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Upper Secondary Students' Task Reasoning.

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Upper secondary students' task solving reasoning was analysed, with a focus on grounds for different strategy choices and implementations. The results indicate that mathematically well-founded considerations were rare. The dominating reasoning types were algorithmic reasoning, where students tried to remember a suitable algorithm, sometimes in a random way, and piloted reasoning, where progress was possible only when essentially all important strategy choices were made by the interviewer.

1. Introduction

As we see it, the Swedish school system¹ cannot help sufficiently many students to reach a desired level of mathematical competence. The syllabus for mathematics at upper secondary school in Sweden mentions the ability to analyse and solve problems as one of the central purposes of the subject². Reasoning is a central component in mathematics and especially in problem solving. Ross [2] claims that "It should be emphasised that the foundation of mathematics is reasoning. [...] If reasoning ability is not developed in the students, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense."

Earlier research [3–5] indicates that in problematic situations undergraduate students tend to rely heavily on their, often mathematically superficial, experiences from school situations. Their strategies are rarely grounded in relevant mathematical concepts, and in the few cases where the students' reasoning is grounded in consistent mathematics it is still dominated by the individuals memory images and familiar routines. This can be argued to be one of the main causes behind learning difficulties, since neither non-routine problem solving nor conceptual understanding are developed through superficial algorithmic reasoning. The purpose of this paper is to investigate in what ways this behaviour is present among upper secondary school students by making a parallel study to Lithner [4].

Upper secondary students in Sweden spend a large part of their time solving different types of mathematical tasks, and mainly textbook exercises [6]. In this activity, the students often meet problematic situations where it is not obvious how to proceed. When a student faces this kind of difficulty she must make a strategy choice concerning what to do in order to solve the problem, and then implement this strategy. This study, which is a summary of the longer pre-print version [7], investigates the bases on which the choices and implementations are made, and the outcomes in terms of success or failure.

2. Theoretical framework

The framework is a summary of [8] which is a theoretical structuring of the outcomes of a series of empirical studies (Section 1) aiming at analysing characteristics the relation between reasoning types and learning

¹including, among others, teachers at all levels, teacher educators and researchers

²"The subject also aims at pupils being able to analyse, critically assess and solve problems in order to be able to independently determine their views on issues important both for themselves and society, covering areas such as ethics and the environment." [1]

difficulties in mathematics.

2.1. *Creative reasoning*

‘Reasoning’ in this paper is the line of thought, the way of thinking, adopted to produce assertions and reach conclusions. It is not necessarily based on formal deductive logic, and may even be incorrect as long as there are some kind of sensible (to the reasoner) arguments that guide the thinking. *Argumentation* is the substantiation, the part of the reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate. In particular, in a task solving situation, which is called *problematic situation* if it is not clear how to proceed, two types of argumentation are central:

- 1) The *strategy choice*, where ‘choice’ is seen in a wide sense (choose, recall, construct, discover, guess, etc.). This choice can be supported by *predictive argumentation*: Will the strategy solve the difficulty?
- 2) The *strategy implementation*, which can be supported by *verificative argumentation*: Did the strategy solve the difficulty?

In this paper, creative reasoning is seen as what you do when you solve non-routine problems [9]. According to Haylock [10] there are at least two major ways in which the term is used: i) Thinking that is divergent and overcomes fixation. ii) The thinking behind a product that is perceived as creative by a large group of people, e.g. works of arts. Central here are the creative aspects of ordinary mathematical task solving thinking, thus notion ii) is not very useful here. Haylock [10] sees two types of fixation. Content universe fixation is in terms of range of elements seen as appropriate for application to a given problem: useful knowledge is not seen as useful. Algorithmic fixation is shown in the repeated use of an initially successful algorithm when this becomes inappropriate. Silver [11] argues that “Although creativity is being associated with the notion of ‘genius’ or exceptional ability, it can be productive for mathematics educators to view creativity instead as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population.” He adds that students hardly experience mathematics as the highly creative intellectual domain it is. Silver sees fluency, flexibility and novelty as the core components of creativity.

In school tasks, one of the goals is also to achieve a high degree of certainty, but one crucial distinction from professional tasks is that within the didactic contract [12] of school it is allowed to guess, to take chances, and use ideas and reasoning that are not completely firmly founded. Even in exams it is accepted to have only, for example, 50% of the answers correct, while it is absurd if the mathematician, the engineer, or the economist are correct only in 50% of their conclusions. This implies that it is allowed, and perhaps even encouraged, within school task solving to use forms of mathematical reasoning with considerably reduced requirements on logical rigour. Pólya [13] stresses the important role of reasoning that is less strict than proof: “In strict reasoning the principal thing is to distinguish a proof from a guess, a valid demonstration from an invalid attempt. In plausible reasoning the principal thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess.”

In this framework, well-founded arguments are anchored in intrinsic properties of components involved in the reasoning. The components one is reasoning about consist of objects, transformations, and concepts. The *object* is the fundamental entity, the ‘thing’ that one is doing something with or the result of doing something. E.g. numbers, variables, functions, graphs, diagrams, matrices, etc. A *transformation* is what is being done to an object (or several), and the outcome is another object (or several). Counting apples is a transformation applied to real-world objects and the outcome is a number. To calculate a determinant is a transformation on a matrix. A *concept* is a central mathematical idea built on a related set of objects, transformations, and their properties. For example the concept of function or the concept of infinity. Since a property of a component may be more or less relevant in a particular context and problematic situation, it is necessary to distinguish between *intrinsic* properties that are central and *surface* properties that have no or little relevance. In deciding which of the fractions $99/120$ and $3/2$ that is largest, the size of the numbers (99, 120, 3, and 2) is a surface property that is insufficient to consider in this particular task while the quotient captures the intrinsic property.

Creative reasoning (CR) fulfils the following conditions:

- i) Novelty:** A new (to the reasoner) sequence of solution reasoning is created, or a forgotten sequence is re-created. To imitate an answer or a solution procedure is not included in CR.
- ii) Flexibility:** It admits different approaches and adaptations to the situation. It does not suffer from fixation that hinders the progress.
- iii) Plausibility:** There are reasons supporting the argumentation in the strategy choice and/or strategy implementation, motivating why the conclusions are true or plausible. This means that pure guesses or vague intuitions are not considered, neither are affective reasons.
- iv) Mathematical foundation:** The argumentation is founded on intrinsic mathematical properties of the components involved in the reasoning.

2.2. Imitative reasoning

More frequent than CR are different versions of *imitative reasoning*, that is, copying or following a model or example without any attempt at originality. Hiebert [14] finds massive amounts of converging data showing that students know some basic elementary skills but there is not much depth and understanding. Students are more proficient in processes like calculating, labelling, and defining than reasoning, communicating, conjecturing, and justifying. Learning difficulties are partly related to a reduction of complexity that appears as a procedural focus on facts and algorithms and a lack of relational understanding [8]. The framework should also capture reasoning that is not justified by mathematical reasons, but has other origins [15]. The definitions below aim at characterising imitative reasoning that may be based on surface clues in non- or semi-cognitive attempts to cope.

Memorised reasoning (MR) is when:

- (i) The strategy choice is founded on recalling by memory an answer.
- (ii) The strategy implementation consists only of writing it down. One can describe any part of the answer without having considered the preceding parts.

An example is to recall every step of a proof.

An *algorithm* is a set of rules that will if followed solve a particular task type. The most common algorithms consist of procedures (sequences of transformations of mathematical objects).

Algorithmic reasoning (AR) is when:

- (i) The strategy choice is founded on recalling by memory, not the whole answer in detail as in MR, but a set of rules that will guarantee that a correct solution will be reached.
- (ii) After this set of rules is given or recalled the reasoning parts that remain in the strategy implementation are trivial for the reasoner and only a careless mistake can hinder that an answer to the task is reached.

Fundamental in AR is how to identify a suitable algorithm. If this can be done, the rest is straightforward. AR based on surface property considerations is common, often dominating, and the studies mentioned in Section 1 have distinguished two (partly overlapping) families of common reasoning:

Familiar AR This reasoning consists of strategy choice attempts to identify a task as being of a familiar type and with a corresponding known solution algorithm. The simplest example is a version of the Key word strategy where the word 'more' in a text task is connected to the addition algorithm and 'less' to subtraction [16]. Another example is in Lithner [3] where students make a holistic but superficial interpretation of the task text and reach a clear but faulty image that it is of a particular familiar type.

Guided AR An individual's reasoning can be guided by a source external to the task. The two main types empirically found are:

- (i) Piloted reasoning, when someone (e.g. a teacher) pilots a student's solution.
- (ii) Identification of similarities, where the strategy choice is founded on identifying similar surface properties in an example, definition, theorem, or some other situation in a text source connected to the task.

3. Research question and method

This study is based on the following research question:

In what ways do students manage or fail to engage in CR, MR, and AR as a means of making progress in problematic situations?

The participating students came from the natural science programme (high mathematical intensity), the social science programme (medium intensity), and the hotel, restaurant and catering programme (low intensity). Three teachers were asked to suggest five students from each program, avoiding students that had extremely high or low grades, and all suggested students agreed to participate. To each programme three to six mathematical tasks, dealing with material recently covered, were selected. The 40-minute sessions were video taped, showing the students' written work and use of a calculator, and the students were asked to think aloud.

The tasks were constructed with the purpose of generating problematic situations. Four criteria were considered: 1) The tasks should not be too easy since then the students would not meet any problematic situations. 2) The tasks should not be too difficult, since then the students would make no progress and there would be no solutions to analyse. In some situations where it was clear that the students would make no progress without aid, the interviewer helped the student with minor hints rather than having the student be forced to leave the solution attempt. This is possible since the purpose of the study is not to see if they managed to solve the tasks, but to see how they reasoned in problematic situations. 3) The solutions of the tasks should be at least partly based on familiar algorithms, in order to make it possible to analyse the balance between AR and CR in the students reasoning. 4) The content and difficulty should be well within the syllabus.

Among the 50 solution attempts made by the students, 7 problematic situations fulfilling criteria 1 and 2 were more extensively analysed than the other, and constitute the data base for this paper. Three of the analyses are presented below. The analysis was, for each student, conducted in three steps: First a *description* of the data from the video recording and the post-interviews. Then an *interpretation* of the data (supported by a post-interview), with the aim to understand the central parts of the reasoning that is not explicit in the data. Finally, in the *analysis* the reasoning was characterised with the help of the framework in Section 2, or perhaps classified as a new (not already in the framework) reasoning type. The purpose was to characterise the reasoning used in each problematic situation.

4. Analysis

The work of four students will be presented, each followed by a summary of the reasoning characteristics. In the quotes, that are translated from Swedish, ... indicates a pause and and [...] omitted irrelevant passages.

4.1. Sally

The natural science student Sally was working on the following task:

Task N1. Find the largest and smallest values of the function $y = 7 + 3x - x^2$ on the interval $[-1, 5]$.

Part 1. Sally reads the task and immediately says "Here you are supposed to differentiate, so that you can find the maximum and minimum points, at least I think so". She differentiates y , finds the zero of the derivative ($x = 1.5$) and evaluates $y = 9.25$. Now she stops. "Hum... I wonder why..." The interviewer asks what it is that makes her uncertain. "I think that I should have got two values, and I don't know why I didn't, what I have done wrong." Sally experiences that her answer doesn't match the question, but only in the sense that the task asks for two values and her method gave one (since she did not complete the method by checking the interval endpoints). She makes no attempt to understand why this is the case, and starts working on another task.

Part 2. When Sally gets back to Task 1 after 20 minutes of work on other tasks, she draws the graph

on her calculator and tries to apply the built-in function for finding the smallest value of a function in a specified interval. "There aren't any of those to look at." She means that there is no local minimum to specify an interval around (since the function does not have one). Again Sally makes no attempt to understand how the method really works or why her attempt failed. She simply avoids the problematic situation by trying another method.

Part 3. Sally's next step is to use the *table*-function, which renders a table with x - and y -values (see Figure 1). The calculator is set for integer steps so Sally finds -3 as the smallest value and 9 as the largest

X	$-1.$	$0.$	$1.$	$2.$	$3.$	$4.$	$5.$
Y_1	$3.$	$7.$	$9.$	$9.$	$7.$	$3.$	$-3.$

Figure 1. Sally's table

value for x from -1 to 5 . She says "Between -1 and 5 ... Then the smallest value should be -3 and the largest value 9 . Here [in Part 1] I got 9.25 . That's... clever." Instead of trying to resolve this contradiction she terminates this solution attempt and tries another algorithm.

Part 4. Sally now tries to solve the task by setting the function equal to zero, and solving the familiar equation (which is a faulty method). She obtains $x_1 \approx 4.54$ and $x_2 \approx -1.54$. She believes that the two values are the answers to the task, but she is still uncertain when the interviewer asks her why she gets the largest and smallest values when solving the equation. She answers "Because... I don't know. I think I get them. But I'm still uncertain because of that differentiation stuff..."

Reasoning characteristics.

Sally's initial difficulty is caused by her inability to correctly recall the complete procedure for this type of optimisation task: To investigate both critical points and endpoints of the interval. Her next difficulty, in trying to resolve this difficulty, seems to be that her conceptual understanding is too weak to help her construct or reconstruct a suitable solution method. She may also be hindered by a belief that she is unable to even try to construct her own solution procedure, a belief that is common among students [17].

Sally makes five major strategy choices. Characteristic of these choices are:

- i) All choices concern trying familiar algorithms.
- ii) There are no signs that any of these choices are based on intrinsic properties of the task or the procedures. There is no explicit motivation behind neither the reasonable nor the faulty choices. It seems that only surface properties are considered, e.g., that the task concerns second degree polynomials.
- iii) It seems likely that all choices have some connection to her prior learning: She has solved many optimisation tasks by solving $f'(x) = 0$, and some of them using graphing calculator functions. She has also solved many quadratic equations, but here the connection to the present task is extremely superficial.

The general characteristics of her strategy implementations are:

- iv) The implementation of each algorithm is carried out by following it step by step. There is no (successful) analysis, evaluation or some other consideration whether the algorithms are suitable and generate any useful knowledge or not.
- v) When progress does not occur as expected (too slow, contradicting results or something else), Sally is quick to abandon the procedure and search for another.

Delimiting AR.

Because of i) and iv), each of the five strategies can be characterised as algorithmic reasoning (AR). In Familiar AR the task is apprehended, wrongly or rightly, by the reasoner as a familiar one and is seen to correspond to a known algorithm. If the task is not sufficiently familiar and the reasoner knows too many algorithms to try them all, one approach is to try to delimit the set of algorithms to try. Based on ii), iii), and v) a new version of AR will be defined as *Delimiting AR* where:

- (i) An algorithm is chosen from a set that is delimited by the reasoner through the included algorithms' surface property relations to the task.
- (ii) The strategy implementation is carried through by following the algorithms. No verificative argumentation is required. If an algorithm does not lead to a (to the reasoner) reasonable conclusion, then the implementation is not evaluated but simply terminated and a new algorithm is tried.

Often Delimiting AR is carried out only in one step, either because the reasoner only knows one algorithm within the delimited set or because the first attempt yields an acceptable conclusion. Trying different procedures is a quite reasonable approach, but Sally's problem is that her conceptual and procedural understanding is too incomplete to help her make proper choices and evaluate them, the choices become mathematically random and the evaluations are simply never done. Relatively elementary CR could have been applied in order to make progress, for example: Sally could have considered some of the circumstances under which second degree polynomials have both a minimum and a maximum. She could have tried to relate the table values or the function or derivative zeros to the graph. In this way she may have come to realise that a second degree polynomial can have only one critical point, so the other extremum must be found elsewhere: at an endpoint.

4.2. Julia

The social science student Julia is working with the following tasks:

Task S1. $y = 3x - 2$ is the equation for a straight line. a) What is the slope of the line? b) Find where the line cuts the y -axis. c) Sketch the line in a coordinate system.

Task S2. Find the equation for the straight line passing through the points $(2, 3)$ and $(5, 9)$.

Part 1, Task S1c. Julia starts by saying "Oh, I can't do stuff like this. The slope of the line... Am I supposed to draw something or?" The interviewer responds affirmatively to Julia's question if it is OK to start with task 1c, and Julia draws a coordinate system. "I wonder if this is right, if you should move step-by-step, counting two ahead and one up or something." She sketches a small picture showing what she means. The interviewer then asks her how she would use that method on Task S1. Julia seems very uncertain, but after a few moments she uses the indicated method and marks some points: $(0, 0)$, $(3, -2)$ and $(6, -4)$. She then draws a line through the three points (see Figure 2). "I don't know, no idea... Something like this maybe." By looking at her faulty graph, she then states that the answer to Task S1b is zero.

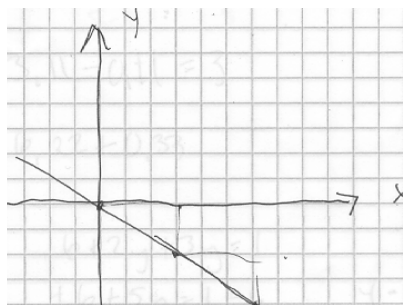


Figure 2. Julia's graph on Task 1c

Part 2, Task S1a. Considering Task S1a, Julia first says "Are you supposed to count... no maybe not... I don't know" and turns to other tasks instead. 25 minutes later she returns to the task. "What is the slope... Are you supposed to calculate k maybe? [...] k is that funny... gradient. I don't know." When the interviewer wants to discuss the k -value Julia says "Yes... because that is delta y delta x [writing $\frac{\Delta y}{\Delta x}$]." The interviewer then gives her Task S2, since that is a familiar task concerning this expression, which Julia solves almost without difficulties. When Julia is asked to compare what she did on Task S2 to Task S1a she can point out the slope on Task S2 (when prompted by the interviewer) and then find the answer to Task S1a. She says "because if you write $y = kx + m$, k is the slope." She concludes the work on the task by saying "But that is too easy".

Reasoning characteristics

Julia meets several problematic situations but the strategy choices are made, or at least influenced, by the interviewer. In some situations Julia is encouraged to try her ideas, like when she hesitantly considers an

algorithm and the interviewer tells her to try it. In other situations the interviewer takes the initiative, like making the connection between slope and the k -value. The strategy implementations are based on surface considerations, e.g. when Julia remembers something about moving step-by-step. When she tries to use the method in Task S1a there is no attempt to analyse how the algorithm works, or why she should start at the origin. When the graph is drawn there is no attempt at verificative argumentation. This lack of reflection may be one of the more important reasons behind Julia's difficulties. The last strategy choices mainly consists of following the interviewers lead, and it is unlikely that she would have solved the task without guidance. The lack of both predicative and verificative argumentation leads to a characterisation as Guided AR.

4.3. Adam

Before Task H6, the hotel, restaurant and catering student Adam had been working on Task H3 and Task H5. He did well on task H5, but had some difficulties on task H3, making an error in his first attempt but solving it after the interviewer told him that he had made a mistake. He also verifies the correct answer.

Task H3. Solve the equation $3x - 2 = 7$

Task H5. Is $x = 7$ a solution to the equation $2x + 3 = 15$?

Task H6. Solve the equation $4 - x = 3x + 14$

Adam uses small figures to indicate changes on both sides of the equal sign:

$$\begin{aligned} 4^{-4} - x &= 3x + 14^{-4} \\ x &= 3x + 14 \\ x^{+x} &= 3x^{+x} + 14 \\ 4x &= 14 \\ \frac{14}{4} &= 3.5 \\ x &= 3.5 \end{aligned}$$

There are several errors but he proceeds methodically through the process. In trying to verify the answer Adam uses the calculator and calculates $4 - 3.5$, but then he stops. He sits quiet, looking at the equation and the calculator, and the interviewer asks why he hesitates. "I didn't know if I should use plus or minus $3x$."

The interviewer points at Task H3, where Adam verified that $x = 3$ was a solution, and asks about the difference between the tasks. "It is easier if you have... if you have the final... the final result alone, and you don't have that here. Here you have... The difference that hinders Adam is that the verification in H3 could be done as a straightforward familiar algorithm, using the calculator to first evaluate $3 \cdot 3 - 2$, and then see if the outcome is 7. In H6 the right side of the equation is not a numerical value, and the algorithm cannot be used.

Reasoning characteristics

Adam relies on his memory when he tries to decide what method to use. He said in the post-interview that he goes to class, listens to the teacher presenting methods, and tries to remember how to solve different types of tasks. He indicates that for him mathematics consists of algorithms and methods. Unlike Sally and Julia, Adam knows what algorithm to choose in the first step, and the reasoning is thus characterised as Familiar AR, but he cannot remember sufficiently well to avoid errors. In the next step, when trying to verify the solution, the limitation of his AR reasoning becomes even clearer: Adam knows an algorithm for verifying the solution to H3 but is unable to carry out the slight modification of it needed for verifying H6. It would take (local) CR to argue that the right and left side should be equal and thus they can be evaluated separately. The verification procedure he uses in H3 seems to be a mathematically meaningless algebraic transformation. He expresses no insights into what the transformations really do and why they are done.

5. Discussion

In almost all problematic situations the students chose their strategies on only or mainly surface property considerations, and they focused on using more or less well mastered algorithms. This may sometimes be a reasonable strategy, but it is often insufficient when meeting different kinds of problematic situations. CR is often suitable in problematic situations, but this was rare in the situations analysed. When the students for some reason failed to carry out the chosen algorithm, two main different approaches were found. One was to quickly change to another algorithm chosen from a 'toolbox' of possible alternatives, and the decision whether an algorithm was appropriate or not was based on surface considerations. The other was to simply stop working. Both approaches were often combined with questions or comments to the interviewer in order to get some kind of hint or guidance about what to do next. There were almost no situations where the students tried to evaluate the chosen algorithm, reconstruct it, or tried to modify the algorithm to the situation at hand.

The students in the Social Science programme and the Hotel, Restaurant and Catering programme to a large extent used Familiar AR or Delimiting AR in their work. They tried to find a suitable algorithm, often by trying to remember or on other superficial grounds. The algorithm was chosen because it had something to do with the situation at hand, and (occasionally) that it seemed possible that it would solve the task. Then the algorithm was carried out, step by step, mostly without attempts to verify or evaluate neither algorithms nor results (see Section 4.3).

Repeated Delimiting AR was found in the work of a few students in the Natural Science programme. They changed quickly between algorithms and appeared rather skillful in their work, which is not the case among the students from the other two programs. The problem was that they rarely had any intrinsic property considerations as a ground for their choices of algorithms, and therefore the possible success depended on two things: if the chosen algorithm would solve the task, and if the algorithm was mastered well enough by the student. For example, Sally (Section 4.1) tried four different algorithms in only a few minutes. The first choice would have solved the task if she had mastered the whole algorithm and not only the first part. The characterisation of Delimiting AR is an outcome of this study.

Among the situations where the students used algorithms that failed, the most notable part was the almost total lack of attempting to understand why the algorithm failed, or if it could be modified to the situation at hand. This lack also seemed to be one major reason behind their difficulties.

There were several situations, mainly in the Social Science programme and in the Natural Science programme, where students relied heavily on their interaction with the interviewer. This was classified as Guided AR, since all important strategy choices were either made by the interviewer, or came as a result of a question or a comment from the interviewer. One example of this can be found in Section 4.2.

Piloted reasoning is well in line with the concept of the didactical contract [12]. The students listen carefully to what the teacher says, and acts according to the conversation. For the students, this is a way to get correct answers in a very large part of the tasks. To the teacher, it means a quick and manageable way to guide almost a whole class through the textbook. Brousseau claims that as long as both teacher and student follow the didactical contract, no learning occurs. However, piloted reasoning can also be something positive, a way to help a student to reach understanding in an area. If the student is uncertain on a specific level, it is possible for the teacher to, by piloted work on a higher level, help the student to strengthen his understanding on the lower level. In that case, it is crucial that the guidance is not too extensive. The guider should not resolve all problematic situations for the student.

CR was absent with a few exceptions (e.g. analysing the shape of a graph and deducing where to find the smallest value). It appears that sometimes students do not attempt CR, and sometimes their conceptual understanding is not sufficient for CR. The two competences conceptual understanding and CR ability are probably connected, since CR requires basic conceptual understanding. CR ability may not be possible to develop by only solving routine, non-CR, exercises where the main goal is to practice algorithms [5]. It should be stressed that the conceptual understanding that is indicated in the analyses as missing are

relatively elementary in relation to the courses the students have taken.

The present paper is a follow-up study to Lithner [4], which had a similar purpose and setting but the participants were undergraduate students. The undergraduate students also focussed on AR in essentially the same ways as the participants of this study. One difficulty at higher educational levels is that the mere amount of known algorithms becomes immense, compared to lower levels. One can hypothesise that there is a progression in superficial reasoning from elementary to more complex mathematical situations: In primary school there are normally few algorithms to choose from, and the keyword strategy (Section 2.2) may work well. In secondary and tertiary education, mathematics becomes more complex and there are more algorithms. It becomes necessary to limit the amount of algorithms to test, and Delimiting AR may be a way to cope if Familiar AR does not work. As the student progress through secondary and perhaps into tertiary mathematics, the mere amount of algorithms may at some stage be insurmountable. This may hypothetically be one of the causes behind the learning collapse we see when students that have managed well seem to suddenly and completely lose contact with mathematics and drop out.

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