

Upwind Schemes, Multigrid and Defect Correction for the Steady Navier-Stokes Equations

Barry Koren

*Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands*

1. NAVIER-STOKES EQUATIONS

The Navier-Stokes equations considered are

$$\frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} - \frac{1}{Re} \left\{ \frac{\partial r(q)}{\partial x} + \frac{\partial s(q)}{\partial y} \right\} = 0, \quad (1)$$

with q the (perfect gas) conservative state vector $(\rho, \rho u, \rho v, \rho e)^T$, $f(q)$ and $g(q)$ the convective flux vectors $(\rho u, \rho u^2 + p, \rho uv, \rho u(e + p/\rho))^T$ respectively $(\rho v, \rho uv, \rho v^2 + p, \rho v(e + p/\rho))^T$, and $r(q)$ and $s(q)$ the diffusive flux vectors $(0, \tau_{xx}, \tau_{xy}, 1/((\gamma - 1)Pr)\partial(c^2)/\partial x)^T$ respectively $(0, \tau_{xy}, \tau_{yy}, 1/((\gamma - 1)Pr)\partial(c^2)/\partial y)^T$.

2. DISCRETIZATION METHOD

To still allow Euler flow ($1/Re=0$) solutions with discontinuities, the equations are discretized in the integral form. A straightforward and simple discretization of the integral form is obtained by subdividing the integration region Ω into quadrilateral finite volumes $\Omega_{i,j}$, and by requiring that the conservation laws hold for each finite volume separately:

$$\oint_{\partial\Omega_{i,j}} (f(q)n_x + g(q)n_y)ds - \frac{1}{Re} \oint_{\partial\Omega_{i,j}} (r(q)n_x + s(q)n_y)ds = 0, \quad \forall i,j. \quad (2)$$

2.1. Evaluation of diffusive fluxes

For the evaluation of the diffusive fluxes at a volume wall, it is necessary to compute $\text{grad}(u)$, $\text{grad}(v)$ and $\text{grad}(c^2)$ at that wall. For this we use the standard technique as outlined in [11]. The technique applied is central, the directional dependence coming from the cross derivative terms is neglected. For sufficiently smooth grids this flux computation is second-order accurate.

2.2. Evaluation of convective fluxes

For convection dominated flows, our objective, a proper evaluation of the convective flux vectors is of paramount importance. Based on our previous experience with the Euler equations (see [3] for an overview), for this we prefer an upwind approach. Along each finite volume wall we assume the convective flux vector to be constant, and to be determined by a constant left and right state only. The 1D Riemann problem thus obtained is solved in an approximate way.

As approximate Riemann solver for the Euler equations, we prefer Osher's scheme [10]. Reasons for this preference are: (i) its continuous differentiability, and (ii) its consistent treatment of boundary conditions. The question arises whether it is still a good choice to use Osher's scheme when typical Navier-Stokes features such as shear, separation and heat conduction also have to be resolved. In [5] we therefore reconsidered the choice of an approximate Riemann solver. Since continuous differentiability is an absolute requirement for the success of our solution method, and since the only known approximate Riemann solvers with this property are Osher's [10] and van Leer's [7], our choice was confined to these two only. The requirement of accurate modelling of physical diffusion determined our choice. In [7], van Leer stated already that his flux vector splitter cannot preserve steady contact discontinuities. Since a discrete shear layer may be interpreted as a layer of contact discontinuities, doubt arose about the suitability of van Leer's scheme for Navier-Stokes codes. Recently, this doubt was confirmed in [9] where van Leer et al. made a qualitative analysis (supplemented with numerical experiments) for various upwind schemes. There, Osher's scheme turned out to be better indeed for the resolution of boundary layer flows. To shed some more light on the difference in quality between both schemes, in [5] a quantitative error analysis is presented for Osher's

and van Leer's scheme. The analysis is confined to the steady, 2D, isentropic Euler equations for a perfect gas with $\gamma=1$; i.e. equation (1) with $1/Re=0$, $q=(\rho, \rho u, \rho e)^T$, $f(q)=(\rho u, \rho(u^2+c^2), \rho uv)^T$, $g(q)=(\rho v, \rho uv, \rho(v^2+c^2))^T$ and $c=\text{constant}$. For both schemes the system of modified equations is derived by considering a first-order accurate, square finite volume discretization, and a subsonic flow with u and v positive and $\rho \approx \text{constant}$. Substituting a boundary layer type solution into the error terms it appears that van Leer's scheme deteriorates compared to Osher's scheme for increasing Re .

The approximation of the left and right state in the 1D Riemann problem determines the accuracy of the convective discretization. First-order accuracy is obtained in the standard way, by taking the left and right state equal to that in the corresponding adjacent volume. Higher-order accuracy is obtained by low-degree piecewise polynomial state interpolation (MUSCL-approach) with van Leer's κ -scheme [8]. For the scalar model equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \right) = 0, \quad (3)$$

in [5] we derive the modified equation by considering a square finite volume discretization. From this we find as the κ that gives the highest possible (i.e. third-order) accuracy

$$\kappa = \frac{1}{3} \left\{ 1 + \epsilon \left(\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^3 \partial y} + 2 \frac{\partial^4 u}{\partial x \partial y^3} + \frac{\partial^4 u}{\partial y^4} \right) / \left(\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} \right) \right\}. \quad (4)$$

Since convection dominated problems are our interest, we neglect the above diffusion dependence and simply take $\kappa=1/3$. To avoid spurious non-monotonicity, in [5] a new limiter is constructed for the $\kappa=1/3$ approximation. Using Sweby's notation [12], it reads

$$\psi(r) = \frac{2r^2+r}{2r^2-r+2}, \quad r \in \mathbb{R}. \quad (5)$$

3. SOLUTION METHOD

To efficiently solve the system of discretized equations, symmetric point Gauss-Seidel relaxation, accelerated by nonlinear multigrid (FAS), is applied. The process is started by nested iteration (FMG). Per finite volume we use a Newton iteration for the collective update of the four state vector components. With again a square finite volume discretization of (3), in [6] it is shown by local mode analysis that symmetric point Gauss-Seidel relaxation accelerated by multigrid converges fast for the first-order discretization for any value of the mesh Reynolds number Re_h . However, it appears to converge very slowly for the higher-order ($\kappa=1/3$) discretization for small and moderately large values of Re_h . It even appears to diverge for large values of Re_h . The cause clearly is the higher-order discretization of the convection operator. No cure can be found in using some other κ . As with the Euler equations [2,3,4], the difficulty in inverting the higher-order operator is circumvented by introducing iterative defect correction (IDeC) as an outer iteration for the nonlinear multigrid cycling. With $F_h(q_h)$ the full, higher-order accurate operator and $\tilde{F}_h(q_h)$ the less accurate operator that can be easily inverted, iterative defect correction can be written as:

$$\begin{aligned} \tilde{F}_h(q_h^1) &= 0, \\ \tilde{F}_h(q_h^{n+1}) &= \tilde{F}_h(q_h^n) - F_h(q_h^n), \quad n=1, 2, \dots, N, \end{aligned} \quad (6)$$

with n referring to the n th iterand. The operator $\tilde{F}_h(q_h)$ necessarily has only first-order accurate convection, but the amount of diffusion can be chosen freely. In [6], this freedom is exploited by analyzing three approximate operators: (i) an operator without physical diffusion (first-order Euler), (ii) an operator with partial physical diffusion (zeroth-order Navier-Stokes), and (iii) an operator with full, second-order accurate physical diffusion (first-order Navier-Stokes). The last approximate operator most closely resembles the higher-order operator, and therefore was supposed to have the best convergence properties. For this operator, theory [1] predicts that already q_h^1 is second-order accurate if F_h is second-order accurate and q_h^1 sufficiently smooth. Theory does not give such a guarantee for the other approximate operators. Local mode analyses with the square finite volume discretization of (3), and experiments with the Navier-Stokes equations show the third approximate operator to have the best convergence properties indeed. Its relative complexity is taken for granted. The numerical results presented hereafter were all obtained with this operator as operator to be inverted.

4. NUMERICAL RESULTS

To evaluate the computational method, we consider: (i) a subsonic flat plate flow at $M=0.5$, $Re=100,400$ and 1600 , (ii) a supersonic flat plate flow with oblique shock wave - boundary layer interaction at $M=2$, $Re=2.96 \cdot 10^5$, and (iii) a hypersonic blunt body flow at $M=8.15$, $Re=1.67 \cdot 10^5$, $\alpha=30^\circ$.

4.1. Approximate Riemann solver

To verify the analytical results obtained for Osher's and van Leer's scheme, we consider the subsonic flat plate flow and perform an experiment with Re -variation. For both schemes we use the first-order accurate discretization, and identical grids and boundary conditions. The results given in Fig. 1 show the predicted deterioration of van Leer's scheme with increasing Re . The numerical results presented hereafter were obtained with Osher's scheme only.

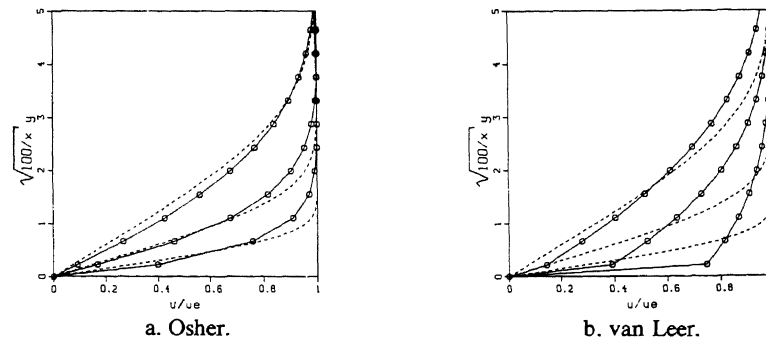


Fig. 1. Velocity profiles subsonic flat plate flow (----: Blasius solution, $Re=100,400,1600$).

4.2. Monotone higher-order accuracy

To evaluate the monotone higher-order accurate discretization derived, we consider the supersonic flat plate flow. At first we compute the Euler flow solution with and without limiter. The inviscid surface pressure distributions obtained (Fig. 2a) clearly show the benefit of the limiter. To show now the benefit of higher-order accuracy we compute the Navier-Stokes flow solution with the limited $\kappa=1/3$ scheme and the first-order scheme. The Navier-Stokes solution is known to have shock-induced separation. The computed viscous surface pressure distributions (Fig. 2b) show that the higher-order solution has shock-induced separation indeed, whereas the first-order solution remains attached. (The latter lacks the plateau in the surface pressure distribution.)

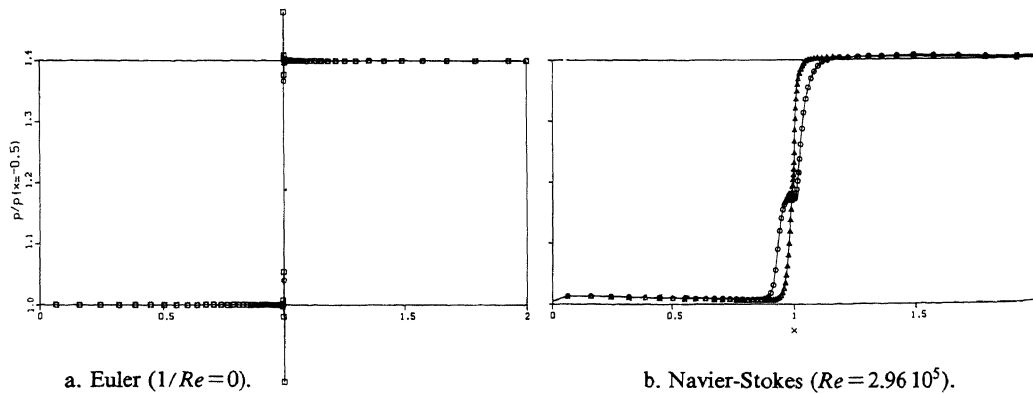


Fig. 2. Surface pressure distributions supersonic flat plate flow, $M=2$.

(\circ : limited $\kappa=1/3$, \square : non-limited $\kappa=1/3$, Δ : first-order).

4.3. Multigrid behaviour

Multigrid convergence results for the subsonic and supersonic flat plate flow are given in Fig. 3. The somewhat worse convergence rates for the supersonic flat plate flow are still satisfactory; for the 80×32 -grid (Fig. 3b) the convergence rate with multigrid is still much faster than that without multigrid.

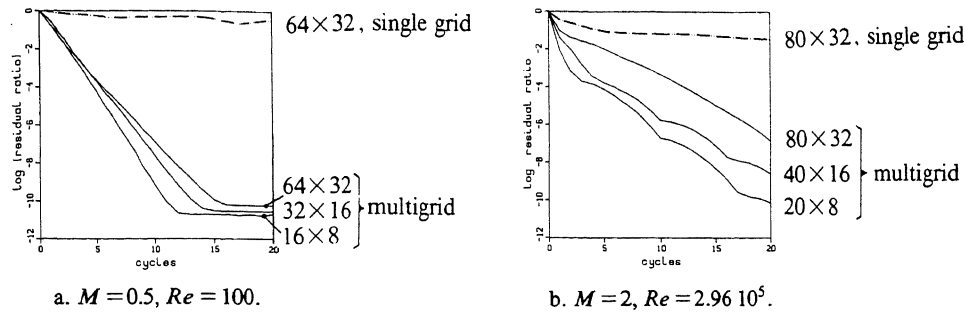


Fig. 3. Multigrid behaviour flat plate flows.

At present the usefulness of multigrid for flows at still higher Mach numbers is open to question. For the hypersonic blunt body flow mentioned (Fig. 4), the multigrid behaviour is shown in Fig. 5. (Notice that the finest grid, the 64×16 -grid, is a locally nested grid and that the single grid comparison is done for the coarser 32×16 -grid.) It appears that multigrid still pays for hypersonics, but it seems that there is a trend indeed of decreasing multigrid effectiveness from subsonic to hypersonic flow problems as well.

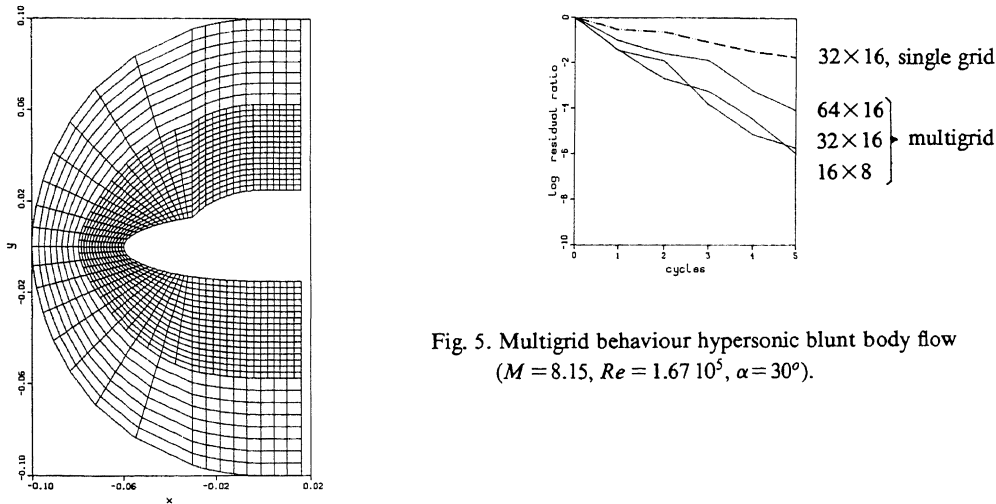


Fig. 4. Blunt body with grid.

5. CONCLUDING REMARKS

Theory and practice show that for sufficiently high Reynolds numbers, Osher's scheme leads to a more accurate resolution of boundary layer flows than van Leer's scheme. The difference in accuracy becomes larger with increasing Reynolds number.

For the first-order accurate discretized Navier-Stokes equations and flow problems with smooth solutions, theory and practice show that point Gauss-Seidel relaxation accelerated by multigrid and applied to the target equations directly, leads to a very fast convergence. For non-smooth problems, practical computations show a slow decline of this fast convergence with increasing Mach number.

For higher-order discretized Navier-Stokes equations, iterative defect correction is introduced, with as approximate solver: multigrid accelerated point Gauss-Seidel relaxation applied to the first-order equations. For smooth problems, both theory and practice show a fast convergence of iterative defect correction. For problems with non-smooth solutions, the convergence is less good though still satisfactory.

The fully implicit solution method applied, imposes very mild computer memory requirements due to the fact that the relaxation is pointwise. The computational method is completely parameter-free; it needs no tuning of parameters.

REFERENCES

1. W. HACKBUSCH (1985). *Multi-Grid Methods and Applications*. Springer, Berlin.
2. P.W. HEMKER (1986). *Defect Correction and Higher Order Schemes for the Multi Grid Solution of the Steady Euler Equations*. Proceedings of the Second European Conference on Multigrid Methods, Cologne 1985. Springer, Berlin.
3. P.W. HEMKER AND B. KOREN (1988). *Defect Correction and Nonlinear Multigrid for the Steady Euler Equations*. Lecture Series on Computational Fluid Dynamics, von Karman Institute for Fluid Dynamics, Rhode-Saint-Genèse.
4. B. KOREN (1988). *Defect Correction and Multigrid for an Efficient and Accurate Computation of Airfoil Flows*. J. Comput. Phys. (to appear).
5. B. KOREN (1988). *Upwind Schemes for the Navier-Stokes Equations*. Proceedings of the Second International Conference on Hyperbolic Problems, Aachen 1988. Vieweg, Braunschweig.
6. B. KOREN (1988). *Multigrid and Defect Correction for the Steady Navier-Stokes Equations*. Proceedings of the Fourth GAMM-Seminar Kiel on Robust Multi-Grid Methods, Kiel 1988. Vieweg, Braunschweig.
7. B. VAN LEER (1982). *Flux-Vector Splitting for the Euler Equations*. Proceedings of the 8th International Conference on Numerical Methods in Fluid Dynamics, Aachen 1982. Springer, Berlin.
8. B. VAN LEER (1985). *Upwind-Difference Methods for Aerodynamic Problems governed by the Euler Equations*. Proceedings of the 15th AMS-SIAM Summer Seminar on Applied Mathematics, Scripps Institution of Oceanography 1983. AMS, Providence, Rhode Island.
9. B. VAN LEER, J.L. THOMAS, P.L. ROE, AND R.W. NEWSOME (1987). *A Comparison of Numerical Flux Formulas for the Euler and Navier-Stokes Equations*. AIAA-paper 87-1104.
10. S. OSHER AND F. SOLOMON (1982). *Upwind-Difference Schemes for Hyperbolic Systems of Conservation Laws*. Math. Comp. 38, 339-374.
11. R. PEYRET AND T.D. TAYLOR (1983). *Computational Methods for Fluid Flow*. Springer, Berlin.
12. P.K. SWEBY (1984). *High Resolution Schemes using Flux Limiters for Hyperbolic Conservation Laws*. SIAM J. Num. Anal. 21, 995-1011.