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Upward and downward continuation as inverse problems

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Summary. The formalism of Backus & Gilbert is applied to the problems of upward and downward continuation of harmonic functions. We first treat downward continuation of a two-dimensional field to a level surface everywhere below the observation locations; the calculation of resolving widths and solution estimates is a straightforward application of Backus–Gilbert theory. The extension to the downward continuation of a three-dimensional field uses a delta criterion giving resolving areas rather than widths. A feature not encountered in conventional Backus–Gilbert problems is the requirement of an additional constraint to guarantee the existence of the resolution integrals. Finally, we consider upward continuation of a two-dimensional field to a level above all observations. We find that solution estimates must be weighted averages of the field not only on this level, but also on a line passing between the observations and sources. Weighting on the lower line may be traded off against resolution on the upper level.

Introduction

Gravity and magnetic data measured above the Earth's surface contain information about the inaccessible structure within the Earth. Such observations, however, are often not made at a constant level; the geologic and topographic signals are then contaminated by the artificial effect of variable path height. This effect can be removed by continuation of the data to a level surface. Continuation is also done to amplify or attenuate certain wavelengths relative to others.

A number of methods have been proposed for continuation from an irregular surface. If the potential field shows a strong lineation, it can usually be assumed to obey the two-dimensional Laplace's equation. For such cases, Parker & Klitgord (1972) demonstrate the use of the Schwarz–Christoffel transformation for continuation to a level line. Although it has seen successful application (e.g. Klitgord *et al.* 1975), the technique suffers from a formal shortcoming, the evaluation of the Dirichlet integral. This requires continuous data on an infinitely long track; actual data, of course, can at best only approximate these conditions. Also, the use of the Schwarz–Christoffel mapping has no extension to the three-dimensional problem of continuing data from an irregular surface to a level plane. Several other techniques have been suggested (Strakhov & Devitsyn 1965; Tsirul'skiy 1968; Dampney 1969; Henderson & Cordell 1971); all result in the construction of one potential function on a plane, compatible with the data.

If a harmonic function is known everywhere on a regular surface, there is indeed a unique solution to the continuation problem (Kellogg 1953, p. 262); this is the fundamental result upon which all the usual methods are founded. In practice, a finite collection of approximate numbers must serve as the data. Therefore, to apply the standard methods, the actual observations must be 'completed', that is, values must be supplied in the gaps either by explicit interpolation or implicitly in the numerical schemes (for example, if trapezium-rule integration is used, piece-wise linear interpolation is implied). There are of course infinitely many different ways of doing this, so that there are infinitely many different solutions. When the solution to the analytic problem (i.e. the one with complete data) is stable in the sense of Hadamard (Parker 1977a), the non-uniqueness is not very important because all these functions will be similar: their differences are bounded by an amount that depends only on the error in the interpolation. This result does not hold for unstable problems; wildly different solutions can then be associated with almost identical data. Downward continuation is well known to be unstable; in an appendix we show that even upward continuation is unstable if data are available only on a finite part of the surface. Therefore it is important to discover continuation methods that do not rely openly or in a hidden way upon data completion.

For this purpose we turn to linear inverse theory, where stability is gained by choosing to extract certain unique properties that all the solutions share. In particular, we apply the theory of Backus & Gilbert (1968), who construct smoothed versions of the solution. An earlier application to the continuation problem is given by Courtillot, Ducruix & Le Mouël (1973) for two-dimensional fields and by Ducruix, Le Mouël & Courtillot (1974) for three-dimensional ones. These authors find the smallest rms solution compatible with the data in the class of band-limited harmonic functions. While based upon some of the ideas of Backus & Gilbert, their treatment does not take full advantage of the concept of resolution and the way that this describes the degree of non-uniqueness. Furthermore, the assumption of band-limitedness of the solution is not only unnecessary, but also actually incorrect.

In this paper we give a treatment that adheres more closely to the theory of Backus & Gilbert. We need no assumptions about the harmonic field beyond those made in the corresponding treatment for exact data. We construct fields that are smoothed versions of the true solution and the smoothing is prescribed in such a way that the field obtained is compatible with every one of the infinite number of possible solutions, including of course the true one. It is assumed the reader is familiar with the Backus–Gilbert method; in addition to the original paper, the discussions of Oldenburg (1976) and Parker (1977a) may be useful.

First, the very straightforward problem of downward continuation in two dimensions is studied. Then the treatment is extended to three-dimensional fields, where a certain amount of ingenuity is required in the construction of a suitable quantitative measure of delta-function quality. Finally we treat upward continuation but in detail only for two-dimensional fields. Somewhat surprisingly (to us anyway) upward continuation is in principle much harder to treat as a linear inverse problem. This is because the relationship between the observations and the model (the field above the observation stations) is not readily described as a continuous linear functional. Our efforts on this problem not only make it accessible to Backus–Gilbert theory, but to more recent sophisticated treatments (Backus 1970a, b; Sabarier 1977a, b; Parker 1977b).

In this paper we have only treated the case of exact data. The formalism for inaccurate data, however, is a straightforward extension (Backus & Gilbert 1970).

Downward continuation: two-dimensional fields

The downward continuation problem, viewed as a linear inverse problem, is a very simple example of Backus–Gilbert theory. We go through the application mainly to remind the reader of that theory. Also, as in the later cases, which are not so easy, the matrix elements arising in the theory can be found analytically. This is very important because, if they could be evaluated only by numerical quadrature, the numerical labour would become very great even with modest numbers of data. There is a certain degree of flexibility in the Backus–Gilbert theory in regard to a choice of criterion measuring the quality of a delta-function approximation; in the later examples we need to invent new criteria, and then a key factor in their formulation is the analytic evaluation of the matrix elements, which in this first problem seems so natural.

Suppose a two-dimensional harmonic function $u(x, z)$ is known everywhere on the line $z = 0$. If the sources of the field lie in the half space $z \leqslant 0$, then at a point (x_i, z_i) with $z_i > 0$

$$\begin{aligned} u(x_i, z_i) &= \int_{-\infty}^{\infty} \frac{z_i}{\pi[(x - x_i)^2 + z_i^2]} u(x, 0)\, dx \\ &= \int_{-\infty}^{\infty} G_i(x) u(x, 0)\, dx. \end{aligned} \tag{1}$$

To apply the Backus–Gilbert theory we take (1) to define a series of observations,

$$u(x_i, z_i), \quad i = 1, 2, \ldots, n,$$

with $u(x, 0)$ as an unknown function, the downward continued field at $z = 0$. We construct $\tilde{u}(x_0)$, an estimate of the field at a fixed point $(x_0, 0)$, but this function is a smooth version of u, and the same $\tilde{u}$ is obtained from every one of the possible solutions to (1). This is achieved by demanding that

$$\begin{aligned} \tilde{u}(x_0) &= \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n} \alpha_i(x_0) G_i(x_0)\right] u(x, 0)\, dx \\ &= \int_{-\infty}^{\infty} \tilde{\delta}(x, x_0) u(x, 0)\, dx \end{aligned}$$

where

$$\tilde{\delta} = \sum_{i=1}^{n} \alpha_i G_i$$

is the function that smooths u and which, by an appropriate choice of α_i, approximates a true delta function $\delta(x - x_0)$. The criterion we apply for optimization of $\tilde{\delta}$ is the quadratic criterion (Backus & Gilbert 1968; Parker 1977a); the solution at a fixed point x_0 is given by minimizing

$$\bar{s}(x_0) = 12 \int_{-\infty}^{\infty} [(x - x_0)\tilde{\delta}(x, x_0)]^2\, dx + \lambda\left[\int_{-\infty}^{\infty} \tilde{\delta}(x, x_0)\, dx - 1\right] \tag{2}$$

by varying $\{\alpha_i\}$ and λ. The first term on the right is a quantitative measure of the resolution of the smoothing function $\tilde{\delta}$ and gives the approximate width of the peak (if any) at x_0; the second term is to ensure that

$$\int_{-\infty}^{\infty} \tilde{\delta}(x, x_0)\, dx = 1,$$

a reasonable (and, for this criterion, essential) demand for an approximate delta function. At the minimum of $\bar{s}$,

$$\frac{\partial \bar{s}}{\partial \alpha_i} = \frac{\partial \bar{s}}{\partial \lambda} = 0.$$

Differentiating (2) with respect to α_i and λ gives

$$\begin{aligned} \frac{\partial \bar{s}}{\partial \alpha_i} &= 24 \sum_{j=1}^{n} \alpha_j \int_{-\infty}^{\infty} G_i(x) G_j(x)\,(x - x_0)^2\, dx + \lambda = 0, \qquad i = 1, 2, \ldots, n \\ \frac{\partial \bar{s}}{\partial \lambda} &= \sum_{j=1}^{n} \alpha_j \int_{-\infty}^{\infty} G_j(x)\, dx - 1 = 0. \end{aligned} \tag{3}$$

These are $n + 1$ linear equations in the $n + 1$ unknowns $\{\alpha_i\}$, λ. The analytic evaluation of the integrals in (3) is straightforward; in particular an application of Parseval's theorem readily yields

$$\int_{-\infty}^{\infty} G_i(x) G_j(x)\,(x - x_0)^2\, dx = \frac{(z_i + z_j)\,[(x_i - x_0)(x_j - x_0) + z_i z_j] + (x_i - x_j)\,[z_i(x_0 - x_j) - z_j(x_0 - x_i)]}{\pi\,[(z_i + z_j)^2 + (x_i - x_j)^2]}.$$

Thus the matrix elements are quickly computable and coefficients α_i are easily found by solution of the linear system.

Numerical example of downward continuation

As a simple example of downward continuation of a two-dimensional field we consider the gravity anomaly from a uniform, semi-infinite slab of material density ρ in the region $0 \leqslant x$, $-h \leqslant z \leqslant 0$ with $h > 0$. The gravitational field at (x, z) with $z \geqslant 0$ is

$$\Delta g(x, z) = 2G\rho\left[\frac{h\pi}{2} + (z + h)\tan^{-1}\left(\frac{x}{z + h}\right) - z\tan^{-1}\left(\frac{x}{z}\right) + \frac{x}{2}\ln\left(\frac{x^2 + (z + h)^2}{x^2 + z^2}\right)\right]. \tag{4}$$

This field was sampled on a constant level at eight points $([n + \tfrac{1}{2}]\,h, 3h)$, $n = -4, -3, \ldots, 2, 3$ (see Fig. 1). These data were then used to compute estimates of the field at several different levels below $z = 3h$, and resolving widths were found for the estimates. Fig. 2

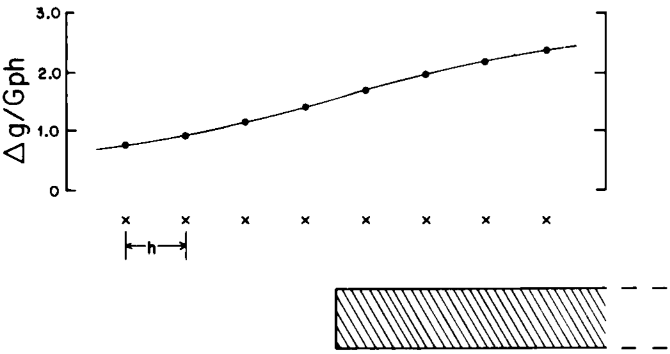

Figure 1. Geometry of example illustrating two-dimensional downward continuation. The crosses mark the observation points above the slab (shown shaded). The gravity anomaly is the continuous curve above the crosses.

shows the true field values and the downward-continued approximations for $x_0 \geqslant 0$ (the curves are symmetric about $x_0 = 0$). As we expect, the downward-continued fields become more and more inaccurate as $\bar{z}$, the distance below the observations, grows.

Somewhat more interesting is the behaviour of the resolution functions (Fig. 3). We see the expected general deterioration of resolution as the attempted depth of continuation increases, and the rapid failure to give narrow functions outside the horizontal range of the

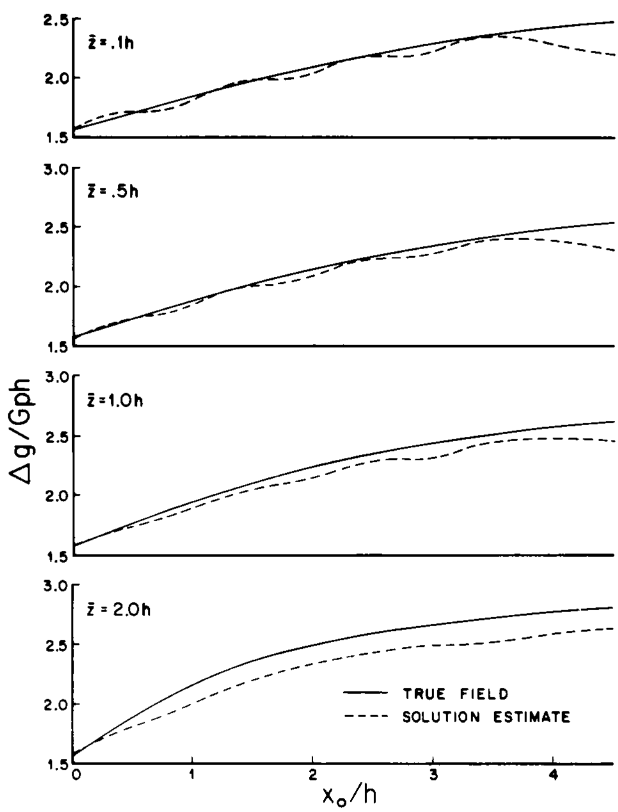

Figure 2. True field values and downward continued estimates at levels $\bar{z}$ below the observation level shown in Fig. 1.

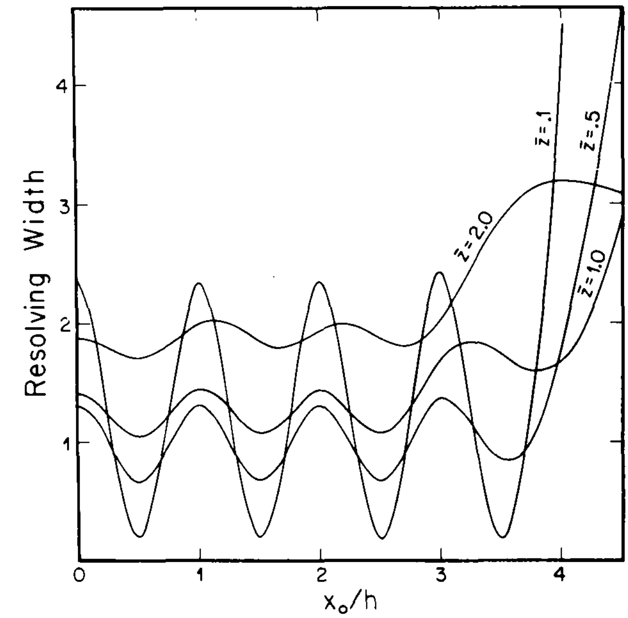

Figure 3. Resolution widths given by quadratic criterion for solution estimates at levels $\tilde{z}$ below the observation level.

data. Less predictable is the very large oscillation in resolution at a level very close to that of the original observations; unless the estimate is made nearly directly below a data sample, the resolution may be very poor. We find that the continuation depth at which the resolution function is uniformly smallest (for x_0 in the horizontal range of the data) is about 0.5 h. This result does not depend on the measured values of Δg at all and is unlikely to be much affected by the number of data, provided they are equally spaced h apart.

Note that the true field (4) is not square-integrable, as apparently required in the Backus–Gilbert formulation. Nonetheless our results remain valid because the data kernels are all $O(x^{-2})$ as $|x| \to \infty$. Therefore (1) is a bounded linear functional even for solutions that grow with x, provided the growth is $O(|x|)$.

Three-dimensional downward continuation

The continuation of three-dimensional potential field data down to a horizontal plane can also be treated as a Backus–Gilbert problem. This extension is not at all straightforward, and must be considered in detail.

In place of (1), the forward problem is now

$$u(x_j, y_j, z_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z_j}{2\pi[(x-x_j)^2 + (y-y_j)^2 + z_j^2]^{3/2}} u(x, y, 0)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_j(x, y) u(x, y, 0)\, dx\, dy$$

with $z_j > 0$.

With observation locations $(x_j, y_j, z_j), j = 1, 2, \ldots n$, the solution estimate at $(x_0, y_0, 0)$ is

$$\tilde{u}(x_0, y_0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\delta}(x, y; x_0, y_0) u(x, y, 0)\, dx\, dy$$

where

$$\tilde{\delta}(x, y; x_0, y_0) = \sum_{j=1}^{n} \alpha_j(x_0, y_0) H_j(x, y). \tag{5}$$

For convenience we will assume $(x_0, y_0) = (0, 0)$. There is no loss of generality, since a change of variables can always make this the case. We seek a criterion for choosing α_j, which not only produces a good approximation to a two-dimensional delta function centred at (0, 0), but also gives an estimate of the resolving area associated with $\tilde{\delta}$. The use of the quadratic criterion above, suggests the criterion

$$T = 3\pi^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2)^2 \tilde{\delta}^2 \, dx \, dy \tag{6}$$

with

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\delta}(x, y) \, dx \, dy = 1.$$

This measure has the dimensions of area; the normalizing factor of $3\pi^2$ is chosen to give a resolving area of πa^2, when the measure is applied to a circular pillbox of radius a and unit volume, centred at (0, 0). From (5) and (6)

$$T = 3\pi^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2)^2 \left[\sum_{j=1}^{n} \alpha_j H_j(x, y) \right] \left[\sum_{k=1}^{n} \alpha_k H_k(x, y) \right] dx \, dy. \tag{7}$$

Normally, we would expect to evaluate T by interchanging the order of integration and summation. The individual integrals that would result within the double sum, however, are divergent; the interchange is invalid without a further constraint in the α_j, to ensure the existence of (7).

Assume there is a set of α_j, such that (7) exists. We may then apply the two-dimensional power theorem to (7) followed by conversion to polar coordinates:

$$T = 3\pi^2 \int_0^{2\pi} \int_0^{\infty} \left\{ \sum_{j=1}^{n} \alpha_j \left[R_j - 2iz_j \Theta_j + \frac{z_j}{2\pi r} \right] \exp\left[-2\pi r(z_j + i\Theta_j)\right] \right\}$$
$$\times \left\{ \sum_{k=1}^{n} \alpha_k \left[R_k + 2iz_k \Theta_k + \frac{z_k}{2\pi r} \right] \exp\left[-2\pi r(z_k - i\Theta_k)\right] \right\} r \, dr \, d\theta$$

where

$$R_j = x_j^2 + y_j^2 - z_j^2, \quad \Theta_j = x_j \cos\theta + y_j \sin\theta.$$

Let us temporarily consider a similar integral:

$$T' = 3\pi^2 \int_0^{2\pi} \int_0^{\infty} \left\{ \sum_{j=1}^{n} \alpha_j \left[(R_j - 2iz_j\Theta_j) \exp\left[-2\pi r(z_j + i\Theta_j)\right] \right. \right.$$
$$\left. \left. + \frac{z_j}{2\pi r} [\![\exp\left[-2\pi r(z_j + i\Theta_j)\right] - \exp(-2\pi r\tilde{z})]\!] \right] \right\}$$
$$\times \left\{ \sum_{k=1}^{n} \alpha_k \left[(R_k + 2iz_k\Theta_k) \exp\left[-2\pi r(z_k - i\Theta_k)\right] \right. \right.$$
$$\left. \left. + \frac{z_k}{2\pi r} [\![\exp\left[-2\pi r(z_k - i\Theta_k)\right] - \exp(-2\pi r\tilde{z})]\!] \right] \right\} r \, dr \, d\theta$$

with $\tilde{z}$ any positive number. For T', we may interchange the order of integration and summation; with this operation safely performed, we force T' to equal T by imposing the auxiliary condition

$$\sum_{j=1}^{n} \alpha_j z_j = 0. \tag{8}$$

The delta approximation is determined by minimizing

$$\bar{T}' = T' + \lambda\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\delta}\, dx\, dy - 1\right) + \mu\left(\sum_{j=1}^{n} \alpha_j z_j\right)$$

where λ and μ are Lagrange multipliers; the resolving area is given by the value of T'. Minimization yields

$$\frac{\partial \bar{T}'}{\partial \alpha_j} = 6\pi^2 \sum_{k=1}^{n} \alpha_k I_{jk} + \lambda + z_j \mu = 0 \qquad j = 1, 2, \ldots n \tag{9a}$$

$$\frac{\partial \bar{T}'}{\partial \lambda} = \sum_{j=1}^{n} \alpha_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_j(x, y)\, dx\, dy - 1 = 0 \tag{9b}$$

$$\frac{\partial \bar{T}'}{\partial \mu} = \sum_{j=1}^{n} \alpha_j z_j = 0 \tag{9c}$$

which forms a system of $n + 2$ linear equation in the $n + 2$ unknowns $\{\alpha_j\}$, λ, μ. The I_{jk} are the individual integrals resulting from the interchange of the order of summation and integration. Each can be evaluated analytically; since the algebra is tedious, we only give the results in Appendix A.

The value of $\tilde{z}$ must be positive but is otherwise arbitrary. Numerical experiments indicate, however, that its choice affects the stability of the matrix inversion. The condition number appears to be a minimum when $\tilde{z}$ is near the average value of z_j.

Finally, this delta criterion places a demand on the data not encountered during two-dimensional continuation: the method fails if all the z_j are equal. In this case (9b) and (9c) are incompatible and the matrix becomes singular. If such a situation arose, we would be forced to choose a different criterion, sacrificing the convenient measure of resolving area and, perhaps, computational ease.

We have applied the formalism to an example of map joining for overlapping aeromagnetic surveys run at two different altitudes. For the sake of brevity we do not present results here, since the main features of the results are present in the two-dimensional problem.

Two-dimensional upward continuation

We now consider the problem of determining solution estimates on a level surface everywhere above a finite number of observations of a two-dimensional potential field; again, the sources are assumed to lie below the measurements. To apply the Backus–Gilbert formulation, we might seek an expression similar to (1) for the forward problem. Unfortunately the relation between each datum and the model is now an unbounded linear functional, the downward continuation operator; this cannot be expressed as an ordinary integral. One way to avoid the difficulty is to invoke a quelling (Backus 1970b), which

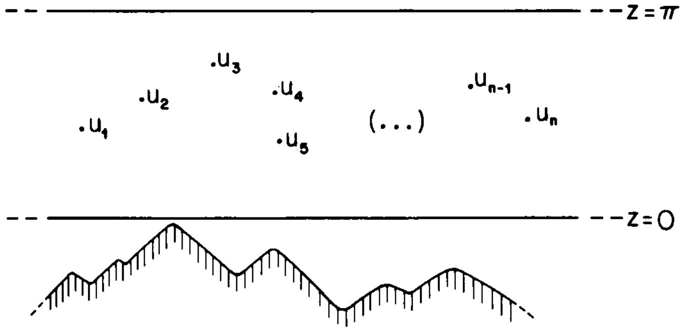

Figure 4. Geometry for two-dimensional upward continuation. Note in this case distances have been scaled to make the separation of the two enclosing levels equal to π.

simultaneously reduces the unbounded data functional and the desired delta-function kernel to more tractable, bounded forms. This is achieved most readily by convolution with a sufficiently smooth function: for example a Gaussian filter or perhaps a band-limited function. The key criterion is super-exponential decay in the wavenumber domain. Gaussian smoothing is presaged remarkably in the early work of Bullard & Cooper (1948); band-limited functions arose in the study of Courtillot *et al.* (1973), but not quite in the role they have here.

We put forward a different approach, in which use is made of the commonly valid condition that the points where data are available lie entirely in a source-free region. This enables us to apply stronger constraints than those available with quelling, because more information about the system has been used. Also, in the simple geometry we have studied, close-form expressions could be obtained for the matrix elements. It is probable, however, that in three-dimensional systems the quelling approach may provide a numerically more attractive procedure; this has not yet been assessed.

We proceed with an example, shown in Fig. 4. Here, the measurements of a potential field, $u(x, z)$, are made between the source region and the level on which we will construct solution estimates. Assume a horizontal line, $z = 0$, can be passed between the sources and the lowest observation. For convenience, we normalize all lengths so that the upper level is $z = \pi$. Between these levels, the field is harmonic; the data are solutions of Dirichlet boundary value problems if the field is known everywhere on $z = 0$ and $z = \pi$.

That is

$$u(x_j, z_j) = \oint_c u(c) \frac{\partial \Gamma_j}{\partial \nu} dc \qquad j = 1, 2, \ldots, n \tag{10}$$

where c consists of $z = 0$ traversed from $-\infty$ to ∞, and $z = \pi$ traversed from ∞ to $-\infty$. The value of the potential on c is $u(c)$; ν is the unit outward normal to c; and Γ_j is the jth Green's function for the interior of c, given by

$$\Gamma_j(x, z) = \frac{1}{4\pi} \ln \left\{ \frac{\operatorname{ch}(x - x_j) - \cos(z - z_j)}{\operatorname{ch}(x - x_j) - \cos(z + z_j)} \right\} \tag{11}$$

where ch is the hyperbolic cosine. Equation (10) is similar to (1), where $u(c)$ is the unknown solution, except that the simple interval is replaced by an integration around c. We can apply the Backus–Gilbert method to this problem, but now the delta approximations are functions defined on c; the solution estimates will be weighted averages not only of the field on the upper level, but also of that on the lower level.

We write (10) as

$$\begin{aligned} u(x_j, z_j) &= \int_{-\infty}^{\infty} u(x, 0) \left.\frac{\partial \Gamma_j}{\partial \nu}\right|_{z=0} dx \\ &+ \int_{-\infty}^{\infty} u(x, \pi) \left.\frac{\partial \Gamma_j}{\partial \nu}\right|_{z=\pi} dx \\ &= \int_{-\infty}^{\infty} u(x, 0) \frac{\sin z_j}{2\pi[\mathrm{ch}(x - x_j) - \cos z_j]} dx \\ &= \int_{-\infty}^{\infty} u(x, \pi) \frac{\sin z_j}{2\pi[\mathrm{ch}(x - x_j) + \cos z_j]} dx. \end{aligned} \tag{12}$$

The data kernels are functions defined on c:

$$G_j(x, z) = \begin{cases} \dfrac{1}{2\pi} \dfrac{\sin z_j}{\mathrm{ch}(x - x_j) - \cos z_j}; & z = 0 \\[2ex] \dfrac{1}{2\pi} \dfrac{\sin z_j}{\mathrm{ch}(x - x_j) + \cos z_j}; & z = \pi. \end{cases}$$

We have not yet made a statement that we are performing upward continuation, (that is, continuation away from the sources). Without further restriction, our formalism allows sources not only below $z = 0$, but also above $z = \pi$. We impose this restriction by defining a new set of data kernels on c, corresponding to upward continuation from the lower level.

$$\bar{G}_j(x) = \begin{cases} \dfrac{1}{\pi} \dfrac{z_j}{(x - x_j)^2 + z_j^2}; & z = 0 \\[2ex] 0; & z = \pi. \end{cases}$$

Then,

$$u(x_j, z_j) = \oint_c u(c) G_j \, dc$$

and

$$u(x_j, z_j) = \oint_c u(c) \bar{G}_j \, dc, \qquad j = 1, 2, \ldots, n.$$

Each observation, $u(x_j, z_j)$, corresponds to two data, associated with kernels G_j and $\bar{G}_j$. The addition of data kernels which are zero on the upper level helps us suppress the averaging of the field on $z = 0$. In effect, we will construct better delta approximations because we have restricted the class of admissible solutions to those harmonic everywhere above $z = 0$, a restriction not implied in (12).

The solution estimate at (x_0, π) is

$$\tilde{u}(x_0, \pi) = \int_{-\infty}^{\infty} \tilde{\delta}(x, \pi) u(x, \pi) \, dx + \int_{-\infty}^{\infty} \tilde{\delta}(x, 0) u(x, 0) \, dx$$

where

$$\tilde{\delta}(x, \pi) = \sum_{j=1}^{n} \alpha_j(x_0) G_j(x; \pi)$$

and

$$\tilde{\delta}(x,0)=\sum_{j=1}^{n}\alpha_j(x_0)G_j(x;0)+\sum_{j=1}^{n}\bar{\alpha}_j(x_0)\bar{G}_j(x;0).$$

The delta approximation should be peaked at x_0 on the upper level, with as little energy as possible on the lower level.

For notational convenience, let $\bar{G}_j \equiv G_{j+n}$, $\bar{\alpha}_j \equiv \alpha_{j+n}$, $x_j \equiv x_{j+n}$, and $z_j \equiv z_{j+n}$.

We then minimize

$$R(x_0)=12\int_{-\infty}^{\infty}(x-x_0)^2\left[\sum_{j=1}^{2n}\alpha_j(x_0)G_j(x;\pi)\right]^2 dx$$

$$+K\int_{-\infty}^{\infty}\left[\sum_{j=1}^{2n}\alpha_j(x_0)G_j(x;0)\right]^2 dx+\lambda\left[\oint_c\tilde{\delta}-1\right];$$

we take our resolving width estimate to be the first term on the right.

The free constant K weights the contribution to R from the power in $\tilde{\delta}$ on the lower level. We may choose any $K \geqslant 0$. For $K=0$, however, the minimum value of R is zero, with $\alpha_j=0, j=1,2,\ldots,n$. That is,

$$\tilde{\delta}=\sum_{j=1}^{2n}\alpha_j(x_0)G_j(x;z)$$

and hence is non-zero only on $z=0$: we are then back to the problem of downward continuation. As K increases, we discriminate more strongly against any averaging on the lower level; the energy of the delta approximation becomes more concentrated on $z=\pi$, but only at the expense of a growing value of R. Generally, K should be chosen as large as possible, while retaining acceptable resolution on $z=\pi$. As the minimum of R,

$$\frac{\partial R}{\partial \alpha_j}=24\int_{-\infty}^{\infty}(x-x_0)^2\left[\sum_{k=1}^{2n}\alpha_k(x_0)G_k(x;\pi)\right]\cdot G_j(x;\pi)\,dx$$

$$+2K\int_{-\infty}^{\infty}\left[\sum_{k=1}^{2n}\alpha_k(x_0)G_k(x;0)\right]\cdot G_j(x;0)\,dx+\lambda=0 \qquad j=1,2,\ldots,2n \tag{13}$$

$$\frac{\partial R}{\partial \lambda}=\sum_{j=1}^{2n}\alpha_j(x_0)-1=0,$$

giving $2n+1$ linear equations to be solved for $\{\alpha_j\}$, λ. All the integrals which will appear in (13) can be performed analytically (Appendix B).

In the principle, to obtain the best possible resolution on the upper level, the lower level of c should drape the top of the source region. We are then being as restrictive as possible about the class of admissible solutions; in the geometry of Fig. 4, we would rule out functions harmonic between $z=0$ and $z=\pi$, but possessing singularities above the sources. This improved resolution, however, is gained by sacrificing the simplicity of the Green's function (11) and the ease of performing the necessary integrations. In certain circumstances this is inevitable, as when some observations occur in valleys.

The problem of three-dimensional upward continuation can be approached in a similar fashion. The added complication of dealing with divergent integrals will again arise; little would be gained by carrying out the unwieldy formalism here.

Numerical example of upward continuation

The data used in the first example were continued to several lines above the observations; we now denote by $\bar{z}$ the height of such a line above the data level. The lower boundary of c naturally coincides with the top of the slab. In this example we have taken $h = 1$. Fig. 5 shows the effect of varying K in (13), for an arbitrarily chosen point, with $\bar{z} = 0.5$: $x_0 = 0.5$, $z_0 = 3.5$. For very small K, the resolving width apparently approaches zero, but this is due to $\tilde{\delta}$ having little energy on $z = 3.5$. The solution estimates are then effectively averages of the field on $z = 0$ only. As K increases, there is more discrimination against energy on $z = 0$; the power in $\tilde{\delta}$ on this level then decreases. This is done with a sacrifice in resolution on $z = 3.5$, but the solution estimates more closely agree with the true values. Because similar behaviour was seen at other points, a value of $K = 10$ would appear to be a good compromise between resolution and the amount of averaging on $z = 0$.

Results for values of $\bar{z}$ ranging from 0.1 to 2.0 showed trends in keeping with those for downward continuation of the same data. Resolution deteriorates as $\bar{z}$ increases, but is also poor for small $\bar{z}$, except near sample points. Again, the height at which resolution is uniformly best is about $\bar{z} = 0.5$. In general, for values of x_0 within the horizontal range of the samples, the accuracy of solution estimates tends to improve as $\bar{z}$ increases from 0.1 to 2.0; for x sufficiently close to the horizontal position of a sample, however, there is an initial degradation of accuracy as $\bar{z}$ increases. If $\bar{z}$ has been increased beyond 2.0, we would expect

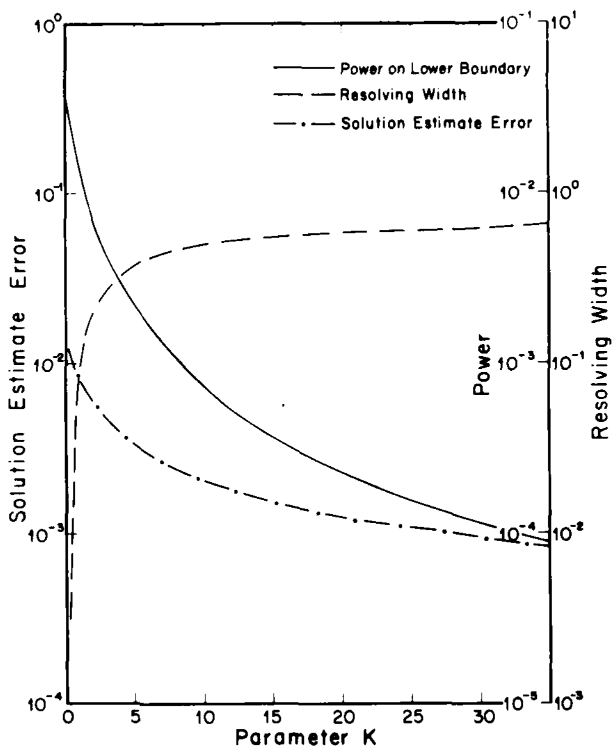

Figure 5. Effect of varying the free parameter K on various measures of solution quality at the point (0.5, 3.5). The power on the lower boundary gives the integral of the square of $\tilde{\delta}$ over the entire lower line, and should of course be as small as possible. Solution estimate error is the difference between the true field at the point and that estimated by linear combinations of the data; like the resolving width this too should be made small.

the accuracy of our predictions eventually to begin to suffer again, as the width of data kernels becomes much larger than the horizontal extent of the samples.

Conclusions

We have shown that upward and downward continuation of harmonic functions can be treated as inverse problems. While it is well known that downward continuation is unstable and must be regularized in some way, we have shown that upward continuation from data on a finite surface is also poorly posed. Therefore it is important in both cases to develop methods that allow one to assess uncertainties. In this paper we have used the Backus–Gilbert concept of resolution to provide the measure of ambiguity. Within the great flexibility of the Backus–Gilbert formalism, we have found procedures that permit the matrix elements to be expressed in closed form so that the very great cost of numerical evaluation of each element is circumvented. This makes the proposed methods practicable for data sets of moderate size, say several hundreds of data.

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Appendix A: analytic expression for I_{jk} (equation (9a))

We express each I_{jk} in (9a) as

$$I_{jk} = A_{jk} + B_{jk} + C_{jk} + D_{jk}$$

and evaluate each term separately.

Define

$$\bar{x} = x_j - x_k$$

$$\bar{y} = y_j - y_k$$

$$\bar{z} = z_j + z_k$$

$$r_j = (x_j^2 + y_j^2)^{1/2}$$

$$\bar{r} = (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2}.$$

Then

$$A_{jk} = \frac{z_j z_k}{2\pi} \ln \left\{ \frac{[z_j + \tilde{z} + \sqrt{r_j^2 + (z_j + \tilde{z})^2}]\,[z_k + \tilde{z} + \sqrt{r_k^2 + (z_k + \tilde{z})^2}]}{4\tilde{z}(\bar{z} + \bar{r})} \right\}.$$

If $\bar{x} \neq 0$ or $\bar{y} \neq 0$,

$$B_{jk} = \frac{1}{2\pi\rho\bar{r}^2} \left\{ F_j[R_k + z_k\bar{r} + 2z_k\rho(y_k + ix_k)] + F_k'[R_j + z_j\bar{r} - 2z_j\rho(y_j + ix_j)] + F_jF_k'\left[\frac{(\bar{z} - 2\bar{r})(\bar{y} + i\bar{x})}{\bar{r}(\bar{r} - \bar{z})}\right]\right\} - \frac{z_j z_k}{\pi}\left[1 + \frac{2(y_j - ix_j)(y_k - ix_k)}{(\bar{y} - i\bar{x})^2}\right]$$

where

$$\rho = \frac{\bar{r} - \bar{z}}{y + i\bar{x}}$$

and

$$F_j = -\rho^2 z_j(y_j + ix_j) + \rho R_j + z_j(y_j - ix_j) \qquad \text{(A1)}$$

$$F_j' = \rho^2 z_j(y_j + ix_j) + \rho R_j - z_j(y_j - ix_j).$$

If $\bar{x} = \bar{y} = 0$

$$B_{jk} = \frac{R_jR_k + \bar{z}(R_jz_k + R_kz_j) + 2z_jz_kr_j^2}{2\pi\bar{z}^2}.$$

When $x_j \neq 0$ or $y_j \neq 0$

$$C_{jk} = \frac{z_k}{\pi}\left[z_j - \frac{F_j}{2\rho_j\sqrt{r_j^2 + (z_j + \tilde{z})^2}}\right]$$

where

$$\rho_j = \frac{-(z_j + \tilde{z}) + \sqrt{r_j^2 + (z_j + \tilde{z})^2}}{y_j + ix_j}$$

and F_j is given by (A1), with $\rho = \rho_j$.

For $x_j = y_j = 0$,

$$C_{jk} = \frac{z_j^2 z_k}{2\pi(z_j + \tilde{z})}.$$

Similarly, when $x_k \neq 0$ or $y_k \neq 0$

$$D_{jk} = \frac{-z_j}{\pi}\left\{z_k - \frac{F_k'}{2\rho_k\sqrt{r_k^2 + (z_k + \tilde{z})^2}}\right\}$$

where

$$\rho_k = \frac{(z_k + \tilde{z}) - \sqrt{r_k^2 + (z_k + \tilde{z})^2}}{y_k + ix_k},$$

and F_k' is given in (A1) with $\rho = \rho_k$. Finally, for $x_k = y_k = 0$,

$$D_{jk} = \frac{-z_j z_k^2}{2\pi(z_k + \tilde{z})}.$$

Further algebraic reduction would demonstrate that all of these terms are, in fact, real.

Appendix B: analytic expressions for the integrals of equation (13)

To evaluate (13), we must perform several types of integrals; all can be expressed in closed form.

The second term on the right side of (13) factors into three types of integrals, which we consider in turn.

Case 1: $k \leqslant n; j \leqslant n$:

$$\int_{-\infty}^{\infty} \frac{\sin z_j \sin z_k}{4\pi^2[\mathrm{ch}(x - x_j) - \cos z_j][\mathrm{ch}(x - x_k) - \cos z_k]}.$$

Letting $y = e^x$, this becomes

$$\int_0^{\infty} \frac{\sin z_j \sin z_k \, y \, dy}{\pi^2 [e^{-x_j}y^2 - 2y\cos z_j + e^{x_j}][e^{-x_k}y^2 - 2y\cos z_k + e^{x_k}]}$$

$$= \frac{e^{x_j + x_k}}{\pi^2 D}\{(e^{2x_k} - e^{2x_j})[(\pi - z_j)\cos z_j \sin z_k - (\pi - z_k)\cos z_k \sin z_j \qquad \text{(B1)}$$

$$+ (x_k - x_j)\sin z_j \sin z_k] + 2(e^{x_j}\cos z_j - e^{x_k}\cos z_k)[(\pi - z_j)\sin z_k \, e^{x_j}$$

$$- (\pi - z_k)\sin z_j \, e^{x_k}]\}$$

where

$$D = 4\, e^{x_j + x_k}[e^{x_k}\cos z_k - e^{x_j}\cos z_j][e^{x_j}\cos z_k - e^{x_k}\cos z_j] + (e^{2x_k} - e^{2x_j}).$$

When $j = k$, (B1) reduces to

$$\frac{1}{2\pi^2}[1 - (\pi - z_j)\cot z_j].$$

Case 2: $k \leqslant n; j > n$:

$$\int_{-\infty}^{\infty} \frac{z_j \sin z_k \, dx}{2\pi^2[\mathrm{ch}(x - x_k) - \cos z_k][(x - x_j)^2 + z_j^2]}.$$

With an application of Parseval's theorem, this integral can be reduced to two integrals of a form given by Erdélyi (1954, p. 15, transform 9).

The result is

$$\frac{1}{2\pi^2} R_e \left\{ \psi\left(\frac{z_k - z_j + i(x_k - x_j)}{2\pi}\right) - \psi\left(\frac{z_k + z_j + i(x_k - x_j)}{2\pi}\right)\right\} + \frac{2\pi(z_k - z_j)}{(z_k - z_j)^2 + (x_k - x_j)^2}$$

where ψ is the digamma function (Abramowitz & Stegun 1965, Ch. 6). If $j = k$, this reduces to

$$-\frac{1}{2\pi^2}\psi\left(\frac{z_j}{\pi}\right) - \gamma$$

where γ is Euler's constant.

For $k > n$; $j \leqslant n$ the same integral occurs with the subscripts reversed.

Case 3: $k > n$; $j > n$:

$$\int_{-\infty}^{\infty} \frac{z_j z_k \, dx}{\pi^2 [(x - x_j)^2 + z_j^2][(x - x_k)^2 + z_k^2]}$$

which is evaluated by residues to give

$$\frac{z_j + z_k}{\pi[(x_k - x_j)^2 + (z_j + z_k)^2]}.$$

The first term on the right of (13) is

$$\int_{-\infty}^{\infty} \frac{(x - x_0)^2 \, dx}{[\mathrm{ch}(x - x_j) + \cos z_j][\mathrm{ch}(x - x_k) + \cos z_k]}.$$

Letting $y = e^{x - x_0}$ we get

$$\int_0^{\infty} \frac{4y(\ln y)^2 \, dy}{[e^{-\delta_j} y^2 + 2y \cos z_j + e^{\delta_j}][e^{-\delta_k} y^2 + 2y \cos z_k + e^{\delta_k}]}$$

where $\delta_j = x_j - x_0$. This is evaluated by residues using standard techniques for powers of logarithms. If $i \neq j$, the result is

$$2i\, e^{\delta_j + \delta_k} \left\{ R_1 + R_2 + R_3 + R_4 + \frac{4i\pi^2}{3} \int_0^{\infty} \frac{y \, dy}{[y^2 + 2y \cos z_j \, e^{\delta_j} + 2\, e^{\delta_j}][y^2 + 2y \cos z_k \, e^{\delta_k} + e^{2\delta_k}]} \right\}$$

where

$$R_1 = \frac{(\cot z_j + i)[(\delta_j + iz_j + i\pi)^2(\delta_j + iz_j - 2i\pi)]}{3[e^{\delta_k + iz_k} - e^{\delta_j + iz_j}][e^{\delta_k - iz_k} - e^{\delta_j + iz_j}]}$$

$$R_2 = \frac{-(\cot z_j - i)[(\delta_j - iz_j + i\pi)^2(\delta_j - iz_j - 2i\pi)]}{3[e^{\delta_k + iz_k} - e^{\delta_j - iz_j}][e^{\delta_k - iz_k} - e^{\delta_j - iz_j}]}$$

$$R_3 = \frac{(\cot z_k + i)[(\delta_k + iz_k + i\pi)^2(\delta_k + iz_k - 2i\pi)]}{3[e^{\delta_j + iz_j} - e^{\delta_k + iz_k}][e^{\delta_j - iz_j} - e^{\delta_k + iz_k}]}$$

and

$$R_4 = \frac{-(\cot z_k - i)\,[(\delta_k - iz_k + i\pi)^2(\delta_k - iz_k - 2i\pi)]}{3\,[e^{\delta_j + iz_j} - e^{\delta_k - iz_k}]\,[e^{\delta_j - iz_j} - e^{\delta_k - iz_k}]}.$$

The integral in the above expression is evaluated as (B1). If $j = k$, the result is

$$\frac{1}{\sin^2 z_j}\left\{-4\pi i(\delta_j + i\pi) + (\delta_j + iz_j + i\pi)^2\left[1 + \frac{\cot z_j}{3}(2\pi - z_j + i\delta_j)\right]\right.$$

$$\left. + (\delta_j - iz_j + i\pi)^2\left[1 - \frac{\cot z_j}{3}(2\pi + z_j + i\delta_j)\right]\right\}$$

$$-\frac{8\pi^2 e^{2\delta_j}}{3}\int_0^\infty \frac{y\,dy}{[y^2 + 2y\cos z_j\, e^{\delta_j} + e^{2\delta_j}]^2}$$

Appendix C: stability of upward continuation

Assume a two-dimensional harmonic function is known everywhere on a line $y = f(x)$, where f is continuous on the interval $-\infty < x < \infty$; the sources lie below y. The upward continuation of this data to a point on a line above $f(x)$ is equivalent to solving a Dirichlet boundary value problem, an integral over $y = f(x)$. Integration, and hence this upward continuation, are mathematically stable processes: the solution depends continuously on the data (Parker 1977a).

As soon as we relax the condition that the field is known everywhere on $y = f(x)$, we lose this stability. Assume now we only know the field over a finite segment of y. This information still defines the field at the upper level uniquely, but no longer as the solution of a boundary value problem. Instead, we now must perform analytic continuation, an unstable process.

As an example, consider the case where we know the value of a harmonic function $\phi(x, z)$ everywhere on a line segment ($|x| \leq a; z = 0$). We analytically continue this function to the entire $z = 0$ line, then continue up to a level $z = h$ with the Dirichlet integral:

$$\phi(x, h) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x', 0)\frac{h}{(x - x')^2 + h^2}\,dx'.$$

Now consider a slightly different data set: let

$$\bar{\phi}(x, 0) = \phi(x, 0) + \frac{Ay}{(x - b)^2 + y^2}; \qquad |x| \leq a.$$

Here $y > 0$, $b > a$ and A is arbitrary. This function is also analytically continued to the entire $z = 0$ line, then upward continued to $z = h$. Hence,

$$\bar{\phi}(x, h) = \phi(x, h) + \frac{A(h + y)}{(x - b)^2 + (h + y)^2}.$$

Define

$$\Delta\phi(x, h) = \bar{\phi}(x, h) - \phi(x, h)$$

$$= \frac{A(h + y)}{(x - b)^2 + (h + y)^2}$$

and

$$\Delta\phi(x, 0) = \bar{\phi}(x, 0) - \phi(x, 0)$$

$$= \frac{Ay}{(x-b)^2 + y^2}; \qquad |x| \leqslant a.$$

Under the uniform norm,

$$\lim_{y \to 0} \|\Delta\phi(x, 0)\| = 0$$

while

$$\lim_{y \to 0} \|\Delta\phi(x, h)\| = A/h.$$

Thus, the solution $\phi(x, h)$ does not depend continuously on the data; arbitrarily small perturbations to the data can give rise to arbitrarily large perturbations in the solution.

Mathematical instability is normally dealt with by applying some type of smoothness criterion to the solution (for example, the filtering used during conventional downward continuation). Indeed, our use of the boundary C in the upward continuation problem is such a criterion, since it forces all sources to be below a specified level. In the example above, the limiting process of letting y approach zero would violate this demand.