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Upper double monophonic number of a graph

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Abstract

A set S of a connected graph G of order n is called a double monophonic set of G if for every pair of vertices x, y in G there exist vertices u, v in S such that x, y lie on a $u-v$ monophonic path. The double monophonic number $dm(G)$ of G is the minimum cardinality of a double monophonic set. A double monophonic set S in a connected graph G is called a minimal double monophonic set if no proper subset of S is a double monophonic set of G. The upper double monophonic number of G is the maximum cardinality of a minimal double monophonic set of G, and is denoted by $dm^+(G)$. Some general properties satisfied by upper double monophonic sets are discussed. It is proved that for a connected graph G of order n, $dm(G) = n$ if and only if $dm^+(G) = n$. It is also proved that $dm(G) = n-1$ if and only if $dm^+(G) = n-1$ for a non-complete graph G of order n with a full degree vertex. For any positive integers $2 \leq a \leq b$, there exists a connected graph G with $dm(G) = a$ and $dm^+(G) = b$.

Key words : *Double monophonic set, double monophonic number, upper double monophonic set, upper double monophonic number.*

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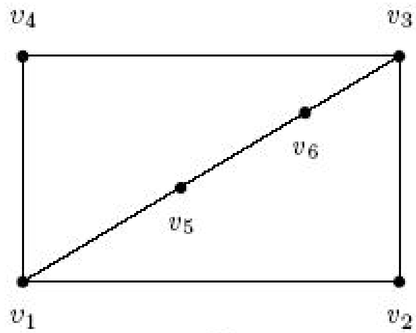
1. Introduction

By a graph $G = V, \;\; E$ we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to [3]. The distance $d(x, y)$ is the length of the shortest $x - y$ path in G. Any $x - y$ path of length $d(x, y)$ is called an $x - y$ geodetic. A subset S of v is called a *geodetic set* of the graph G if every vertex x of G lies on a $u - v$ geodesic for some vertices u, v in S. A geodetic set of minimum cardinality is a *minimum geodetic set*. The cardinality of a minimum geodetic set is the *geodetic number* of G and is denoted by $g(G)$. The geodetic number of a graph was introduced and studied in [1,2,4]. Denote by $I[x, y]$ the set of all vertices lying on some $x - y$ geodesic of G. A vertex v in a connected graph G is called *weak extreme* if there exists a vertex u in G such that u, $v \in I[x, y]$ for a pair of vertices x, y in G, then v=x or v=y. It is easy to see that each extreme vertex of a graph is weak extreme. For the graph G in Figure 1.1, it is clear that the pair v_2, v_5 lies only on the $v_2 - v_5$ geodesic and so v_2 and v_5 are weak extreme vertices of G. It is easily seen that each vertex of G is weak extreme. Weak extreme vertices are introduced in [6]. A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. A subset S of v is called a *monophonic set* of G if every vertex v of G lies on a $x - y$ monophonic path for some vertices x and y in S. The minimum cardinality of a monophonic set of G is called the *monophonic number* of G and is denoted by $m(G)$. Let G be a connected graph with at least two vertices. A set S of vertices of G is called a *double geodetic set* of G if for each pair of vertices x, y in G, there exist vertices u, v in S such that x, y lie on a $u - v$ geodesic. The *double geodetic number* $dg(G)$ of G is the minimum cardinality of a double geodetic set. Any double geodetic set of cardinality $dg(G)$ is called *dg-set* of G. The double geodetic number of a graph was introduced and studied in [6]. A set S of vertices of G is called a *double monophonic set* of G if for each pair of vertices x,y in G there exist vertices u, v in S such that x,y lie on a u-v monophonic path. The *double monophonic number* $dm(G)$ of G is the minimum cardinality of a double monophonic set. Any double monophonic set of cardinality $dm(G)$ is called a *dm-set*. The double monophonic number of a graph was introduced and studied in [7]. A double geodetic set S in a connected graph G is called a *minimal double geodetic* set if no proper subset of S is a double geodetic set of G. The *upper double geodetic number* $dg^+(G)$ of G is the maximum cardinality of a minimal double geodetic set of G. The upper double geodetic number of a graph was introduced and studied in [5]. The following theorems will be used in the sequel.



G

Figure 1.1

A graph with all its vertices weak extreme

Theorem 1.1. [7] Each extreme vertex of a connected graph G belongs to every double monophonic set of G. In particular, if the set of all extreme vertices of G is a double monophonic set, then it is the unique minimum double monophonic set of G.

Theorem 1.2. [7] Let G be a connected graph with a cut-vertex v. Then each double monophonic set of G contains at least one vertex from each component of G-v.

Theorem 1.3. [7] No cut-vertex of a connected graph G belongs to any minimum double monophonic set of G.

Theorem 1.4. [7] For the complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$, $dm(G) = min\{m, n\}$.

2. Upper double monophonic number of a graph

Definition 2.1. Let G be a connected graph with at least two vertices. A double monophonic set S in a connected graph G is a *minimal double* *monophonic set* if no proper subset of S is a double monophonic set of G. The upper double monophonic number of a graph G is the maximum cardinality of a minimal double monophonic set of G, denoted by $dm^+(G)$.

Example 2.2. For the graph G in Figure 2.1, $S = \{v_2, v_4\}$ is the only double monophonic set of G so that $dm(G) = 2$. The sets $S_1 = \{v_1, v_2, v_4\}$, $S_2 = \{v_1, v_3, v_5\}$, $S_3 = \{v_2, v_3, v_4\}$ and $S_4 = \{v_2, v_4, v_5\}$ are the only double monophonic sets of cardinality 3. Hence $S_2 = \{v_1, v_3, v_5\}$ is the only minimal double monophonic set of cardinality 3. It is easily verified that all 4-element subsets are double monophonic, and none of them is minimal. Thus $dm^+(G) = 3$.

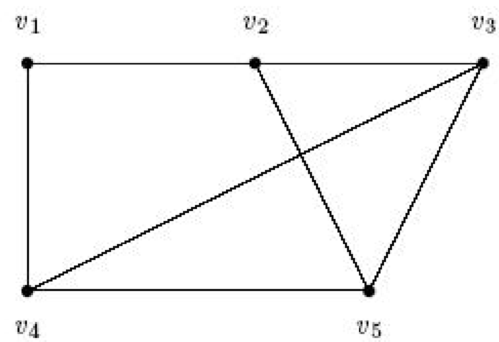


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Figure 2.1

It is clear that every minimum double monophonic set of G is a minimal double monophonic set of G. The converse need not be true. For the graph G given in Figure 2.1, $S_2 = \{v_1, v_3, v_5\}$ is a minimal double monophonic set, and not a minimum double monophonic set.

Theorem 2.3. For any connected graph G of order $n, 2 \leq dm(G) \leq dm^+(G) \leq n$.

Proof. Since every double monophonic set contains at least two vertices, it follows that $dm(G) \geq 2$. Also every minimal double monophonic set is a double monophonic set so that $dm(G) \leq dm^+(G)$. Thus $2 \leq dm(G) \leq dm^+(G) \leq n$.

The bounds in Theorem 2.3 are sharp. For any non-trivial path $P, dm(P) = 2$. By Theorems 1.1 and 1.2, it is clear that $dm(T) = dm^+(T)$ for any tree T and $dm^+(K_n) = n (n \geq 2)$. Further, all the inequalities in the theorem are strict. For a complete bipartite graph $G = K_r, s (3 \leq r \leq s), dm(G) = r, dm^+(G) = s$ and $n = r + s$.

Corollary 2.4. For any connected graph G, $dm(G) = n$ if and only if $dm^+(G) = n$.

Proof. If $dm^+(G) = n$, then the vertex set v is the only minimal double monophonic set of G. Hence it follows that v is the only minimum double monophonic set of G so that $dm(G) = n$. If $dm(G) = n$, then the result follows from Theorem 2.3

Corollary 2.5. If G is a connected graph of order n with $dm(G) = n - 1$, then $dm^+(G) = n - 1$.
Proof. Since $dm(G) = n-1$, it follows from Theorem 2.3 that $dm^+(G) = n$ or $dm^+(G) = n - 1$. It follows from Corollary 2.4 that $dm^+(G) = n - 1$.

As a consequence of this theorem, the following corollary is clear.

Corollary 2.6. For the complete graph $G = K_n, (n \geq 2)$, $dm^+(G) = n$.

Remark 2.7. It is proved in [6] that every double geodetic set of a connected graph G contains all the weak extreme vertices of G. This result is not true for the case of a double monophonic set. That is, a double monophonic set of a connected graph need not contain all the weak extreme vertices of G. For the graph G in Figure 1.1, all the vertices are weak extreme. The set $S = v_1, v_3$ is a double monophonic set. Thus $dm(G) = 2$. However, the vertices v_2, v_4, v_5, v_6 do not belong to S. We now compute the upper double monophonic number of G. Clearly,

$S_1 = \{v_1, v_3\}$, $S_2 = \{v_1, v_6\}$ and $S_3 = \{v_3, v_5\}$ are the only three double monophonic sets, so that $dm(G) = 2$. It is easily verified that the sets $T_1 = \{v_1, v_2, v_3\}, T_2 = \{v_1, v_2, v_6\}$, $T_3 = \{v_1, v_3, v_4\}, T_4 = \{v_1, v_3, v_5\}$, $T_5 = \{v_1, v_3, v_6\}, T_6 = \{v_1, v_4, v_6\}$, $T_7 = \{v_1, v_5, v_6\}, T_8 = \{v_2, v_3, v_5\}$,

$T_9 = \{v_2, v_4, v_5\}, T_{10} = \{v_2, v_4, v_6\}$, $T_{11} = \{v_3, v_4, v_5\}$ and $T_{12} = \{v_3, v_5, v_6\}$ are double monophonic sets of G. Out of these, $T_9 = \{v_2, v_4, v_5\}$ and $T_{10} = \{v_2, v_4, v_6\}$ are the only two minimal double monophonic sets. It is verified that all the 4-element sets are double monophonic, and none of them is minimal. It is also verified that all 5-element sets are double monophonic, and none of them is minimal. Hence for this graph $m(G) = 2$, $dm(G) = 2$ and $dm^+(G) = 3$.

A vertex in a graph G of order n is called a *full degree vertex* if its degree is $n - 1$. The *monophonic closed interval* $I_m[x, y]$ consists of all vertices lying on some $x - y$ monophonic path of G.

Theorem 2.8. Let G be a non-complete connected graph. Then a full degree vertex does not belong to any minimal double monophonic set of G.

Proof. Let S be a minimal double monophonic set of G containing a full degree vertex v_1. Let $S' = S - \{v_1\}$. We claim that S' is double monophonic set of G. Let $x, y \in v$.

Case 1. $x, y \in S$. If $v_1 \neq x, y$, then x, $y \in S'$ and so S' is double monophonic set of G. So assume that $x = v_1$. If y is not a full degree vertex, then there exists $y' \neq y$ such that y and y' are non-adjacent and so $x, y \in I_m[y', y]$ with $y', y \in S'$. Now , if y is a full degree vertex, then since the subgraph induced by S is not complete, there exist non-adjacent vertices y', y'' in S such that $x, y \in I_m[y', y'']$. Thus S' is a double monophonic set of G, which is a contraction to S a minimal double monophonic set.

Case 2. $x \notin S$ or $y \notin S$. Since S is a double monophonic set, there exists $u, v \in S$ such that $x, y \in I_m[u, v]$. Since v_1 is a full degree vertex, it follows that $u \neq v_1$ and $v \neq v_1$. Thus $u, v \in S'$ and so S' is a double monophonic set of G, which is again a contradiction to S a minimal double monophonic set of G. Thus the proof is complete.

Theorem 2.9. Let G be a non-complete graph of order n with a full degree vertex v. Then $dm^+(G) = n - 1$ if and only if $dm(G) = n - 1$.

Proof. If $dm(G) = n - 1$, then by Corollary 2.5, $dm^+(G) = n - 1$. Let $dm^+(G) = n - 1$. Let S be a minimal double monophonic set of cardinality $n - 1$. By Theorem 2.8, $v \notin S$. Suppose that $dm(G) \leq n - 2$. Let S' be a minimum double monophonic set of G. Then it follows from Theorem 2.8 that $v \notin S'$ and $S' \subseteq S$, which is a contradiction to S a minimal double monophonic set of G. Hence $dm(G) = n - 1$.

Theorem 2.10. Let G be a connected graph with a cut vertex v. Then every minimal double monophonic set of G contains at least one vertex from each component of $G - v$.

Proof. This follows from Theorem 1.2

Theorem 2.11. No cut vertex of a connected graph G belongs to any minimal double monophonic set of G.

Proof. Let S be any minimal double monophonic set of G. Suppose that S contains a cut vertex w of G. Let $G_1, G_2, ..., G_k (k \geq 2)$ be the components of $G - w$. Let $S_1 = S - \{w\}$. We show that S_1 is a double monophonic set of G. Let u, v be any two vertices of G. Since S is a double monophonic set, there exist $x, y \in S$ such that $u, v \in I_m[x, y]$. If $w \notin \{x, y\}$, then $x, y \in S_1$ and so S_1 is a double monophonic set of G, which is a contradiction to the minimality of S. Now, assume that $w \in \{x, y\}$, say $w = x$. Assume without loss of generality that y belongs to S_1. By theorem 2.10, we can choose a vertex z in $G_l (l \neq 1)$ such that $z \in S$. Now, since w is a cut vertex of G, it follows that $I_m[w, y] \subseteq I_m[z, y]$. Hence $u, v \in I_m[z, y]$, where $z, y \in S_1$ so that S_1 is a double monophonic set of G, which is a contradiction to the minimality of S. Thus no cut vertex belongs to any minimal double monophonic set of G.

In the following we present the upper double monophonic number of some standard graphs.

Theorem 2.12. For any tree T with k end-vertices $dm(T) = k = dm^+(T)$.

Proof. This follows from Theorems 1.1 and 2.11

Theorem 2.13. For the complete bipartite graph $G = K_{m,n}$,

(i) $dm^+(G) = 2$ if $m = n = 1$.
(ii) $dm^+(G) = n$ if $m = 1, n \geq 2$.
(iii) $dm^+(G) = max\{m, n\}$ if $m, n \geq 2$.

Proof. Results (i) and (ii) follow from Theorem 2.12. (iii) Let X and Y be the partite sets of $K_{m,n}$. Let S be a double monophonic set of $K_{m,n}$. We claim that $X \subseteq S$ or $Y \subseteq S$. Otherwise, there exist vertices x, y such that $x \in X, y \in Y$ and $x, y \notin S$. It is clear the pair of vertices x, y lie only on the intervals $I_m[x, y]$, $I_m[x, t]$ and $I_m[s, y]$ for some $t \in X$ and $s \in Y$. Therefore $x \in S$ or $y \in S$, which is a contradiction. Thus $X \subseteq S$ or $Y \subseteq S$. Since both X and Y are double monophonic sets of $K_{m,n}$ the result follows.

Theorem 2.14.
(i) For the cycle $G = C_n(n \geq 4), dm(G) = dm^+(G) = 2$.
(ii) For the wheel $G = W_1, n-1, dm(G) = dm^+(G) = 2$.
(iii) For the graph $G = K_n - e, dm^+(G) = 2$.

Proof. (i) It is clear that any set S of vertices consisting of two non-adjacent vertices is a double monophonic set so that $dm(G) = 2$. Now let T be any double monophonic set of vertices such that $|T| \geq 3$. Then S contains at least two non-adjacent vertices so that T is not minimal. It follows that $dm^+(G) = 2$.
(ii) Let $v = \{v, v_1, v_2, ..., v_{n-1}\}$ with v the central vertex and $v_1, v_2, ..., v_{n-1}$ the cycle C_{n-1}. Let S be any set consisting of two non-adjacent vertices on the cycle C_{n-1}. It is clear that S is a double monophonic set of G so that $dm(G) = 2$. Now, let T be any double monophonic set of vertices such that $|T| \geq 3$. Then S contains at least two non-adjacent vertices so that T is not minimal . It follows that $dm^+(G) = 2$.
(iii) Let e be the edge $e = uv$. Then u and v are the only extreme vertices of G and it is clear that $S = \{u, v\}$ is a double monophonic set so that $dm(G) = 2$. Let T be any double monophonic set such that $|T| \geq 3$. Since u and v are extreme vertices, by Theorem 1.1, $u, v \in T$ so that T is not minimal. Hence $dm^+(G) = 2$.

The following theorem is a realization result with regard to Theorem 2.3

Theorem 2.15. For any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $dm(G) = a$ and $dm^+(G) = b$.

Proof. For $a = b$, it follows from Theorem 2.12 that $dm(G) = dm^+(G) = a$, where $G = K_{1,a}$. For $a < b$, it follows from Theorem 2.13 that $dm(G) = a$ and $dm^+(G) = b$, where $G = K_a, b$.

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