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Upstream Attenuation and Quasi-Steady Rotor Lift Fluctuations in Asymmetric Flows in Axial Compressors

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This paper examines the quasi-steady lift variations experienced by a compressor rotor working in a circumferential inlet flow distortion. It is shown that for a given axial velocity distortion at the compressor face, the phase and magnitude of the lift fluctuations are strongly dependent not only on the geometry of the specified rotor, but on the total compressor configuration in which the rotor is operated. A simple numerical example is presented to illustrate this point. It is then demonstrated that the differences in the rotor lift fluctuation occur due to the upstream circumferential velocity component which is associated with the upstream attenuation of inlet velocity distortion. It is also pointed out that proper consideration of the circumferential component resolves an apparent discrepancy between two previous analyses of this problem. Finally, arguments are presented concerning the influence of the bound (blading) and downstream shed (wake) vorticity on the flow field upstream of the compressor. For the cases considered, it is shown that the induced velocity field associated with the upstream attenuation of inlet flow distortion is due equally to the bound and shed vorticity.

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INTRODUCTION

When a compressor rotor moves through a steady flow that is nonuniform in the direction of rotor rotation, the blades see an unsteady velocity disturbance, which sweeps by with a speed equal in magnitude to the rotor velocity. As a result of this velocity fluctuation, the circulation and the lift on a given blade will vary with time. The problem of quantitatively understanding these variations is an important part of predicting compressor behavior in asymmetric flows.

Different authors have taken varying approaches to this topic. One of the most basic has been to assume that the blading responds in a quasi-steady fashion to a time-varying inlet flow. The central approximation involved in this approach is that the ratio of blade chord to disturbance wavelength is vanishingly small, and, hence, the reduced frequencies associated with any velocity disturbance are low enough such that the quasi-steady concept is valid. As examples of its use, this approach has been applied by Whitehead (1)[1] to the description of compressor blade flutter, and by Rannie and Marble (2), Katz (3), and Dunham (4) to investigate the response of axial flow compressors to steady, circumferential inlet flow distortion. In addition, more sophisticated treatments of this latter topic, such as that of Seidel (5), have used the quasi-steady formulation as a framework on which to base an analysis that includes a further approximation to the unsteady rotor response.

The present paper is directed toward a physical understanding of some of the general features of asymmetric flow in axial compressors. It was undertaken to elucidate several aspects of this topic which do not seem to have been completely clarified and is, therefore, primarily

[1] Underlined numbers in parentheses designate References at end of paper.

NOMENCLATURE

- A = amplitude of axial velocity disturbance
- b = blade chord
- $\vec{C}$ = absolute velocity
- C_u = absolute circumferential velocity component
- C_x = axial velocity component
- l = disturbance wave length
- L = lift per unit span
- L_x = lift component in axial direction (per unit span)
- L_y = lift component in circumferential direction (per unit span)
- N = number of compressor stages
- P = static pressure
- R = mean radius of compressor annulus
- S = blade gap
- U = blade rotational speed
- W_u = relative circumferential velocity component
- $\vec{W}$ = relative velocity
- X = axial coordinate
- Y = absolute circumferential coordinate
- Y_R = relative circumferential coordinate
- α = absolute flow angle (Fig. 1)
- β = relative flow angle (Fig. 1)
- Γ = circulation per unit length of actuator disk (measured counterclockwise)
- ν = disturbance frequency
- ρ = density

Superscripts

- (') = denotes perturbation quantity
- (-) = denotes uniform flow quantity

Subscripts

- m = mean, across the blade row
- 10 = rotor leading edge
- 20 = rotor trailing edge

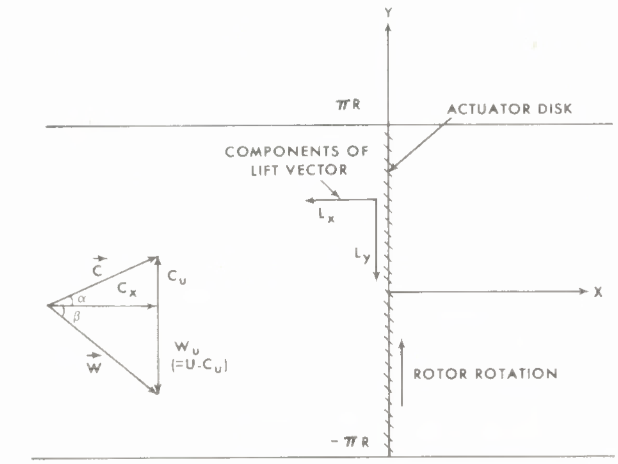

Fig. 1 Flow field geometry showing actuator disk location

focused on two facets of the problem that have not been discussed in the foregoing references. The first is an examination of the relationship between the inlet velocity distortion and the quasi-steady variation in rotor lift and circulation. The other is a discussion of the relative influence of the bound (blade) and downstream shed (wake) vorticity on the asymmetric velocity field upstream of the compressor. The arguments presented in connection with the first topic also resolve an apparent discrepancy between the results of Whitehead (1) and the quasi-steady limit considered by Henderson and Horlock (6), and a simple example is worked out to illuminate this point.

ANALYSIS

We will consider non-axisymmetric flow in an axial flow compressor of high enough hub/tip ratio such that a two-dimensional analysis can be used. The two-dimensional flow field is then obtained by "unwrapping" the compressor annulus at some mean radius, R, say. The working fluid is inviscid and incompressible. The blade and vane rows will be represented by actuator disks (perhaps more precisely they should be called actuator strips in the present context) which have infinitesimal thickness in the axial direction. Across these disks, the axial velocity component is continuous, but the circumferential (y) velocity component and the pressure can be discontinuous. The resulting geometry and notation is shown in Fig. 1, where, for simplicity, the axial position of the compressor has been indicated by an isolated rotor only. Finally, the flows which will be examined consist of a basic uniform flow upon which are superposed nonuniformities of sufficiently small magnitude such that a linearized description may be adopted.

Let us derive the expression for the variations in lift on a compressor rotor blade in terms of the perturbations in axial and relative circumferential velocity components. As in the uniform flow case, the components of the lift force on the blades per unit span are given by[2]

$$L_x = (\rho S/2)(W_{u_{10}}^2 - W_{u_{20}}^2) \qquad (1)$$

and

$$L_y = (\rho S C_{x_{10}})(W_{u_{10}} - W_{u_{20}}) \qquad (2)$$

If we write the velocity components in terms of mean (barred) and perturbation (primed) quantities and linearize, we find:

$$\bar{L}_x + L_x' = (\rho S/2)[\bar{W}_{u_{10}}^2 - \bar{W}_{u_{20}}^2 + 2\bar{W}_{u_{10}}W_{u_{10}}' - 2\bar{W}_{u_{20}}W_{u_{20}}'] \qquad (3)$$

$$\bar{L}_y + L_y' = \rho S[\bar{C}_{x_{10}}(\bar{W}_{u_{10}} - \bar{W}_{u_{20}}) + \bar{C}_{x_{10}}(W_{u_{10}}' - W_{u_{20}}') + C_{x_{10}}'(\bar{W}_{u_{10}} - \bar{W}_{u_{20}})] \qquad (4)$$

Within the context of the actuator disk approximation, one can define a circulation (measured counterclockwise) per unit length of the disk. This is numerically equal to the discontinuity in circumferential velocity across the disk. Denoting the uniform flow value of the quantity by $\bar{\Gamma}$ and the perturbation by Γ', the expressions for L_x and L_y can be written as:

$$L_x = \rho S(\bar{\Gamma}\bar{W}_{u_m} + \bar{W}_{u_m}\Gamma' + W_{u_m}'\bar{\Gamma}) \qquad (5)$$

$$L_y = \rho S(\bar{\Gamma}\bar{C}_{x_{10}} + \bar{C}_{x_{10}}\Gamma' + C_{x_{10}}'\bar{\Gamma}) \qquad (6)$$

where W_{u_m}, is the mean value of the circumferential velocity across the actuator disk. We can define a local relative mean vector velocity across the disk, W_m, and call the relative flow angle associated with this velocity β_m. The local lift, which is the only force on the blade, is in a direction perpendicular to this local mean vector velocity.

The equation for the local lift/unit span

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[2] These force components are defined such that if the compressor blades have a positive lift on them in the conventional sense, L_x and L_y are both positive (Fig. 1).

is,

$$L = L_x \sin\beta_m + L_y \cos\beta_m$$

Substituting from equations (5) and (6) and expressing the perturbations in β_m which occur as a result of the velocity distortion, we obtain

β_m

$$L/s = \bar{L}/s + L'/s = \rho\bar{W}_m\bar{\Gamma} + \rho(\bar{W}_m\Gamma' + W_m'\bar{\Gamma}) \quad (7)$$

or

$$L'/s = \rho(\Gamma W_m)' \quad (8)$$

Since the quantities on the right-hand side of equation (8) are seen to involve both the circumferential and the axial velocity components, the fluctuation in quasi-steady lift, L', is more properly regarded as being in phase with variations in the quantity (ΓW_m) rather than in either one or the other of the velocity components. In addition, we can note that not only the magnitude, but the direction of the lift vector on a given blade will fluctuate as the blade moves.

Let us now examine a numerical example using these results to investigate the lift fluctuations on a compressor rotor which is working in a steady, asymmetric inlet flow. In keeping with the principal aim of the paper, which is to achieve a basic physical understanding of the flow with a minimum of algebraic complexity, we can adopt several simplifying assumptions concerning the compressor configurations to be studied. To facilitate comparison with the results presented by Henderson and Horlock (6), consider a rotor consisting of flat plate airfoils with stagger angle of 45 deg and gap/chord ratio of unity. The basic uniform flow, which is axial in the absolute coordinate system, has zero degrees angle of attack on the blades, and there is no mean total pressure rise across the rotor. The rotor we are examining can be operated either by itself or as part of a single- or multi-stage compressor. In these latter instances, the flat plate stators have a stagger angle of zero (i.e., they are set axially) so that, again, there is no mean total pressure rise across the machine. In addition, we make the further approximations that the relative leaving angle from any blade row can be taken as constant (at the stagger angle) and that, in the case of the compressors with multiple blade rows, the very weak effect of the small axial gaps between the rows can be neglected [ample justification for this has been given by Dunham (4)].

Under this set of conditions, the perturbation in lift per unit length on the actuator disk representing the rotor, which is given in equation (8), is simplified to

$$L'/s = \rho\bar{W}_m\Gamma' \quad (9)$$

since there is zero circulation around each blade in the uniform flow situation. Expressed in terms of the local absolute velocity components, equation (9) can be rewritten as

$$L'/s = -\rho U\sqrt{2}\,(C'_{x_{10}} + C'_{u_{10}}) \quad (10)$$

where the assumption of constant leaving angle has been used.

Suppose that at the compressor inlet there is an axial velocity distribution of the form

$$C_{x_{10}} = \bar{C}_x + A\sin(2\pi y/\ell) \quad (11)$$

In the relative coordinate system, this will appear to the rotor as an axial velocity "wave" of the form

$$C_{x_{10}} = \bar{C}_x + A\sin[\nu(t - y_R/U)] \quad (12)$$

In equation (12), Y_R is the circumferential coordinate in the relative system taken as positive in the direction of the circumferential velocity component of the uniform relative flow, t is time, and, as in reference (6),

$$\nu = 2\pi U/\ell.$$

In practice, there are several ways in which this steady nonuniformity could be created. Two of the most common causes, however, are: (a) the presence of an upstream total pressure nonuniformity (or "distortion") which we can assume here to be imposed at upstream infinity, and (b) an asymmetric back pressuring of the compressor, from downstream struts, for example. In the former situation, the flow upstream of the compressor is rotational, while in the latter, the non-uniform axial velocity can arise in a flow that is irrotational upstream of the compressor.

In this latter instance, the specification of the axial velocity component at the face of the compressor (along with the relevant boundary conditions at $y = \pm\pi R$ and $x = -\infty$) is enough to determine the upstream velocity field. However, in the first situation, we need additional infor-

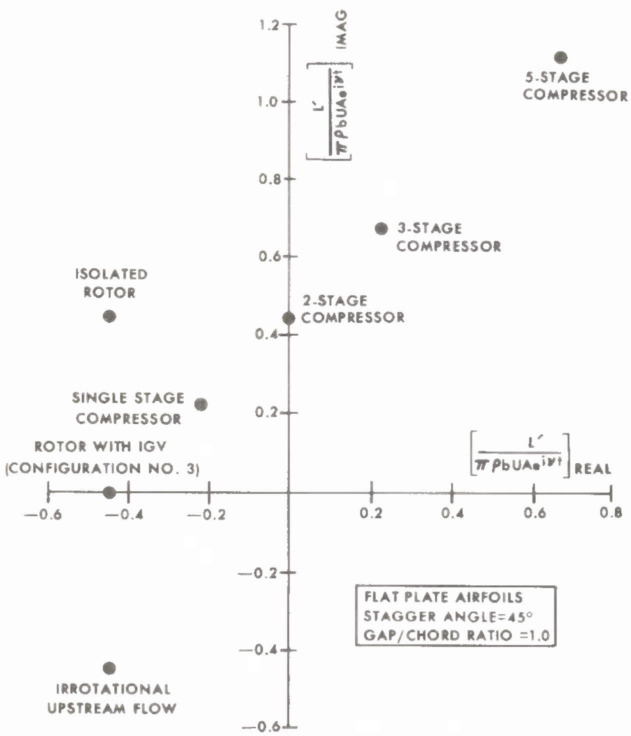

Fig. 2 Quasi-steady rotor lift

mation to calculate the upstream flow. This is due to that fact that if the upstream flow is rotational, the inlet axial velocity perturbation which the compressor sees will be the sum of a rotational and an irrotational velocity field, and knowledge of the axial velocity alone does not provide sufficient information to determine each of these separately. Therefore, in order to fully describe the velocity field, we must also specify the particular compressor configuration in use. Once this is done, we can apply the techniques which have been developed for analyzing asymmetric flow in axial flow compressors in order to compute the velocity field and find the lift on the rotor. The method used for the numerical examples given in the following is a slight modification of that presented by Katz (3). This procedure enables one to calculate the velocity and pressure perturbations upstream and downstream of each blade row in the compressor if the axial velocity at the compressor inlet and the cascade properties (loss and turning) of the individual blade rows are specified. The analysis, therefore, provides the value of $C'_{u_{10}}$ in equation (10) which is necessary to determine the local lift on the rotor. The actual calculation procedure is straightforward and will not be reproduced, although a brief summary is given in Appendix A.

Let us list several different compressor configurations in which our rotor could be utilized. As will be seen, the lift can vary substantially depending on the specific configuration. In all of these, the same axial velocity perturbation occurs at the compressor face. In the first three, an axial velocity (or total pressure) distortion, which is not the same for the different configuration, exists at upstream infinity. The configurations are:

1 As an isolated rotor
2 As the rotor in the first stage of an N (identical) stage compressor without inlet guide vanes. As mentioned in the foregoing, there is no mean pressure rise through the compressor.
3 As a rotor in one of the rear N-1 stages in configuration 2 or equivalently as the first rotor but with an axial inlet guide vane present.

In addition to these (as configuration 4), we can include the situation where the rotor operates in an upstream irrotational flow, and again there are no inlet guide vanes.

The resulting expressions for the perturbation in lift per length of the rotor actuator disk as a function of the absolute circumferential coordinate are shown in the table. In all of the configurations the rotor blades have the same axial velocity distribution, given by equation (12), incident on them. Therefore, the lift variations are due solely to the difference in the circumferential velocity components at the rotor inlet.

Configuration	Lift per unit length (and unit span) of the actuator disk as a function of absolute circumferential position.
1	$-\sqrt{2}\rho UA[\sin(2\pi Y/\ell) - \cos(2\pi Y/\ell)]$
2	$\sqrt{2}\rho UA[(N/2-1)\sin(2\pi Y/\ell) + (N/2)\cos(2\pi Y/\ell)]$
3	$-\sqrt{2}\rho UA[\sin(2\pi Y/\ell)]$
4	$-\sqrt{2}\rho UA[\sin(2\pi y/\ell) + \cos(2\pi y/\ell)]$

The results shown in this tabulation can also be presented graphically. Let us represent the axial velocity perturbation seen by a given rotor blade (i.e., at a fixed value of Y_R) as the real part of the quantity, $Ae^{i\nu t}$, where A is real. Since the perturbations in lift are generally not in phase with those in axial velocity, a similar representation of the lift (for a specified configuration and axial velocity perturbation) as the real part of the product of a constant, say B, times $e^{i\nu t}$ implies that B will have both real and imaginary parts to reflect this difference in phase. This is shown in Fig. 2, where the real and imaginary parts of the non-dimensional lift associated with an axial velocity perturbation of the form given in equation (12) are plotted for several different situations. As in reference (6), the axial velocity perturbation has been used in the nondimensionalization. The single- and multi-stage points that are plotted are for configuration 2. It can be seen that the magnitude and phase of the lift vary considerably for the different configurations investigated.

DISCUSSION

It may be instructive to inquire in more detail into the cause of these changes in the rotor lift vector. In addition, such an examination will enable us to point out the source of the apparent discrepancy between the results of Whitehead (1) and Henderson and Horlock (6).

Consider first the reason for the lift vector's increase in magnitude and swing toward a phase angle of 45 deg (Fig. 2) as N is increased in configuration 2. As the number of identical stages is increased any "far upstream" [i.e., far enough upstream such that the upstream irrotational velocity components, which decay strongly in the upstream direction, are negligible and the static pressure is constant circumferentially — see reference (7)], axial velocity nonuniformity is increasingly attenuated. The attenuation is a result of an upstream irrotational velocity perturbation induced by the bound vorticity associated with the compressor blading and the downstream shed vorticity. The sum of the (rotational) far upstream axial velocity distribution and the axial component of the induced irrotational velocity field must combine to give the specified disturbance at the compressor inlet.

The increased velocity attenuation due to the addition of more stages means that the ratio of the amplitude of the far upstream axial velocity distortion to that of the axial velocity distribution at the compressor face has increased. Likewise the ratio of the amplitude of the induced axial velocity perturbation at the compressor face to the actual inlet axial velocity has also increased. However, the presence of the induced axial velocity perturbation implies (through irrotationality) that there will be an induced circumferential velocity component of equal magnitude. For the many stage compressor, then, this circumferential velocity perturbation is much larger than the axial velocity perturbation which is seen by the compressor. Hence, the lift on the rotor is due mainly to the inlet circumferential velocity, C'_{u10}, rather than the inlet axial velocity, C'_{x10}. The calculation procedure described in the Appendix shows that the phase difference between these two velocity components is 225 deg (i.e., the circumferential velocity perturbation leads the axial velocity perturbation by 225 deg). Thus, as can be inferred from equation (10), if N is increased and C'_{u10} becomes much larger than C'_{x10}, the phase of the lift vector will tend toward an angle of 45 deg relative to the axial velocity perturbation.

The presence of an inlet guide vane, or a stator, in front of the rotor changes this picture drastically. Within the approximation of constant relative leaving angle from each blade row, the stationary set of vanes can be said to remove the circumferential velocity component at the rotor leading edge. Consequently, wherever in the compressor the rotor is operated, or however many stages there are, the lift will be 180 deg out of phase with the perturbation in axial velocity. Since the perturbations examined by Henderson and Horlock (6) are restricted to being solely in the axial velocity component (which is equivalent to the use of an axially discharging inlet guide vane or stator in front of the rotor), they find this 180-deg difference for quasi-steady flow, whereas the Whitehead (1) analysis, which makes no such restriction, does not.

We can also briefly comment on the circulation variation on the rotor blades. In the present situation, the perturbation in circulation is directly proportional to the fluctuation in lift. The circumferential variation in circulation is thus not necessarily in phase with either the inlet axial velocity perturbation or the far upstream velocity nonuniformity (as would be the case for an isolated airfoil). This is important in the context of understanding attenuation of inlet total pressure distortion, since the local quasi-steady total pressure rise across a rotor is directly proportional to the local circulation/unit length.

SHED VORTICITY AND UPSTREAM INDUCED VELOCITY

An important concept in the preceding arguments is that the far upstream axial velocity nonuniformity can be substantially different from the axial velocity distortion which the compressor actually sees. This is perhaps the most fundamental aspect of the problem of inlet distortion in axial flow compressors. It is, therefore, interesting to consider, from a basic aerodynamic viewpoint, the nature of the irrotational flow redistribution upstream of a compressor which is operating in an asymmetric flow. Let us examine a relatively simple yet relevant situation. As

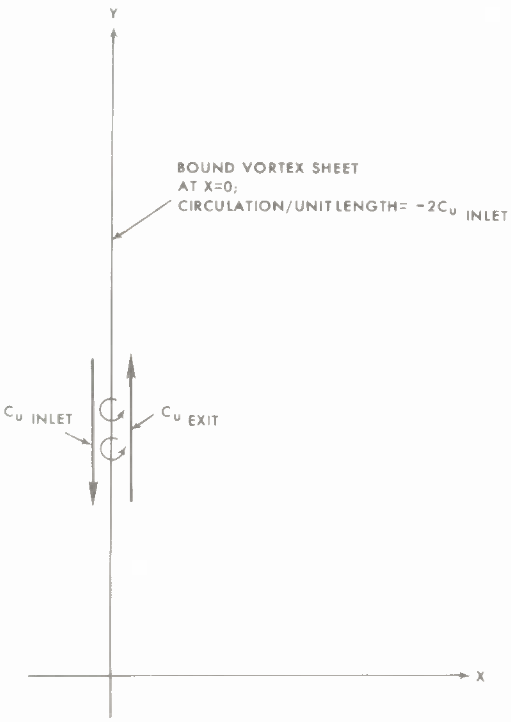

Fig. 3(a) Bound vortex sheet at x = 0 showing circumferential velocity components

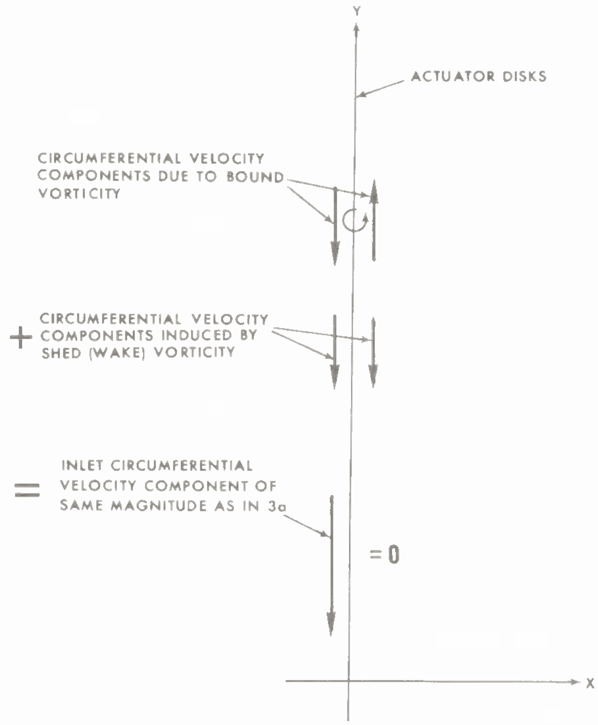

Fig. 3(b) Actuator disks representing compressor at x = 0 showing circumferential velocity components due to bound and shed vorticity

is usual in practice, the uniform flow upstream of the compressor is taken as axial. Upon this uniform flow is superposed a steady (rotational) perturbation in axial velocity which can be viewed as originating at upstream infinity. Nothing is specified about the number and geometry of the blade rows comprising the compressor (which are again approximated as actuator disks) except for the fact that there is an exit stator row with axial discharge. Thus, since the flow exits axially, the net value of the local bound circulation per unit length associated with all the actuator disks representing the compressor blading is just equal to the negative of the local absolute circumferential velocity component at the inlet face of the compressor.

Within the linearized theory, the vorticity associated with the far upstream velocity nonuniformity is convected unchanged from upstream to downstream infinity, along the axial, uniform streamlines. There are no irrotational velocity perturbations induced by this vorticity distribution, and the irrotational velocity field that arises upstream of the compressor is due only to the net bound vorticity on the actuator disks and the shed vorticity in the wakes.

Consider the flow if we replaced the "compressor" (i.e., all these actuator disks) plus the distributed shed vorticity downstream by a bound vortex sheet of nonuniform strength, which induced the same irrotational velocity perturbation at the inlet. As shown by Sears (8), the local circulation per unit circumferential length of the sheet necessary to do this has a magnitude twice that of the local circumferential component of velocity at the front of the disks. Therefore, the net distributed vorticity on the actuator disks which represent the compressor is just one-half of that needed to induce the actual upstream irrotational velocity perturbation. In addition, for the case we are examining — constant (zero) leaving angle from the exit stator row — there are no downstream irrotational velocity perturbations, since the downstream flow is a parallel shear flow which is a purely rotational distortion. However, if we considered only the bound vorticity on the compressor blading, there would be circumferential velocity components in front of and in back of the compressor which were of equal magnitude but were opposite in sign, and there would be downstream irrotational velocity perturbations. What has occurred is that the shed vorticity downstream of the compressor has induced a circumferential velocity which is of the proper strength at the axial location of the compressor to cancel the circumferential velocity induced by the bound vorticity at the rear of the compressor and to increase, by a factor of two, the circumferential velocity at the front, as shown schematically in Figs. 3(a) and 3(b).

In other words, to obtain the proper upstream flow without the presence of the shed vorticity, the bound vorticity at the actuator disk location would have to be increased by a factor of two. For the axial exit case, therefore, the effect (on the upstream flow field) of the shed vorticity in the downstream wakes is equal to that of the bound vorticity on all the blade rows of the compressor, and each can be viewed as being equally responsible for the upstream flow redistribution. For the downstream field, on the other hand, the circumferential velocities induced by the former and those induced by the latter are equal and opposite, and there is no circumferential velocity.

CONCLUSION

An analysis has been presented of the quasi-steady lift and circulation on a compressor rotor in an inlet flow distortion. It has been shown that for a given inlet axial velocity distortion, the fluctuating lift or circulation on the rotor can depend strongly on the specific compressor configuration in which the rotor appears, due to the presence of the upstream circumferential velocity components. This is also true for the lift associated with a specified far upstream velocity (or total pressure) distortion because the upstream velocity attenuation depends on the performance of the compressor as a whole.

In view of these results, it is concluded that for a given rotor blade row, there is no unique phase or magnitude relation between either of the inlet velocity components and the rotor lift which holds for all compressor configurations. Calculations based on an isolated rotor will, therefore, not, in general, describe the behavior of the same rotor when it is working as part of a multi-row machine.

In addition, a discussion has been given concerning the relative influence of the bound and shed vorticity. It has been shown that for a compressor in which the last stator row discharges axially, the upstream irrotational velocity field, which is associated with attenuation of inlet flow distortion, can be regarded as being equally due to the bound and shed vortex elements.

APPENDIX A

ASYMMETRIC FLOW FIELD CALCULATION PROCEDURE

This appendix gives a brief summary of the technique for calculating the performance of an axial flow compressor in a steady asymmetric flow which has been used to derive the results in this paper. For a more detailed exposition of the basic method, the reader is referred to the papers by Rannie and Marble (2) or Katz (3). In this procedure, the linearized, two-dimensional equations of motion upstream and downstream of the various blade rows are rewritten in the equivalent form:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \qquad (A1)$$

$$\frac{\partial F}{\partial x} = -\frac{\partial G}{\partial y} \qquad (A2)$$

$$\left(\bar{C}_x \frac{\partial}{\partial x} + \bar{C}_u \frac{\partial}{\partial y}\right) H = 0 \qquad (A3)$$

The quantities F, G, and H are defined by:

$$F = \bar{C}_x C_u' - \bar{C}_u C_x' \qquad (A4)$$

$$G = P'/\rho \qquad (A5)$$

$$H = P'/\rho + \bar{C}_x C_x' + \bar{C}_u C_u' \qquad (A6)$$

and, in this appendix, the coordinates x and y have been nondimensionalized using the mean radius, R. It can be seen that H is proportional to the perturbation in total pressure, G is proportional to the perturbation in static pressure, and F is proportional to the perturbation in absolute flow angle, α. To represent the flow in an annulus, all of these functions must be periodic with period 2π.

Equations (A1) and (A2) are the Cauchy-Rieman conditions, and, therefore, F and G are conjugate harmonic functions. Equation (A3) states that H is convected unchanged along the uniform flow streamlines.

Across the actuator disks which represent the blade rows, the uniform flow variables can change discontinuously. Thus, the field of flow, which is a strip that is infinite in the x-direction and extends from $-\pi$ to π in the y-direction, is broken into different regions each with its own representation for F, G, and H. The first region extends from upstream infinity to the leading edge of the first actuator disk, the second region from the trailing edge of the first actuator disk to the leading edge of the second, etc., with the last region extending from the trailing edge of the last actuator disk to downstream infinity.

To connect the flow quantities on the upstream and downstream sides of the actuator disks, three matching conditions are required at each disk. The boundary conditions that are applied are that the axial velocity is continuous across the disk, the relative leaving angle is constant and equal to the flat plate stagger angle, and there is no loss in relative total pressure across an actuator disk. The remaining necessary boundary conditions are that the static pressure perturbations at $X = \pm \infty$ are zero (or equivalently that the flow is parallel) and that the axial velocity profile at the inlet to the compressor is given. This last boundary condition differs from the more usual one applied in analyses of asymmetric flows in compressors, which is the specification of the total pressure distortion at upstream infinity, but it is more pertinent to the discussion in the paper.

Because of the periodicity, it is natural to express F, G, and H using a Fourier series representation. Since F and G obey Laplace's equation, the forms of these functions are simply products of sines and cosines (of y) and exponentials in x. For example, the solutions for F and G in the region upstream of the first actuator disk take the form:

$$F_1 = \sum_{j=1}^{\infty} (f_{1j} \cos jy + g_{1j} \sin jy) e^{jx} \qquad (A7)$$

$$G_1 = \sum_{j=1}^{\infty} (g_{1j} \cos jy - f_{1j} \sin jy) e^{jx} \qquad (A8)$$

where the f_{1j} and g_{1j} are constants which will be determined using the boundary conditions. The functional form of the solution for H_1 also has a simple form:

$$H_1 = \sum_{j=1}^{\infty} [a_{1j} \cos j(y - x\bar{C}_{u_1}/\bar{C}_{x_1}) + b_{1j} \sin j(y - x\bar{C}_{u_1}/\bar{C}_{x_1})] \quad (A9)$$

where a_{1j} and b_{1j} are likewise unknown constants.

In a similar manner, the form of the solutions for F, G, and H can be written down for the other regions. To determine the constants in these expressions, the matching conditions, together with the specification of the inlet axial velocity, must be satisfied simultaneously. This leads to a non-homogeneous system of algebraic equations which can be solved in a standard manner.

As an illustration, we can write down the relevant forms of F, G, and H, and the matching conditions for an isolated workless rotor. In this case, there are only two regions, and the uniform flow is axial in both. We denote the station at the rotor leading edge by the subscript 10 and that at the trailing edge by 20. From equations (A5) and (A6), it can be seen that because $\bar{C}_u$ is zero in both regions, the axial velocity perturbation, C'_x, can be expressed as

$$C'_x = (H - G)/\bar{C}_x \qquad (A10)$$

Therefore, if the specified axial velocity perturbation at the rotor inlet is given by

$$C'_x(0,y) = V(y)/\bar{C}_x \qquad (A11)$$

then,

$$H_{10} - G_{10} = V(y). \qquad (A12)$$

In addition, continuity of axial velocity across the disk requires

$$H_{10} - G_{10} = H_{20} - G_{20} \qquad (A13)$$

From consideration of the exit velocity triangles and the use of equation (A13), it can be shown that the condition of constant relative leaving angle is

$$U C'_{x_{20}} = -F_{20} \qquad (A14)$$

This leads to the matching condition

$$(\bar{C}_x/U) F_{20} = G_{10} - H_{10}. \qquad (A15)$$

Finally, the condition of no loss leads to an equation for the change in absolute total pressure across the rotor

$$\rho(H_{20} - H_{10}) = \rho U (C'_{u_{20}} - C'_{u_{10}}) \qquad (A16)$$

Using the definition of F, this becomes

$$(\bar{C}_x/U)(H_{20} - H_{10}) = F_{20} - F_{10} \qquad (A17)$$

We can now plug the functional forms of F, G, and H into equations (A12), (A13), (A15), and (A17). If the Fourier series for V(y) is given by

$$V(y) = \sum_{j=1}^{\infty} (m_j \cos jy + n_j \sin jy), \qquad (A18)$$

and the linear independence of the sine and the cosine is invoked, it is found that for each of the j harmonics, the unknown coefficients in F, G, and H form the set of equations

$$\begin{array}{llllllll}
a_{1j} & & & -g_{1j} & & & = m_j & (A19) \\
& b_{1j} & +f_{1j} & & & & = n_j & (A20) \\
a_{1j} & & & -g_{1j} & -a_{2j} & -g_{2j} & = 0 & (A21) \\
& b_{1j} & +f_{1j} & & -b_{2j} & +f_{2j} & = 0 & (A22)
\end{array}$$

$$a_{1j} - g_{1j} - \left(\frac{\bar{c}_x}{U}\right) f_{2j} = 0 \quad \text{(A23)}$$

$$b_{1j} + f_{1j} + \left(\frac{\bar{c}_x}{U}\right) g_{2j} = 0 \quad \text{(A24)}$$

$$\left(\frac{\bar{c}_x}{U}\right) a_{1j} - f_{1j} - \left(\frac{\bar{c}_x}{U}\right) a_{2j} + f_{2j} = 0 \quad \text{(A25)}$$

$$\left(\frac{\bar{c}_x}{U}\right) b_{1j} - g_{1j} - \left(\frac{\bar{c}_x}{U}\right) b_{2j} + g_{2j} = 0 \quad \text{(A26)}$$

Solution of equations (A19) through (A26) determines the velocity and pressure fields both upstream and downstream of the rotor.

The analysis of configurations that have more than a single blade row is carried out in a similar fashion. Each additional blade row will involve six more coupled equations because of the additional boundary conditions that are imposed.

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