise we have $f|_Z = 0$ and as above

$$j_Z^*\left(\frac{\partial \alpha_2}{\partial t}\right) = \left(\frac{\partial f}{\partial t} \circ j_Z\right) d(x_J \circ j_Z) = 0 \ .$$

By linearity, we conclude that

$$j_Z^*(\partial_t \alpha_2) = 0 \tag{4.2}$$

for any arbitrary $(k-1)$-form α_2.

Next, let d_B and δ_B be the exterior derivative and the codifferential on $B = B^{n-1}(0, \varepsilon)$ respectively. We have

$$d\omega = d_B\alpha_1 + d_B\alpha_2 \wedge dt + (-1)^k \frac{\partial \alpha_1}{\partial t} \wedge dt \ , \tag{4.3}$$

$$\delta\omega = \delta_B\alpha_1 + \delta_B\alpha_2 \wedge dt + (-1)^k \frac{\partial \alpha_2}{\partial t} \ . \tag{4.4}$$

We show that the pointwise inner product

$$\langle \alpha_1, \alpha_2 \wedge dt \rangle = 0 \ . \tag{4.5}$$

By definition, $\langle \alpha_1, \alpha_2 \wedge dt\rangle \mu = \alpha_1 \wedge *(\alpha_2 \wedge dt)$. To begin with, let $\alpha_1 = f dx_I$ and $\alpha_2 = g dx_J$. There must be a factor 'dx_i' in dx_I that does not belong to the basis dx_J, because α_1 and α_2 are k and $(k-1)$-form on B respectively. Since $dx_i \neq dt$, dx_i belongs to the basis $*(dx_J \wedge dt)$. So, we have $f dx_I \wedge *(g dx_J \wedge dt) = 0$. By linearity, we have $\alpha_1 \wedge *(\alpha_2 \wedge dt) = 0$ for arbitrary forms α_1 and α_2. More generally, we have

$$dx_I \wedge * dx_J = 0 \Longleftrightarrow I \neq J, \text{ and } dx_I \wedge * dx_I = \mu \tag{4.6}$$

for all indexes I and J (written in an increasing order).

Next, we compute $\{|d\omega|^2 + |\delta\omega|^2\}\mu$. We drop the volume element μ from our notation in the following computations,

$$|d\omega|^2 = |d_B\alpha_1|^2 + |d_B\alpha_2 \wedge dt|^2 + |\frac{\partial \alpha_1}{\partial t} \wedge dt|^2 + 2(-1)^k \langle \frac{\partial \alpha_1}{\partial t} \wedge dt, d_B\alpha_2 \wedge dt\rangle \ , \tag{4.7}$$

since $\langle d_B\alpha_1, d_B\alpha_2 \wedge dt\rangle = 0$ and $\langle d_B\alpha_1, \partial_t\alpha_1 \wedge dt\rangle = 0$ as in (4.5), and the symmetry of the pointwise inner product [Section 2.4]. Similarly, we get

$$|\delta\omega|^2 = |\delta_B\alpha_1|^2 + |\delta_B\alpha_2 \wedge dt|^2 + |\frac{\partial\alpha_2}{\partial t}|^2 + 2(-1)^k\langle\delta_B\alpha_1, \frac{\partial\alpha_2}{\partial t}\rangle \ . \tag{4.8}$$

We want to show that the integrals of the cross derivative terms in (4.7) and (4.8) cancel out. From (4.4), we see that $\delta\alpha_1 = \delta_B\alpha_1$. Integrating the cross derivative term on the right hand side of (4.8) without the constant $2(-1)^k$, we have

$$\int_{T(\varepsilon)} \langle\delta_B\alpha_1, \frac{\partial\alpha_2}{\partial t}\rangle = \int_{T(\varepsilon)} \langle\delta\alpha_1, \frac{\partial\alpha_2}{\partial t}\rangle = \int_{T(\varepsilon)} \langle\alpha_1, d(\frac{\partial\alpha_2}{\partial t})\rangle \ , \tag{4.9}$$

where we have applied Green's formula [Theorem 3.1.3] to the second equality. Note that the boundary term $\int_{\partial T(\varepsilon)} j^*(\partial_t\alpha_2) \wedge j^*(*\alpha_1)$ is zero because $j^*(*\alpha_1) = 0$ on $E = B \times \{-L/2, L/2\}$ (since $*\alpha_1$ contains dt) and $j^*(\partial_t\alpha_2) = 0$ on Z (4.2). Now, since d commutes with ∂_t [Appendix Equation 7.8],

$$\int_{T(\varepsilon)} \langle\alpha_1, d(\frac{\partial\alpha_2}{\partial t})\rangle = \int_{T(\varepsilon)} \langle\alpha_1, \frac{\partial}{\partial t}(d\alpha_2)\rangle = \int_{T(\varepsilon)} \langle\alpha_1, \frac{\partial}{\partial t}(d_B\alpha_2)\rangle \ , \tag{4.10}$$

The latter equality in (4.10) holds because α_1 has no factor dt in the basis and $d\alpha_2 = d_B\alpha_2 + (-1)^{k-1}\partial_t\alpha_2 \wedge dt$. We like to evaluate $\partial_t(d_B\alpha_2)$. Let $\omega = d_B\alpha_2 \wedge dt$ and substitute ω into (4.4), we get $\partial_t(d_B\alpha_2) = (-1)^{k+1}\delta(d_B\alpha_2 \wedge dt) + (-1)^k\delta_B d_B\alpha_2 \wedge dt$. Hence, by Green's formula

$$\int_{T(\varepsilon)} \langle\alpha_1, \frac{\partial}{\partial t}(d_B\alpha_2)\rangle = (-1)^{k+1}\int_{T(\varepsilon)} \langle\alpha_1, \delta(d_B\alpha_2 \wedge dt)\rangle = (-1)^{k+1}\int_{T(\varepsilon)} \langle d\alpha_1, d_B\alpha_2 \wedge dt\rangle \ , \tag{4.11}$$

where the boundary term $\int_{\partial T(\varepsilon)} j^*\alpha_1 \wedge j^*(*d_B\alpha_2 \wedge dt)$ is zero (since $j^*\alpha_1 = 0$ on $T(\varepsilon)$). It follows from (4.3) applied to α_1 that

$$(-1)^{k+1}\int_{T(\varepsilon)} \langle d\alpha_1, d_B\alpha_2 \wedge dt\rangle = -\int_{T(\varepsilon)} \langle\frac{\partial\alpha_1}{\partial t} \wedge dt, d_B\alpha_2 \wedge dt\rangle \ . \tag{4.12}$$

Combining (4.9), (4.10), (4.11), and (4.12), we have

$$\int_{T(\varepsilon)} \langle\delta_B\alpha_1, \frac{\partial\alpha_2}{\partial t}\rangle = -\int_{T(\varepsilon)} \langle\frac{\partial\alpha_1}{\partial t} \wedge dt, d_B\alpha_2 \wedge dt\rangle \ . \tag{4.13}$$

That is, the integrals of the cross derivative terms in (4.7) and (4.8) cancel out. Thus,

$$\int_{T(\varepsilon)} |d\omega|^2 + |\delta\omega|^2 = \int_{T(\varepsilon)} |d_B\alpha_1|^2 + |d_B\alpha_2 \wedge dt|^2 + |\frac{\partial\alpha_1}{\partial t} \wedge dt|^2$$

$$+ \int_{T(\varepsilon)} |\delta_B\alpha_1|^2 + |\delta_B\alpha_2 \wedge dt|^2 + |\frac{\partial\alpha_2}{\partial t}|^2 \ . \tag{4.14}$$

We now give the proof of Lemma 4.2.1.

Proof. Let ω be a test k-form on $T(\varepsilon)$, $j^*\omega = 0$. Rewrite ω as $\omega = \alpha_1 + \alpha_2 \wedge dt$, then α_1 and α_2 are forms on B, $j_Z^*\alpha_1 = j_Z^*\alpha_2 = 0$. Let $k < n-1$. Then by Proposition 3.4.3, α_1 and α_2 are not $\Delta_B^{(k)}$-harmonic and $\Delta_B^{(k-1)}$-harmonic respectively. We show that α_i is a test form on B, $i = 1, 2$. Since $j_Z^*\alpha_i = 0$, it follows that the $j_{\partial B\times\{t\}}^*\alpha_i = 0$ for any fixed $t \in [-L/2, L/2]$. Here $j_{\partial B\times\{t\}}^*$ is the restriction of j_Z^* to $\partial B \times \{t\}$. Since there is no relative harmonic k-form on B for $k < n-1$, α_i is orthogonal to the relative harmonic space. Hence, α_i is a test form on B. By Corollary 4.1.5 and the min-max principle (4.1), we have the Poincaré inequalities

$$0 < \int_B |\alpha_1|^2 \leq \varepsilon^2 \int_B |d_B\alpha_1|^2 + |\delta_B\alpha_1|^2 \ , \tag{4.15}$$

and

$$0 < \int_B |\alpha_2|^2 \leq \varepsilon^2 \int_B |d_B\alpha_2|^2 + |\delta_B\alpha_2|^2 \ . \tag{4.16}$$

From (4.6), we have $d\theta_I \wedge *d\theta_I = \mu$ and $(d\theta_I \wedge dt) \wedge *(d\theta_I \wedge dt) = \mu$ for all bases $d\theta_I$ on B^{n-1}. Hence, it follows that $\int_{T(\varepsilon)} |\alpha_2|^2 = \int_{T(\varepsilon)} |\alpha_2 \wedge dt|^2$. So inequality (4.16) extends to

$$0 < \int_B |\alpha_2 \wedge dt|^2 \leq \varepsilon^2 \int_B |d_B\alpha_2 \wedge dt|^2 + |\delta_B\alpha_2 \wedge dt|^2 \ . \tag{4.17}$$

Integrating (4.15) and (4.17) over $[-L/2, L/2]$ and using (4.14),

$$0 < \int_{T(\varepsilon)} |\omega|^2 \leq \varepsilon^2 \int_{T(\varepsilon)} |d_B\alpha_1|^2 + |\delta_B\alpha_1|^2 + |d_B\alpha_2 \wedge dt|^2 + |\delta_B\alpha_2 \wedge dt|^2$$

$$\leq \varepsilon^2 \int_{T(\varepsilon)} |d\omega|^2 + |\delta\omega|^2 \ .$$

Therefore, we have proved the Poincaré inequality on $T(\varepsilon)$ for $k < n-1$.

For ω an n-form or $(n-1)$-form, we can construct ω that does not verify the Poincaré inequality on $T(\varepsilon)$. The construction is similar to the example below. That is, $\omega = \sin(2\pi t/L)\mu_B$ for the case $k = n-1$ and $\omega = \cos(2\pi t/L)\mu_B \wedge dt$ for the case $k = n$; here μ_B is the volume $(n-1)$-form on B. $\square$

Next, observe that the lemma (for test forms) also follows from the min-max principle (4.1) if we can show that the first relative eigenvalue $\lambda_1^{(k)}(T(\varepsilon))$ on $T(\varepsilon)$ is greater or equal to $1/\varepsilon^2$. Before giving the proof, let us look at a concrete example.

Example. Let $T(\varepsilon) = B^2(0,\varepsilon) \times [-L/2, L/2] \subset \mathbb{R}^3$ be a 3-dimensional tube. We use cylindrical coordinates on $T(\varepsilon)$. Suppose $\omega \in H^2\Omega^1(T(\varepsilon))$ is an eigenform corresponding to the relative eigenvalue λ, and ω is of the form $f dt$. We show that $\lambda \geq 1/\varepsilon^2$. Applying (7.5) to ω, we get

$$\Delta^{(1)}_{T(\varepsilon)}\omega = \left\{ \Delta^{(0)}_{B^2(0,\varepsilon)} f - \frac{\partial^2 f}{\partial t^2} \right\} dt = \lambda f dt \tag{4.18}$$

where $\Delta^{(0)}_{B^2(0,\varepsilon)}$ is the Laplacian on $B^2(0,\varepsilon)$. Consider the relative boundary conditions $j^*\omega = j^*\delta\omega = 0$. The first condition $j^*\omega = 0$ implies $f|_{\partial B^2(0,\varepsilon)\times[-L/2,L/2]} = 0$. The second condition $j^*\delta\omega = j^*(-\partial_t f) = 0$ implies $\partial_t f|_{\partial T(\varepsilon)} = 0$. Let $f = G(r,\theta)T(t)$. Then (4.18) separates into two boundary problems:

$$\Delta^{(0)}_{B^2(0,\varepsilon)} G = \lambda' G, \ G|_{\partial B^2(0,\varepsilon)} = 0 \ , \tag{4.19}$$

and

$$\frac{\partial^2 T}{\partial t^2} + \lambda'' T = 0, \ \frac{\partial T}{\partial t}|_{\{t=-L/2,L/2\}} = 0 \ , \tag{4.20}$$

where $\lambda' + \lambda'' = \lambda$. We solve the former problem by separation of variables technique. Recall that

$$\Delta^{(0)}_{B^2(0,\varepsilon)} G = -\frac{\partial^2 G}{\partial r^2} - \frac{1}{r}\frac{\partial G}{\partial r} - \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} \ . \tag{4.21}$$

Let $G = \Theta R$. Substituting G into (4.21) gives

$$\Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2}\Theta'' R + \lambda' \Theta R = 0.$$

or

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\lambda' r^2=-\frac{\Theta''}{\Theta}=c.$$

Solving the equation $\Theta''+c\Theta=0$, we get $\Theta=a\sin(\sqrt{c}\theta)+b\cos(\sqrt{c}\theta)$. Since Θ must be a periodic function of period 2π (otherwise Θ is not single-valued), it follows that $\sqrt{c}$ must be a nonnegative integer n. Thus we obtain $\Theta=a\sin(n\theta)+b\cos(n\theta)$. For R, the differential equation reduces to

$$r^2R''+rR'+(\lambda' r^2-n^2)R=0.$$

We observe that the solution regular at the origin is a Bessel function of integral order

$$R=J_n(\sqrt{\lambda'}r)\ .$$

The boundary condition $R(\varepsilon)=0$ implies $J_n(\sqrt{\lambda'}\varepsilon)=0$. Let x_0 be the first positive zero of J_n. Then the first positive λ' is x_0^2/ε^2. It follows that $n=0$ gives the smallest such positive eigenvalue with numerical approximation $x_0>2$. To solve for λ'', we substitute T into (4.20). It follows that $\lambda''=(2n\pi)^2/L^2$ for nonnegative integer n. Thus, we have $\lambda\geq x_0^2/\varepsilon^2+\lambda''>1/\varepsilon^2$.

For $k=2$, let $\omega=\sin(2\pi t/L)\,rd\theta\wedge dr$. Then ω is a test form on $T(\varepsilon)$. Further,

$$\int_{T(\varepsilon)}|\omega|^2=|B^2|\int_{-L/2}^{L/2}\sin^2(2\pi t/L)dt=\frac{L|B^2|}{2}\ ,$$

and

$$\int_{T(\varepsilon)}|d\omega|^2+|\delta\omega|^2=\frac{4\pi^2|B^2|}{L^2}\int_{-L/2}^{L/2}\cos^2(2\pi t/L)dt=\frac{2\pi^2|B^2|}{L}\ ,$$

where $|B^2|$ is the volume of $B^2(0,\varepsilon)$. Hence, Lemma 4.2.1 does not hold. Similarly, we let $\omega=\cos(2\pi t/L)\,rd\theta\wedge dr\wedge dt$ for the case $k=3$. $\diamond$

We give a second proof of Lemma 4.2.1 that involves the relative boundary conditions. This proof mimics the example given above. We first show that the relative boundary conditions break into a set of boundary conditions. Then we use separation of variables technique to give a lower bound for $\lambda_1^{(k)}(T(\varepsilon))$.

Proof. Let $\omega = \alpha_1 + \alpha_2 \wedge dt$ be an eigenform of the relative eigenvalue problem with eigenvalue λ and degree $k < n-1$; where α_1, α_2 are k, $(k-1)$ forms on B respectively. Then the Hodge Laplaican $\Delta^{(k)}_{T(\varepsilon)}$ can be written in term of the Hodge Laplacian on B by the formula [Appendix equation (7.5)]

$$\Delta^{(k)}_{T(\varepsilon)}\omega = \Delta^{(k)}_{B}\alpha_1 - \frac{\partial^2\alpha_1}{\partial t^2} + \left(\Delta^{(k-1)}_{B}\alpha_2 - \frac{\partial^2\alpha_2}{\partial t^2}\right) \wedge dt \ .$$

We consider the boundary conditions $j^*\omega = j^*\delta\omega = 0$. Recall that $E = B \times \{-L/2, L/2\}$, $Z = \partial B \times [-L/2, L/2]$, and $\partial T(\varepsilon) = E \cup Z$. Denote j^*_E and j^*_Z be the restriction of j^* on E and Z respectively. Then as in the previous proof, $j^*(\alpha_1 + \alpha_2 \wedge dt) = 0$ implies $j^*\alpha_1 = 0$ and $j^*_Z\alpha_2 = 0$. Applying j^* to (4.4), we get $j^*(\delta\alpha_1 + (-1)^k\partial_t\alpha_2) = 0$ and $j^*(\delta_b\alpha_2 \wedge dt) = 0$ since the forms are independent. Furthermore, this implies $j^*_Z\delta_B\alpha_2 = 0$. The former equation implies $j^*_Z\delta\alpha_1 + (-1)^k j^*_Z(\partial_t\alpha_2) = 0$. Since $j^*_Z(\partial_t\alpha_2) = 0$ as in the preliminary calculation (4.2), we have $j^*_Z\delta\alpha_1 = 0$. Note that $\delta\alpha_1 = \delta_B\alpha_1$ by (4.4).

On E, we have $j^*_E(\delta_B\alpha_1 + (-1)^k\partial_t\alpha_2) = 0$. A similar argument as in the preliminary calculation (4.2) shows that $j^*_E\alpha_1 = 0$ implies $j^*_E\delta_B\alpha_1 = 0$. Hence, $j^*_E(\partial_t\alpha_2) = 0$. Thus we have the following boundary conditions: $j^*_Z\alpha_1 = j^*_Z\delta_B\alpha_1 = 0$, $j^*_Z\alpha_2 = j^*_Z\delta_B\alpha_2 = 0$, $j^*_E\alpha_1 = 0$, and $j^*_E(\partial_t\alpha_2) = 0$.

Since k-forms with dt and k-forms without dt are independent, the equation $\Delta^{(k)}_{T(\varepsilon)}\omega = \lambda\omega$ separates into two equations

$$\Delta^{(k)}_{B}\alpha_1 - \frac{\partial^2\alpha_1}{\partial t^2} = \lambda\alpha_1 \ , \tag{4.22}$$

$$\left(\Delta^{(k-1)}_{B}\alpha_2 - \frac{\partial^2\alpha_2}{\partial t^2}\right) \wedge dt = \lambda\alpha_2 \wedge dt \ , \tag{4.23}$$

where λ is the eigenvalue corresponding to ω. Equation (4.23) reduces to

$$\Delta^{(k-1)}_{B}\alpha_2 - \frac{\partial^2\alpha_2}{\partial t^2} = \lambda\alpha_2 \ , \tag{4.24}$$

with relative boundary conditions $j_Z^*\alpha_2 = j_Z^*\delta_B\alpha_2 = 0$ and $\partial_t\alpha_2|_{\{t=-L/2,L/2\}} = 0$. We solve (4.24) by separation of variables. Let $\alpha_2 = \sum_J G_J T_J dx_J$, where G_J and T_J are functions on B and $[-L/2, L/2]$ respectively. Equation (4.24) becomes

$$\sum_J \left\{ T_J \Delta_B^{(k-1)}(G_J dx_J) - \frac{\partial^2 T_J}{\partial t^2} G_J dx_J \right\} = \lambda \sum_J G_J T_J dx_J \ . \tag{4.25}$$

Since the bases dx_J's are independent, we have $T_J\Delta_B^{(k-1)}(G_J dx_J) - T_J'' G_J dx_J = \lambda G_J T_J dx_J$. Assume G_J and T_J are nonzero almost everywhere. Then dividing the latter equation by $G_J T_J$, we get $\Delta_B^{(k-1)}(G_J dx_J)/G_J - T_J''/T_J dx_J = \lambda dx_J$. Summing over J,

$$\sum_J \Delta_B^{(k-1)}(G_J dx_J)/G_J - \sum_J T_J''/T_J dx_J = \lambda \sum_J dx_J \ . \tag{4.26}$$

Equation (4.26) separates into two equations $\sum_J \Delta_B^{(k-1)}(G_J dx_J)/G_J = \lambda' \sum_J dx_J$ and $\sum_J T_J''/T_J dx_J + \lambda'' \sum_J dx_J = 0$, where $\lambda' + \lambda'' = \lambda$. Let $\alpha_2' = \sum_J G_J dx_J$ and $\alpha_2'' = \sum_J T_J dx_J$. Then the above two equations can be rewrite as $\Delta_B^{(k-1)}\alpha_2' = \lambda'\alpha_2'$ and $\partial_t^2\alpha_2'' = \lambda''\alpha_2''$. We show that the boundary condition $j_Z^*\alpha_2 = 0$ implies $j_B^*\alpha_2' = 0$ almost everywhere. Since the bases dx_J's are independent, we only need to show $j_B^*(G_J dx_J) = 0$. Assume dx_J does not contain dr, otherwise we're done. Then $j_Z^*(G_J T_J dx_J) = 0$ implies $(G_J T_J)|_Z = T_J(t) G_J|_{\partial B} = 0$ for all $t \in [-L/2, L/2]$. Since $T_J(t) \neq 0$ almost everywhere, we have $G_J|_{\partial B} = 0$ almost everywhere. Thus, $j_B^*\alpha_2' = 0$. Similarly, the boundary conditions $j_Z^*\delta_B\alpha_2 = 0$ implies $j_B^*\delta_B\alpha_2' = 0$, and the boundary condition $j_E^*(\partial_t\alpha_2) = 0$ implies $\partial_t\alpha_2''|_{\{t=-L/2,L/2\}} = 0$. By Corollary **??**, we have $\lambda' \geq \lambda_1^{(k-1)}(B) \geq \pi^2/(2\varepsilon)^2$. We can solve for λ'' explicitly, that is, $\lambda'' = (n\pi)^2/L^2$ for $n = 0, 1, ...$ Thus, $\lambda = \lambda' + \lambda'' \geq \pi^2/(2\varepsilon)^2$.

Similarly, we can break the (4.22) into two equations $\Delta_B^{(k)}\alpha_1' = \tilde{\lambda}'\alpha_1'$ and $\partial_t^2\alpha_1'' + \tilde{\lambda}''\alpha_1'' = 0$, where $\alpha_1' = \sum_I G_I dx_I$, $\alpha_1'' = \sum_I T_I dx_I$, and $\tilde{\lambda}' + \tilde{\lambda}'' = \lambda$. The boundary conditions are $j_B^*\alpha_1' = j_B^*\delta_B\alpha_1' = 0$ and $\alpha_1''|_{\{t=-L/2,L/2\}} = 0$. Hence, $\lambda \geq \pi^2/(4\varepsilon^2)$ as before. Since $\lambda \geq \pi^2/(4\varepsilon^2)$ for all eigenvalues λ, we have $\lambda_1^{(k)} \geq \pi^2/(4\varepsilon^2) \geq 1/\varepsilon^2$. The lemma follows from this fact together with min-max principle (4.1) for all test forms.

Next, from equation (4.14) we see that if $\omega = \alpha_1 + \alpha_2 \wedge dt$ is a relative harmonic form on $T(\varepsilon)$, then α_i's are relative harmonic forms on B^{n-1}. Since $\mathcal{H}^k_R(B^{n-1}) = 0$ for all $k < n-1$, we have $\mathcal{H}^k_R(T(\varepsilon)) = 0$. So the conditions $\omega \in H^1\Omega^k(M)$, $j^*\omega = 0$, and $k < n-1$ imply that ω is a test form on $T(\varepsilon)$.

Finally for the case $k \geq n-1$, we can construct k-forms that does not satisfy the Lemma as in the previous proof. □

4.3 Exponential Decay of Eigenforms

In this section, we show that the relative eigenform on $M_1(\varepsilon)$ decay exponentially along the tube $T(\varepsilon)$ (defined in Section 4.2) in the L^2-sense. Then using the fact that eigenform are regular (locally) on $T(\varepsilon)$, we obtain the pointwise exponential decay. Let $\lambda_1^{(k)}(\varepsilon)$ be the first relative eigenvalue on $M_1(\varepsilon)$. For ε small enough, Assumptions 1, 2, and Proposition 6.2.2 imply $\lambda_1^{(k)}(\varepsilon)$ is simple and positive for $k < n$. Let ω be the corresponding unique eigenform on $M_1(\varepsilon)$ with $\|\omega\|_{L^2\Omega^k(M_1(\varepsilon))} = 1$. Now for a fixed constant $d \in (0,1)$, let

$$\psi(t) = \begin{cases} 0 & t \leq -L/2 + 2\varepsilon \\ (1-d)(t + L/2 - 2\varepsilon) & -L/2 + 2\varepsilon \leq t \leq L/2 - 2\varepsilon \\ (1-d)(L - 4\varepsilon) & L/2 - 2\varepsilon \leq t \leq L/2 \end{cases}.$$

Observe that ψ is Lipschitz continuous on $M_1(\varepsilon)$ and that $|\partial_t \psi|^2 \leq (1-d)^2 \leq (1-d)$ almost everywhere. We smooth ψ to get a smooth function, also called ψ, with the same property. That is, $\partial_t\psi(\pm L/2) = 0$, and $|\partial_t \psi|^2 \leq (1-d)$ almost a.e., see Appendix for smooth approximation. Let χ be a cutoff function, $\chi(t) = 1$ for all $t \in [-L/2 + 2\varepsilon, L/2]$ and $\chi(t) = 0$ for $t \leq -L/2 + \varepsilon$ with $|\partial_t \chi| \leq C\varepsilon^{-1}$ on $\operatorname{supp} \partial_t \chi$. Define $f = \chi e^{\psi/\varepsilon}$. We have the following proposition.

Proposition 4.3.1 *Let $\omega \in H^1\Omega^k(M_1(\varepsilon))$ be an eigenform corresponding to the relative eigenvalue $\lambda_1^{(k)}(\varepsilon)$ on $M_1(\varepsilon)$ such that $\|\omega\|_{L^2} = 1$. For $k < n-1$ and any*

$0 < d < 1$, there exists $\varepsilon_0(d)$ such that for all $0 < \varepsilon < \varepsilon_0(d)$,

$$\int_{T(\varepsilon)} f^2|\omega|^2 < c$$

for some constant c depending on d and χ.

Proof. Observe that $f\omega$ is localized on $T(\varepsilon)$, i.e., $\operatorname{supp} f\omega \subset T(\varepsilon)$. We first show that $f\omega$ satisfies the hypothesis of Lemma 4.2.1. So we write $f\omega$ as $f\alpha_1 + f\alpha_2 \wedge dt$. Let $E_1 = B \times \{-L/2\}$, $E_2 = B \times \{L/2\}$, and as before, $Z = \partial B \times [-L/2, L/2]$. Then the relative boundary conditions on ω imply $j^*\omega = j^*\delta\omega = 0$ on $Z \cup E_2$. Since $f(-L/2) = 0$, we have $j^*(f\omega) = 0$ on $\partial T(\varepsilon)$. Consequently, from Lemma 4.2.1 we have

$$\int_{T(\varepsilon)} |f\omega|^2 \leq \varepsilon^2 \left\{ \int_{T(\varepsilon)} |d(f\omega)|^2 + |\delta(f\omega)|^2 \right\} \tag{4.27}$$

Next, we show that $j^*\delta(f\omega) = 0$ on $\partial T(\varepsilon)$. Applying (4.4) to $f\omega$, we get

$$\delta(f\omega) = \delta_B(f\alpha_1) + \delta_B(f\alpha_2) \wedge dt + (-1)^k \partial_t(f\alpha_2) \ .$$

Since f depends only on t, we have

$$\delta(f\omega) = f\delta_B\alpha_1 + f\delta_B\alpha_2 \wedge dt + (-1)^k(\partial_t f)\alpha_2 + (-1)^k f\partial_t\alpha_2 \ . \tag{4.28}$$

Replacing $\omega = \alpha_2 \wedge dt$ in (4.4) and solving for $\delta_B\alpha_2 \wedge dt$ gives

$$\delta_B\alpha_2 \wedge dt = \delta(\alpha_2 \wedge dt) + (-1)^{k+1}\partial_t\alpha_2 \ .$$

Substituting $\delta_B\alpha_2 \wedge dt$ back into equation (4.28) yields

$$\delta(f\omega) = f\delta\omega + (-1)^k(\partial_t f)\alpha_2 \ , \tag{4.29}$$

where we have use the fact that $\delta\alpha_1 = \delta_B\alpha_1$. Hence by (4.29), we have

$$j^*\delta(f\omega) = (f \circ j)j^*\delta\omega + (-1)^k(\partial_t f \circ j)j^*\alpha_2 \ .$$

The term $(f \circ j)j^*\delta\omega$ is zero because $j^*\delta\omega = 0$ on $Z \cup E_2 \subset \partial M_1(\varepsilon)$ and $f \circ j = 0$ on E_1. The term $(\partial_t f \circ j)j^*\alpha_2$ is zero because $j^*\alpha_2 = 0$ on Z (as in the proof of Lemma 4.2.1) and $\partial_t f \circ j = 0$ on $E_1 \cup E_2$. Together, $j^*\delta(f\omega) = 0$ on $\partial T(\varepsilon)$ as claimed.

As a consequence, $f\omega$ satisfies the relative boundary conditions on $T(\varepsilon)$. So we can apply Corollary 3.2.1 to $f\omega$ and by (4.27) we obtain

$$\int_{T(\varepsilon)} |f\omega|^2 \leq \varepsilon^2 \int_{T(\varepsilon)} \langle \Delta^{(k)}_{T(\varepsilon)}(f\omega), f\omega \rangle \ , \tag{4.30}$$

where the two boundary terms vanished because $j^*(f\omega) = j^*\delta(f\omega) = 0$. We want to evaluate the right hand side of (4.30). Using (7.6) and the fact that f is a function of t,

$$\Delta^{(k)}_{T(\varepsilon)}(f\omega) = f\Delta^{(k)}_{T(\varepsilon)}\omega - 2\frac{\partial f}{\partial t}\frac{\partial \omega}{\partial t} - \frac{\partial^2 f}{\partial t^2}\omega \ .$$

Hence

$$\langle \Delta^{(k)}_{T(\varepsilon)}(f\omega), f\omega \rangle = \lambda^{(k)}_1(\varepsilon) f^2 |\omega|^2 - 2\langle \frac{\partial f}{\partial t}\frac{\partial \omega}{\partial t}, f\omega \rangle - \langle \frac{\partial^2 f}{\partial t^2}\omega, f\omega \rangle \ . \tag{4.31}$$

The last term in (4.31) can be written as

$$\langle \frac{\partial^2 f}{\partial t^2}\omega, f\omega \rangle = \langle \frac{\partial}{\partial t}(\frac{\partial f}{\partial t}\omega), f\omega \rangle - \langle \frac{\partial f}{\partial t}\frac{\partial \omega}{\partial t}, f\omega \rangle \ . \tag{4.32}$$

We want to show that the integral of the right hand side of (4.31) reduces to a better form such that its integrand involves only the first derivative of f. Integrating equation (4.32) by parts with respect to the t-variable gives

$$\begin{aligned} \int_{T(\varepsilon)} \langle \frac{\partial^2 f}{\partial t^2}\omega, f\omega \rangle &= -\int_{T(\varepsilon)} \langle \frac{\partial f}{\partial t}\omega, \frac{\partial}{\partial t}(f\omega) \rangle - \int_{T(\varepsilon)} \langle \frac{\partial f}{\partial t}\frac{\partial \omega}{\partial t}, f\omega \rangle \\ &= -\int_{T(\varepsilon)} |\frac{\partial f}{\partial t}\omega|^2 - 2\int_{T(\varepsilon)} \langle \frac{\partial f}{\partial t}\frac{\partial \omega}{\partial t}, f\omega \rangle \ , \end{aligned}$$

where the boundary term vanished because $\partial_t f(\pm L/2) = 0$. Substituting this into (4.31) we obtain

$$\int_{T(\varepsilon)} \langle \Delta^{(k)}_{T(\varepsilon)}(f\omega), f\omega \rangle = \lambda^{(k)}_1(\varepsilon) \int_{T(\varepsilon)} |f\omega|^2 + \int_{T(\varepsilon)} |\frac{\partial f}{\partial t}\omega|^2 \ .$$

This equality together with (4.30) give

$$\int_{T(\varepsilon)} |f\omega|^2 \le \varepsilon^2 \lambda_1^{(k)}(\varepsilon) \int_{T(\varepsilon)} |f\omega|^2 + \varepsilon^2 \int_{T(\varepsilon)} |\frac{\partial f}{\partial t}\omega|^2 \ . \tag{4.33}$$

Recall that $f = \chi e^{\psi/\varepsilon}$, so we have $\partial_t f = (\partial_t \psi/\varepsilon) f + (\partial_t \chi) e^{\psi/\varepsilon}$. Since $|\partial_t \chi| \le C\varepsilon^{-1}$,

$$|\partial_t f|^2 \le |\partial_t \psi|^2 |f|^2/\varepsilon^2 + \{c_1 |\partial_t \psi| \varepsilon^{-2} + c_2 \varepsilon^{-2}\} e^{2\psi/\varepsilon} \ .$$

Since $|\partial_t \psi|^2 \le (1-d)$, equation (4.33) becomes

$$\int_{T(\varepsilon)} |f\omega|^2 \le \{\varepsilon^2 \lambda_1^{(k)}(\varepsilon) + (1-d)\} \int_{T(\varepsilon)} |f\omega|^2$$

$$+ \ \{c_1(1-d) + c_2\} \int_{B\times[-L/2+\varepsilon, -L/2+2\varepsilon]} e^{2\psi/\varepsilon} |\omega|^2. \tag{4.34}$$

Since $e^{2\psi/\varepsilon} = 1$ for $t < -L/2 + 2\varepsilon$, the last integral on the right hand side of (4.34) is bounded above by 1. So we have

$$\{d - \varepsilon^2 \lambda_1^{(k)}(\varepsilon)\} \int_{T(\varepsilon)} |f\omega|^2 \le c_1(1-d) + c_2 \ .$$

By Corollary 6.2.3, $\lambda_1^{(k)}(\varepsilon) \to \lambda_1^{(k)}(\mathcal{C})$ as $\varepsilon \to 0$. Taking ε sufficiently small so that $\varepsilon^2 \lambda_1^{(k)}(\varepsilon) \le d/2$, we get

$$(d/2) \int_{T(\varepsilon)} |f\omega|^2 \le c_1(1-d) + c_2 \ .$$

That is,

$$\int_{T(\varepsilon)} |f\omega|^2 \le 2\{c_1(1-d) + c_2\}/d \ .$$

This completes the proof. □

Corollary 4.3.2 *Let ω be as described in Proposition 4.3.1. Let $\chi_{T(\varepsilon)}$ be the characteristic function on $T(\varepsilon)$. Then*

$$\chi^2 |\omega|^2 \le c e^{-2\psi/\varepsilon} \chi_{T(\varepsilon)}$$

for some constant c depending on d; where ψ is a Lipschitz continuous function defined previously.

This fact follows from the regularity of ω on $T(\varepsilon)$ [see Remark after Corollary 3.4.2] and a proof similar to that in Section 3.5 of Hislop and Sigal [11].

Chapter 5 Gap Estimate

We prove the main theorem [Theorem 4.1.1] in this chapter using the exponential decay results of Chapter 4. To begin with, we give two key L^2-estimates. The first one is an estimate of the eigenforms on $M_i(\varepsilon)$ near the end of the tube. The second one is an estimate of the commutators, also near the end of the tube [Section 5.1]. We use these estimates to compute the matrix representation for the Hodge Laplacian restricted to a suitable 2-dimensional subspace [Section 5.2]. Consequently, we obtain the gap estimate [Section 5.3]. Finally, we sharpen this gap estimate in Section 5.4.

5.1 Preliminary Lemmas

We recall that $M_1(\varepsilon)$ is the set $\mathcal{C} \cup \hat{T}(\varepsilon)$, and $R(x,t) = (x,-t)$ is the reflection operator [Section 4.1]. Define $M_2(\varepsilon) = RM_1(\varepsilon)$ and $M(\varepsilon) = M_1(\varepsilon) \cup M_2(\varepsilon)$. Let ω_1 be the eigenform corresponding to the relative eigenvalue $\lambda_1^{(k)}(\varepsilon)$ on $M_1(\varepsilon)$ with $\|\omega_1\|_{M_1(\varepsilon)} = 1$. Here $\|\cdot\|_M$ is the shorthand notation for the L^2-norm $\|\cdot\|_{L^2\Omega^k(M)}$. Let $\omega_2 = \omega_1 \circ R$, that is, $\omega_2(p) = \omega_1(R(p))$ for all $p \in M_2(\varepsilon)$. Then ω_2 is the eigenform corresponding to the same relative eigenvalue $\lambda_1^{(k)}(\varepsilon)$ on $M_2(\varepsilon)$ with $\|\omega\|_{M_2(\varepsilon)} = 1$. Now for $\varepsilon > 0$, let $U_1'(\varepsilon) = B^{n-1} \times [L/2 - 3\varepsilon, L/2]$ be a small portion of the tube $T(\varepsilon)$, and $U_2'(\varepsilon) = RU_1'(\varepsilon)$. The purpose of $U_i'(\varepsilon)$ will be clear later. First, we need a preliminary result in order to estimate the L^2-norm of the commutator $[\Delta_{M(\varepsilon)}^{(k)}, \chi_i]\omega_i$ in Lemma 5.1.3.

Lemma 5.1.1 *Let $U_i'(\varepsilon)$ and ω_i be described as above, $i = 1, 2$. Then for $k < n-1$,*

$$\int_{U_i'(\varepsilon)} |\omega_i|^2 \leq C\varepsilon^n e^{-2(1-d)L/\varepsilon}$$

for some constant C depending on d and n but independent of ε.

Proof. By Corollary 4.3.2,

$$\int_{U_1'(\varepsilon)} |\omega_1|^2 \leq c \int_{U_1'(\varepsilon)} e^{-2\psi/\varepsilon} \ .$$

Computing the right hand side (RHS) of the above inequality,

$$\begin{aligned} RHS = c'\varepsilon^{n-1} \int_{L/2-3\varepsilon}^{L/2} e^{-2\psi/\varepsilon} &\leq c'\varepsilon^{n-1} \sup(e^{-2\psi/\varepsilon}) \int_{L/2-3\varepsilon}^{L/2} dt \\ &= 3c'\varepsilon^n e^{-2(1-d)(L-5\varepsilon)/\varepsilon} \leq C\varepsilon^n e^{-2(1-d)L/\varepsilon} \ . \end{aligned}$$

By definition,

$$\int_{U_2'(\varepsilon)} |\omega_2(p)|^2 = \int_{U_2'(\varepsilon)} |\omega_1(R(p))|^2 = \int_{U_1'(\varepsilon)} |\omega_1(p')|^2 \ ,$$

where $p' = R(p) \in U_1'(\varepsilon)$. □

Next, let $\chi_1(t)$ be the cutoff function on $M(\varepsilon)$ satisfying $\chi_1 = 1$ for $t \leq L/2 - 2\varepsilon$, $\chi_1 = 0$ for $t \geq L/2 - \varepsilon$, $|\partial_t \chi_1| \leq C\varepsilon^{-1}$ and $|\partial_t^2 \chi_1| \leq C'\varepsilon^{-2}$ on $\operatorname{supp} \partial_t \chi_1$. We extend ω_i to $M(\varepsilon)$ by taking $\omega_i = 0$ on $M(\varepsilon)\backslash M_i(\varepsilon)$, $i = 1, 2$. Let $\chi_2 = \chi_1 \circ R$, and let $\eta_i = \chi_i \omega_i$. Note that η_i belongs to domain of the Hodge Laplacian $\Delta^{(k)}_{M(\varepsilon)}$ on $M(\varepsilon)$.

Observe that χ_i is defined in such a way that $\operatorname{supp} \partial_t \chi_i \subset U_i'(\varepsilon)$. We use Lemma 5.1.1 to get an L^2-estimate of the commutator $[\Delta^{(k)}_{M(\varepsilon)}, \chi_i]\omega_i$, which lives on the support of $\partial_t \chi_i$. In order to do so, we need the next lemma. Let $U_1(\varepsilon) = B^{n-1} \times [L/2 - 2\varepsilon, L/2]$, and $U_2(\varepsilon) = RU_1(\varepsilon)$. Note that $\operatorname{supp} \partial_t \chi_i \subset U_i(\varepsilon) \subset U_i'(\varepsilon)$.

Lemma 5.1.2 *Let $\omega_1 \in H^1\Omega^k(M_1(\varepsilon))$ be an eigenform corresponding to the relative eigenvalue $\lambda_1^{(k)}(\varepsilon)$ on $M_1(\varepsilon)$, $k < n-1$. Then*

$$\int_{U_1(\varepsilon)} |d\alpha_1|^2 + |\delta\alpha_1|^2 \leq \{\lambda_1^{(k)}(\varepsilon) + c\varepsilon^{-2}\} \int_{U_1'(\varepsilon)} |\alpha_1|^2$$

and

$$\int_{U_1(\varepsilon)} |d(\alpha_2 \wedge dt)|^2 + |\delta(\alpha_2 \wedge dt)|^2 \leq \{\lambda_1^{(k)}(\varepsilon) + c\varepsilon^{-2}\} \int_{U_1'(\varepsilon)} |\alpha_2 \wedge dt|^2$$

for some constant c, where $\omega_1 = \alpha_1 + \alpha_2 \wedge dt$ is localized on $T(\varepsilon) := B^{n-1} \times [-L/2, L/2]$.

Remark. This lemma also holds for ω_2 on the sets $\operatorname{supp} \partial_t \chi_2 \subset U_2(\varepsilon) \subset U_2'(\varepsilon)$.

Proof. Let $\zeta(t)$ be a cutoff function on $M(\varepsilon)$ such that $\zeta(t) = 0$ for $t \leq L/2 - 3\varepsilon$, $\zeta(t) = 1$ for $t \geq L/2 - 2\varepsilon$, $|\partial_t \zeta| \leq C\varepsilon^{-1}$, and $|\partial_t^2 \zeta| \leq C'\varepsilon^{-2}$. Taking the inner product of $\zeta^2 \alpha_1$ and $\Delta^{(k)}_{M_1(\varepsilon)} \alpha_1$ on $U_1'(\varepsilon)$,

$$(\zeta^2\alpha_1, \Delta^{(k)}_{M_1(\varepsilon)}\alpha_1)_{U_1'(\varepsilon)} = (d(\zeta^2\alpha_1), d\alpha_1)_{U_1'(\varepsilon)} + (\delta(\zeta^2\alpha_1), \delta\alpha_1)_{U_1'(\varepsilon)}$$

$$+ \int_{\partial U_1'(\varepsilon)} j^*_{U_1'} \delta\alpha_1 \wedge j^*_{U_1'}(*\zeta^2\alpha_1) - \int_{\partial U_1'(\varepsilon)} j^*_{U_1'}(\zeta^2\alpha_1) \wedge j^*_{U_1'}(*d\alpha_1) \ ,$$

where $j^*_{U_1'}$ is the restriction of j^* on $U_1'(\varepsilon)$. As in the second proof of Lemma 4.2.1, the first boundary term is zero because $j^*\delta\alpha_1 = 0$ on $Z \cup E_2$ and $\zeta^2 = 0$ on $B \times \{L/2 - 3\varepsilon\}$. The second boundary term is zero because $j^*\alpha_2 = 0$ on $Z \cup E_2$ and $\zeta^2 = 0$ on $B \times \{L/2 - 3\varepsilon\}$. Thus,

$$(\zeta^2\alpha_1, \Delta^{(k)}_{M_1(\varepsilon)}\alpha_1)_{U_1'(\varepsilon)} = (d(\zeta^2\alpha_1), d\alpha_1)_{U_1'(\varepsilon)} + (\delta(\zeta^2\alpha_1), \delta\alpha_1)_{U_1'(\varepsilon)} \ . \tag{5.1}$$

Applying (4.3) to $\omega = \zeta^2\alpha_1$, we have

$$d(\zeta^2\alpha_1) = d_B(\zeta^2\alpha_1) + (-1)^k \partial_t(\zeta^2\alpha_1) \wedge dt$$

$$= \zeta^2 d_B\alpha_1 + (-1)^k \zeta^2 \partial_t \alpha_1 \wedge dt + (-1)^k (\partial_t \zeta^2)\alpha_1 \wedge dt$$

$$= \zeta^2 d\alpha_1 + (-1)^k (\partial_t\zeta^2)\alpha_1 \ .$$

Applying (4.4) to $\omega = \zeta^2\alpha_1$, we have $\delta(\zeta^2\alpha_1) = \delta_B(\zeta^2\alpha_1) = \zeta^2\delta\alpha_1$. Also,

$$(-1)^k((\partial_t\zeta^2)\alpha_1 \wedge dt, d\alpha_1)_{U_1'(\varepsilon)} = (-1)^k((\partial_t\zeta^2)\alpha_1 \wedge dt, (-1)^k \partial_t\alpha_1 \wedge dt)_{U'(\varepsilon)}$$

$$= ((\partial_t\zeta^2)\alpha_1, \partial_t\alpha_1)_{U'(\varepsilon)} \ .$$

Hence

$$(\zeta^2\alpha_1, \Delta^{(k)}_{M_1(\varepsilon)}\alpha_1)_{U_1'(\varepsilon)} = (\zeta^2 d\alpha_1, d\alpha_1)_{U_1'(\varepsilon)} + (\zeta^2\delta\alpha_1, \delta\alpha_1)_{U_1'(\varepsilon)} + \left(\frac{\partial\zeta^2}{\partial t}\alpha_1, \frac{\partial\alpha_1}{\partial t}\right)_{U_1'(\varepsilon)} . \tag{5.2}$$

Integrating the last term of (5.2) by parts with respect to variable t gives

$$\left(\frac{\partial\zeta^2}{\partial t}\alpha_1, \frac{\partial\alpha_1}{\partial t}\right)_{U_1'(\varepsilon)} = -\left(\frac{\partial^2\zeta^2}{\partial t^2}\alpha_1, \alpha_1\right)_{U_1'(\varepsilon)} - \left(\frac{\partial\zeta^2}{\partial t}\frac{\partial\alpha_1}{\partial t}, \alpha_1\right)_{U_1'(\varepsilon)} .$$

Here the boundary term vanishes because $\partial_t\zeta^2 = 2\zeta\zeta' = 0$ for both $t = L/2 - 3\varepsilon$ and $t = L/2$. Therefore,

$$(\zeta^2\alpha_1, \Delta^{(k)}_{M_1(\varepsilon)}\alpha_1)_{U_1'(\varepsilon)} = (\zeta^2 d\alpha_1, d\alpha_1)_{U_1'(\varepsilon)} + (\zeta^2\delta\alpha_1, \delta\alpha_1)_{U_1'(\varepsilon)} - \frac{1}{2}\left(\frac{\partial^2\zeta^2}{\partial t^2}\alpha_1, \alpha_1\right)_{U_1'(\varepsilon)} .$$

So,

$$(\zeta^2 d\alpha_1, d\alpha_1)_{U_1'(\varepsilon)} + (\zeta^2\delta\alpha_1, \delta\alpha_1)_{U_1'(\varepsilon)} = \frac{1}{2}\left(\frac{\partial^2\zeta^2}{\partial t^2}\alpha_1, \alpha_1\right)_{U_1'(\varepsilon)} + \lambda_1^{(k)}(\varepsilon)\|\alpha_1\|^2_{U_1'(\varepsilon)} .$$

Since the derivative $\partial_t^2\zeta^2$ is bounded by $C'\varepsilon^{-2}$, we get the desired result

$$\int_{U_1(\varepsilon)} |d\alpha_1|^2 + |\delta\alpha_1|^2 \leq \{\lambda_1^{(k)}(\varepsilon) + c\varepsilon^{-2}\}\int_{U_1'(\varepsilon)} |\alpha_1|^2 ,$$

where we replaced the left hand side by $U_1(\varepsilon)$ because $\zeta = 1$ on $U_1(\varepsilon)$. Similar argument holds for $\alpha_2 \wedge dt$. This completes the proof. $\square$

With this lemma, we can estimate the L^2-norm of the commutator $[\Delta^{(k)}_{M(\varepsilon)}, \chi_i]\omega_i$ on ω_i, where $[\Delta^{(k)}_{M(\varepsilon)}, \chi_i]\omega_i = \Delta^{(k)}_{M(\varepsilon)}(\chi_i\omega_i) - \chi_i\Delta^{(k)}_{M(\varepsilon)}\omega_i$ for $i = 1, 2$. This estimate plays a crucial role in our matrix representation for the Hodge Laplacian restricted to a suitable 2-dimension subspace, where the gap of the eigenvalues follows.

Lemma 5.1.3 *Let $r_i = [\Delta^{(k)}_{M(\varepsilon)}, \chi_i]\omega_i$ for $i = 1, 2$. Then*

$$\|r_i\|_{U_i(\varepsilon)} \leq C\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon}$$

for some constant C.

Proof. From the definition of r_i, we see that $\operatorname{supp} r_i = \operatorname{supp} \partial_t\chi_i$. So r_i lives on $U_i(\varepsilon) \subset T(\varepsilon)$. Applying equation (7.6),

$$\Delta^{(k)}_{T(\varepsilon)}(\chi_i\omega_i) = \chi_i\Delta^{(k)}_{T(\varepsilon)}\omega_i - 2\frac{\partial\chi_i}{\partial t}\frac{\partial\omega_i}{\partial t} - \frac{\partial^2\chi_i}{\partial t^2}\omega_i .$$

So

$$r_i = -2\frac{\partial \chi_i}{\partial t}\frac{\partial \omega_i}{\partial t} - \frac{\partial^2 \chi_i}{\partial t^2}\omega_i \ . \tag{5.3}$$

We estimate the L^2-norm of r_1 on $U_1(\varepsilon)$. By Minkowski's inequality,

$$\|r_1\|_{U_1(\varepsilon)} \leq 2\left\|\frac{\partial \chi_1}{\partial t}\frac{\partial \omega_1}{\partial t}\right\|_{U_1(\varepsilon)} + \left\|\frac{\partial^2 \chi_1}{\partial t^2}\omega_1\right\|_{U_1(\varepsilon)} \ .$$

Since $|\partial_t^j \chi_1| \leq C\varepsilon^{-j}$ for $j = 1, 2$, we have

$$\|r_1\|_{U_1(\varepsilon)} \leq c_1\varepsilon^{-1}\|\partial_t \omega_1\|_{U_1(\varepsilon)} + c_2\varepsilon^{-2}\|\omega_1\|_{U_1(\varepsilon)} \ . \tag{5.4}$$

To estimate $\|\partial_t \omega_1\|_{U_1(\varepsilon)}$, we replace $\omega = \alpha_1$ into equations (4.3) and (4.4) to obtain:

$d\alpha_1 = d_B\alpha_1 + (-1)^k\partial_t\alpha_1$ and

$\delta\alpha_1 = \delta_B\alpha_1$.

So $|d\alpha_1|^2 + |\delta\alpha_1|^2 = |d_B\alpha_1|^2 + |\delta_B\alpha_1|^2 + |\partial_t\alpha_1 \wedge dt|^2$, and hence

$|\partial_t\alpha_1 \wedge dt|^2 \leq |d\alpha_1|^2 + |\delta\alpha_1|^2$.

Since $|\partial_t\alpha_1 \wedge dt|^2 = |\partial_t\alpha_1|^2$, integrating over $U_1(\varepsilon)$ gives

$$\int_{U_1(\varepsilon)} |\partial_t\alpha_1|^2 \leq \int_{U_1(\varepsilon)} |d\alpha_1|^2 + |\delta\alpha_1|^2 \ . \tag{5.5}$$

The same argument hold for $\alpha_2 \wedge dt$. That is,

$$\int_{U_1(\varepsilon)} |\partial_t\alpha_2 \wedge dt|^2 \leq \int_{U_1(\varepsilon)} |d(\alpha_2 \wedge dt)|^2 + |\delta(\alpha_2 \wedge dt)|^2 \ . \tag{5.6}$$

Equations (5.5), (5.6), and Lemma 5.1.2 together give

$$\int_{U_1(\varepsilon)} |\partial_t\alpha_1 \wedge dt|^2 \leq \{\lambda_1^{(k)}(\varepsilon) + c\varepsilon^{-2}\} \int_{U_1'(\varepsilon)} |\alpha_1|^2$$

and

$$\int_{U_1(\varepsilon)} |\partial_t\alpha_2|^2 \leq \{\lambda_1^{(k)}(\varepsilon) + c\varepsilon^{-2}\} \int_{U_1'(\varepsilon)} |\alpha_2 \wedge dt|^2 \ .$$

Thus combining together,

$$\|\partial_t\omega_1\|_{U_1(\varepsilon)} \leq c_3\varepsilon^{-1}\|\omega_1\|_{U_1'(\varepsilon)}$$

for some constant c_3. Hence, we have

$$\|r_1\|_{U_1(\varepsilon)} \leq c\varepsilon^{-2}\|\omega_1\|_{U_1'(\varepsilon)} \ . \tag{5.7}$$

Using Lemma 5.1.1, we get $\|r_1\|_{U_1(\varepsilon)} \leq C\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon}$. The same estimate hold for r_2 on $U_2(\varepsilon)$. □

5.2 Matrix Representation

In this section, we give a matrix representation for the Hodge Laplacian restricted to a suitable 2-dimensional subspace of $L^2\Omega^k(M(\varepsilon))$. Let F be a 2-dimensional subspace spanned by the eigenforms corresponding to the relative eigenvalues $\lambda_1^{(k)}(M(\varepsilon))$ and $\lambda_2^{(k)}(M(\varepsilon))$ on $M(\varepsilon)$. By Corollary 6.2.4, F is a 2-dimensional subspace of $L^2\Omega^k(M(\varepsilon))$. Furthermore, F is an invariant subspace for $\Delta^{(k)}_{M(\varepsilon)}$.

Next, let $\pi_F : L^2\Omega^k(M(\varepsilon)) \to F$ be the orthogonal projection. Then π_F has a Riesz integral representation defined as follows [11]. Let $I(\varepsilon) = [\alpha(\varepsilon), \beta(\varepsilon)]$ be an interval centered on $\lambda_1^{(k)}(\mathcal{C})$, where $\alpha(\varepsilon) = \lambda_1^{(k)}(\mathcal{C}) - \varepsilon^{1/2}$ and $\beta(\varepsilon) = \lambda_1^{(k)}(\mathcal{C}) + \varepsilon^{1/2}$. From Proposition 6.2.2 and Corollary 6.2.4, we see that

$$\sigma(\Delta^{(k)}_{M(\varepsilon)}) \cap \sigma(\Delta^{(k)}_{M_i(\varepsilon)}) \cap I(\varepsilon) = \{\lambda_1^{(k)}(M(\varepsilon)), \lambda_2^{(k)}(M(\varepsilon)), \lambda_1^{(k)}(\varepsilon)\} \ .$$

Let $a = \{\lambda_2^{(k)}(\mathcal{C}) - \lambda_1^{(k)}(\mathcal{C})\}/8$ be a fixed constant. Then for ε small so that $\varepsilon^{1/2} \leq a$, Corollary 6.2.3 and Corollary 6.2.5 imply $\Delta^{(k)}_{M(\varepsilon)}$ and $\Delta^{(k)}_{M_i(\varepsilon)}$ have no spectrum in the intervals $[\alpha(\varepsilon) - 2a, \alpha(\varepsilon))$ and $(\beta(\varepsilon), \beta(\varepsilon) + 2a]$. Define

$$\pi_F = (2\pi i)^{-1} \int_\gamma (z - \Delta^{(k)}_{M(\varepsilon)})^{-1} dz \ ,$$

where γ is a counter clockwise oriented boundary of $[\alpha(\varepsilon) - a, \beta(\varepsilon) + a] \times i[-R, R]$ with $R > 0$ a positive number. We prove $\|\pi_F \eta_i\|_{M(\varepsilon)}$ converges to a finite constant greater than zero as $\varepsilon \to 0$.

Lemma 5.2.1 *Let π_F be defined as above. Then there exists $\varepsilon_0 > 0$ such that $\|\pi_F\eta_i\|_{M(\varepsilon)} \geq 1/2$ for all $\varepsilon < \varepsilon_0$.*

Proof. From the fact that $r_i = [\Delta^{(k)}_{M(\varepsilon)}, \chi_i]\omega_i$ as in Lemma 5.1.3, it follows that

$$(z - \Delta^{(k)}_{M(\varepsilon)})\eta_i = (z - \lambda^{(k)}_1(\varepsilon))\eta_i - r_i \ .$$

So for $z \notin I(\varepsilon)$, we have

$$(z - \Delta^{(k)}_{M(\varepsilon)})^{-1}\eta_i = (z - \lambda^{(k)}_1(\varepsilon))^{-1}\eta_i + (z - \Delta^{(k)}_{M(\varepsilon)})^{-1}(z - \lambda^{(k)}_1(\varepsilon))^{-1} r_i \ .$$

Hence integrating over the contour γ,

$$\pi_F \eta_i = \eta_i + \frac{1}{2\pi i}\int_\gamma (z - \Delta^{(k)}_{M(\varepsilon)})^{-1}(z - \lambda^{(k)}_1(\varepsilon))^{-1} r_i dz \ . \tag{5.8}$$

Let η'_i be the integral on the right hand side of (5.8). We estimate $\|\eta'_i\|_{M(\varepsilon)}$. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where

$\gamma_1 = (\beta(\varepsilon) + a) + it$, $-R \le t \le R$

$\gamma_2 = (\alpha(\varepsilon) + \beta(\varepsilon) - t) + iR$, $\alpha(\varepsilon) - a \le t \le \beta(\varepsilon) + a$

$\gamma_3 = (\alpha(\varepsilon) - a) - it$, $-R \le t \le R$

$\gamma_4 = t - iR$, $\alpha(\varepsilon) - a \le t \le \beta(\varepsilon) + a$

On γ_2 and γ_4, we have $|z - \lambda^{(k)}_1(\varepsilon)|^{-1} \le R^{-1}$, and $\|(z - \Delta^{(k)}_{M(\varepsilon)})^{-1}\| \le |\mathrm{Im}(z)|^{-1} = R^{-1}$. So

$$\left\| \frac{1}{2\pi i}\int_{\gamma_2+\gamma_4} (z - \Delta^{(k)}_{M(\varepsilon)})^{-1}(z - \lambda^{(k)}_1(\varepsilon))^{-1} r_i dz \right\|_{M(\varepsilon)} \le \frac{\|r_i\|_{U_i(\varepsilon)}}{\pi R^2}\int_{\alpha(\varepsilon)-a}^{\beta(\varepsilon)+a} dt \ .$$

Hence for $R \to \infty$, the integrals on γ_2 and γ_4 approach zero since $\|r_i\|_{U_i(\varepsilon)}$ is small [Lemma 5.1.3].

On γ_1 and γ_3, we have $|z - \lambda^{(k)}_1(\varepsilon)|^{-1} \le 1/\sqrt{a^2+t^2}$. By the choice of a, we have $\|(z - \Delta^{(k)})^{-1}\| \le \mathrm{dist}(z, \sigma(\Delta^{(k)}_{M(\varepsilon)}))^{-1} \le 1/\sqrt{a^2+t^2}$. Thus using Lemma 5.1.3,

$$\|\eta'_i\|_{M(\varepsilon)} \le \frac{\|r_i\|_{U_i(\varepsilon)}}{\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+t^2} dt = \frac{\|r_i\|_{U_i(\varepsilon)}}{a} \le C\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon} \ . \tag{5.9}$$

Since $\eta_i = \pi_F \eta_i - \eta'_i$, it follows that

$$\|\eta_i\|_{M(\varepsilon)} \le \|\pi_F \eta_i\|_{M(\varepsilon)} + \|\eta'_i\|_{M(\varepsilon)} \ . \tag{5.10}$$

We give a lower bound for $\|\eta_i\|_{M(\varepsilon)}$. Recall that $\eta_i = \chi_i\omega_i$ with $\|\omega_i\|_{M_i(\varepsilon)} = 1$. Hence

$$\|\eta_i\|_{M(\varepsilon)} = \|\omega_i - (1-\chi_i)\omega_i\|_{M_i(\varepsilon)} \geq 1 - \|(1-\chi_i)\omega_i\|_{M_i(\varepsilon)}$$

$$= 1 - \|(1-\chi_i)\omega_i\|_{U_i(\varepsilon)} \ .$$

Since $\|(1-\chi_i)\omega_i\|_{U_i(\varepsilon)} \leq \|\omega_i\|_{U_i(\varepsilon)} \leq C\varepsilon^{n/2}e^{-(1-d)L/\varepsilon}$ [Lemma 5.1.3], we have

$$\|\eta_i\|_{M(\varepsilon)} \geq 1 - C\varepsilon^{n/2}e^{-(1-d)L/\varepsilon} \geq 3/4 \tag{5.11}$$

for ε small. So there exists $\varepsilon_1 > 0$ such that $\|\eta_i\|_{M(\varepsilon)} \geq 3/4$ for all $\varepsilon < \varepsilon_1$. Hence, (5.9), (5.10), and (5.11) imply that there exists $\varepsilon_0 > 0$ ($\varepsilon_0 < \varepsilon_1$) such that $\|\pi_F\eta_i\|_{M(\varepsilon)} \geq 1/2$ for all $\varepsilon < \varepsilon_0$. □

Next, we want to show that η_1 and η_2 are linearly independent.

Lemma 5.2.2 *Let $\eta_i = \chi_i\omega_i$ be as described previously, $i = 1, 2$. Then η_1 and η_2 are linearly independent.*

Proof. Since $\eta_2 = \eta_1 \circ R$, we have

$$(\eta_1, \eta_2)_{M(\varepsilon)} = 2(\eta_1, \eta_2)_{T_2(\varepsilon)} \ ,$$

where $T_2(\varepsilon) = B \times [0, L/2]$. On $B \times [0, L/2 - 2\varepsilon]$,

$$|(\eta_1, \eta_2)_{B\times[0,L/2-2\varepsilon]}| \leq \int_{B\times[0,L/2-2\varepsilon]} |\langle\omega_1, \omega_2\rangle| \leq \int_{B\times[0,L/2-2\varepsilon]} |\omega_1||\omega_2|$$

$$\leq c\varepsilon^{n-1}\int_0^{L/2-2\varepsilon} e^{-\psi/\varepsilon}e^{-\psi(-t)/\varepsilon} = c\varepsilon^{n-1}\int_0^{L/2-2\varepsilon} e^{-(1-d)(L-4\varepsilon)/\varepsilon} \leq C\varepsilon^{n-1}e^{-(1-d)L/\varepsilon} \ .$$

On $B \times [L/2 - 2\varepsilon, L/2]$, Lemma 5.1.1 gives

$$(\eta_1, \eta_2)_{B\times[L/2-2\varepsilon,L/2]} \leq \|\omega_1\|_{B\times[L/2-2\varepsilon,L/2]} \leq \|\omega_1\|_{U_1(\varepsilon)} \leq C\varepsilon^{n/2}e^{-(1-d)L/\varepsilon} \ .$$

Therefore, we have

$$(\eta_1, \eta_2)_{M(\varepsilon)} \leq C\varepsilon^{n/2}e^{-(1-d)L/\varepsilon} \ . \tag{5.12}$$

To prove η_1 and η_2 are linearly independent, we assume the contrary. That is, assume $\eta_1 = c\eta_2$ for some constant c. Then $\|\eta_1\|_{M(\varepsilon)} = |c|\|\eta_2\|_{M(\varepsilon)}$. Since the norms of η_1 and η_2 are equal, $c = \pm 1$. Now,

$$\|\eta_1\|^2_{M(\varepsilon)} = (\eta_1, c\eta_2)_{M(\varepsilon)} = c(\eta_1, \eta_2) \to 0$$

as $\varepsilon \to 0$. This contradicts the fact that $\|\eta_1\|_{M(\varepsilon)}$ is bounded below by 3/4 (5.11). Therefore η_1 and η_2 are linearly independent. $\square$

We now define a basis for F and calculate the matrix representation of $\Delta^{(k)}_{M(\varepsilon)}$ restricted to F. We show that $\{\pi_F\eta_1, \pi_F\eta_2\}$ is a basis of F. Assume to the contrary that $\pi_F\eta_1 = c\pi_F\eta_2$ for some constant c. By Lemma 5.2.1,

$$1 \geq \|\pi_F\eta_1\|_{M(\varepsilon)} = |c|\|\pi_F\eta_2\|_{M(\varepsilon)} \geq 1/2 \ .$$

Since $\|\pi_F\eta_2\|_{M(\varepsilon)}$ is also bounded below by 1/2, $|c|$ is bounded below by 1/2. Recall from the proof of Lemma 5.2.1 that $\pi_F\eta_2 = \eta_2 + \eta_2'$ with $\|\eta_2'\|$ exponentially small as in (5.9). Furthermore, $|(\eta_1, \eta_2)|$ is exponentially small as in (5.12). Consequently, we have

$$1/2 \leq \|\pi_F\eta_1\|^2_{M(\varepsilon)} = c(\eta_1, \pi_F\eta_2)_{M(\varepsilon)} = c(\eta_1, \eta_2)_{M(\varepsilon)} + c(\eta_1, \eta_2')_{M(\varepsilon)} \to 0$$

as $\varepsilon \to 0$. Thus, a contradiction. Therefore $\pi_F\eta_1$ and $\pi_F\eta_2$ are linearly independent.

Let $\beta_1 = \pi_F\eta_1/\|\pi_F\eta_1\|_{M(\varepsilon)}$ and $\beta_2 = \pi_F\eta_2/\|\pi_F\eta_2\|_{M(\varepsilon)}$. We determine the matrix representation of $\Delta^{(k)}_{M(\varepsilon)}$ restricted to F with respect to the basis $\{\beta_1, \beta_2\}$.

Proposition 5.2.3 *The matrix representation for $\Delta^{(k)}_{M(\varepsilon)}$ restricted to F with the basis $\{\beta_1, \beta_2\}$ for F is*

$$\Delta^{(k)}_{M(\varepsilon)}|_F = \begin{pmatrix} \lambda_1^{(k)}(\varepsilon) & 0 \\ 0 & \lambda_1^{(k)}(\varepsilon) \end{pmatrix} + (w_{ij}) \ ,$$

where $w_{12} = w_{21} = O(\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon})$, and $w_{11} = w_{22} = O(\varepsilon^{n-4} e^{-2(1-d)L/\varepsilon})$.

Proof. Note that η_1 and η_2 belong to the domain of $\Delta^{(k)}_{M(\varepsilon)}$, and π_F commutes with $\Delta^{(k)}_{M(\varepsilon)}$ [11, Proposition 6.9]. Hence,

$$\Delta^{(k)}_{M(\varepsilon)}\beta_i = \Delta^{(k)}_{M(\varepsilon)}\pi_F\eta_i/\|\pi_F\eta_i\|_{M(\varepsilon)} = \pi_F\Delta^{(k)}_{M(\varepsilon)}\eta_i/\|\pi_F\eta_i\|_{M(\varepsilon)}$$

$$= \lambda_1^{(k)}(\varepsilon)\beta_i + \pi_F r_i/\|\pi_F\eta_i\|_{M(\varepsilon)} \ .$$

So

$$(\Delta^{(k)}_{M(\varepsilon)}\beta_i, \beta_i)_{M(\varepsilon)} = \lambda_1^{(k)}(\varepsilon) + (\pi_F r_i, \pi_F\eta_i)_{M(\varepsilon)}/\|\pi_F\eta_i\|^2_{M(\varepsilon)} \ .$$

Define $w_{ii} := (\pi_F r_i, \pi_F\eta_i)_{M(\varepsilon)}/\|\pi_F\eta_i\|^2_{M(\varepsilon)}$ for $i = 1, 2$. Since $\|\pi_F\eta_i\|_{M(\varepsilon)}$ is bounded below by $1/2$ for ε small, we have $|w_{ii}| \leq 4|(\pi_F r_i, \pi_F\eta_i)_{M(\varepsilon)}| = 4|(r_i, \pi_F\eta_i)_{M(\varepsilon)}|$. Since $\pi_F\eta_i = \eta_i + \eta_i'$,

$$|w_{ii}| \leq 4\|r_i\|_{U_i(\varepsilon)}\|\omega_i\|_{U_i(\varepsilon)} + 4\|r_i\|_{U_i(\varepsilon)}\|\eta_i'\|_{U_i(\varepsilon)} \ .$$

Lemma 5.1.1, Lemma 5.1.3, and inequality (5.9) together imply

$$|w_{ii}| \leq c\varepsilon^{n-2}e^{-2(1-d)L/\varepsilon} + c'\varepsilon^{n-4}e^{-2(1-d)L/\varepsilon} \leq C\varepsilon^{n-4}e^{-2(1-d)L/\varepsilon} \ .$$

Next, for $i \neq j$

$$(\Delta^{(k)}_{M(\varepsilon)}\beta_i, \beta_j)_{M(\varepsilon)} = \lambda_1^{(k)}(\varepsilon)(\pi_F\eta_i, \pi_F\eta_j)_{M(\varepsilon)}/(\|\pi_F\eta_1\|_{M(\varepsilon)}\|\pi_F\eta_2\|_{M(\varepsilon)})$$

$$+ (\pi_F r_i, \pi_F\eta_j)_{M(\varepsilon)}/(\|\pi_F\eta_1\|_{M(\varepsilon)}\|\pi_F\eta_2\|_{M(\varepsilon)}) \ .$$

Define $w_{ij} := (\Delta^{(k)}_{M(\varepsilon)}\beta_i, \beta_j)_{M(\varepsilon)}$ for $i \neq j$. Hence,

$$|w_{ij}| \leq c_1|(\pi_F\eta_i, \pi_F\eta_j)_{M(\varepsilon)}| + c_2|(\pi_F r_i, \pi_F\eta_j)_{M(\varepsilon)}| \tag{5.13}$$

for some constant c_1 and c_2. By Lemma 5.1.3, the second term on the right hand side of (5.13)

$$|(\pi_F r_i, \pi_F\eta_j)_{M(\varepsilon)}| = |(r_i, \pi_F\eta_j)_{M(\varepsilon)}| \leq c\|r_i\|_{U_i(\varepsilon)} \leq C\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon} \ .$$

We estimate the first term on the right hand side of (5.13). Since $\pi_F\eta_j = \eta_j + \eta_j'$,

$$(\pi_F\eta_i, \pi_F\eta_j)_{M(\varepsilon)} = (\eta_i, \pi_F\eta_j)_{M(\varepsilon)} = (\eta_i, \eta_j)_{M(\varepsilon)} + (\eta_i, \eta_j')_{M(\varepsilon)} \ .$$

By inequality (5.9), $|(\eta_i, \eta_j')_{M(\varepsilon)}| \leq c\|\eta_j'\|_{M(\varepsilon)} \leq C\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon}$. By inequality (5.12), $|(\eta_i, \eta_j)_{M(\varepsilon)}| \leq C\varepsilon^{n/2}e^{-(1-d)L/\varepsilon}$. Hence, $|(\pi_F\eta_i, \pi_F\eta_j)_{M(\varepsilon)}| \leq C\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon}$. Together, we have $|w_{ij}| \leq C\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon}$. □

5.3 Estimation of the Gap of Eigenvalues

In this section, we prove the main theorem as stated in Section 4.1. We restate the necessary hypothesis on the cavity $\mathcal{C}$. Assumption 1: $\mathcal{C} \subset \mathbb{R}^n$ is a compact set (with nonempty interior) that is homotopy equivalent to a closed ball. Assumption 2: the relative eigenvalue $\lambda_1^{(k)}(\mathcal{C})$ on $\mathcal{C}$ is nondegenerate.

Theorem 5.3.1 *Let $M(\varepsilon)$ be a symmetric region with Assumption 1, 2 on the cavity $\mathcal{C}$. Then for $k \neq n-1, n$, and for all ε sufficiently small and any $d \in (0,1)$, there exists a constant $c > 0$ depending only on d and n such that*

$$0 \leq \lambda_2^{(k)}(M(\varepsilon)) - \lambda_1^{(k)}(M(\varepsilon)) \leq c\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon} \ .$$

Proof. The eigenvalues of the interaction matrix in Proposition 5.2.3 are given by

$$\lambda_{2,1}^{(k)}(M(\varepsilon)) = \lambda_1^{(k)}(\varepsilon) + w_{11} \pm |w_{12}| \ .$$

So

$$\lambda_2^{(k)}(M(\varepsilon)) - \lambda_1^{(k)}(M(\varepsilon)) = 2|w_{12}| \leq C\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon} \ .$$

□

Corollary 5.3.2 *Let $M(\varepsilon)$ be as described in Theorem 5.3.1. Then for $k \neq 0, 1$, and for all ε sufficiently small and any $d \in (0,1)$, there exists a constant $c > 0$ depends only on d and n such that*

$$0 \leq \mu_2^{(k)}(M(\varepsilon)) - \mu_1^{(k)}(M(\varepsilon)) \leq c\varepsilon^{(n-4)/2}e^{-(1-d)L/\varepsilon}.$$

Proof. From the definition of $*$ and δ, it follows that $*\Delta^{(k)}_{M(\varepsilon)} = \Delta^{(n-k)}_{M(\varepsilon)}*$. Hence if $\Delta^{(k)}_{M(\varepsilon)}\omega = \lambda\omega$, then $\Delta^{(n-k)}_{M(\varepsilon)}(*\omega) = *\Delta^{(k)}_{M(\varepsilon)}\omega = \lambda(*\omega)$. We show that $*\omega$ satisfies the absolute boundary conditions whenever ω satisfies the relative boundary conditions.

Let ν be an inward unit normal vector field defined almost everywhere on the boundary $\partial M(\varepsilon)$. Let $U \subset M(\varepsilon)$ be a small neighborhood of $\partial M(\varepsilon)$. Extend ν to be a unit vector field $\tilde{\nu}$ a.e. on U such that $\tilde{\nu}|_{\partial M(\varepsilon)} = \nu$, and let $\{d\tilde{\nu}, d\tilde{x}_1, ..., d\tilde{x}_{n-1}\}$ be an orthonormal coframe on U [see discussion after (6.1)]. We show that $j^*\omega = 0$ implies $j^*i_\nu(*\omega) = 0$.

First, assume $\omega = fd\tilde{x}_I$ with $\operatorname{supp}\omega \subset U$ and $j^*\omega = (f \circ j)dx_I = 0$, where $dx_I = d\tilde{x}_I|_{\partial M(\varepsilon)}$. Then $*\omega = \operatorname{sgn}(I, J')fd\tilde{\nu} \wedge d\tilde{x}_J$, and hence $i_{\tilde{\nu}}(*\omega) = \operatorname{sgn}(I, J')fd\tilde{x}_J$ for some indexes J, J'. So $j^*i_\nu(*\omega) = \operatorname{sgn}(I, J')(f \circ j)dx_J = 0$. Next, assume $\omega = fd\tilde{\nu} \wedge d\tilde{x}_I$ with $\operatorname{supp}\omega \subset U$ and (then) $j^*\omega = 0$. We have

$$i_{\tilde{\nu}}(*\omega) = \operatorname{sgn}(I', J)i_{\tilde{\nu}}(fd\tilde{x}_J) = 0 \ .$$

Thus, $j^*i_\nu(*\omega) = 0$. By linearity of j^* and compactness of $\partial M_1(\varepsilon)$, $j^*i_\nu(*\omega) = 0$ for any arbitrary k-form ω on $M_1(\varepsilon)$. It follows that $j^*\delta\omega = 0$ implies $j^*i_\nu(*\delta\omega) = 0$. Since $*\delta = (-1)^n d*$, we have $j^*i_\nu d(*\omega) = 0$. Therefore, $*\omega$ satisfies the absolute boundary conditions. □

5.4 Another Estimate

We give a better estimate for the splitting of the eigenvalues, with the prefactor ε^{n-2}. From the proof of Theorem 5.3.1, we need to have a better estimate for $w_{12} = w_{21}$. Recall that $w_{ij} := (r_i, \eta_j)_{M(\varepsilon)}$ for $i \neq j$. We first prove a lemma.

Lemma 5.4.1 *Let $r_i = [\Delta_{M(\varepsilon)}, \chi_i]\omega_i$ be the commutator on ω_i. Then for $i \neq j$,*

$$w_{ij} = (d\chi_i \wedge \omega_i, \chi_j d\omega_j)_{T(\varepsilon)} - (\chi_j d\omega_i, d\chi_i \wedge \omega_j)_{T(\varepsilon)} - (i_{\nabla\chi_i}\omega_i, \chi_j\delta\omega_j)_{T(\varepsilon)} + (\chi_j\delta\omega_i, i_{\nabla\chi_i}\omega_j)_{T(\varepsilon)} \ .$$

Proof. We want to estimate $(r_i, \eta_j)_{M(\varepsilon)}$ for $i \neq j$. Since $\operatorname{supp} r_i = \operatorname{supp} D\chi_i$, we have $(r_i, \eta_j)_{M(\varepsilon)} = (r_i, \eta_j)_{T(\varepsilon)}$. So all calculation will be localize on $T(\varepsilon)$. Let $\{f_1 d\theta_1, ..., f_{n-2} d\theta_{n-2}, dr, dt\}$ be the orthonormal coframe on $T(\varepsilon)$. Write $\omega_1 = \alpha_1 + \alpha_2 \wedge dt$ and $\omega_2 = \beta_1 + \beta_2 \wedge dt$ on $T(\varepsilon)$. Hence using (4.3) and (4.4), we get

$$d\eta_i = \chi_i d\omega_i + d\chi_i \wedge \omega_i \ , \tag{5.14}$$

$$\delta\eta_i = \chi_i \delta\omega_i - i_{\nabla\chi_i}\omega_i \ . \tag{5.15}$$

Back to the estimation of w_{ij},

$$(r_i, \eta_j)_{T(\varepsilon)} = (\Delta_{M(\varepsilon)}\eta_i, \eta_j)_{T(\varepsilon)} - (\lambda_1^{(k)}(\varepsilon)\eta_i, \eta_j)_{T(\varepsilon)} \ . \tag{5.16}$$

Calculating the first term on the right hand side of (5.16),

$$(\Delta_{M(\varepsilon)}\eta_i, \eta_j)_{T(\varepsilon)} = (d\eta_i, d\eta_j)_{T(\varepsilon)} + (\delta\eta_i, \delta\eta_j)_{T(\varepsilon)} + \int_{\partial T(\varepsilon)} j^*(\delta\eta_i \wedge *\eta_j) - \int_{\partial T(\varepsilon)} j^*(\eta_j \wedge *d\eta_i) \ ,$$

where j^* is the pullback induced by the inclusion $j : \partial T(\varepsilon) \to T(\varepsilon)$. The boundary terms are zero because: $j^*\eta_i = j^*\delta\eta_i = 0$ on Z (5.15) for $i = 1, 2$, and χ_1, $\chi_2 = 0$ on E_1, E_2 respectively. So,

$$(\Delta_{M(\varepsilon)}\eta_i, \eta_j)_{T(\varepsilon)} = (d\eta_i, d\eta_j)_{T(\varepsilon)} + (\delta\eta_i, \delta\eta_j)_{T(\varepsilon)} \ . \tag{5.17}$$

Similarly, the second term on the right hand side of (5.16)

$$\begin{aligned}(\lambda_1^{(k)}(\varepsilon)\eta_i, \eta_j)_{T(\varepsilon)} &= (d\omega_i, d(\chi_i\eta_j))_{T(\varepsilon)} + (\delta\omega_i, \delta(\chi_i\eta_j))_{T(\varepsilon)} \\ &\quad + \int_{\partial T(\varepsilon)} j^*(\delta\omega_i \wedge *(\chi_i\eta_j)) - \int_{\partial T(\varepsilon)} j^*(\chi_i\eta_j \wedge *d\omega_i) \ .\end{aligned}$$

$$= (\delta\omega_i, \delta(\chi_i\eta_j))_{T(\varepsilon)} + (d\omega_i, d(\chi_i\eta_j))_{T(\varepsilon)} \ . \tag{5.18}$$

Together,

$$(r_i, \eta_j)_{M(\varepsilon)} = (d\eta_i, d\eta_j)_{T(\varepsilon)} + (\delta\eta_i, \delta\eta_j)_{T(\varepsilon)} - (d\omega_i, d(\chi_i\eta_j))_{T(\varepsilon)} - (\delta\omega_i, \delta(\chi_i\eta_j))_{T(\varepsilon)} \ . \tag{5.19}$$

Using (5.14) and (5.15), we calculate each term on the right hand side of (5.19) separately. The first term,

$$(d\eta_i, d\eta_j)_{T(\varepsilon)} = (\chi_i d\omega_i, \chi_j d\omega_j)_{T(\varepsilon)} + (d\chi_i \wedge \omega_i, d\chi_j \wedge \omega_j)_{T(\varepsilon)}$$

$$+(\chi_i d\omega_i, d\chi_j \wedge \omega_j)_{T(\varepsilon)} + (d\chi_i \wedge \omega_i, \chi_j d\omega_j)_{T(\varepsilon)} \ .$$

The second term,

$$(\delta\eta_i, \delta\eta_j)_{T(\varepsilon)} = (\chi_i \delta\omega_i, \chi_j \delta\omega_j)_{T(\varepsilon)} + (i_{\nabla\chi_i}\omega_i, i_{\nabla\chi_j}\omega_j)_{T(\varepsilon)}$$

$$-(\chi_i \delta\omega_i, i_{\nabla\chi_j}\omega_j)_{T(\varepsilon)} - (i_{\nabla\chi_i}\omega_i, \chi_j \delta\omega_j)_{T(\varepsilon)} \ .$$

The third term,

$$(d\omega_i, d(\chi_i\eta_j))_{T(\varepsilon)} = (\chi_i d\omega_i, \chi_j d\omega_j)_{T(\varepsilon)} + (\chi_i d\omega_i, d\chi_j \wedge \omega_j)_{T(\varepsilon)} + (\chi_j d\omega_i, d\chi_i \wedge \omega_j)_{T(\varepsilon)} \ .$$

The fourth term,

$$(\delta\omega_i, \delta(\chi_i\eta_j))_{T(\varepsilon)} = (\chi_i \delta\omega_i, \chi_j \delta\omega_j)_{T(\varepsilon)} - (\chi_i \delta\omega_i, i_{\nabla\chi_j}\omega_j)_{T(\varepsilon)} - (\chi_j \delta\omega_i, i_{\nabla\chi_i}\omega_j)_{T(\varepsilon)} \ .$$

Since $(d\chi_i \wedge \omega_i, d\chi_j \wedge \omega_j)_{T(\varepsilon)} = 0$ and $(i_{\nabla\chi_i}\omega_i, i_{\nabla\chi_j}\omega_j)_{T(\varepsilon)} = 0$, putting all together we have

$$(r_i, \eta_j)_{M(\varepsilon)} = (d\chi_i \wedge \omega_i, \chi_j d\omega_j)_{T(\varepsilon)} - (\chi_j d\omega_i, d\chi_i \wedge \omega_j)_{T(\varepsilon)} - (i_{\nabla\chi_i}\omega_i, \chi_j \delta\omega_j)_{T(\varepsilon)}$$

$$+(\chi_j \delta\omega_i, i_{\nabla\chi_i}\omega_j)_{T(\varepsilon)} \ . \tag{5.20}$$

This completes the proof. $\square$

We see in the proof of Lemma 5.4.1 that $(\Delta_{M(\varepsilon)}\eta_j, \eta_j)_{T(\varepsilon)} = (d\eta_i, d\eta_j)_{T(\varepsilon)} + (\delta\eta_i, \delta\eta_j)_{T(\varepsilon)} = (\eta_i, \Delta_{M(\varepsilon)}\eta_j)$ for $i \neq j$; so $w_{ij} = w_{ji}$. We estimate w_{21} using equation (5.20). Observe that in equation (5.20), all terms on the right hand side involve the derivative of χ_2. Thus we can just integrate over half of the tube $T(\varepsilon)$. More precisely, define $T_1(\varepsilon) := B \times [-L/2, 0]$ and $T_2(\varepsilon) := B \times [0, L/2]$. That is, we will integrate over $T_1(\varepsilon)$ because the support of $D\chi_2$ is a subset of $T_1(\varepsilon)$. Note also that $\chi_1 = 1$ on $T_1(\varepsilon)$. Hence equation (5.20) gives

$$(r_2, \eta_1)_{M(\varepsilon)} = (d\chi_2 \wedge \omega_2, d\omega_1)_{T_1(\varepsilon)} - (d\omega_2, d\chi_2 \wedge \omega_1)_{T_1(\varepsilon)} - (i_{\nabla\chi_2}\omega_2, \delta\omega_1)_{T_1(\varepsilon)}$$

$$+ (\delta\omega_2, i_{\nabla\chi_2}\omega_1)_{T_1(\varepsilon)} \ . \tag{5.21}$$

Using (5.14) and (5.15), the first term on the right hand side of (5.21)

$$(d\chi_2 \wedge \omega_2, d\omega_1)_{T_1(\varepsilon)} = (d(\chi_2\omega_2), d\omega_1)_{T_1(\varepsilon)} - (\chi_2 d\omega_2, d\omega_1)_{T_1(\varepsilon)}$$

$$= (\chi_2\omega_2, \delta d\omega_1)_{T_1(\varepsilon)} - (\chi_2 d\omega_2, d\omega_1)_{T_1(\varepsilon)} + \int_{B\times\{0\}} j_1^*(\chi_2\omega_2 \wedge *d\omega_1) \ .$$

where j_1^* is the restriction of j^* to $T_1(\varepsilon)$. The third term on RHS of (5.21),

$$-(i_{\nabla\chi_2}\omega_2, \delta\omega_1)_{T_1(\varepsilon)} = (\delta(\chi_2\omega_2), \delta\omega_1)_{T_1(\varepsilon)} - (\chi_2\delta\omega_2, \delta\omega_1)_{T_1(\varepsilon)}$$

$$= (\chi_2\omega_2, d\delta\omega_1)_{T_1(\varepsilon)} - (\chi_2\delta\omega_2, \delta\omega_1)_{T_1(\varepsilon)} - \int_{B\times\{0\}} j_1^*(\delta\omega_1 \wedge *(\chi_2\omega_2)) \ .$$

The second term on the right hand side of (5.21),

$$-(d\omega_2, d\chi_2 \wedge \omega_1)_{T_1(\varepsilon)} = -(d\omega_2, d(\chi_2\omega_1))_{T_1(\varepsilon)} + (d\omega_2, \chi_2 d\omega_1)_{T_1(\varepsilon)}$$

$$= -(\delta d\omega_2, \chi_2\omega_1)_{T_1(\varepsilon)} + (d\omega_2, \chi_2 d\omega_1)_{T_1(\varepsilon)} - \int_{B\times\{0\}} j_1^*(\chi_2\omega_1 \wedge *d\omega_2) \ .$$

The fourth term on the right hand side of (5.21),

$$(\delta\omega_2, i_{\nabla\chi_2}\omega_1)_{T_1(\varepsilon)} = -(\delta\omega_2, \delta(\chi_2\omega_1))_{T_1(\varepsilon)} + (\delta\omega_2, \chi_2\delta\omega_1)_{T_1(\varepsilon)}$$

$$= -(d\delta\omega_2, \chi_2\omega_1)_{T_1(\varepsilon)} + (\delta\omega_2, \chi_2\delta\omega_1)_{T_1(\varepsilon)} + \int_{B\times\{0\}} j_1^*(\delta\omega_2 \wedge *(\chi_2\omega_1)) \ .$$

Using the fact that $\omega_1 = R \circ \omega_2$, adding all together

$$w_{21} = \int_{B\times\{0\}} j_1^*(\omega_1 \wedge *(d\omega_1 - d\omega_2)) - j_1^*((\delta\omega_1 - \delta\omega_2) \wedge *\omega_1) \ . \tag{5.22}$$

Recall that $\omega_1 = \alpha_1 + \alpha_2 \wedge dt$ and $\omega_2 = \beta_1 + \beta_2 \wedge dt$ on $T(\varepsilon)$. Now on $B \times \{0\}$,

$$d\omega_1 - d\omega_2 = (d_B\alpha_1 - d_B\beta_1) + (d_B\alpha_2 - d_B\beta_2) \wedge dt + \ (-1)^k(\partial_t\alpha_1 - \partial_t\beta_1) \wedge dt$$

$$= 2(-1)^k \partial_t \alpha_1 \wedge dt$$

and

$$\delta\omega_1 - \delta\omega_2 = (\delta_B \alpha_1 - \delta_B \beta_1) + (\delta_B \alpha_2 - \delta_B \beta_2) \wedge dt + (-1)^k (\partial_t \alpha_2 - \partial_t \beta_2)$$

$$= 2(-1)^k \partial_t \alpha_2 \ .$$

So

$$w_{21}/2 = \int_{B \times \{0\}} j_1^*(\omega_1 \wedge *(-1)^k \partial_t \alpha_1 \wedge dt) - j_1^*((-1)^k \partial_t \alpha_2 \wedge *\omega_1) \ . \tag{5.23}$$

On $B \times \{0\}$, we have

$j_1^* \omega_1 = j_1^* \alpha_1,$

$j_1^*(*(-1)^k \partial_t \alpha_1 \wedge dt) = j_1^*(*d\alpha_1 - *d_B \alpha_1) = j^*(*d\alpha_1),$

$j_1^*((-1)^k \partial_t \alpha_2) = j_1^*(\delta(\alpha_2 \wedge dt) - \delta_B \alpha_2 \wedge dt) = j_1^* \delta(\alpha_2 \wedge dt),$

$j_1^*(*\omega_1) = j_1^*(*(\alpha_2 \wedge dt)).$

Substituting into (5.23), we get

$$w_{21}/2 = \int_{B \times \{0\}} j_1^*(\alpha_1 \wedge *d\alpha_1) - j_1^*(\delta(\alpha_2 \wedge dt) \wedge *(\alpha_2 \wedge dt))$$

$$+ \int_{B \times \{0\}} j_1^*((\alpha_2 \wedge dt) \wedge *d(\alpha_2 \wedge dt)) - j_1^*(\delta\alpha_1 \wedge *\alpha_1) \ .$$

Note that we have added the second integral to $w_{21}/2$, which is zero. Next, let j_2^* be the restriction of j^* to $T_2(\varepsilon)$. As in the second proof of Lemma 4.2.1, we have $j_2^* \delta\alpha_1 = j_2^* \delta(\alpha_2 \wedge dt) = 0$ on E_2. Hence,

$$\int_{\partial B \times [0, L/2] \cup E_2} j_2^*(\alpha_1 \wedge *d\alpha_1) - j_2^*(\delta(\alpha_2 \wedge dt) \wedge *(\alpha_2 \wedge dt)) = 0$$

and

$$\int_{\partial B \times [0, L/2] \cup E_2} j_2^*((\alpha_2 \wedge dt) \wedge *d(\alpha_2 \wedge dt)) - j_2^*(\delta\alpha_1 \wedge *\alpha_1) = 0 \ .$$

Thus,

$$w_{21}/2 = \int_{\partial T_2(\varepsilon)} j_2^*(\alpha_1 \wedge *d\alpha_1) - j_2^*(\delta(\alpha_2 \wedge dt) \wedge *(\alpha_2 \wedge dt))$$

$$+ \int_{\partial T_2(\varepsilon)} j_2^*((\alpha_2 \wedge dt) \wedge *d(\alpha_2 \wedge dt)) - j_2^*(\delta\alpha_1 \wedge *\alpha_1) \ . \tag{5.24}$$

Again, apply Green's formula to (5.24),

$$w_{21}/2 = \|d\alpha_1\|^2_{T_2(\varepsilon)} - (\alpha_1, \delta d\alpha_1)_{T_2(\varepsilon)} - (d\delta(\alpha_2 \wedge dt), \alpha_2 \wedge dt)_{T_2(\varepsilon)} + \|\delta(\alpha_2 \wedge dt)\|^2_{T_2(\varepsilon)}$$

$$+ \|d(\alpha_2 \wedge dt)\|^2_{T_2(\varepsilon)} - (\alpha_2 \wedge dt, \delta d(\alpha_2 \wedge dt))_{T_2(\varepsilon)} - (d\delta\alpha_1, \alpha_1)_{T_2(\varepsilon)} + \|\delta\alpha_1\|^2_{T_2(\varepsilon)} \ .$$

That is,

$$w_{21}/2 = \|d\alpha_1\|^2_{T_2(\varepsilon)} + \|\delta\alpha_1\|^2_{T_2(\varepsilon)} - (\Delta\alpha_1, \alpha_1)_{T_2(\varepsilon)}$$

$$+ \|d(\alpha_2 \wedge dt)\|^2_{T_2(\varepsilon)} + \|\delta(\alpha_2 \wedge dt)\|^2_{T_2(\varepsilon)} - (\Delta(\alpha_2 \wedge dt), \alpha_2 \wedge dt)_{T_2(\varepsilon)} \ . \tag{5.25}$$

From the proof of Lemma 5.1.2, we get the estimate

$$w_{21}/2 \le c\varepsilon^{-2}\|\omega_1\|^2_{T_2'(\varepsilon)} \ ,$$

where $T_2'(\varepsilon) = B \times [-\varepsilon, 0]$. Hence,

$$w_{21} \le c'\varepsilon^{n-2}e^{-2\psi(-\varepsilon)/\varepsilon} \le C\varepsilon^{n-2}e^{-(1-d)L/\varepsilon} \ . \tag{5.26}$$

With this estimate, we can restate Theorem 5.3.1 and Corollary 5.3.2,

Theorem 5.4.2 *Let $M(\varepsilon)$ be a symmetric region as described in Section 4.1 together with Assumption 1. Then for $k < n-1$, and for all ε sufficiently small and any $d \in (0,1)$, there exists a constant $c > 0$ depending only on d and n such that*

$$0 \le \lambda_2^{(k)}(M(\varepsilon)) - \lambda_1^{(k)}(M(\varepsilon)) \le c\varepsilon^{n-2}e^{-(1-d)L/\varepsilon}.$$

Corollary 5.4.3 *Let $M(\varepsilon)$ be as described in Theorem 5.3.1. Then for $k \neq 0, 1$, and for all ε sufficiently small and any $d \in (0,1)$, there exists a constant $c > 0$ depending only on d and n such that*

$$0 \le \mu_2^{(k)}(M(\varepsilon)) - \mu_1^{(k)}(M(\varepsilon)) \le c\varepsilon^{n-2}e^{-(1-d)L/\varepsilon}.$$

Chapter 6 Stability of Eigenvalues

In this chapter, we show that the effect of adding a thin tube $T(\varepsilon)$ to the cavity $\mathcal{C}$ is to shift the relative eigenvalues of $\Delta_{\mathcal{C}}^{(k)}$ by a vanishing small order of ε. We then draw a couple necessary corollaries for our work. We use Hislop and Martinez [1] as our main reference.

6.1 Preliminaries

In this section, we provide the preliminary tools to prove the convergence of eigenvalues. We pick up the material in Section 3.2. Let A be a linear operator on a Hilbert space $\mathcal{H}$ with domain $D(A) \subset \mathcal{H}$. Recall that the spectrum $\sigma(A)$ of A is the set of all points $z \in \mathbb{C}$ such that $z - A$ is not invertible. The resolvent set $\rho(A)$ is the set of all points $z \in \mathbb{C}$ such that $z - A$ is invertible. Here $z - A$ is said to be invertible if there exists a bounded operator $(z - A)^{-1} : \mathcal{H} \to D(A)$ such that $(z - A)(z - A)^{-1} = 1_{\mathcal{H}}$ and $(z - A)^{-1}(z - A) = 1_{D(A)}$. For $z \in \rho(A)$, the operator $(z - A)^{-1}$ is called the resolvent of A at z. We state the second resolvent identity [11].

Theorem 6.1.1 (Second resolvent identity) *Let A and B be two closed operators with $z \in \rho(A) \cap \rho(B)$. Then*

$$R_A(z) - R_B(z) = R_A(z)(A - B)R_B(z) = R_B(z)(B - A)R_A(z)$$

where $R_A(z) := (z - A)^{-1}$ for $z \in \rho(A)$.

Up until now, it has been sufficient to work on a real Hilbert space with real valued forms. The use of the resolvent necessitates that we now work on a complex Hilbert space so our forms may be complex valued. Let M be a compact connected set in $\mathbb{R}^n$ as before. We now define the L^2-inner product for $\omega, \eta \in L^2\Omega^k(M)$ by

$$(\omega, \eta)_M = \int_M \omega \wedge *\bar{\eta} = \int_M \langle \omega, \eta \rangle \mu \ .$$

This inner product is linear in the first form and conjugate-linear in the second form. Recall that $D(\Delta_M^{(k)}) = \{\omega \in H^2\Omega^k(M) : j_M^*\omega = j_M^*\delta\omega = 0\}$ is the domain of $\Delta_M^{(k)}$, where j_M^* is induced by the inclusion map $j_M : \partial M \to M$. With this domain, $\Delta_M^{(k)}$ is a self-adjoint operator (see the discussion of Theorem 3.2.3). Hence, $(\Delta_M^{(k)}\omega, \eta)_M = (\omega, \Delta_M^{(k)}\eta)_M$ for all $\omega, \eta \in D(\Delta_M^{(k)})$.

Next, the pointwise inner product $\langle \cdot, \cdot \rangle$ induces a pointwise inner product on the boundary ∂M of M by restriction. Let $\omega \in L^2\Omega^k(M)$ and $\eta \in L^2\Omega^{k+1}(M)$. We show that

$$\int_{\partial M} j^*\omega \wedge j^*(*\bar{\eta}) = \int_{\partial M} \langle T\omega, i_\nu T\eta \rangle \mu_{\partial M} \ , \tag{6.1}$$

where $T : H^1\Omega^k(M) \to L^2\Omega^k(M)|_{\partial M}$ is the trace operator defined in Section 3.3 and $\mu_{\partial M}$ is the volume element on ∂M with orientation induced by μ. Recall that ν is the inward unit normal field sitting on the boundary ∂M. Let $U \subset M$ be a small neighborhood of ∂M. Extend ν to a unit vector field $\tilde{\nu}$ on U such that $\tilde{\nu}|_{\partial M} = \nu$ [7]. Choose an orthonormal frame $\{\tilde{\nu}, E_1, ..., E_{n-1}\}$ on U such that $E_j|_{\partial M} \in T(\partial M)$, and let $\{d\tilde{\nu}, d\tilde{x}_1, ..., d\tilde{x}_{n-1}\}$ be the corresponding dual orthonormal coframe. Then the volume element on ∂M is $\mu_{\partial M} = dx_1 \wedge \cdots \wedge dx_{n-1}$, where $dx_j = d\tilde{x}_j|_{\partial M}$.

We prove $j^*\omega \wedge j^*(*\bar{\eta}) = \langle T\omega, i_\nu T\eta \rangle \mu_\partial$ pointwise on ∂M for $\omega = f d\tilde{x}_I$ and $\eta = g\, d\tilde{\nu} \wedge d\tilde{x}_I$, where $d\tilde{x}_I$ is some k-basis without the factor $d\tilde{\nu}$. Note that if ω contains a factor $d\tilde{\nu}$ in the basis or η without a factor $d\tilde{\nu}$ in the basis, both sides of the latter equation equal zero. Thus is the reason why we choose such ω and η above. We compute the right hand side,

$$\langle T\omega, i_\nu T\eta \rangle \mu = \omega|_{\partial M} \wedge *i_\nu \bar{\eta}|_{\partial M} = f_{\partial M} \bar{g}_{\partial M} \mu \ .$$

The left hand side,

$$\omega \wedge *\bar{\eta} = \mathrm{sgn}(I', J)\omega \wedge \bar{g} dx_J = \mathrm{sgn}(I', J)\mathrm{sgn}(I, J) f\bar{g}\mu = f\bar{g}\mu \ ,$$

where $\operatorname{sgn}(I', J) = \operatorname{sgn}(I, J)$ because the index for $d\tilde{\nu}$ is fixed. Hence, both sides agree on the boundary ∂M. By linearity, $j^*\omega \wedge j^*(*\bar{\eta}) = \langle T\omega, i_\nu T\eta\rangle \mu_{\partial M}$ on ∂M for all arbitrary ω and η. Thus, equation (6.1) holds.

6.2 Stability of eigenvalues

We prove the convergence $\lambda_1^{(k)}(\varepsilon) \to \lambda_1^{(k)}(\mathcal{C})$ as $\varepsilon \to 0$ in this section. To begin, let us recall that $T(\varepsilon)$ is a tube of length L [Section 4.2], $\tilde{T}(\varepsilon)$ is the extension of $T(\varepsilon)$ into the interior of the cavity $\mathcal{C}$, and $\hat{T}(\varepsilon)$ is the closure of $\tilde{T}(\varepsilon)\backslash\mathcal{C}$ [Section 4.1]. Here $T(\varepsilon) \subset \hat{T}(\varepsilon) \subset \tilde{T}(\varepsilon)$ and $T(\varepsilon)$, $\tilde{T}(\varepsilon)$ are tubes. Moreover, $M_1(\varepsilon) = \mathcal{C} \cup \hat{T}(\varepsilon)$ and $\mathcal{C} \cap \hat{T}(\varepsilon)$ has measure zero. Define the operator $\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)} : L^2\Omega^k(\mathcal{C}) \oplus L^2\Omega^k(\hat{T}(\varepsilon)) \to L^2\Omega^k(\mathcal{C}) \oplus L^2\Omega^k(\hat{T}(\varepsilon))$ by

$$\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)}(\omega_1 \oplus \omega_2) = \Delta_{\mathcal{C}}^{(k)}\omega_1 \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)}\omega_2 \ .$$

Here $\omega_1 \oplus \omega_2 \in D(\Delta_{\mathcal{C}}^{(k)}) \cup D(\Delta_{\hat{T}(\varepsilon)}^{(k)})$, which is the domain of $\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)}$. Let $R(z)$, $\hat{R}(z)$ be the resolvents of the operators $\Delta_{M_1(\varepsilon)}^{(k)}$ and $\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)}$ respectively. Both resolvent sets contain $\mathbb{C}\backslash\mathbb{R}$ and hence intersect. Let $z \in \rho(\Delta_{M_1(\varepsilon)}^{(k)}) \cap \rho(\Delta_{\mathcal{C}} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)})$. We want to establish an identity for $R(z) - J^*\hat{R}(z)J$, where $J : L^2\Omega^k(M_1(\varepsilon)) \to L^2\Omega^k(\mathcal{C}) \oplus L^2\Omega^k(\hat{T}(\varepsilon))$ is the identification operator defined by

$$J\omega = \omega|_{\mathcal{C}} \oplus \omega|_{\hat{T}(\varepsilon)} \ ,$$

and J^* is the adjoint of J.

Let $\alpha, \beta \in L^2\Omega^k(M_1(\varepsilon))$, and let z be in the intersection for the resolvent sets of $\Delta_{M_1(\varepsilon)}^{(k)}$ and $\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)}$. By the second resolvent identity [Theorem 6.1.1],

$$
\begin{aligned}
&(\alpha, (R(z) - J^*\hat{R}(z)J)\beta)_{M_1(\varepsilon)} \\
&= (\alpha, R(z)\Delta_{M_1(\varepsilon)}^{(k)} J^*\hat{R}(z)J\beta)_{M_1(\varepsilon)} - (\alpha, R(z)J^*\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)}\hat{R}(z)J\beta)_{M_1(\varepsilon)} \\
&= (\Delta_{M_1(\varepsilon)}^{(k)} R(z)^*\alpha, J^*\hat{R}(z)J\beta)_{M_1(\varepsilon)} - (R(z)^*\alpha, J^*\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{\hat{T}(\varepsilon)}^{(k)}\hat{R}(z)J\beta)_{M_1(\varepsilon)} \ . \quad (6.2)
\end{aligned}
$$

We apply corollary to Green's formula [Corollary 3.2.1] to the last two terms of (6.2). The first term,

$$(\Delta^{(k)}_{M_1(\varepsilon)} R(z)^*\alpha, J^*\hat{R}(z)J\beta)_{M_1(\varepsilon)} = \mathcal{D}(R(z)^*\alpha, J^*\hat{R}(z)J\beta)$$

$$+ \int_{\partial M_1(\varepsilon)} j^*\delta R(z)^*\alpha \wedge j^*(*J^*\hat{R}J\beta) - \int_{\partial M_1(\varepsilon)} j^*(J^*\hat{R}J\beta) \wedge j^*(*dR\alpha) \ ,$$

where $\mathcal{D}$ is the Dirichlet integral [Section 3.1]. Since $R(z)^*\alpha = R(\bar{z})\alpha$ and $J^*\hat{R}(z)J\beta$ are both in the domain $D(\Delta^{(k)}_{M_1(\varepsilon)})$, the two boundary terms vanish. Hence, we have

$$(\Delta^{(k)}_{M_1(\varepsilon)} R(z)^*\alpha, J^*\hat{R}(z)J\beta)_{M_1(\varepsilon)} = \mathcal{D}(R(z)^*\alpha, J^*\hat{R}(z)J\beta) \ . \tag{6.3}$$

From now on, we suppress the operators J and J^*. We compute the second term on the right hand side of (6.2),

$$(R(z)^*\alpha, \Delta^{(k)}_{\mathcal{C}} \oplus \Delta^{(k)}_{\hat{T}(\varepsilon)} \hat{R}(z)\beta)_{M_1(\varepsilon)} = \mathcal{D}(R(z)^*\alpha, \hat{R}(z)\beta)_{\mathcal{C}\oplus\hat{T}(\varepsilon)}$$

$$+ \int_{\partial\mathcal{C}} j^*_{\mathcal{C}}\delta\hat{R}(z)\beta \wedge j^*_{\mathcal{C}}(*R(z)^*\alpha) - \int_{\partial\mathcal{C}} j^*_{\mathcal{C}}R(z)^*\alpha \wedge j^*_{\mathcal{C}}(*d\hat{R}(z)\beta)$$

$$+ \int_{\partial\hat{T}(\varepsilon)} j^*_{\hat{T}(\varepsilon)}\delta\hat{R}(z)\beta \wedge j^*_{\hat{T}(\varepsilon)}(*R(z)^*\alpha) - \int_{\partial\hat{T}(\varepsilon)} j^*_{\hat{T}(\varepsilon)}R(z)^*\alpha \wedge j^*_{\hat{T}(\varepsilon)}(*d\hat{R}(z)\beta) \ .$$

Since $\hat{R}(z)\beta \in D(\Delta^{(k)}_{\mathcal{C}}) \cup D(\Delta^{(k)}_{\hat{T}(\varepsilon)})$, the two "plus" boundary terms vanish. So,

$$(R(z)^*\alpha, \Delta^{(k)}_{\mathcal{C}} \oplus \Delta^{(k)}_{\hat{T}(\varepsilon)} \hat{R}(z)\beta)_{M_1(\varepsilon)} = \mathcal{D}(R(z)^*\alpha, \hat{R}(z)\beta)_{M_1(\varepsilon)}$$

$$- \int_{\partial\mathcal{C}} j^*_{\mathcal{C}}R(z)^*\alpha \wedge j^*_{\mathcal{C}}(*d\hat{R}(z)\beta) - \int_{\partial\hat{T}(\varepsilon)} j^*_{\hat{T}(\varepsilon)}R(z)^*\alpha \wedge j^*_{\hat{T}(\varepsilon)}(*d\hat{R}(z)\beta) \ . \tag{6.4}$$

Combining (6.2), (6.3), and (6.4) together, we get

$$(\alpha, (R(z) - \hat{R}(z))\beta)_{M_1(\varepsilon)} = \int_{\partial\mathcal{C}} j^*_{\mathcal{C}}R(z)^*\alpha \wedge j^*_{\mathcal{C}}(*d\hat{R}(z)\beta)$$

$$+ \int_{\partial\hat{T}(\varepsilon)} j^*_{\hat{T}(\varepsilon)}R(z)^*\alpha \wedge j^*_{\hat{T}(\varepsilon)}(*d\hat{R}(z)\beta) \ . \tag{6.5}$$

We want to combine the boundary terms in (6.5) into a single integral. Let $N(\varepsilon) \subset M_1(\varepsilon)$ be a small neighborhood of $D(\varepsilon) := \mathcal{C} \cap \hat{T}(\varepsilon)$. Choose an orthonormal frame

$\{\tilde{\nu}, E_1, ..., E_{n-1}\}$ on $N(\varepsilon)$ such that $\tilde{\nu}|_{\partial\mathcal{C}\cap N(\varepsilon)} = \nu$, where ν is the inward normal unit vector field sitting on the boundary of $\mathcal{C}$. Let $\{d\tilde{\nu}, dx_1, ..., dx_{n-1}\}$ be the corresponding dual orthonormal coframe. Then we can use equation (6.1) to rewrite (6.5). That is,

$$(\alpha, (R(z) - \hat{R}(z))\beta)_{M_1(\varepsilon)} = \int_{D(\varepsilon)} \langle TR(z)^*\alpha, B\hat{R}(z)\beta\rangle \mu_\partial \ , \tag{6.6}$$

where $T : H^1\Omega^k(M_1(\varepsilon)) \to L^2\Omega^k(M_1(\varepsilon))|_{D(\varepsilon)}$ is the trace operator defined for each k and $B : H^2\Omega^k(\mathcal{C}) \oplus H^2\Omega^k(\hat{T}(\varepsilon)) \to L^2\Omega^k(D(\varepsilon))$ is defined by

$$B(\omega_1 \oplus \omega_2) = i_\nu T d(\omega_1 + \omega_2) \ .$$

The right hand side of (6.6) becomes

$$\begin{aligned}\int_{D(\varepsilon)} \langle TR(z)^*\alpha, B\hat{R}(z)\beta\rangle_\partial \mu_\partial &= \int_{M_1(\varepsilon)} \langle R(z)^*\alpha, T^*B\hat{R}(z)\beta\rangle \mu \\ &= \int_{M_1(\varepsilon)} \langle \alpha, R(z)T^*B\hat{R}(z)\beta\rangle \mu \ .\end{aligned}$$

where $T^* : L^2\Omega^k(D(\varepsilon')) \to H^{-1}\Omega^k(M_1(\varepsilon'))$ is the adjoint of T. Therefore,

$$R(z) - \hat{R}(z) = R(z)T^*B\hat{R}(z) \ . \tag{6.7}$$

From (6.7), we have $R(\bar{z}) - \hat{R}(\bar{z}) = R(\bar{z})T^*B\hat{R}(\bar{z})$. Furthermore, $T\hat{R}(\bar{z})\beta = 0$ because $\hat{R}(\bar{z})\beta$ belongs to $D(\Delta_{\mathcal{C}}^{(k)}) \cup D(\Delta_{\hat{T}(\varepsilon)}^{(k)})$. Thus, $TR(z)^* = TR(z)^*T^*B\hat{R}(z)^*$. Take the adjoint of $TR(z)^*$ and substitute into (6.7) gives,

$$R(z) - \hat{R}(z) = \hat{R}(z)B^*TR(z)T^*B\hat{R}(z) \ . \tag{6.8}$$

We give the first lemma.

Lemma 6.2.1 *Let $\lambda_1^{(k)}(\mathcal{C}) \in \sigma(\Delta_{\mathcal{C}}^{(k)})$ with $n \geq 3$ and $k < n-1$. Let γ_ε be a simple closed contour about $\lambda_1^{(k)}(\mathcal{C})$ of radius ε^b with $b > 0$. Then there exists ε' such that*

$$\|B\hat{R}(z)\|_{L^2\Omega^k(M_1(\varepsilon)),L^2\Omega^k(D(\varepsilon))} = O(\varepsilon^{1/2-b}) \tag{6.9}$$

for all $\varepsilon < \varepsilon'$ and $z \in \gamma_\varepsilon$.

Proof. We first consider the case on $\mathcal{C}$. Let $\omega \in H^1\Omega^k(\mathcal{C})$. Applying the boundary trace [Theorem 3.3.3], we have

$$H^1\Omega^k(\mathcal{C}) \hookrightarrow L^r\Omega^k(\partial\mathcal{C}) \ ,$$

where $r = \frac{2(n-1)}{n-2}$ and the trace operator is given by $j^*_{\mathcal{C}}$. So $j^*_{\mathcal{C}}\omega \in L^r\Omega^k(\partial\mathcal{C})$, and $\|j^*_{\mathcal{C}}\omega\|_{L^r\Omega^k(\partial\mathcal{C})} \leq C\|\omega\|_{H^1\Omega^k(\mathcal{C})}$. Let $p = n-1$ and $q = (n-1)/(n-2)$, then $\frac{1}{p} + \frac{1}{q} = 1$. Using Holder's inequality, we have

$$\int_{\partial\mathcal{C}} |j^*_{\mathcal{C}}\omega|^2 \leq \left(\int_{\partial\mathcal{C}} |\chi_{\mathcal{C}}|^{2p}\right)^{1/p} \left(\int_{\partial\mathcal{C}} |j^*_{\mathcal{C}}\omega|^{2q}\right)^{1/q} \ .$$

Here $\chi_{\mathcal{C}}$ is the characteristic function on $\mathcal{C}$. So

$$\|i_\nu T\omega\|_{L^2\Omega^k(D(\varepsilon))} \leq \|\chi_{D(\varepsilon)}\|_{L^{2p}(\partial\mathcal{C})}\|j^*_{\mathcal{C}}\omega\|_{L^{2q}\Omega^k(\partial\mathcal{C})} \leq c\varepsilon^{1/2}\|\omega\|_{H^1\Omega^k(\mathcal{C})} \ , \tag{6.10}$$

where we approximate $D(\varepsilon)$ by a ball of radius ε in $\mathbb{R}^{n-1}$. Also since d is bounded on $D(\Delta^{(k)}_{\mathcal{C}})$,

$$\|d(z-\Delta^{(k)}_{\mathcal{C}})^{-1}\|_{L^2\Omega^k(\mathcal{C}),H^1\Omega^k(\mathcal{C})} \leq C\|(z-\Delta_{\mathcal{C}})^{-1}\|_{L^2\Omega^k(\mathcal{C}),L^2\Omega^k(\mathcal{C})} \leq C\varepsilon^{-b} \ .$$

This together with (6.10) imply $\|B(z-\Delta^{(k)}_{\mathcal{C}})^{-1}\|_{L^2\Omega^k(\mathcal{C}),L^2\Omega^k(D(\varepsilon))} = O(\varepsilon^{1/2-b})$.

Next, let ω be an eigenform of degree $k < n-1$ corresponding to the first relative eigenvalue on $\hat{T}(\varepsilon)$. Extend ω to $\tilde{\omega}$ on $\tilde{T}(\varepsilon)$ such that $\tilde{\omega} = 0$ on $\tilde{T}(\varepsilon)\backslash\hat{T}(\varepsilon)$. Then $\tilde{\omega}$ is a test form on $\tilde{T}(\varepsilon)$ and

$$c\varepsilon^{-2} \leq \mathcal{R}(\tilde{\omega}) = \mathcal{R}(\omega) = \lambda^{(k)}_1(\hat{T}(\varepsilon)) \ ,$$

where $\mathcal{R}$ is the Rayleigh quotient in (4.1). So $\|(z-\Delta^{(k)}_{\hat{T}(\varepsilon)})^{-1}\|_{L^2\Omega^k(\hat{T}(\varepsilon)),L^2\Omega^k(\hat{T}(\varepsilon))} \leq C\varepsilon^2$. Hence, the lemma follows from a similar estimate as (6.10). □

We prove the stability of eigenvalues in the following proposition. For the general case, let us assume that the first relative eigenvalue $\lambda^{(k)}_1(\mathcal{C})$ has multiplicity N_0. See Section 7.3 for further discussion on higher multiplicity.

Proposition 6.2.2 *Let $\lambda_1^{(k)}(\mathcal{C}) \in \sigma(\Delta_{\mathcal{C}}^{(k)})$ with multiplicity N_0. Then for $n \geq 3$ and $k < n-1$, there exist $\varepsilon' > 0$, $c > 0$ such that for all $\varepsilon < \varepsilon'$, $\Delta_{M_1(\varepsilon)}^{(k)}$ has N_0 eigenvalues (counting multiplicity) $\lambda_1^{(k)}(\varepsilon), ..., \lambda_{N_0}^{(k)}(\varepsilon)$ satisfying*

$$|\lambda_1^{(k)}(\mathcal{C}) - \lambda_j^{(k)}(\varepsilon)| \leq \varepsilon^{1/2}$$

for all $j = 1, ..., N_0$.

Proof. Let $\lambda_1^{(k)}(\mathcal{C}) \in \sigma(\Delta_{\mathcal{C}}^{(k)})$ with multiplicity N_0. Let γ_ε be a simple closed contour about $\lambda_1^{(k)}(\mathcal{C})$ of radius ε^b, $0 < b < 1/2$. Choose ε' such that (6.9) holds for $\varepsilon < \varepsilon'$. We prove that on γ_ε,

$$\|R(z)\|_{H^{-1}\Omega^k(M_1(\varepsilon)),H^1\Omega^k(M_1(\varepsilon))} = O(\varepsilon^{-b}) \ . \tag{6.11}$$

Let z be in the intersection of the resolvent sets of $\Delta_{M_1(\varepsilon)}^{(k)}$ and $\Delta_{\mathcal{C}}^{(k)} \oplus \Delta_{T(\varepsilon)}^{(k)}$. Equation (6.7) gives

$$R(z) = \hat{R}(z) + (1 + \Delta_{M_1(\varepsilon)}^{(k)})^{1/2} R(z) (1 + \Delta_{M_1(\varepsilon)}^{(k)})^{-1/2} T^* B \hat{R}(z) \ . \tag{6.12}$$

Since T^* and $(1 + \Delta_{M_1(\varepsilon)}^{(k)})^{-1/2}$ are bounded operators, there exists $c > 0$ such that

$$\|(1 + \Delta_{M_1(\varepsilon)})^{-1/2} T^*\|_{L^2\Omega^k D(\varepsilon), L^2\Omega^k M_1(\varepsilon)} < c \ . \tag{6.13}$$

Let us consider the norm $\|(1 + \Delta_{M_1(\varepsilon)})^{1/2} R(z)\|_{L^2\Omega^k(M_1(\varepsilon)),H^1\Omega^k(M_1(\varepsilon))}$. We drop the subscripts in our calculation,

$$(1 + \Delta)^{1/2} R(z) = (1 + \Delta)^{-1/2} \{(1 + z) R(z) - 1\} \ .$$

Thus,

$$\|(1 + \Delta_{M_1(\varepsilon)}^{(k)})^{1/2} R(z)\| \leq C\{(1 + |z|)\|R(z)\| + 1\} \ . \tag{6.14}$$

It follows from (6.12) and (6.13) that

$$\|R(z)\| \leq \|\hat{R}(z)\| + C'(z)\|R(z)\| + C(z) \ , \tag{6.15}$$

where $C'(z), C(z) \sim \|B\hat{R}(z)\|_{L^2\Omega^k(M_1(\varepsilon')),L^2\Omega^k D(\varepsilon')} = O(\varepsilon^{1/2-b})$ [by equation (6.9)]. Hence, we have

$$\|R(z)\|_{L^2,L^2} = O(\varepsilon^{-b}) \tag{6.16}$$

for $z \in \gamma_\varepsilon \subset \rho(\Delta^{(k)}_{M_1(\varepsilon)})$. Now for $\omega \in H^{-1}\Omega^k(M_1(\varepsilon))$, we can use Gaffney's inequality [Theorem 3.2.2] to show

$$\|R(z)\omega\|_{H^1} \leq C\{1 + (1+|z|)\|R(z)\omega\|_{L^2}\}$$

for some constant $C > 0$. It follows that

$$\|R(z)\|_{H^{-1},H^1} \leq C\{1 + (1+|z|)\|R(z)\|_{L^2,L^2}\} \; . \tag{6.17}$$

So (6.11) follows from (6.16) and (6.17). By equation (6.8),

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_{\gamma_\varepsilon}(R(z)-\hat{R}(z))dz\right\| &\leq c\varepsilon^b\|R(z)\|_{H^{-1},H^1}(\sup_{z\in\gamma_\varepsilon}\|B\hat{R}(z)\|^2)\\ &\leq c\varepsilon^{2(1/2-b)} \; .\end{aligned}$$

So the spectrum of $\Delta^{(k)}_{M_1(\varepsilon)}$ intersects the interior of γ_ε. It follows [11] that the dimension of Ran $[(2\pi i)^{-1}\int_{\gamma_\varepsilon} R(z)dz]$ is equal to the dimension of Ran $[(2\pi i)^{-1}\int_{\gamma_\varepsilon} \hat{R}(z)dz]$. Hence, $\Delta^{(k)}_{M_1(\varepsilon)}$ has N_0 eigenvalues $\lambda^{(k)}_1(\varepsilon), ..., \lambda^{(k)}_{N_0}(\varepsilon)$ satisfying $|\lambda^{(k)}_1(\mathcal{C}) - \lambda^{(k)}_j(\varepsilon)| \leq \varepsilon^{1/2}$. □

We draw a few corollaries from Proposition 6.2.2.

Corollary 6.2.3 *Let $\lambda^{(k)}_2(\mathcal{C})$ be the second relative eigenvalue on $\mathcal{C}$ with multiplicity N_0, $k < n-1$. Then $\lambda^{(k)}_{2_j}(\varepsilon) \to \lambda^{(k)}_2(\mathcal{C})$ as $\varepsilon \to 0$ for $j = 1, ..., N_0$.*

Proof. Since the spectrum of the self-adjoint operator Hodge Laplacian on $\mathcal{C}$ is discrete, we can choose a contour γ_ε of radius ε^b such that $\lambda^{(k)}_2(\varepsilon)$ is the only relative eigenvalue in the interior of γ_ε. Also, since $\lambda^{(k)}_1(\hat{T}(\varepsilon) \geq c\varepsilon^{-2}$, the spectrum of of the Hodge Laplacian on $\hat{T}(\varepsilon)$ is away from $\lambda^{(k)}_2(\mathcal{C})$. Hence Lemma 6.2.1 holds for the second relative eigenvalue on $\mathcal{C}$. Replacing $\lambda^{(k)}_1(\mathcal{C})$ by $\lambda^{(k)}_2(\mathcal{C})$ in Proposition 6.2.2 proves the corollary. □

Corollary 6.2.4 *Let $\mathcal{A} = \mathcal{C} \cup R\mathcal{C}$ be the union of the two cavities, and $\lambda_1^{(k)}(\mathcal{A})$ be the first relative eigenvalue on $\mathcal{A}$ with multiplicity $2N_0$. Then for $n \geq 3$ and $k < n-1$, there exists $\varepsilon_0 > 0$, $c > 0$ such that for all $\varepsilon < \varepsilon_0$, $\Delta^{(k)}_{M(\varepsilon)}$ has $2N_0$ eigenvalues (counting multiplicity) $\lambda_1^{(k)}(M(\varepsilon))$, ..., $\lambda_{2N_0}^{(k)}(M(\varepsilon))$ satisfying*

$$|\lambda_1^{(k)}(\mathcal{A}) - \lambda_j^{(k)}(M(\varepsilon))| \leq \varepsilon^{1/2}$$

for $j = 1, ..., 2N_0$.

Proof. Let $\alpha, \beta \in L^2\Omega^k M(\varepsilon)$. Let $R(z)$ and $\hat{R}(z)$ be the resolvents of $\Delta^{(k)}_{M(\varepsilon)}$ and $\Delta^{(k)}_{\mathcal{A}} \oplus \Delta^{(k)}_{\hat{T}(\varepsilon)}$ respectively, where $\widehat{T}(\varepsilon) = M(\varepsilon) \backslash (\mathcal{C} \cup R\mathcal{C})$. For z in the intersection of the resolvent sets of $\Delta^{(k)}_{M(\varepsilon)}$ and $\Delta^{(k)}_{\mathcal{A}} \oplus \Delta^{(k)}_{\hat{T}(\varepsilon)}$, apply the second resolvent identity [Theorem 6.1.1] and corollary to Green's formula [Corollary 3.2.1]

$$\begin{aligned}
(\alpha, (R(z) - \hat{R}(z))\beta) &= (\alpha, R(z)(\Delta^{(k)}_{M(\varepsilon)} - \Delta^{(k)}_{\mathcal{A}} \oplus \Delta^{(k)}_{\hat{T}(\varepsilon)})\hat{R}(z)\beta) \\
&= (\Delta^{(k)}_{M(\varepsilon)} R(z)^*\alpha, \hat{R}(z)\beta) - (R(z)^*\alpha, \Delta^{(k)}_{\mathcal{A}} \oplus \Delta^{(k)}_{\hat{T}(\varepsilon)})\hat{R}(z)\beta) \\
&= \int_{D_1(\varepsilon)} \langle T_1 R(z)^*\alpha, B\hat{R}(z)\beta\rangle_\partial \mu_\partial + \int_{D_2(\varepsilon)} \langle T_2 R(z)^*\alpha, B\hat{R}(z)\beta\rangle \mu_\partial \ ,
\end{aligned}$$

where $D_1(\varepsilon) = \mathcal{C} \cap \hat{T}(\varepsilon)$, $D_2(\varepsilon) = RD_1(\varepsilon)$, T_1 and T_2 are the trace operators into $D_1(\varepsilon')$ and $D_2(\varepsilon')$ respectively; and $B : H^1\Omega^k(\mathcal{A}) \oplus H^1\Omega^k(\hat{T}(\varepsilon)) \to L^2\Omega^k(D_1(\varepsilon) \cup D_2(\varepsilon))$,

$$B(\omega_1 \oplus \omega_2) = (i_\nu T_1 + i_{\hat{\nu}} T_2) d(\omega_1 + \omega_2)$$

with $\hat{\nu} = \nu \circ R$. Let $T = T_1 + T_2$. Then

$$(\alpha, (R(z) - \hat{R}(z))\beta) = \int_{D_1(\varepsilon) \cup D_2(\varepsilon)} \langle TR(z)^*\alpha, BR^D(z)\beta\rangle \mu_\partial \ .$$

So $R(z) - \hat{R}(z) = R(z)T^*B\hat{R}(z)$. With a minor change, repeating the proof of Proposition 6.2.2 yields the desired result. □

Corollary 6.2.5 *Let $\lambda_2^{(k)}(\mathcal{C})$ be the second relative eigenvalue on $\mathcal{C}$ with multiplicity N_0, $k < n-1$. Then $\lambda_{2_j}^{(k)}(M(\varepsilon)) \to \lambda_2^{(k)}(\mathcal{C})$ as $\varepsilon \to 0$ for $j = 1, ..., 2N_0$.*

Chapter 7 Further Discussion

7.1 On the Smoothness of the boundary of $\mathcal{C}$

Throughout the dissertation, we have assumed the boundary $\partial\mathcal{C}$ of our cavity $\mathcal{C}$ to be smooth. In this section, we like to replace a weaker assumption on the smoothness of $\partial\mathcal{C}$. That is, we let $\mathcal{C}$ be a compact region in $\mathbb{R}^n$ with nonempty interior and non-smooth boundary $\partial\mathcal{C}$.

First of all, we need the boundary to be regular enough so that (smooth) solutions of the relative eigenvalue problem exist. In this case, we just need a cavity $\mathcal{C}$ with boundary such that both the Dirichlet and the Neumann eigenvalue problems are solvable. Next, we want the boundary of $\mathcal{C}$ smooth enough for the homotopy assumption. Finally, we need $\mathcal{C}$ such that the eigenvalues stable when we attach a thin tube to it. Here the boundary trace theorem is needed.

Recall that in Section 3.3, we stated the boundary trace theorem [Theorem 3.3.3] for domains with piecewise smooth boundary. We restate another general version of the boundary trace theorem here. This theorem is a generalization of the trace embedding [17] theorem on function (0-forms).

Proposition 7.1.1 *Let $M \subset \mathbb{R}^n$ be a compact region with Lipschitz boundary ∂M. Then for $p \in [1, n)$, there is a continuous embedding $W^{1,p}\Omega^k(M) \hookrightarrow L^{\frac{(n-1)p}{n-p}}\Omega^k(\partial M)$.*

Is it possible to replace to replace the smoothness of $\mathcal{C}$ by Lipschitz condition in our domains? Here we only conjecture that the cavity $\mathcal{C}$ can be taken to have piecewise smooth boundary.

7.2 On the cavity $\mathcal{C}$ with simple relative eigenvalue

In this section, we give a brief discussion on generic cavities such that their first relative eigenvalues are simple. We begin with some calculation and then give a conjecture on such cavities.

First, we take $\mathcal{C}$ to be a rectangular box of dimension $l_1 \times l_2 \times l_3$. More precisely, let $Rec = \{(x, y, z) : x \in [0, l_1], y \in [0, l_2], z \in [0, l_3]\}$. We calculation the first relative eigenvalue $\lambda_1^{(1)}$ on Rec. Take $\{dx, dy, dz\}$ be the orthonormal coframe. Let ω be a 1-form such that $\Delta^{(1)}_{Rec}\omega = \lambda\omega$, $j^*\omega = j^*\delta\omega = 0$. Now suppose $\omega = fdx + gdy + hdz$ for some smooth functions f, g, h on Rec. Then

$$\Delta^{(1)}_{Rec}\omega = (\Delta^{(0)}_{Rec}f)dx + (\Delta^{(0)}_{Rec}g)dy + (\Delta^{(0)}_{Rec}h)dz \ ,$$

where $\Delta^{(0)}_{Rec}$ is the usual Laplacian $-\Delta$ on functions. Thus, the problem reduced to solving a system of three equations

$$-\Delta f = \lambda f;\ -\Delta g = \lambda g;\ -\Delta h = \lambda h$$

with the following boundary conditions: $f|_{\{y=0,l_2;z=0,l_3\}} = 0$, $g|_{\{x=0,l_1;z=0,l_3\}} = 0$, $h|_{\{x=0,l_1;y=0,l_2\}} = 0$, and $\partial_x f + \partial_y g + \partial_z h = 0$ on ∂Rec. We use separation of variables technique. Assume that $f = X_1Y_1Z_1$, $g = X_2Y_2Z_2$, $h = X_3Y_3Z_3$. The first equation and the boundary conditions imply

$$X_1'' + a_1X_1 = 0; X_1'(0) = X_1'(l_1) = 0 \ ,$$

$$Y_1'' + b_1Y_1 = 0; Y_1(0) = Y_1(l_2) = 0 \ ,$$

$$Z_1'' + c_1Z_1 = 0; Z_1(0) = Z_1(l_3) = 0 \ ,$$

where $a_1 + b_1 + c_1 = \lambda$. Hence $f = \cos(\sqrt{a_1}x)\sin(\sqrt{b_1}y)\sin(\sqrt{c_1}z)$ with $a_1 = (n\pi/l_1)^2$, $b_1 = (n\pi/l_2)^2$, $c_1 = (n\pi/l_3)^2$. Similarly, $g = \sin(\sqrt{a_2}x)\cos(\sqrt{b_2}y)\sin(\sqrt{c_2}z)$ and $h = \sin(\sqrt{a_3}x)\sin(\sqrt{b_3}y)\cos(\sqrt{c_3}z)$ with $a_3 = a_2 = a_1$, $b_3 = b_2 = b_1$, $c_3 = c_2 = c_1$. For

f, the minimum value of λ is achieved when $a_1 = 0$, $b_1 = (\pi/l_2)^2$ and $c_1 = (\pi/l_3)^2$; i.e., $\lambda = (\pi/l_2)^2 + (\pi/l_3)^2$. For g, the smallest eigenvalue is $\lambda = (\pi/l_1)^2 + (\pi/l_3)^2$; and for h, the smallest eigenvalue is $\lambda = (\pi/l_1)^2 + (\pi/l_2)^2$. Hence we see that if l_1, l_2, l_3 are all distinct, then the first relative eigenvalue $\lambda_1^{(1)}$ has multiplicity 1. If $l_i = l_j$ for some $i, j \in \{1, 2, 3\}$, $i \neq j$, then $\lambda_1^{(1)}$ has multiplicity 2; and if $l_1 = l_2 = l_3$, $\lambda_1^{(1)}$ has multiplicity 3. We remark that this calculation can be generalize to an n-dimensional rectangular box.

We now take the domain to be $B_R(0)$, a ball of radius R in $\mathbb{R}^3$. Let $\{rd\theta, r\sin\theta d\varphi, dr\}$ be an orthonormal coframe on $B_R(0)$. Let $\omega = frd\theta + gr\sin\theta d\varphi + hdr$. Then, with some calculation

$$|d\omega|^2 = \left\{ \frac{1}{r^2\sin^2\theta}\left(\frac{\partial(g\sin\theta)}{\partial\theta} - \frac{\partial f}{\partial\varphi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h}{\partial\theta} - \frac{\partial(fr)}{\partial r}\right)^2 \right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial h}{\partial\varphi} - \frac{\partial(gr)}{\partial r}\sin\theta\right)^2 \right\}\mu \; .$$

and

$$|\delta\omega|^2 = \left(\frac{1}{r\sin\theta}\frac{\partial(f\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial g}{\partial\varphi} + \frac{1}{r^2}\frac{\partial(hr^2)}{\partial r}\right)^2\mu \; .$$

For $g = h = 0$ and f a function of r, we have

$$|d\omega|^2 + |\delta\omega|^2 = \left\{\frac{1}{r^2}\left(\frac{\partial(fr)}{\partial r}\right)^2 + \left(\frac{f\cos\theta}{r\sin\theta}\right)^2\right\}\mu \; ,$$

and for $f = h = 0$ and g a function of r

$$|d\omega|^2 + |\delta\omega|^2 = \left\{\left(\frac{g\cos\theta}{r\sin\theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial(gr)}{\partial r}\right)^2\right\}\mu \; .$$

Taking $f = g$, we see that the corresponding relative eigenvalue has multiplicity at least 2.

In general, we would like to classify all cavities (smooth or non-smooth) that satisfy Assumption 2, i.e., cavities with first relative eigenvalues simple. In analogous to the rectangular box and the ball examples, we conjecture that a (solid) three dimensional ellipsoid with principal axes $0 < a_1 < a_2 < a_3$ will satisfy Assumption 2? More generally, all convex cavities with the John ellipsoids having distinct principal axes will have simple first relative eigenvalues? All non-convex cavities without some conditions on symmetry?

7.3 On the Multiplicity Assumption

We imposed Assumption 2, simple first relative eigenvalue, on $\mathcal{C}$ in order to have a (2×2) matrix representation for the Hodge Laplacian restricted to the eigenspace F [Section 5.2]. In general, if the first relative eigenvalue on $\mathcal{C}$ has multiplicity m, then we would expect to have a $(2m \times 2m)$ matrix representation. See Dimassi and Sjöstrand [9] for the treatment of arbitrary multiplicity via the interaction matrix.

More precisely, we let η_{s_i} be the approximate eigenforms on $M(\varepsilon)$ for $s = 1, 2$ and $i = 1, ..., m$. Here $\eta_{s_i} = \chi_s \omega_{s_i}$ lives on $M(\varepsilon) \backslash M_s(\varepsilon)$ [Section 5.1]. Then we have a set of $2m$ approximate eigenforms on $M(\varepsilon)$. Let F be the space spanned by the eigenforms corresponding to the relative eigenvalues $\lambda_1^{(k)}(M(\varepsilon)), ..., \lambda_{2m}^{(k)}(M(\varepsilon))$, where $\lambda_i^{(k)}(M(\varepsilon)) \to \lambda_1^{(k)}(\mathcal{C})$ as $\varepsilon \to 0$ for all $i = 1, ..., 2m$. Define π_F the projection onto F in a similar manner as in Section 5.2. We show that $\{\pi_F \eta_{1_i}, \pi_F \eta_{2_i}\}_{i=1}^m$ forms a basis for F. First, observe that $\{\omega_{s_i}\}_{i=1}^m$ is an orthogonal set of eigenforms on $M_s(\varepsilon)$, $s = 1, 2$. It follows that

$$(\eta_{s_i}, \eta_{s_j})_{M(\varepsilon)} = (\omega_{s_i}, \omega_{s_j})_{M(\varepsilon)} - \int_{M(\varepsilon)} (1 - \chi_s^2) \langle \omega_{s_i}, \omega_{s_j} \rangle = - \int_{M(\varepsilon)} (1 - \chi_s^2) \langle \omega_{s_i}, \omega_{s_j} \rangle \ .$$

Hence,

$$|(\eta_{s_i}, \eta_{s_j})_{M(\varepsilon)}| \leq C \varepsilon^{n-2} e^{-2(1-d)L/\varepsilon} \ . \tag{7.1}$$

We estimate $(\pi_F \eta_{1_i}, \pi_F \eta_{1_j})_{M(\varepsilon)}$ and $(\pi_F \eta_{1_i}, \pi_F \eta_{2_j})_{M(\varepsilon)}$ for $i \neq j$.

$$(\pi_F \eta_{1_i}, \pi_F \eta_{1_j})_{M(\varepsilon)} = (\eta_{1_i}, \pi_F \eta_{1_j})_{M(\varepsilon)} = (\eta_{1_i}, \eta_{1_j})_{M(\varepsilon)} + (\eta_{1_i}, \eta'_{1_j})_{M(\varepsilon)} \ ,$$

where the norm of η'_{1_j} is small as in (5.9). By (7.1) and (5.9),

$$|(\pi_F \eta_{1_i}, \pi_F \eta_{1_j})_{M(\varepsilon)}| \leq C \varepsilon^{n-2} e^{-2(1-d)L/\varepsilon} \ . \tag{7.2}$$

Similarly, we have

$$(\pi_F \eta_{1_i}, \pi_F \eta_{2_j})_{M(\varepsilon)} = (\eta_{1_i}, \eta_{2_j})_{M(\varepsilon)} + (\eta_{1_i}, \eta'_{2_j})_{M(\varepsilon)} \ ,$$

and hence Lemma 5.2.2 and (5.9) imply

$$|(\pi_F \eta_{1_i}, \pi_F \eta_{2_j})_{M(\varepsilon)}| \leq C\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon} \ . \tag{7.3}$$

Inequalities (7.2) and (7.3) imply $\{\pi_F \eta_{1_i}, \pi_F \eta_{2_i}\}_{i=1}^m$ is linearly independent.

Normalizing to get a new basis $\{\beta_{1_i}, \beta_{2_i}\}_{i=1}^m$ with $\beta_{s_i} = \pi_F \eta_{s_i} / \|\pi_F \eta_{s_i}\|_{M(\varepsilon)}$ for $s = 1, 2$. From Chapter 5, (7.2), and (7.3), we deduce the following estimates:

$(\Delta^{(k)}_{M(\varepsilon)} \beta_{s_i}, \beta_{s_i})_{M(\varepsilon)} = \lambda^{(k)}_1(\varepsilon) + O(\varepsilon^{n-2} e^{-2(1-d)L/\varepsilon})$,

$(\Delta^{(k)}_{M(\varepsilon)} \beta_{s_i}, \beta_{t_i})_{M(\varepsilon)} = O(\varepsilon^{n-2} e^{-(1-d)L/\varepsilon})$ for $s \neq t$,

$(\Delta^{(k)}_{M(\varepsilon)} \beta_{s_i}, \beta_{s_j})_{M(\varepsilon)} = O(\varepsilon^{n-2} e^{-2(1-d)L/\varepsilon})$ for $i \neq j$, and

$(\Delta^{(k)}_{M(\varepsilon)} \beta_{s_i}, \beta_{t_j})_{M(\varepsilon)} = O(\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon})$ for $s \neq t$ and $i \neq j$.

Hence, we get a $(2m \times 2m)$ matrix representation

$$\Delta^{(k)}_{M(\varepsilon)}|_F = \lambda^{(k)}_1(\varepsilon) I + W \ , \tag{7.4}$$

where $W_{s_i t_j} = O(\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon})$.

Now, let α_j be the eigenform corresponding to the relative eigenvalue $\lambda^{(k)}_j(M(\varepsilon))$ on $M(\varepsilon)$ with $\|\alpha_j\|_{M(\varepsilon)} = 1$, $j = 1, ..., 2m$. Then

$$\lambda^{(k)}_j(M(\varepsilon)) = (\Delta^{(k)}_{M(\varepsilon)}|_F \alpha_j, \alpha_j)_{M(\varepsilon)} = \lambda^{(k)}_1(\varepsilon) + (W\alpha_j, \alpha_j)_{M(\varepsilon)} \ .$$

Thus

$$|\lambda^{(k)}_j(M(\varepsilon)) - \lambda^{(k)}_1(\varepsilon)| \leq \|W\| \ ,$$

and hence

$$|\lambda^{(k)}_j(M(\varepsilon)) - \lambda^{(k)}_l(M(\varepsilon))| \leq C\varepsilon^{(n-4)/2} e^{-(1-d)L/\varepsilon}$$

for $1 \leq j, l \leq 2m$.

Finally, one may ask about a lower bound for the gap of the first two relative eigenvalues. Is there exist a lower bound for this gap? If not, what are the counter examples?

Appendix

Calculation on Laplacian

We first prove equation (7.5) and its implication which were used throughout the dissertation. That is, let $\{f_1 d\theta_1, ..., f_{n-2}d\theta_{n-2}, dr, dt\}$ be an orthonormal coframe on $T(1) = B^{n-1}(0,1) \times [-L/2, L/2]$. Let $\omega = \alpha_1 + \alpha_2 \wedge dt$ be a k-form on $T(1)$. We show

$$\Delta_{T(1)}\omega = \Delta_B\alpha_1 - \frac{\partial^2\alpha_1}{\partial t^2} + \left(\Delta_B\alpha_2 - \frac{\partial^2\alpha_2}{\partial t^2}\right) \wedge dt \ , \tag{7.5}$$

where Δ_B is the Laplacian on B^{n-1}. Recall that d is exterior derivative and $\delta = (-1)^{nk+n+1} * d*$ is the codifferential. We calculate $d\delta\omega$ and $\delta d\omega$ separately,

$$d\omega = d_B\alpha_1 + d_B\alpha_2 \wedge dt + (-1)^k \frac{\partial\alpha_1}{\partial t} \wedge dt \ ,$$

$$\delta d\omega = \delta_B d_B\alpha_1 + \delta_B d_B\alpha_2 \wedge dt + (-1)^{k+1}\frac{\partial}{\partial t}(d_B\alpha_2) - \frac{\partial^2\alpha_2}{\partial t^2} + (-1)^k\delta_B(\frac{\partial\alpha_1}{\partial t}) \wedge dt \ ,$$

$$\delta\omega = \delta_B\alpha_1 + \delta_B\alpha_2 \wedge dt + (-1)^k\frac{\partial\alpha_2}{\partial t} \ ,$$

$$d\delta\omega = d_B\delta_B\alpha_1 + (-1)^{k-1}\frac{\partial}{\partial t}(\delta_B\omega) + d_B\delta_B\alpha_2 \wedge dt - \frac{\partial^2\alpha_2}{\partial t^2} \wedge dt + (-1)^k d_B(\frac{\partial\alpha_2}{\partial t}) \ .$$

Since d_B, δ_B commute with ∂_t, combining $d\delta\omega$ and $\delta d\omega$ gives the above result.

Next we want to show the following formula, which was used in the proof of Proposition 4.3.1 and Lemma 5.1.3 :

$$\Delta_{T(1)}(f\omega) = f\Delta_{T(1)}\omega - 2\frac{\partial f}{\partial t}\frac{\partial\omega}{\partial t} - \frac{\partial^2 f}{\partial t^2}\omega \ , \tag{7.6}$$

where f depends only on t. Apply (7.5), we get

$$\Delta_{T(1)}(f\omega) = \Delta_B(f\alpha_1) - \frac{\partial^2(f\alpha_1)}{\partial t^2} + \left(\Delta_B(f\alpha_2) - \frac{\partial^2(f\alpha_2)}{\partial t^2}\right) \wedge dt$$

$$= f\Delta_B\alpha_1 - f\frac{\partial^2\alpha_1}{\partial t^2} - 2\frac{\partial f}{\partial t}\frac{\partial\alpha_1}{\partial t} - \frac{\partial^2 f}{\partial t^2}$$

$$+\left(f\Delta_B\alpha_2 - f\frac{\partial^2\alpha_2}{\partial t^2} - 2\frac{\partial f}{\partial t}\frac{\partial\alpha_2}{\partial t} - \frac{\partial^2 f}{\partial t^2}\alpha_2\right)\wedge dt$$

$$= f\Delta_{T(1)}\omega - 2\frac{\partial f}{\partial t}\frac{\partial\omega}{\partial t} - \frac{\partial^2 f}{\partial t^2}\omega \ .$$

Thus, we have established equation (7.6).

Mollifier

We construct a family of smooth functions which converges to f, where $f = \chi e^{\psi/\varepsilon}$. Recall that

$$\psi(t) = \begin{cases} 0 & t \leq -L/2 + h \\ (1-d)(t + L/2 - h) & -L/2 + h \leq t \leq L/2 - h \\ (1-d)(L-2h) & L/2 - h \leq t \leq L/2 \end{cases} ,$$

where $h = 2\varepsilon$. We extend the domain of $e^{\psi/\varepsilon}$ to $[-L/2-h, L/2+h]$ by letting ψ to be

$$\psi(t) = \begin{cases} 0 & -L/2 - h \leq t \leq -L/2 + h \\ (1-d)(t + L/2 - h) & -L/2 + h \leq t \leq L/2 - h \\ (1-d)(L-2h) & L/2 - h \leq t \leq L/2 + h \end{cases} .$$

Let η_ϵ be the standard mollifier defined as follows. Define $\eta \in C^\infty(\mathbb{R}^n)$ by $\eta(x) = C\exp(\frac{1}{|x|^2-1})$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geq 1$. Here the constant C is selected so that $\int_{\mathbb{R}} \eta = 1$. For $\epsilon > 0$, we set $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$.

Next for $\epsilon < h$, define

$$\psi_\epsilon(t) := \eta_\epsilon * \psi(t) = \int_{-L/2-h}^{L/2+h} \eta_\epsilon(t-y)\psi(y)dy$$

for $t \in (-L/2 - h + \epsilon, L/2 + h - \epsilon)$. With some calculation [16],

$$\partial_t\psi_\epsilon = \int_{-L/2-h}^{L/2+h} \partial_t\eta_\epsilon(t-y)\psi(y)dy = \int_{-\epsilon}^{\epsilon} \partial_t\eta_\epsilon(y)\psi(t-y)dy \ .$$

Let j be a positive integer such that $1/j < \epsilon$. Define $\varphi_n = \psi_{1/(j+n)}$, $n = 1, 2, ...$ Then $\varphi_n \to \psi$ as $n \to \infty$ (see [16]). Also, $\partial_t\varphi_n = \eta_{1/(j+n)} * \partial_t\psi$ converges to $\partial_t\psi$ and

$$\partial_t\varphi_n(\pm L/2) = \int_{-1/(j+n)}^{1/(j+n)} \partial_t\eta_{1/(j+n)}(y)\psi(\pm L/2 - y)dy$$

$$= \psi(\pm L/2) \int_{-1/(j+n)}^{1/(j+n)} \partial_t \eta_{1/(j+n)}(y) dy = 0 \ .$$

That is, we have shown there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of smooth bounded functions such that $\varphi_n \to \psi$ and $\partial_t \varphi_n \to \partial_t \psi$ pointwise on $[-L/2, L/2]$, and $\partial_t \varphi_n$ vanishes on the boundary for each n. Now, let $\chi(t)$ be a cutoff function and let $f_n = \chi e^{\varphi_n/\varepsilon}$. Then $f_n \to \chi e^{\psi/\varepsilon} = f$. Also, $\partial_t f_n = \chi \partial_t e^{\varphi_n/\varepsilon} + (\partial_t \chi) e^{\varphi_n/\varepsilon} \to \chi \partial_t e^{\psi/\varepsilon} + (\partial_t \chi) e^{\psi/\varepsilon} = \partial_t f$.

Next, let $g \in L^2[-L/2, L/2]$. We show that $f_n g \to fg$ in L^2. Let $Z = \{x \in [-L/2, L/2] : g(x) = \infty\}$. Then Z has measure zero. Let $x \in [-L/2, L/2] - Z$. Then $g(x) \leq M_x$ for some constant M_x. Hence

$$|f_n g(x) - fg(x)| \leq M_x |f_n(x) - f(x)| \to 0$$

as $n \to \infty$. Thus $f_n g \to fg$ a.e. on $[-L/2, L/2]$. Let $\phi = |g| \|f\|_{L^\infty} \in L^2$. We see that $|f_n g|^2 \leq |\phi|^2$ a.e. for all n. By dominated convergence theorem, $f_n g \to fg$ in L^2. Similar argument shows that $(\partial_t f_n) g \to (\partial_t f) g$ in L^2.

Integration by parts

Let $\omega = \alpha_1 + \alpha_2 \wedge dt$ and $\eta = \beta_1 + \beta_2 \wedge dt$ be smooth k-forms on $T(1)$ satisfying $j^*\omega = j^*\eta = 0$ and either $j^*\beta_2 = 0$ or $j^*\alpha_2 = 0$. We prove the integration by parts formula

$$\int_{T(1)} \langle \partial_t \omega, \eta \rangle = - \int_{T(1)} \langle \omega, \partial_t \eta \rangle \ . \tag{7.7}$$

We show $(\partial_t \alpha_1, \beta_1)_{T(1)} = -(\alpha_1, \partial_t \beta_1)_{T(1)}$. By (4.3), $\partial_t \alpha_1 \wedge dt = (-1)^k d\alpha_1 + (-1)^{k+1} d_B \alpha_1$. So,

$$\begin{aligned}(\partial_t \alpha_1, \beta_1)_{T(1)} &= (\partial_t \alpha_1 \wedge dt, \beta_1 \wedge dt)_{T(1)} = (-1)^k (d\alpha_1, \beta_1 \wedge dt)_{T(1)} \\ &= (-1)^k (\alpha_1, \delta(\beta_1 \wedge dt))_{T(1)} + \int_{\partial T(1)} j^*(\alpha_1 \wedge *(\beta \wedge dt)) \ .\end{aligned}$$

Since $j^*\alpha_1 = 0$ and $\delta(\beta_1 \wedge dt) = \delta_B \beta_1 \wedge dt + (-1)^{k+1} \partial_t \beta_1$ (4.4), we have $(\partial_t \alpha_1, \beta_1)_{T(1)} = -(\alpha_1, \partial_t \beta_1)_{T(1)}$. A similar calculation show that $(\partial_t \alpha_2 \wedge dt, \beta_2 \wedge dt)_{T(1)} = (\alpha_2 \wedge dt, \partial_t \beta_2 \wedge dt)_{T(1)}$. Therefore, we have proved (7.7).

Finally, we want to show d commutes with ∂_t on $H^1\Omega^k(M)$. By linearity, it is sufficient to show $d(\partial_t \omega) = \partial_t(d\omega)$ for some $\omega = f dx_I$. By the definition of d, it is enough to show $\partial_t(\partial_{x_i} f) = \partial_{x_i}(\partial_t f)$ for all $i = 1, ..., n$. Let ϕ be a smooth function supported on the interior of M. Then $\partial_{x_i}(\partial_t \phi) = \partial_t(\partial_{x_i}\phi)$. Integration by part,

$$\int_M \partial_{x_i} f \partial_t \phi = -\int_M f \partial_{x_i}(\partial_t \phi) = -\int_M f \partial_t(\partial_{x_i}\phi) = \int_M \partial_t f \partial_{x_i} \phi \ .$$

Hence,

$$\int_M \partial_t(\partial_{x_i} f)\phi = \int_M \partial_{x_i}(\partial_t f)\phi \ .$$

It follows that

$$d(\partial_t \omega) = \partial_t(d\omega) \tag{7.8}$$

for all arbitrary $\omega \in H^1\Omega^k(M)$.

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