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UPPER LIMITS IN EXPERIMENTS WITH BACKGROUND OR MEASUREMENT ERRORS

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ABSTRACT

It is shown that upper limits can be calculated in classical statistics for measurements contaminated by background or distorted owing to the finite resolution of the apparatus. The a posteriori probability distributions, as fixed by the experimental results, are used for the background and the measurement errors, respectively. Contrary to the Bayesian approach, assumptions on prior distributions of unknown parameters are avoided.

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1. INTRODUCTION

The search for rare processes often leads to the observation of a few events that can be explained by background. If the expected background can be calculated with high precision, an upper limit for the physical process can be derived [1-3], using Bayes theorem. For a confidence level $c = 1 - \varepsilon$, N observed events, and an expected number of background events b, the limit for the signal s is given by:

$$\varepsilon = \int_s^\infty P(n;s'+b)ds' \Big/ \int_0^\infty P(n;s'+b)ds' \ , \qquad (1)$$

where $P(n;\mu)$ is the Poisson distribution with the expectation value μ.

The confidence limit is interpreted as the probability that the true signal has a value smaller than s_0. The derivation of Eq. (1) assumes a uniform a priori distribution for s. Both the assumption and the interpretation of the result are questionable and are unacceptable to many physicists [3, 4]. In order to avoid the problems related to the Bayesian approach, upper limits are usually defined through the frequency of obtaining a result smaller than or equal to the observed one, if the experiment were repeated many times with a signal of mean s [5]. The result (1) remains correct with the modified definition, but the coincidence is accidental. Below we will demonstrate the validity of Eq. (1) in classical statistics and discuss its interpretation. A similar approach to that for discrete variables can be used in the analogous case of a continuous variable distorted by measurement errors. We indicate, for a simple example, how an upper limit can be computed without using Bayes theorem.

The arguments and results presented in this paper are certainly not new to everybody and have been applied in the past. But it is also true that doubtful limits have been published and a variety of methods have been used in different experiments [6].

2. POISSON DISTRIBUTED SIGNALS WITH BACKGROUND

For given average values of the signal s and the background b, the probability of observing n events is given by the Poisson distribution

$$P(n;s+b) = \frac{e^{-(s+b)}(s+b)^n}{n!} \ . \qquad (2)$$

This formula is the product of the Poisson probabilities for the signal and the background numbers for all possible combinations that give a total of n:

$$P(n;s+b) = \sum_{n_b=0}^{n} \sum_{n_s=0}^{n-n_b} P(n_b;b)P(n_s;s) \ .$$

The relation (2) cannot be used directly to compute an upper limit on s. This can be seen easily by considering the special case with large b and no observed event, where a negative unphysical limit would be obtained.

In an experiment where b background events are expected and N events have been found, $P(n_b;b)$ no longer corresponds to our improved knowledge of the background distribution. Since n_b can only take the numbers $n_b \leq N$, P_b has to be renormalized to the new range of n_b:

$$P'(n_b;b) = P(n_b;b) \Big/ \sum_{n_b=0}^{N} P(n_b;b) \ .$$

Therefore the combined distribution (2) is changed into

$$W(n) = P(n;s+b) \Big/ \sum_{n_b=0}^{N} P(n_b;b) \ .$$

The probability ε of observing N events or less, knowing that the background is N events or less for a signal s, is then given by the sum

$$\varepsilon = \sum_{n=0}^{N} W(n)$$

$$\varepsilon = \sum_{n=0}^{N} P(n;s+b) \Big/ \sum_{n_b=0}^{N} P(n_b;b) \ . \qquad (3)$$

Relation (3) can be used to compute an upper limit s with confidence level $c = 1 - \varepsilon$. The formulae (1) and (3) are mathematically identical, but the intrepretations are different. The limit in the 'frequency interpretation' can be stated as follows: for an infinitely large number of experiments, looking for a signal with expectation s and Poisson distributed background with mean b, where the background is restricted to values of less or equal to N, the frequency of observing N or less events is ε.

Uncertainties on the mean b can be included easily in Eq. (3). For a given probability distribution g(b) we obtain:

$$\varepsilon = \sum_{n=0}^{N} \int g(b)P(n;s+b)db \Big/ \sum_{n_b=0}^{N} \int g(b)P(n_b;b)db \ . \qquad (4)$$

3. CONTINUOUS VARIABLES WITH MEASUREMENT ERRORS

The frequency definition can also be applied to continuous variables distorted by measurement errors. A general treatment of the problem leads to clumsy formulae which obscure the essential arguments. We prefer to present a simple example, leaving the generalization to the reader.

We propose to calculate an upper limit on a particle lifetime $\tau = 1/\lambda$, from the observation of a single decay at time T. The measurement error is assumed to follow a Gaussian G with variance σ^2 independent of the true decay time t. The a priori distribution $f(t';\lambda)$ of observed times t' is given by the convolution

$$f(t';\lambda) = \int_0^\infty G(t'-t;\sigma)\lambda e^{-\lambda t} dt \ ,$$

and the probability ε of observing t' smaller than T is the integral

$$\varepsilon = \int_{-\infty}^{T} f(t';\lambda) dt' \ . \qquad (5)$$

The formula (5) cannot be used naîvely to calculate an upper limit on λ, because, once T has been found, the measurement fluctuations are restricted to values smaller than T, and the error distribution has to be renormalized to this range:

$$\varepsilon = \int_{-\infty}^{T} f(t';\lambda) dt' \Big/ \int_{-\infty}^{T} G(x;\sigma) dx \ . \qquad (6)$$

Relation (6) gives the correct upper limit for λ with confidence $1 - \varepsilon$.

The same problem would lead to a different result in the Bayesian philosophy and depend on the choice of the parameter. The upper limit for τ would be different from $1/\lambda$, the inverse of the upper limit of the decay width λ.

4. CONCLUSION

The calculation of confidence limits is very unsatisfactory in methods where a priori probabilities for unknown constants are needed. This difficulty is avoided in classical statistics. However a naîve application to experiments with background or measurement errors can result in unphysical values for the parameters, such as negative rates or lifetimes. Experiments suffering from background or large measurement errors would on the average produce limits as good as those of clean and precise experiments. This problem is avoided by using a posteriori distributions for the background and the measurement errors. These are obtained using the restrictions given by the experimental data. The method is well defined and gives reasonable limits. We propose to adopt it as a standard and to include it in the definition of an upper limit in classical statistics.

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