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UPPER SEMICONTINUOUS DIFFERENTIAL INCLUSIONS WITHOUT CONVEXITY

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ABSTRACT. We prove existence of solutions to the Cauchy problem for the differential inclusion $\dot{x} \in A(x)$, when A is cyclically monotone and upper semicontinuous.

INTRODUCTION

In this paper we deal with the problem of existence of absolutely continuous solutions to differential inclusions with a right-hand side F upper semicontinuous. For this class of inclusions, it is well known that existence holds under the additional assumption of convexity of the values of F (see for instance the chapter 2 of [1]), while it is easy to give counterexamples to the existence of solutions when the assumption of convexity is dropped.

The simplest example of a differential inclusion with upper semicontinuous right-hand side such that the Cauchy problem

$$(1) \qquad \dot{x}(t) \in -F(x(t)), \qquad x(0) = 0,\ t \geq 0,$$

has no solutions is given by the monotonic map F defined as

$$F(x) = \begin{cases} +1 & x > 0 \\ \{-1, +1\} & x = 0 \\ -1 & x < 0. \end{cases}$$

We remark that the above map F, although monotone, is not maximal, since the values are not convex. For the same F, the problem

$$(2) \qquad \dot{x}(t) \in F(x(t)), \qquad x(0) = 0,\ t \geq 0,$$

has exactly two solutions, namely $x_1 = t$ and $x_2 = -t$. Hence this is an example of a differential inclusion with an upper semicontinuous, nonconvex valued right-hand side such that the corresponding Cauchy problem has a closed nonempty set of solutions.

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From another point of view, consider any closed set $K \subseteq \mathbf{R}^n$, and the projector of best approximation on K from x, $\pi_K(x)$,

(3) $$\pi_K(x) = \{y \in K : d(x,y) = d(x,K)\}.$$

In the special case of $X = \mathbf{R}$ and $K = \{-1, +1\}$, example (2) is the problem

$$\dot{x}(t) \in \pi_K(x(t)), x(0) = 0.$$

Purpose of the present note is to show that existence of solutions holds in general for any Cauchy problem of the form

$$\dot{x}(t) \in A(x(t)), \qquad x(0) = \xi \in \mathbf{R}^n,$$

with A an upper semicontinuous, cyclically monotone map with closed nonempty values.

The map $x \to \pi_K(x)$ affords an example of such an operator.

The argument used in the proof is based on showing that in the present case the convergence of a sequence of approximate solutions implies the strong convergence of their derivatives.

Main result

We recall the definition and some properties of a cyclically monotone map.

Definition. A multifunction $A : \mathbf{R}^n \to \mathbf{R}^n$ is called cyclically monotone if for every cyclical sequence

$$x_0, x_1, \dots, x_N = x_0 \quad (N \text{ arbitrary})$$

and every sequence $y_i \in A(x_i)$, $i = 1, \dots, N$, we have

$$\sum_{i=1}^{N} \langle x_i - x_{i-1}, y_i \rangle \geq 0.$$

Proposition 1. [2, Theorem 2.5, p. 38] *A is cyclically monotone if and only if there exists a proper convex lower semicontinuous function $V : \mathbf{R}^n \to \mathbf{R}$ such that*

$$A(x) \subseteq \partial V(x),$$

where ∂V is the subdifferential of V.

We denote by B the open unit ball of $\mathbf{R}^n$. A map A is called upper semicontinuous if for every x and every $\varepsilon > 0$ there exists $\delta > 0$ such that x' in $x + \delta B$ implies $A(x') \subset A(x) + \varepsilon B$. Recall that an upper semicontinuous map with closed values has closed graph.

The following is our main result.

Theorem. *Let A be a map from $\mathbf{R}^n$ into the compact nonempty subsets of $\mathbf{R}^n$, upper semicontinuous and cyclically monotone. Then there exists $\delta > 0$ such that on $[0, \delta]$ the Cauchy problem*

(CP) $$\dot{x} \in A(x), \qquad x(0) = \xi,$$

admits a nonempty closed set of solutions.

Proof. By Proposition 1 there exists a proper lower semicontinuous convex function V such that $A(x) \subset \partial V(x)$. Since A is locally bounded (see [1, Proposition 1.1.3]), the same holds for ∂V. In fact, suppose that for every x in some open set U we have that $\sup\{|y| : y \in A(x)\}$ is bounded by $M > 0$ and assume, by contradiction, that there exist $x^* \in U$ and $y^* \in \partial V(x^*)$ such that $|y^*| > M$. For a sufficiently small positive λ, the point $x = x^* + \lambda y^*$ belongs to U. Choose $y \in A(x)$. Then

$$\langle y - y^*, x - x^* \rangle = \lambda \langle y - y^*, y^* \rangle < 0,$$

which contradicts the monotonicity of the multifunction ∂V. Hence we can assume that there exists an open ball about ξ, $B[\xi, R]$ and a $M < \infty$ such that V is Lipschitz continuous with constant M on $B[\xi, R]$, and A is bounded by M on $B[\xi, R]$. By choosing δ less that R/M we have that no Lipschitzian function x with Lipschitz constant M and such that $x(0) = \xi$ can leave $B[\xi, R]$ on $[0, \delta]$.

Our purpose is to define on $[0, \delta]$ a family of polygonals and to show that a subsequence converges to a solution to (CP). Define the nth polygonal by setting

$$x_n(0) = \xi,$$
$$x_n\left((i+1)\frac{\delta}{n}\right) = x_n\left(i\frac{\delta}{n}\right) + \frac{\delta}{n} y_i, \qquad i = 0, \dots, n-1,$$

where y_i belongs to $A(x_n(i\delta/n))$, and linearly between the nodal points $i\delta/n$, $(i+1)\delta/n$. The x_n are Lipschitzian with Lipschitz constant M. The sequence of pairs $((x_n, \dot{x}_n))_n$ is precompact in $C \times L^2$, the first space with the sup norm and the second with the weak topology. Consider a subsequence (that we denote with the same indexes) converging to $(x, \dot{x})$.

We claim that $\|\dot{x}\|_2 = \lim \|\dot{x}_n\|_2$, so that $\dot{x}_n$ converges to $\dot{x}$ in L^2-norm [3, p. 124].

Let us remark that, from known results (see [1, Theorem 1.4.1]), x is a solution to

$$\dot{x}(t) \in \operatorname{co} A(x(t)) \subseteq \partial V(x(t)), \qquad x(0) = \xi.$$

Both the maps $t \to x(t)$ and $t \to V(x(t))$ are Lipschitzian, hence differentiable a.e. By Lemma 3.3 in [2, p. 73],

$$\frac{d}{dt}(V(x(t))) = |\dot{x}(t)|^2 \text{ a.e. on } [0, \delta].$$

By integrating

$$V(x(\delta)) - V(\xi) = \int_0^\delta |\dot{x}(\tau)|^2 \, d\tau. \tag{4}$$

On the other hand, for each polygonal line on each interval $(i\delta/n, (i+1)\delta/n)$, the convexity of V implies

$$V\left(x_n\left((i+1)\frac{\delta}{n}\right)\right) \geq V\left(x_n\left(i\frac{\delta}{n}\right)\right) + \left\langle \dot{x}_n(t)\frac{\delta}{n}, y \right\rangle$$

for every y in $\partial V(x_n(i\delta/n))$; hence in particular, for $y \equiv \dot{x}_n$ on each $(i\delta/n, (i+1)\delta/n)$,

$$V(x_n(\delta)) - V(\xi) \geq \int_0^\delta |\dot{x}_n(\tau)|^2 \, d\tau.$$

By passing to the limit for $n \to \infty$ and using the continuity of V at the point $x(\delta)$,

$$\limsup \int_0^\delta |\dot{x}_n(\tau)|^2 \, d\tau \leq \int_0^\delta |\dot{x}(\tau)|^2 \, d\tau.$$

By the weak lower semicontinuity of the norm, we have that

$$\liminf \int_0^\delta |\dot{x}_n(\tau)|^2 d\tau \geq \int_0^\delta |\dot{x}(\tau)|^2 \, d\tau,$$

so that the claim is proved.

A subsequence of $\dot{x}_n$ converges pointwise almost everywhere.

By our construction,

$$d((x_n(t), \dot{x}_n(t)), \text{ graph}(A)) \leq \frac{\delta}{n} M;$$

since graph(A) is closed and, on the complement of a null set, $(x_n(t), \dot{x}_n(t))$ converges to $(x(t), \dot{x}(t))$,

$$\dot{x}(t) \in A(x(t)) \text{ a.e.}$$

This proves that the set of solutions to (CP) is nonempty. Let (y_m) be solutions converging to y in $C([0, \delta])$. By taking a subsequence, we can assume that $(\dot{y}_m)$ converges weakly in L^2. We apply (4) directly to y_m and to y to obtain that $\dot{y}_m$ converges to $\dot{y}$ in the norm topology of L^2. The same argument as before shows that y is a solution to (CP). Hence, the set of solutions to (CP) is closed in $C([0, \delta])$. □

AN APPLICATION

Proposition 2. *Let K be a closed nonempty subset of $\mathbf{R}^n$ with the Euclidean norm and define the projection π_K as in (3). Then there exists a convex function $V: \mathbf{R}^n \to \mathbf{R}$ such that*

$$\pi_K(x) \subseteq \partial V(x), \qquad \forall x \in \mathbf{R}^n.$$

Proof. For every $u \in \mathbf{R}^n$, consider the functional

$$\varphi_u(x) = \tfrac{1}{2}|u|^2 + \langle u, x - u \rangle,$$

whose graph is the hyperplane tangent to the graph of $x \to |x|^2/2$ at the point $(u, \frac{1}{2}|u|^2)$. Observe that, for every $x, u, v \in \mathbf{R}^n$, $u \neq v$, one has

$$|x - u| \leq |x - v| \text{ iff } \varphi_u(x) \geq \varphi_v(x). \tag{5}$$

Indeed, the set $H = \{x; \varphi_u(x) = \varphi_v(x)\}$ is the affine hyperplane

$$\{x \in \mathbf{R}^n : \langle u - v, x \rangle = \tfrac{1}{2}(|u|^2 - |v|^2)\}$$

which is orthogonal to $u-v$. Moreover, the midpoint $(u+v)/2$ of the segment joining u and v lies on H. Therefore $\varphi_u(x) = \varphi_v(x)$ iff $|x-u| = |x-v|$. Since $\varphi_u(u) \geq \varphi_v(u)$, the linearity of φ_u and φ_v implies (5). After these preliminaries, define

$$V(x) = \sup\{\varphi_u(x) \colon u \in K\}.$$

Clearly V is convex, everywhere defined and locally bounded. More precisely:

$$V(x) \leq \tfrac{1}{2}|x|^2 = \sup\{\varphi_u(x) \colon u \in \mathbf{R}^n\}.$$

In order to prove that $\pi_K(x) \subseteq \partial V(x)$, for every $u \in \pi_K(x)$ it suffices to show that $V(x) = \varphi_u(x)$, i.e. $\varphi_v(x) \leq \varphi_u(x)$ for every $v \in K$. Since $|u-x| \leq |v-x|$, this is a consequence of (5). □

Corollary. *Let $K \subseteq \mathbf{R}^n$ be closed. Then the Cauchy problem*

$$\dot{x} \in \pi_K(x), \qquad x(0) = \xi,$$

admits a closed nonempty set of solutions defined on $[0, +\infty)$.

Proof. Combining our main Theorem with Proposition 2, one obtains the local existence of forward solutions. Since there exist constants a and b such that

$$|y| \leq a|x| + b \qquad \text{for every } y \in \pi_K(x),$$

every local solution admits an extension to $[0, +\infty)$. □

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