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UPPER SEMI CONTINUITY OF ATTRACTORS OF DELAY DIFFERENTIAL EQUATIONS IN THE DELAY

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It is shown that if a retarded delay differential equation has a global attractor $\mathcal{A}_{\tau_0}$ in the space $C([-\tau_0, 0], \mathbb{R}^d)$ for a given nonzero constant delay τ_0, then the equation has an attractor $\mathcal{A}_\tau$ in the space $C([-\tau, 0], \mathbb{R}^d)$ for nearby constant delays τ. Moreover the attractors $\mathcal{A}_\tau$ converge upper semi continuously to $\mathcal{A}_{\tau_0}$ in $C([-\tau_0, 0], \mathbb{R}^d)$ in the sense that they are identified through corresponding segments of entire trajectories in $\mathbb{R}^d$ with nonempty compact subsets $I_{\tau_0}(\mathcal{A}_\tau)$ of $C([-\tau_0, 0], \mathbb{R}^d)$ which converge upper semi continuously to $\mathcal{A}_{\tau_0}$ in $C([-\tau_0, 0], \mathbb{R}^d)$.

1. INTRODUCTION

Consider a retarded delay differential equation

$$\dot{x}(t) = f\big(x(t), x(t-\tau)\big), \quad t \geqslant 0, \tag{1}$$

in $\mathbb{R}^d$ with a constant delay $\tau > 0$ and assume that $f \in C_b^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$, where C_b^1 is the space of continuously differentiable functions from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$ which are bounded and have bounded first derivatives. Then, for any given continuous function $\phi : [-\tau, 0] \to \mathbb{R}^d$ as initial value, the delay differential equations (1) possesses a unique solution $x(t, \phi)$ which exists for all future time (see, for example, Driver [1] or Hale and Verduyn Lunel [3]).

The solutions of initial value problems for the delay differential equations (1) can be reformulated as a semi-dynamical system or semi-flow on the Banach space $\mathcal{C}_\tau$ of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^d$ equipped with the norm

$$\|\phi\|_\tau = \max\Big\{\|\phi(s)\| \mid s \in [-\tau, 0]\Big\}.$$

For any $y \in C\big([-\tau, \infty), \mathbb{R}^d\big)$ and fixed $t \geqslant 0$ define $y_t \in \mathcal{C}_\tau$ by $y_t(s) := y(t+s)$ for $s \in [-\tau, 0]$. Then, the initial value problem for the delay differential equations (1) with

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solution $x(t,\phi) \in \mathbb{R}^d$ for $t \geqslant 0$ generates a semi-flow $\Phi_t : \mathcal{C}_\tau \to \mathcal{C}_\tau$ with $\phi \to \Phi_t(\phi)$ defined by

$$\Phi_t(\phi)(s) := x(t+s,\phi), \quad s \in [-\tau, 0], \quad t \geqslant 0. \tag{2}$$

Let

$$M := \sup\Big\{ \|f(x,y)\| \mid (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \Big\} < \infty. \tag{3}$$

It is easy to see that $\Phi_t(\phi)$ is Lipschitz continuous with constant M for any initial function $\phi \in \mathcal{C}_\tau$ and $t \geqslant 0$, which means the family $\{\Phi_t\}_{t \geqslant 0}$ defines a semi-flow on the subspace

$$\mathrm{Lip}_{\tau,L} := \Big\{ \psi \in \mathcal{C}_\tau \mid |\psi(s_1) - \psi(s_2)| \leqslant L|s_1 - s_2|, \quad s_1, s_2 \in [-\tau, 0] \Big\}$$

of $\mathcal{C}_\tau$ for any $L \geqslant M$. The operator Φ_t is compact for each $t > 0$ by the Arzela–Ascoli theorem and the semi-flow will thus have an attractor if it has a closed and bounded absorbing set.

Since different delays will be considered in the sequel, we shall denote by $\Phi_t^{(\tau)}$ the semi-flow on $\mathcal{C}_\tau$ corresponding to the τ-delay differential equations (1), that is, with delay $\tau > 0$. A nonempty compact subset $\mathcal{A}_\tau$ of $\mathcal{C}_\tau$ is called the global attractor of the semiflow Φ_t of the delay differential equations (1) if it is invariant, that is, $\Phi_t^{(\tau)}(\mathcal{A}_\tau) = \mathcal{A}_\tau$ for all $t \geqslant 0$ and attracts all nonempty closed and bounded subsets B of $\mathcal{C}_\tau$, that is, for every such set B and each $\varepsilon > 0$ there exists $T_{\tau,B,\varepsilon} \geqslant 0$ such that

$$H_\tau^*\big(\Phi_t^{(\tau)}(B), \mathcal{A}_\tau\big) < \varepsilon \quad \text{for} \quad t \geqslant T_{\tau,B,\varepsilon}, \tag{4}$$

where H_τ^* is the Hausdorff semi-distance between closed and bounded subsets of $\mathcal{C}_\tau$. A global attractor is uniformly asymptotically stable [2] and can thus be characterised by a Lyapunov function (see Theorem 2.2 later).

Let $\mathcal{S}_\tau \subset C(\mathbb{R}, \mathbb{R}^d)$ denote the set of all entire bounded solutions of the τ-delay differential equations (1), that is, those which exist and are bounded for all $t \in \mathbb{R}$. An entire solution is obviously Lipschitz continuous with Lipschitz constant M. The corresponding entire trajectories of the semi-flow $\Phi_t^{(\tau)}$ thus belong to $\mathrm{Lip}_{\tau,L}$ for any $L \geqslant M$ (we henceforth choose and then hold fixed such an L) and are contained in the global attractor $\mathcal{A}_\tau$ of the semi-flow, if it exists. Indeed, by invariance, there exists an entire bounded trajectory through every point of an attractor $\mathcal{A}_\tau$, which is thus the image in $\mathrm{Lip}_{\tau,L}$ of all of the entire bounded trajectories of the semi-flow $\Phi_t^{(\tau)}$. Consequently, $\mathcal{A}_\tau$ can be represented by or identified with a compact subset $I_{\tau_0}(\mathcal{A}_\tau)$ in the space $\mathcal{C}_{\tau_0}$ for any other delay $\tau_0 \neq \tau$, where

$$I_{\tau_0}(\mathcal{A}_\tau) := \{\phi \in \mathcal{C}_{\tau_0} \mid \phi(s) = x^{(\tau)}(t+s), \; s \in [-\tau_0, 0], \; \text{for some} \;\; x^{(\tau)} \in \mathcal{S}_\tau \;\; \text{and} \;\; t \in \mathbb{R}\}$$

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2. The main result

The following theorem is the main result of this paper. The underlying assumption that $f \in C_b^1$ is not a major restriction since the long term dynamics takes values in a closed and bounded subset of the function space $\mathcal{C}_\tau$, with the corresponding functions thus taking values in a closed and bounded, hence compact, subset of $\mathbb{R}^d \times \mathbb{R}^d$.

Theorem 2.1. *Suppose that the semi-flow $\Phi_t^{(\tau_0)}$ has a global attractor $\mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$. Then the semi-flow $\Phi_t^{(\tau)}$ has an attractor $\mathcal{A}_\tau$ (possibly only local) for all $\tau > 0$ with $|\tau - \tau_0|$ sufficiently small. Moreover, the $\mathcal{A}_\tau$ converge to $\mathcal{A}_{\tau_0}$ upper semi continuously in $\mathcal{C}_{\tau_0}$ in the sense that*

$$H^*_{\tau_0}\big(I_{\tau_0}(\mathcal{A}_\tau), \mathcal{A}_{\tau_0}\big) \to 0 \quad \text{as } \tau \to \tau_0. \tag{1}$$

The upper semi continuous convergence (1) is obviously equivalent to the upper semi continuous convergence of the compact $\mathcal{S}_\tau$ to $\mathcal{S}_{\tau_0}$ in the Hausdorff semi metric based on the metric

$$d_{unif}(x, y) := \sum_{j=-\infty}^{\infty} 2^{-|j|} \max_{-\tau_0 \leqslant s \leqslant 0} \big\| x(j\tau_0 + s) - y(j\tau_0 + s) \big\|$$

on $C(\mathbb{R}, \mathbb{R}^d)$, that is, corresponding to uniform convergence on compact subsets. (The choice of the length τ_0 of the intervals here is suggestive, but not critical – it need only be finite).

2.1. Proof of Theorem 2.1: Theorem 2.1 is essentially a consequence of the total stability of the global attractor $\mathcal{A}_{\tau_0}$, in particular of the fact that the uniform asymptotical stability of such attractor can be characterised by a Lipschitz continuous Lyapunov function. The proof which follows is similar to those, in different contexts, in Kloeden and Lorenz [**4**, Theorem 1.1] and in Kloeden and Siegmund [**6**, Theorem 1].

Yoshizawa [**7**] provides various necessary and sufficient conditions which characterise the uniform asymptotical stability of a compact sets in terms of Lyapunov functions for ordinary differential equations. The following theorem is a straightforward generalisation of Yoshizawa in [**7**, Theorem 22.5] to retarded delay differential equations, see also [**7**, Theorem 35.2] as well as [**5**]. □

Theorem 2.2. *Suppose that the solution flow $\Phi_t^{(\tau_0)}(\phi)$ in $\mathcal{C}_{\tau_0}$ of τ_0-delay differential equations (1) has a global attractor $\mathcal{A}_{\tau_0}$ in $\mathcal{C}_{\tau_0}$, which is thus uniformly asymptotically stable. Then there exists a function $V : \mathcal{C}_{\tau_0} \to [0, \infty)$ satisfying*

(1) $\big|V(\phi) - V(\psi)\big| \leqslant L_V \|\phi - \psi\|_{\tau_0}$ *for all $\phi, \psi \in \mathcal{C}_{\tau_0}$*

(2) *There exist strongly monotone increasing functions $a, b : [0, \infty) \to [0, \infty)$ with $a(0) = b(0) = 0$ and $a(r) < b(r)$ for $r > 0$ such that*

$$a\big(\mathrm{dist}_{\tau_0}(\phi, \mathcal{A}_{\tau_0})\big) \leqslant V(\phi) \leqslant b\big(\mathrm{dist}_{\tau_0}(\phi, \mathcal{A}_{\tau_0})\big), \quad \phi \in \mathcal{C}_{\tau_0}.$$

(3) *There exists a positive real number c such that*

$$V\big(\Phi_t^{(\tau_0)}(\phi)\big) \leqslant e^{-ct}\, V(\phi), \quad t \geqslant 0,\ \phi \in \mathcal{C}_{\tau_0}.$$

Here $\operatorname{dist}_{\tau_0}(\phi, \mathcal{B})$ denotes the distance between a point ϕ and a closed and bounded subset $\mathcal{B}$ in $\mathcal{C}_{\tau_0}$.

Consider the solution $x^{(\tau)}(t, \phi_0) \in \mathbb{R}^d$ for $t \geqslant -\tau$ of the τ–delay differential equations (1) with initial value $\phi_0 \in \mathcal{C}_\tau$. If $\tau < \tau_0$, extend $x^{(\tau)}(t, \phi_0)$ to $t \geqslant -\tau_0$ by defining $x^{(\tau)}(t, \phi_0) = \phi_0(-\tau)$ for $-\tau_0 \leqslant t \leqslant -\tau$. For $\phi_0 \in \mathcal{C}_\tau$ and $t \geqslant 0$ define

$$S_t^{(\tau,\tau_0)}(\phi_0)(-s) := x^{(\tau)}(t - s, \phi_0), \quad 0 \leqslant s \leqslant \tau_0,$$

so $S_t^{(\tau,\tau_0)}(\phi_0) \in \mathcal{C}_{\tau_0}$ for all $t \geqslant 0$ with $S_0^{(\tau,\tau_0)}(\phi_0) = E_\tau^{\tau_0}(\phi_0)$, where

$$E_\tau^{\tau_0}(\phi_0) := \begin{cases} \phi_0(-s) & \text{for} \quad 0 \leqslant s \leqslant \min\{\tau, \tau_0\} \\ \phi_0(-\tau) & \text{for} \quad \tau \leqslant s \leqslant \tau_0 \ \text{ if } \ \tau < \tau_0 \end{cases}$$

for each $\phi_0 \in \mathcal{C}_\tau$. Note that $S_t^{(\tau,\tau_0)}$ is not a semi-flow.

The following Lyapunov inequality, which will be proved in the appendix, is fundamental to the proof.

Lemma 2.3. *Suppose that $|\tau - \tau_0| \leqslant \min\{\tau, \tau_0\}$. There exists a constant K independent of τ and τ_0 such that*

$$V\big(S_{s+t}^{(\tau,\tau_0)}(\phi_0)\big) \leqslant e^{-cs}\, V\big(S_t^{(\tau,\tau_0)}(\phi_0)\big) + L_V K |\tau - \tau_0|^2 \tag{2}$$

for all $s \in \big[0, |\tau - \tau_0|\big]$, $t \geqslant 0$ and all $\phi_0 \in \operatorname{Lip}_{\tau_0,L}$.

From Lemma 2.3 we have

$$V\big(S_{(n+1)\Delta}^{(\tau,\tau_0)}(\phi_0)\big) \leqslant e^{-c\Delta}\, V\big(S_{n\Delta}^{(\tau,\tau_0)}(\phi_0)\big) + L_V K \Delta^2$$

for $n \geqslant 1$, where $\Delta := |\tau - \tau_0| > 0$ with $|\tau - \tau_0| \leqslant \min\{\tau, \tau_0\}$. Define

$$\eta(\Delta) := \frac{2 L_V K \Delta^2}{1 - e^{-c\Delta}} \approx \frac{2 L_V K}{c} \Delta$$

and

$$\Lambda_{\tau_0}\big[\eta(\Delta)\big] := \big\{\phi \in \operatorname{Lip}_{\tau_0,L} \mid V(\phi) \leqslant \eta(\Delta)\big\},$$

Then, as in the proof of [4, Theorem 1.1] it can be shown that the closed and bounded set $\Lambda_{\tau_0}\big[\eta(\Delta)\big]$ is positively invariant and absorbing for the sequence of mappings $S_{n\Delta}^{(\tau,\tau_0)}$, provided Δ is sufficiently small, in the sense that $S_{n\Delta}^{(\tau,\tau_0)}(\phi_0) \in \Lambda_{\tau_0}\big[\eta(\Delta)\big]$ for $n \geqslant 1$ and $\phi_0 \in \mathcal{C}_\tau$ with $E_\tau^{\tau_0}(\phi_0) \in \Lambda_{\tau_0}\big[\eta(\Delta)\big]$ and that for any finite $R_0 > 0$ there exists an $N_{R_0,\tau,\Delta} \in \mathbb{N}$ such that

$$S_{n\Delta}^{(\tau,\tau_0)}(\phi_0) \in \Lambda_{\tau_0}\big[\eta(\Delta)\big]$$

for $n \geqslant N_{R_0,\tau,\Delta}$ and all $\phi_0 \in \mathcal{C}_\tau$ with $E_\tau^{\tau_0}(\phi_0) \in \mathcal{B}_{\tau_0}[\mathcal{A}_{\tau_0}, R_0]$, the closed and bounded ball of radius R_0 about the compact subset $\mathcal{A}_{\tau_0}$ in $\mathrm{Lip}_{\tau_0,L}$ defined by

$$\mathcal{B}_{\tau_0}[\mathcal{A}_{\tau_0}, R_0] := \{\phi \in \mathrm{Lip}_{\tau_0,L} \mid \mathrm{dist}_{\tau_0}(\phi, \mathcal{A}_{\tau_0}) \leqslant R_0\}.$$

Note that

$$\Lambda_{\tau_0}[\eta(\Delta)] \subset \mathcal{B}_{\tau_0}[\mathcal{A}_{\tau_0}, a^{-1}(\eta(\Delta))]$$

by property (2) of the Lyapunov function.

To show positive invariance it suffices to consider a single iterate $S_\Delta^{(\tau,\tau_0)}(\phi_0)$ for $\phi_0 \in \mathcal{C}_\tau$ with $E_\tau^{\tau_0}(\phi_0) \in \Lambda_{\tau_0}[\eta(\Delta)]$. Then

$$\begin{aligned} V(S_\Delta^{(\tau,\tau_0)}(\phi_0)) &\leqslant e^{-c\Delta}\, V(S_0^{(\tau,\tau_0)}(\phi_0)) + L_V K \Delta^2 \\ &\leqslant e^{-c\Delta}\, \eta(\Delta) + \frac{1}{2}(1 - e^{-c\Delta})\, \eta(\Delta) \\ &= \frac{1}{2}(1 + e^{-c\Delta})\, \eta(\Delta) \leqslant \eta(\Delta), \end{aligned}$$

since $V(S_0^{(\tau,\tau_0)}(\phi_0)) \leqslant \eta(\Delta)$. Hence $S_\Delta^{(\tau,\tau_0)}(\phi_0) \in \Lambda_{\tau_0}[\eta(\Delta)]$.

To show the absorbing property first note that there exists a $\gamma > 0$ such that

$$1 + e^{-c\gamma} = 2e^{-c\gamma/2} \qquad \text{and} \qquad 1 + e^{-c\Delta} < 2e^{-c\Delta/2}$$

for all $0 < \Delta < \gamma$. Suppose that

$$\Delta = |\tau - \tau_0| < \Delta^* := \min\{\tau, \tau_0, \gamma\}$$

and consider $S_\Delta^{(\tau,\tau_0)}(\phi_0)$ for $\phi_0 \in \mathrm{Lip}_{\tau,L}$ with $E_\tau^{\tau_0}(\phi_0) \notin \Lambda_{\tau_0}[\eta(\Delta)]$. Then

$$\begin{aligned} V(S_\Delta^{(\tau,\tau_0)}(\phi_0)) &\leqslant e^{-c\Delta}\, V(S_0^{(\tau,\tau_0)}(\phi_0)) + L_V K \Delta^2 \\ &= e^{-c\Delta}\, V\left(S_0^{(\tau,\tau_0)}(\phi_0)\right) + \frac{1}{2}\left(1 - e^{-c\Delta}\right)\, \eta(\Delta) \\ &\leqslant \frac{1}{2}\left(1 + e^{-c\Delta}\right)\, V\left(S_0^{(\tau,\tau_0)}(\phi_0)\right) \\ &\leqslant e^{-c\Delta/2} V\left(S_0^{(\tau,\tau_0)}(\phi_0)\right), \end{aligned}$$

since $V\left(S_0^{(\tau,\tau_0)}(\phi_0)\right) > \eta(\Delta)$. Repeating this argument,

$$V\left(S_{n\Delta}^{(\tau,\tau_0)}(\phi_0)\right) < e^{-cn\Delta/2}\, V\left(S_0^{(\tau,\tau_0)}(\phi_0)\right)$$

as long as $S_0^{(\tau,\tau_0)}(\phi_0)$, $S_\Delta^{(\tau,\tau_0)}(\phi_0)$, $\ldots$, $S_{(n-1)\Delta}^{(\tau,\tau_0)}(\phi_0) \notin \Lambda_{\tau_0}[\eta(\Delta)]$. Now

$$V(S_0^{(\tau,\tau_0)}(\phi_0)) \leqslant b\Big(\mathrm{dist}_{\tau_0}(S_0^{(\tau,\tau_0)}(\phi_0), \mathcal{A}_{\tau_0})\Big) \leqslant b(R_0),$$

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so

$$V\left(S_{n\Delta}^{(\tau,\tau_0)}(\phi_0)\right) \leqslant e^{-cn\Delta/2}\, b(R_0).$$

Finally, define $N_{R_0,\tau,\Delta}$ to be the smallest integer n for which

$$e^{-cn\Delta/2}\, b(R_0) \leqslant \eta(\Delta) < e^{-c(n-1)\Delta/2}\, b(R_0).$$

This proves that $\Lambda_{\tau_0}\left[\eta(\Delta)\right]$ is an absorbing set for the $S_{n\Delta}^{(\tau,\tau_0)}$.

Hence the semi-flow $\Phi_t^{(\tau)}(\phi_0)$ will be contained in a corresponding positive invariant, closed and bounded subset $\mathcal{B}_\tau$ of $\mathcal{C}_\tau$ for all $t \geqslant N_{R_0,\tau,\Delta}\Delta$. Since the operators $\Phi_t^{(\tau)}$ are compact for $t > 0$, the semi-flow $\Phi^{(\tau)}$ has an attractor (possibly only local) $\mathcal{A}_\tau$ in $\mathcal{B}_\tau$.

By invariance, for any $\phi_0 \in \mathcal{A}_\tau$ and $T > 0$, there is a $\psi_0 \in \mathcal{A}_\tau$ such that $\Phi_T^{(\tau)}(\psi_0) = \phi_0$. Taking

$$T = n\Delta \geqslant \max\{\tau, \tau_0, N_{R_0,\tau,\Delta}\Delta\},$$

it follows that

$$S_{n\Delta}^{(\tau,\tau_0)}(\psi_0) \in \Lambda_{\tau_0}\left[\eta(\Delta)\right],$$

which means that

$$\operatorname{dist}_{\tau_0}\left(S_{n\Delta}^{(\tau,\tau_0)}(\psi_0), \mathcal{A}_{\tau_0}\right) \leqslant a^{-1}\left(\eta(\Delta)\right)$$

for any $\psi_0 \in \mathcal{A}_\tau$. Since $S_{n\Delta}^{(\tau,\tau_0)}(\psi_0)$ corresponds in $\mathrm{Lip}_{\tau_0,L}$ to part of an entire solution of the τ-delay differential equations through $\phi_0 = \Phi_T^{(\tau)}(\psi_0) \in \mathcal{A}_\tau$, this implies that

$$H_{\tau_0}^*\left(I_{\tau_0}(\mathcal{A}_\tau), \mathcal{A}_{\tau_0}\right) \leqslant a^{-1}\left(\eta(\Delta)\right) \to 0 \quad \text{as} \quad \Delta \to 0,$$

which completes the proof of Theorem 2.1. ∎

3. Appendix: Proof of Lemma 2.3

Consider the semi-flow representation $\Phi_s^{(\tau_0)}\left(S_t^{(\tau,\tau_0)}(\phi_0)\right)$ solution of the τ_0–delay differential equations starting at $S_t^{(\tau,\tau_0)}(\phi_0)$, where $\phi_0 \in \mathrm{Lip}_{\tau,L}$. Then

$$\begin{aligned} V\left(S_{s+t}^{(\tau,\tau_0)}(\phi_0)\right) &\leqslant V\left(\Phi_s^{(\tau_0)}\left(S_t^{(\tau,\tau_0)}(\phi_0)\right)\right) + \left|V\left(S_{s+t}^{(\tau,\tau_0)}(\phi_0)\right) - V\left(\Phi_s^{(\tau_0)}\left(S_t^{(\tau,\tau_0)}(\phi_0)\right)\right)\right| \\ &\leqslant e^{-cs}\, V\left(S_t^{(\tau,\tau_0)}(\phi_0)\right) + L_V\left\|S_{s+t}^{(\tau,\tau_0)}(\phi_0) - \Phi_s^{(\tau_0)}\left(S_t^{(\tau,\tau_0)}(\phi_0)\right)\right\|_{\tau_0} \\ &\leqslant e^{-cs}\, V\left(S_t^{(\tau,\tau_0)}(\phi_0)\right) + L_V K|\tau - \tau_0|^2. \end{aligned}$$

Here we have used the exponential decay of the Lyapunov function along the semi-flow $\Phi^{(\tau_0)}$ and the Lipschitz property of the Lyapunov function on $\mathcal{C}_{\tau_0}$. In the last line we have used the following lemma,

Lemma 3.1. *Suppose that* $|\tau - \tau_0| \leqslant \min\{\tau, \tau_0\}$. *There exists a constant* K *independent of* τ *and* τ_0 *such that*

$$\left\| S_{s+t}^{(\tau,\tau_0)}(\phi_0) - \Phi_s^{(\tau_0)}\left(S_t^{(\tau,\tau_0)}(\phi_0)\right)\right\|_{\tau_0} \leqslant K|\tau - \tau_0|^2 \tag{3}$$

for all $s \in \big[0, |\tau - \tau_0|\big], t \geqslant 0$ *and all* $\phi_0 \in \mathrm{Lip}_{\tau,L}$.

Proof: Let $x(t)$ be the solution of the τ-delay differential equations and $y(t)$ be the solution of the τ_0-delay differential equations for an initial value which is the same on the interval $\big[-\min\{\tau, \tau_0\}, 0\big]$, so in particular $x(0 = y(0)$. Subtracting interval versions of the differential equations gives

$$x(t) - y(t) = \int_0^t \Big(f\big(x(s), x(s-\tau)\big) - f\big(y(s), y(s-\tau_0)\big)\Big)\, ds$$

and hence

$$\big|x(t) - y(t)\big| \leqslant L_f \int_0^t \Big(\big|x(s) - y(s)\big| + \big|x(s-\tau) - y(s-\tau_0)\big|\Big)\, ds,$$

where L_f is the Lipschitz constant of the vector field function f.

Assume that $\tau_0 \leqslant \tau$ (otherwise interchange the roles of x and y in what follows). Then

$$\begin{aligned}
\big|x(t) - y(t)\big| &\leqslant L_f \int_0^t \big|x(s) - y(s)\big|\, ds + L_f \int_0^t \big|x(s-\tau) - y(s-\tau_0)\big|\, ds, \\
&\leqslant L_f \int_0^t \big|x(s) - y(s)\big|\, ds + L_f \int_0^t \big|x(s-\tau_0) - y(s-\tau_0)\big|\, ds \\
&\qquad\qquad + L_f \int_0^t \big|x(s-\tau) - x(s-\tau_0)\big|\, ds, \\
&\leqslant L_f \int_0^t \big|x(s) - y(s)\big|\, ds + L_f \int_0^t 0\, ds + L_f \int_0^t \big|x(s-\tau) - x(s-\tau_0)\big|\, ds, \\
&\leqslant L_f M|\tau - \tau_0|^2 + L_f \int_0^t \big|x(s) - y(s)\big|\, ds
\end{aligned}$$

for $0 \leqslant t \leqslant |\tau - \tau_0|$.

Here we have used the facts that $x(s - \tau_0) = y(s - \tau_0)$ for $0 \leqslant s \leqslant \tau_0$ and

$$\big|x(s-\tau) - x(s-\tau_0)\big| \leqslant L|\tau - \tau_0|$$

for all $s \geqslant 0$ and initial values $\phi_0 \in \mathrm{Lip}_{\tau,L}$. The latter follows from the inequality

$$\big|x(s-\tau) - x(s-\tau_0)\big| = \int_{s-\tau_0}^{s-\tau} \Big|f\big(x(s), x(s-\tau)\big)\Big|\, ds \leqslant M|\tau - \tau_0| \leqslant L|\tau - \tau_0|$$

for all $s \geqslant \tau$ and by the L-Lipschitz property of ϕ_0 for $0 \leqslant s \leqslant \tau$ (we omit the details).

An application of the Gronwall inequality and the reinterpretation of the solutions as elements of the space $\mathcal{C}_{\tau_0}$ yield the asserted inequality (3). ∎

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