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USE OF A NEW TECHNIQUE IN HOMOGENEOUS CONTINUA

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Recently Ungar has employed a theorem due to Effros [3] in the study of homogeneous continua. It is the purpose of this paper to use the same theorem to clarify and simplify Bing's proof that the only *plane* homogeneous continuum containing an arc is a simple closed curve [1].

Preliminary definitions and lemmas. Let M denote a continuum (= compact, connected, metric space). Then M is said to be homogeneous if any point x of M can be moved to any other point y of M by a homeomorphism of M onto M. The simplest examples are the circle, the torus and the Hilbert cube.

Effros has recently proved a theorem which yields the following useful corollary:

If M is a homogeneous continuum and ϵ is a positive number, there exists a positive number δ such that if $x, y \in M$ and $d(x,y) < \delta$ there exists a homeomorphism h of M onto M such that h takes x to y and moves no point more than ϵ[i.e., $d(h(p),p) \leqslant \epsilon$ for $p \in M$].

This gives us a precision that Bing lacked and almost makes the proof of his theorem trivial. However, some care is still required and the following well known (or easily established) properties of plane homogeneous indecomposable continua are helpful.

Let M denote a homogeneous indecomposable continuum lying in the plane E.

(a) M is hereditarily unicoherent (otherwise M would contain a proper subcontinuum separating the plane and hence each of the uncountably many disjoint composants would contain a homeomorphic copy).

(b) M is atriodic (for much the same reason as in (a) since the plane cannot contain uncountably many disjoint triods).

(c) If two arcs A and B of M intersect then $A \cup B$ is an arc, and in particular where they have a common end point and intersect somewhere else, then one of them is a subset of the other.

(d) If ab is an arc of M and T is the union of all arcs ax of M containing ab, then T is the one-to-one continuous image of the non-negative real numbers. Furthermore every point of ab is a limit point of T-ab.

This last statement requires a little proof. If W = cl(T-ab) contains a but not all of ab, then $W \cup ab$ is not unicoherent. So if the statement is false we may assume that W does not contain a. Let c be the first point of T (if there is one) which is a limit point of T-ae for some point e beyond c in T. Let $2\epsilon < \min\{d(a,W), d(e,ac)\}$. There is a point c' of ac-c close enough to c so that there is a homeomorphism of M onto M which moves c to c' and moves no point of M more than ϵ. Then c' has the properties possessed by c and hence c was not the first point of T with these properties. Hence no such point c exists.

But T is not closed; so $\overline{T} - T \neq \phi$. If $\overline{T} - T$ were a single point then $\overline{T}$ would be an arc and be a subset of T by definition. So $\overline{T} - T$ is a nondegenerate continuum. Let p,q, and r be three points (with $p,q \in \overline{T} - T$ and $r \in T$) close enough to each other so that there exist homeomorphisms of M onto M taking any one of them to any other one without moving any point of M by as much as $\frac{1}{2}d(a,\overline{T} - T)$. Then T union with the image of T under two such homeomorphisms taking r to each of p and q would contain a triod.

(e) If ab is an accessible arc in M, then every point of $ab - \{a,b\}$ is accessible from the same complementary domain and $ab - \{a,b\}$ is not accessible from both sides.

PROOF. There is a homeomorphism of the plane E onto itself which takes ab onto the interval [0,1] of the x-axis. Suppose that ab is accessible from below at the point p = (½,0), i.e., there exists an arc pq lying (except for p) below the x-axis and in E - M. There is no loss in generality in assuming that pq is vertical.

If p is a limit point of M from below the x-axis, let 4ϵ be a positive number less than $\min\{1, d(p,q)\}$ and let p' be a point of M below the x-axis so near to p that there is a homeomorphism of M onto M taking p to p' so that no point is moved more than ϵ. Since the image of neither of the intervals $[p,p+2\epsilon]$ and $[p-2\epsilon,p]$ of the x-axis can intersect $pq \cup ab$, this homeomorphism imposes an impossible motion on one of them. So p is not a limit point of M from below the x-axis.

Hence there exists a positive number r_p such that the r_p-neighbourhood U of p contains no point of M below the x-axis. Hence each point of U∩ab is accessible from below. Actually the endpoints of this interval are also (using lines inclined 45°). Using this fact, it is easy to see that all points of ab (even a and b) are accessible from below.

Now if some point w of ab - {a,b} were accessible from above, then w could not be a limit point of M from above the x-axis either. But w is a limit point of every composant of M.

(f) If p is a point of M and D is an open set in E with $p \notin \overline{D}$, then there exists a positive number ϵ such that no arc abc in M has the property that both a and c belong to D and the ϵ-neighbourhood of the subarc ab contains both p and the subarc bc.

PROOF. Let D_1 be an open set such that $p \notin \overline{D}_1$ and $\overline{D} \subset D_1$, and suppose that the lemma is false. Then there exists an arc abc in M and a homeomorphism of M onto M which takes b to p but leaves the images q_1 and q_2 of a and c respectively in D. Let q_1pq_2 denote the image of abc and let p_1 and p_2 denote points of q_1pq_2 between q_1 and p and p and q_2 respectively such that $\overline{D}_1 \cap p_1p\,p_2$ of $q_1\,pq_2 = \phi$.

Now let ϵ be a positive number such that

$$4\epsilon < d(q_1p_1,p_2q_2),\ d(p_1pp_2, \overline{D}_1, \text{ and } d(\overline{D}, E - D_1).$$

Since the lemma is false there exists an arc a′b′c′ such that both a′ and c′ belong to D, the ϵ-neighbourhood of a′b′ contains both p and b′c′, and b′ is close enough to b so that there exists a homeomorphism of M to M that moves b′ to p but moves no point more than ϵ. Since b′c′ is in the ϵ-neighbourhood of a′b′, the image of b′c′ must be in the 3ϵ-neighbourhood of the image of a′b′. This is impossible since the 3ϵ-neighbourhood of q_1p_1 contains no point of p_2q_2 (and conversely).

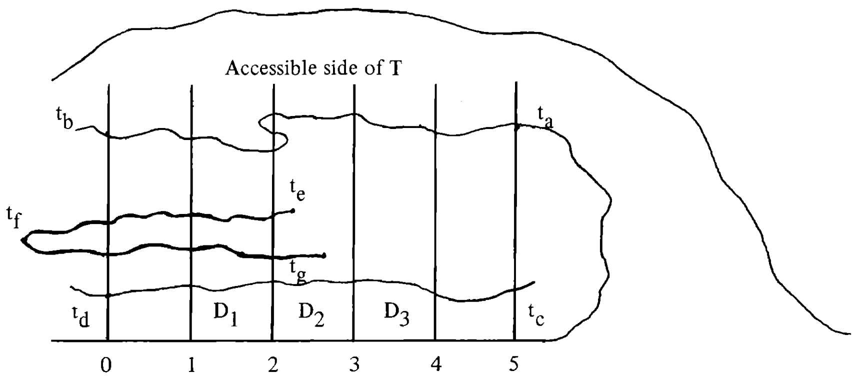
THEOREM. *If M is a homogeneous plane continuum containing an arc, then M is a simple closed curve.*

PROOF. If M is decomposable then M is a simple closed curve [2]. So from here on in the argument we suppose that M is indecomposable.

Let T denote a maximal ray in M parameterized by $[0,\infty]$ such that t_0 is its end (or emanation) point. Furthermore let us assume that T is accessible from the complement of M and that the interval $[t_0,t_5]$ of T lies on the x-axis with $t_n = (n,0)$ for n = 0,...,5 such that $[t_0,t_5]$ is accessible from below. There exists a positive integer η such that no arc of T lying between $y = 1/\eta$ and the x-axis, has an interior point in only one of the lines x = n(n = 1,2,3,4) and has both end points either on x = n+1 or on x = n-1. This follows from property (f).

Now let a and b be the smallest positive numbers with $a < b$ such that $[t_a,t_b]$ is irreducible from x = 5 to x = 0 and lies between $y = 1/\eta$ and the x-axis. Since every point of [0,5] is a limit point $[t_6,t_\infty]$ such numbers a and b must exist. The situation is roughly as indicated in the figure with the accessible side of $[t_a,t_b]$ lying above. This selection of a and b is possible because if the order of t_a and t_b were reversed, that is, t_b belonged to x = 5 and t_a belonged to x = 0 we would take the next arc of T below $[t_a,t_b]$ and appeal to the Jordan curve theorem to get the proper order.

For each i(i = 1,...4) let z_i be the lowest point of $[t_a,t_b]$ on x = i and let D_i be the simple domain bounded by [i,i+1] of the x-axis, $[(i,0), z_i]$ of x = i, $[z_i,z_{i+1}]$ of $[t_a,t_b]$ and $[(z+1,0), z_{i+1}]$ of x = i+1.



Let ϵ be a positive number less than ½ such that if two points of M are no farther apart than ϵ and one of them is between $[t_a,t_b]$ and $[t_0,t_5]$ and between lines x = 1 and x = 4 then the other it also between $[t_a,t_b]$ and $[t_0,t_5]$. By Effros' theorem there exists a positive number δ such that if p and q are points of M and $d(p,q) \leqslant \delta$ then there is a homeomorphism of M onto M taking p to q and moving no point more than ϵ.

Now let c and d denote positive numbers with $c < d$ so that $[t_c,t_d]$ lies above the x-axis, is irreducible from x = 5 to x = 0, and no point of $[t_c,t_d]$ lies above $y = \delta$. Let h be a homeomorphism of M onto M which moves no point more than ϵ and takes t_0 to t_d. This makes h take $[t_0, t_d]$ onto (approximately) $[t_d,t_0]$. So there is a fixed point t_f on $[t_0,t_d]$. Clearly t_f cannot lie in $\bar{D}_1 \cup \bar{D}_2 \cup \bar{D}_3$ without violating the selection of η. So there exist numbers e and g with $e < f < g$ such that t_e and t_g lie in

D_2, $h[t_e,t_f] = [t_g,\dot{t}_f]$, and $h(t_e') = t_g$. By appropriately selecting a sequence of values $\epsilon_1, \epsilon_2, \ldots$ of ϵ converging to zero so that the corresponding points t_{ϵ_1}, $t_{\epsilon_2}, \ldots$ converge to a point p it is easy to see that $p \notin \overline{D}_2$ and property (f) is violated.

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