Acta Physica Academiae Scientiarum Hungaricae, Tomus 24 (4), pp. 325—330 (1968)

USE OF DELTA FUNCTIONS IN GENERAL RELATIVITY FOR DETERMINATION OF THE INTEGRATION CONSTANTS

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(Presented by A. Kónya. — Received 4. IV. 1967)

During the integration of the Einstein—Maxwell equations, integration constants appear and their interpretation is often very difficult. There is therefore, a requirement for a calculus of delta functions which will automatically relate integration constants to sources. In this paper a calculus of this kind is developed and applied to the spherically symmetric problem. In this way we get a method of distinction between pure mathematical and physical singularities.

I. Calculus of delta functions

In this paper a calculus of delta functions suitable for certain physical problems is presented. The work here extends the use of delta functions to be found in a paper by ROSEN and SHAMIR [1] and a book by INFELD and PLEBANSKY [2].

A general delta tensor of the rank k in space-time is defined by

$$\delta^{n_1\ldots n_k}(x^j, \xi^j) = \delta^{n_1\ldots n_k}(\xi^j, x^j) = \begin{cases} 0 & \text{for } x^j \neq \xi^j, \\ \infty & \text{for } x^j = \xi^j, \end{cases} \tag{1}$$

(k is an integer between 1 and 4). The strength of the singularity is of such a kind that

$$i \int\limits_{V_{4-k}} \delta^{n_1\ldots n_k}(x^j, \xi^j)\, df_{n_1\ldots n_k}(x^j) = 1 \tag{2}$$

is valid if the singularity is situated in the $(4 - k)$-dimensional subspace over which the integral is taken. $df_{n_1\ldots n_k}$ is the surface-element pseudo-tensor of rank k. A consequence of (1) and (2) is the equation

$$i \int\limits_{V_{4-k}} F(x^j)\, \delta^{n_1\ldots n_k}(x^j, \xi^j)\, df_{n_1\ldots n_k}(x^j) = F(\xi^j). \tag{3}$$

(The factor i in the last two equations is a consequence of definition of $df_{n_1\ldots n_k}$) Two cases are particularly interesting: namely if the delta tensor is of the rank 0 ($k = 0$) we get

$$\int\limits_{V_4} F(x^j)\, \delta(x^j, \xi^j)\, d^{(4)}f(x^j) = F(\xi^j) \tag{4}$$

and if the delta tensor is of the rank 1 ($k = 1$), we get

$$i \int_{V_3} F(x^j)\, \delta^k(x^j, \xi^j)\, df_k(x^j) = F(\xi^j)\,. \tag{5}$$

In the first case the delta tensor is called Dirac's delta function. In the second case it is called a surface delta function. Our further investigations refer to Dirac's delta function.

First we study the 1-dimensional case. Heaviside's step function is defined by

$$\Theta(x) = \begin{cases} -\dfrac{1}{2} & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ \dfrac{1}{2} & \text{for } x > 0 \end{cases} \tag{6}$$

so that the relations

$$a)\ \Theta(x)' = \delta(x),\ b)\ \Theta(x) + \Theta(-x) = 0\,,\ c)\ \Theta(x)\,\Theta(-x) = \begin{cases} -\dfrac{1}{4} \text{ for } x < 0 \\ 0 \text{ for } x = 0 \\ -\dfrac{1}{4} \text{ for } x > 0 \end{cases} \tag{7}$$
$$d)\ \Theta(x)^2 = -\,\Theta(x)\ \ \Theta(-x)$$

hold. Dirac's delta function satisfies the differential equation

$$x\delta'(x) + \delta(x) = 0\,. \tag{8}$$

The 3-dimensional delta function is given by

$$\delta(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z) \quad \left(r = \sqrt{x^2 + y^2 + z^2}\right). \tag{9}$$

This quantity allows the definition of the 3-dimensional radial delta function

$$\delta(r) = 4\,\pi r^2\, \delta(\vec{r})\,, \tag{10}$$

for which the equations

$$a)\ \delta(r) = \begin{cases} \infty & \text{for } r = 0 \\ 0 & \text{for } r > 0 \end{cases}\,,\quad b)\ \int_0^\infty \delta(r)\, dr = 1 \tag{11}$$

are valid. The relation

$$\delta(r) = 2\,\bar{\Theta}(r)' \tag{12}$$

exists between the delta function and the radial step function

$$\bar{\Theta}(r) = \begin{cases} 0 & \text{for } r = 0\,, \\ \dfrac{1}{2} & \text{for } r > 0\,. \end{cases} \tag{13}$$

By differentiation one verifies the important equation

$$\Delta\left(\frac{\bar{\Theta}(r)}{2\pi r}\right) = -\delta(\vec{r})\,, \tag{14}$$

where Δ is the Laplace operator in spherical polar coordinates. In distribution theory literature, this is replaced by the notation

$$\Delta\left(\frac{1}{4\pi r}\right) = -\delta(\vec{r})\,.$$

The 2-dimensional radial delta function is defined by

$$\delta(\sigma) = 2\pi\sigma\,\delta(x)\,\delta(y) \qquad \left(\sigma = \sqrt{x^2+y^2}\right). \tag{15}$$

For this the relations

$$a)\ \delta(\sigma) = \begin{cases} \infty & \text{for } \sigma = 0 \\ 0 & \text{for } \sigma > 0 \end{cases}, \quad b)\ \int_0^\infty \delta(\sigma)\,d\sigma = 1 \tag{16}$$

are valid. By calculation one verifies that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(-\frac{\bar{\Theta}(\sigma)\ln\sigma}{2\pi}\right) = -\delta(x)\,\delta(y)\,. \tag{17}$$

II. Investigation of Reissner—Weyl—Schwarzschild field

The square of the line element is used in the form

$$(ds)^2 = e^{\mu(r)}(dr)^2 + r^2[\sin^2\Theta(d\varphi)^2 + (d\Theta)^2] - e^{\nu(r)}(dx^4)^2, \quad (x^1 = r, x^2 = \varphi, x^3 = \Theta)\,. \tag{18}$$

We write the field equations in the form

$$a)\ R_i^j = \varkappa\left(T_i^j - \frac{1}{2}g_i^j T_m^m\right), \quad \text{(Einstein equation)}$$

$$b)\ (E^\mu\sqrt{-\overset{3}{g}})_{|\mu} = \varrho\sqrt{-\overset{3}{g}}, \tag{19}$$

$$c)\ (E_3\sqrt{-g_{44}})_{,1} = (E_1\sqrt{-g_{44}})_{,3} = 0\,, \quad (E_2\sqrt{-g_{44}})_{,1} = (E_1\sqrt{-g_{44}})_{,2}\,, \quad \text{(Maxwell}$$

$$(E_3\sqrt{-g_{44}})_{,2} = (E_2\sqrt{-g_{44}})_{,3}\,, \qquad (\overset{4}{g} = -|g_{ij}|\,,\ \overset{3}{g} = -|g_{\mu\nu}|) \qquad \text{equation)}$$

taking into account that we have to treat a static problem. (Greek indices run from 1 to 3, latin indices from 1 to 4.) $E^\mu = g^{\mu\nu}E_\nu$ is the 3-dimensional electric field strength, and ϱ is the 3-dimensional charge density.

The Einstein equations can be written in the following way:

$$\begin{aligned}
&a)\ \frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\mu'}{r}-\frac{\nu'\mu'}{4}=\varkappa e^{\mu}\left(T_1^1-\frac{1}{2}T_m^m\right),\\
&b)\ \frac{\nu'-\mu'}{2}r+1-e^{\mu}=\varkappa r^2 e^{\mu}\left(T_2^2-\frac{1}{2}T_m^m\right),\\
&c)\ \frac{\nu'-\mu'}{2}r+1-e^{\mu}=\varkappa r^2 e^{\mu}\left(T_3^3-\frac{1}{2}T_m^m\right)\\
&d)\ \frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\nu'}{r}-\frac{\nu'\mu'}{4}=\varkappa e^{\mu}\left(T_4^4-\frac{1}{2}T_m^m\right).
\end{aligned} \tag{20}$$

From the second and third it follows that

$$T_2^2=T_3^3, \qquad (21($$

while the first and the fourth yield

$$\frac{\nu'+\mu'}{r}=\varkappa e^{\mu}(T_4^4-T_1^1). \tag{22}$$

Using the abbreviation

$$\gamma=e^{-\mu} \tag{23}$$

we get from (20) after some calculation

$$T_m^m=-\frac{\gamma}{\varkappa}\left[\nu''+\frac{\nu'^2}{2}+\frac{2\nu'}{r}\right]+\frac{1}{\varkappa}\left[\frac{2}{r^2}(1-\gamma)-\frac{\nu'\gamma'}{2}-\frac{2\gamma'}{r}\right], \tag{24}$$

$$T_4^4=\frac{1}{\varkappa}\left[-\frac{\gamma'}{r}+\frac{1}{r^2}(1-\gamma)\right], \tag{25}$$

$$T_1^1=\frac{1}{\varkappa}\left[\frac{1}{r^2}(1-\gamma)-\frac{\nu'\gamma}{r}\right], \tag{26}$$

$$T_2^2=-\frac{\gamma}{2\varkappa}\left[\nu''+\frac{\nu'^2}{2}+\frac{\nu'}{r}\right]-\frac{\gamma'}{2\varkappa}\left[\frac{1}{r}+\frac{\nu'}{2}\right]. \tag{27}$$

The energy tensor is now split into an electromagnetic part E_i^j and a remainder part Θ_i^j:

$$T_i^j=E_i^j+\Theta_i^j, \tag{28}$$

where

$$(E_i^j)=\left(\begin{array}{c|c} E_{\mu}E^{\nu}-\frac{1}{2}g_{\mu}^{\nu}E_{\lambda}E^{\lambda} & 0\\ \hline 0 & \frac{1}{2}E_{\lambda}E^{\lambda}\end{array}\right),\qquad (\Theta_i^j)=\begin{pmatrix}\alpha&0&0&0\\0&\beta&0&0\\0&0&\beta&0\\0&0&0&\mu c^2\end{pmatrix}. \tag{29}$$

An electrical point singularity with the charge e must be described by

$$\varrho(r) = \frac{e}{4\pi r^2 e^{\mu/2}} \delta(r) \tag{30}$$

because the equation

$$\int \varrho d^{(3)} f = \int \varrho \sqrt{-\overset{3}{g}}\, d^{(3)} x = e \int_0^\infty \delta(r)\, dr = e \tag{31}$$

must be valid.

The integration of the Maxwell equations yields

$$E_2 = E_3 = 0, E_1 = \frac{e\bar{\theta}(r)\, e^{\mu/2}}{2\pi r^2} \tag{32}$$

so that

$$\begin{aligned} E_1^1 = E_4^4 &= \frac{1}{2} E_1 E^1 = \frac{e^2 \bar{\theta}(r)^2}{8\pi^2 r^4}, \\ E_2^2 = E_3^3 &= -\frac{1}{2} E_1 E^1 = -\frac{e^2 \bar{\theta}(r)^2}{8\pi^2 r^4} \end{aligned} \tag{33}$$

follows.

For integration of (25) we choose for a point singularity with the mass M the rest mass density

$$\mu = \frac{M\delta(r)}{4\pi r^2}. \tag{34}$$

Using (33) we find

$$(\gamma r)' = 1 - \frac{\varkappa M c^2 \delta(r)}{4\pi} - \frac{\varkappa e^2 \bar{\theta}(r)^2}{8\pi^2 r^2} \tag{35}$$

and further by integration

$$\gamma = 1 - \frac{\varkappa M c^2 \bar{\theta}(r)}{2\pi r} + \frac{\varkappa e^2 \bar{\theta}(r)^2}{8\pi^2 r^2} + \frac{\varkappa e^2}{8\pi^2 r} \int_0^r \bar{\theta}(r)\, \delta'(r)\, dr. \tag{36}$$

Outside the singularity this is the well known result

$$\gamma = 1 - \frac{\varkappa M c^2}{4\pi r} + \frac{\varkappa e^2}{32\pi^2 r^2}. \tag{37}$$

Up to this point the stress distribution in the singularity is not fixed. Further assumptions about $\nu(r)$ would be necessary. The formula $\nu + \mu = 0$, which holds outside the singularity, leads to non-physical stress inside the singularity.

REFERENCES

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ПРИМЕНЕНИЕ δ-ФУНКЦИЙ В ТЕОРИИ ОБЩЕЙ ОТНОСИТЕЛЬНОСТИ ДЛЯ ОПРЕДЕЛЕНИЯ ПОСТОЯННЫХ ИНТЕГРИРОВАНИЯ

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Резюме

При интегрировании уравнений Эйнштейна—Максвелла появляются постоянные интегрирования, интерпретация которых часто представляется очень трудной. Отсюда возникает потребность применения в вычислениях δ-функций, которые автоматически указывают на происхождение этих постоянных. В данной работе развивается метод такого характера, дается его применение в случае проблемы, обладающей сферической симметрией. Таким путем нами дается метод для различия между чисто математической и физической сингулярностями.