$\mathcal{R}^{n_{i2}\times n_{o1}}$ exists such that $N_{12}^I N_{12} = I$, then $N_{12}^I P_{12} = D_{22}^{-1}$ implies $D_{22}N_{12}^I P_{12} = I$; hence, $D_{22}N_{12}^I \in \mathcal{M}(\mathcal{R})$ is a left-inverse of P_{12}. It can be shown similarly that $P_{21} = \tilde{D}_{22}^{-1}\tilde{N}_{21}$ has an $\mathcal{R}$-stable right-inverse if and only if $\tilde{N}_{21} \in \mathcal{R}^{n_{o2}\times n_{i1}}$ has a right-inverse $\tilde{N}_{21}^I \in \mathcal{R}^{n_{i1}\times n_{o2}}$. Construct $Q_{11}\in\mathcal{R}^{r\times r}$ with q_{jj} chosen as above; since $\text{rank}P_{12}(s) = n_{i2}$ and $\text{rank}P_{21}(s) = n_{o2}$ for all $s\in\mathcal{U}$, P_{12} and P_{21} have no $\mathcal{U}$-zeros. Therefore, Q_{11} is chosen so that the nondiagonal entries $q_{ij}\neq 0$ are constants, and $q_{jj}\in\mathcal{R}$ are such that $(v_j - q_{jj}\lambda_j)(\infty)\neq 0$. To guarantee that $\tilde{N}_{C2}$ has no real blocking $\mathcal{U}$-zeros, let

$$Q_2 = -\hat{A} + M^{-1}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}\end{bmatrix}\tilde{M}^{-1} \in \mathcal{R}^{n_{i2}\times n_{o2}}$$

$Q_{12}\in\mathcal{R}^{r\times(n_{o2}-r)}$, $Q_{21}\in\mathcal{R}^{(n_{i2}-r)\times r}$, $Q_{22}\in\mathcal{R}^{(n_{o2}-r)\times(n_{o2}-r)}$ can be arbitrary if both $n_{o2}>1$ and $n_{i2}>1$; if $n_{i2}=1$, let $Q_{12}\in\mathbb{R}^{1\times(n_{o2}-1)}$ be nonzero real; if $n_{o2}=1$, let $Q_{21}\in\mathbb{R}^{(n_{i2}-1)\times 1}$ be nonzero real. Let $Q_1 := \tilde{N}_{21}^I\hat{Q}_1 N_{12}^I \in \mathcal{M}(\mathcal{R})$. Since $\tilde{N}_{C2}$ has no real blocking $\mathcal{U}$-zeros, $\hat{Q}_1\in\mathcal{M}(\mathcal{R})$ exists such that $\tilde{D}_{22} + \tilde{N}_{21}\tilde{N}_{21}^I\hat{Q}_1N_{12}^IN_{12}\tilde{N}_{C2} = \tilde{D}_{22}+\hat{Q}_1\tilde{N}_{C2}$ is $\mathcal{R}$-unimodular, i.e., (14) holds. Since $\tilde{D}_{C2}$ is biproper, C_2 is proper. There is $\hat{Q}_1\in\mathcal{M}(\mathcal{R})$ such that $\tilde{D}_{22}+\hat{Q}_1\tilde{N}_{C2}$ is $\mathcal{R}$-unimodular if and only if $\tilde{Q}_1\in\mathcal{M}(\mathcal{R})$ exists such that $\tilde{D}_{22}+(s+a)^{-1}\tilde{Q}_1\tilde{N}_{C2}$ is $\mathcal{R}$-unimodular; choosing $Q_1 = (s+a)^{-1}\tilde{Q}_1\in\mathcal{M}(R_s)$ implies C_1 is proper.

4) Let C_S be any $\mathcal{R}$-stable $\mathcal{R}$-stabilizing controller for P_{22}. Without loss of generality, let the RCF $P_{22} = N_{22}D_{22}^{-1}$ satisfy $D_{22} + C_SN_{22} = I$; hence, $N_{22}D_{22} = (I - N_{22}C_S)N_{22}$. Then $U_2 = C_S + T(I - N_{22}C_S)$, $V_2 = I - TN_{22}$, $\tilde{U}_2 = C_S + D_{22}T$, $\tilde{V}_2 = I - N_{22}T$, $T\in\mathcal{M}(\mathcal{R})$ satisfy (10). By assumption, $P_{21} = \tilde{D}_{22}^{-1}\tilde{N}_{21}$ implies $\tilde{N}_{21}$ has a right-inverse $\tilde{N}_{21}^I\in\mathcal{M}(\mathcal{R})$. Also, $\tilde{L}P_{12} = \tilde{L}N_{12}D_{22}^{-1} = P_{22} = N_{22}D_{22}^{-1}$ implies $\tilde{L}N_{12} = N_{22}$. Let C_1, C_2 be given by (11) and (12), $Q_1 = \tilde{N}_{21}^I\tilde{L}\in\mathcal{R}^{n^{i1}\times n_{o1}}$, $Q_2 = -T$. Then (13) becomes $D_{22}+C_SN_{22} = I$. Since P_{11}, $P_{12}\in\mathcal{M}(R_s)$, $(N_{11} - N_{12}Q_2\tilde{N}_{21})\in\mathcal{M}(R_s)$ implies $\tilde{D}_{C1}$ is biproper, i.e., $C_1\in\mathcal{M}(R_p)$. Since $C_2 = C_S\in\mathcal{M}(\mathcal{R})$, (C_1, C_2) is a reliable decentralized controller pair.

5) Let C_S be as in 4); let $D_{22}+C_SN_{22} = I$. By assumption, $P_{12} = N_{12}D_{22}^{-1}$ implies N_{12} has a left-inverse $N_{12}^I\in\mathcal{M}(\mathcal{R})$. Also, $P_{21}\tilde{R} = \tilde{D}_{22}^{-1}\tilde{N}_{21}\tilde{R} = P_{22} = \tilde{D}^{-1}\tilde{N}_{22}$ implies $\tilde{N}_{21}\tilde{R} = \tilde{N}_{22} = N_{22}$. Let C_1, C_2 be given by (11) and (12), $Q_1 = \tilde{R}N_{12}^I\in\mathcal{R}^{n_{i1}\times n_{o1}}$, $Q_2 = -T$. The conclusion follows as in 4).

6) Choose C_S as in 4). By assumption, $\tilde{L}P_{12} = P_{22}$ implies $\tilde{L}N_{12} = N_{22}$, and $P_{21}\tilde{R} = P_{22}$ implies $\tilde{N}_{21}\tilde{R} = \tilde{N}_{22} = N_{22}$. Let $Q_2 = -T$, $Q_1 = \tilde{R}\hat{Q}_1\tilde{L}$, $\hat{Q}_1 = \sum_{m=2}^k r_mk^{-m}(C_SN_{22})^{m-2}C_S$; k is any integer such that $k > \|C_SN_{22}\|$ and r_m are the binomial coefficients. By (13) $D_{22} + C_SN_{22}\hat{Q}_1N_{22} = I - C_SN_{22} + \sum_{m=2}^k r_mk^{-m}(C_SN_{22})^m = (I - k^{-1}C_SN_{22})^k$ is $\mathcal{R}$-unimodular. Then $C_1\in\mathcal{M}(R_s)$ since $\hat{Q}_1$, $Q_1\in\mathcal{M}(R_s)$. Since $C_2 = C_S$ is proper, (C_1, C_2) is a reliable decentralized controller pair. □

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Useful Nonlinearities and Global Stabilization of Bifurcations in a Model of Jet Engine Surge and Stall

Miroslav Krstić, Dan Fontaine, Petar V. Kokotović, and James D. Paduano

***Abstract*— Compressor stall and surge are complex nonlinear instabilities that reduce the performance and can cause failure of aircraft engines. We design a feedback controller that globally stabilizes a broad range of possible equilibria in a nonlinear compressor model. With a novel backstepping design we retain the system's useful nonlinearities which would be cancelled in a feedback linearizing design. The design control law is simple and, moreover, it is optimal with respect to a meaningful nonquadratic cost functional. As in a previous bifurcation-theoretic design, we change the character of the bifurcation at the stall inception point from subcritical to supercritical. However, since we do not approach bifurcation control using a normal form but using Lyapunov tools, our controller achieves not only local but also global stability. The controller requires minimal modeling information (bounds on the slope of the stall characteristic and the B-parameter) and simpler sensing (rotating stall is stabilized without measuring its amplitude).**

***Index Terms*— Axial flow compressors, backstepping, bifurcation control, jet engines, rotating stall, surge.**

I. Introduction

In control engineering the importance of qualitative low-order nonlinear models is twofold. First, they can capture the dominant dynamic phenomena; second, they are testbeds which help refine new nonlinear design methods. One such model, the Moore–Greitzer

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(MG3) [16] compressor model, is used in this paper to present further improvements of a newly proposed backstepping design [12].

MG3 describes the compression system *rotating stall* and *surge* instabilities which are among the main challenges in the design and operation of jet engines (for a general introduction to the problem, see [10]). Rotating stall manifests itself as a region of severely reduced flow that rotates at a fraction of the rotor speed and causes a drop in performance; surge is a pumping oscillation that can cause flameout and engine damage. MG3 is a Galerkin approximation of a higher order partial differential equation (PDE) model and is given by

$$\dot{\Phi} = -\Psi + \frac{1}{2\pi}\int_0^{2\pi} \Psi_C(\Phi + A\sin\theta)\,d\theta \tag{1}$$

$$\dot{\Psi} = \frac{1}{\beta^2}(\Phi - \Phi_T) \tag{2}$$

$$\dot{A} = \frac{\sigma}{3\pi}\int_0^{2\pi} \Psi_C(\Phi + A\sin\theta)\sin\theta\,d\theta \tag{3}$$

where $\Psi_c(\Phi)$ is the compressor characteristic, to be defined later, (Φ, Ψ, A) are the scaled annulus-averaged flow, plenum pressure rise, and rotating stall amplitude, respectively, Φ_T is the flow through the throttle, $\beta = 2BH/W$ where B is Greitzer's stability parameter and H and W are defined in [16], σ is a parameter proportional to the length of inlet duct, t is time scaled by rotor frequency, and $\Phi_T(\Psi)$ is the throttle characteristic

$$\Psi = \frac{1}{\gamma^2}(1 + \Phi_T)^2 \tag{4}$$

where γ is the throttle opening which is used as a control variable. Further details about this model and a thorough analysis of its complex nonlinear dynamics are given by McCaughan [15].

The emergence of MG3 triggered interest in model-based nonlinear feedback control of jet engines. Liaw and Abed [14] developed a nonlinear controller that changes the character of the bifurcation at the stall inception point, from hard subcritical to soft supercritical, thus avoiding an abrupt transition into rotating stall. This design was experimentally validated by Badmus *et al.* [3]. Eveker *et al.* [7] observed that, although it reduces rotating stall, this design is ineffective against surge. They combined the rotating stall controller of [14] with a surge controller of Badmus *et al.* [4], and considerably extended the operating regime. Alternative approaches to control of rotating stall and surge have been developed by Paduano *et al.* [17], Behnken *et al.* [6], Baillieul *et al.* [5], and in papers referenced therein. Table I gives a catalog of MG3 controllers including the backstepping controller designed in this paper. In addition to extending the operating region and employing for feedback variables that are easier to measure, our backstepping controller retains the simplicity of the controllers [14], [7].

Useful Nonlinearities and Global Stabilization of Bifurcated Equilibria: Dynamics of systems designed to operate in large-signal regimes are almost always dominated by nonlinearities which may act as *harmful* or *useful*. Feedback linearization designs cancel nonlinearities (and their derivatives), and, as a result, not only may waste control effort but may also introduce feedback of destabilizing character with consequences on robustness in the case of inexact cancellations. A drastic example is the scalar system $\dot{x} = x - x^3 + u$ discussed by Freeman and Kokotović [8] where a feedback linearizing design would introduce a destabilizing term x^3. Every meaningful optimal design would safeguard against such destabilizing terms but would, unfortunately, require a solution of a Hamilton–Jacobi PDE.

In the design for the Moore–Greitzer model, we retain the nonlinearities that are useful for stability. The avoidance of cancellation results in a simpler, less nonlinear controller, that uses lower control effort for large signals. It also allows us to design a *partial-state* feedback controller with a simpler sensing, which requires minimal modeling knowledge—only a bound on Greitzer's β-parameter [9] and the maximal slope of the stall characteristic are needed.

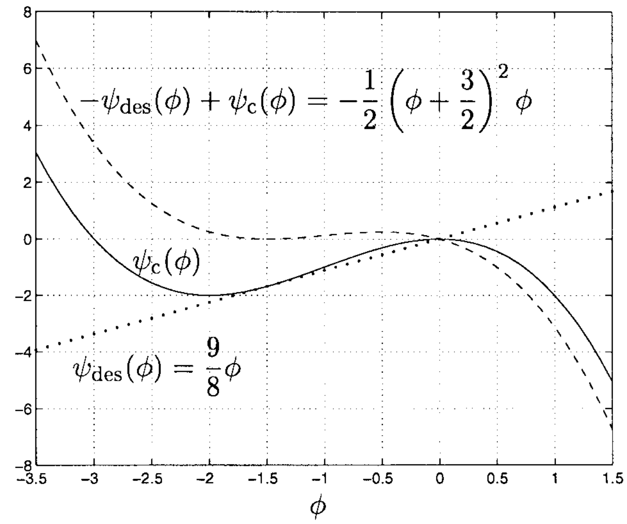

Fig. 1. Linear feedback $\psi_{\text{des}}(\phi) = \frac{9}{8}\phi$ eliminates the destabilizing third-quadrant part of the function $\psi_c(\phi)$.

Simplicity and the absence of modeling requirements are among the advantages of bifurcation-theoretic designs [14], [7], [1], [2]. However, these designs guarantee only *local* stability because they are based on local normal forms. By exploiting the structure of the system, the insight into the role of nonlinearities, and by using backstepping to build a Lyapunov function, we design a *global* bifurcation controller for the Moore–Greitzer model. A single parameter Γ is employed which, when varied, takes the system through all of its possible open-loop equilibria, and each of these equilibria is globally asymptotically stabilized.

II. Surge Control: Useful Nonlinearities

Restricting (1)–(3) to the invariant manifold $A = 0$ (the integral in (3) is zero when $A = 0$), we obtain the "surge model"

$$\dot{\Phi} = -\Psi + \Psi_C(\Phi) \tag{5}$$

$$\dot{\Psi} = \frac{1}{\beta^2}(\Phi + 1 - \gamma\sqrt{\Psi}) \tag{6}$$

which was originally introduced by Greitzer and shown to be a good qualitative description of surge dynamics [9]. Using γ as the control variable, our task is to regulate the state of the system (5) and (6) to the equilibrium $(\Phi, \Psi) = (\Phi_0, \Psi_C(\Phi_0))$. In error coordinates $\phi = \Phi - \Phi_0$, $\psi = \Psi - \Psi_C(\Phi_0)$, and with $\psi_c(\phi) = \Psi_C(\Phi) - \Psi_C(\Phi_0)$ system (5) and (6) is rewritten as

$$\dot{\phi} = -\psi + \psi_c(\phi) \tag{7}$$

$$\dot{\psi} = \frac{1}{\beta^2}(\phi + \Phi_0 + 1 - \gamma\sqrt{\Psi}). \tag{8}$$

Our backstepping design which avoids cancellation of useful nonlinearities is in two steps.

Step 1: We seek the *virtual control law* $\psi = \psi_{\text{des}}(\phi)$ to globally asymptotically stabilize the system

$$\dot{\phi} = -\psi_{\text{des}}(\phi) + \psi_c(\phi). \tag{9}$$

We avoid the choice of $\psi_{\text{des}}(\phi) = \psi_c(\phi) + \phi$ because it requires the knowledge of the compressor characteristic $\psi_c(\phi) = \Psi_C(\Phi) - \Psi_C(\Phi_0)$. Instead, we examine in Fig. 1 a typical characteristic

$$\Psi_C(\Phi) = \Psi_{C0} + 1 + \frac{3}{2}\Phi - \frac{1}{2}\Phi^3 \tag{10}$$

introduced by Moore and Greitzer [16] where Ψ_{C0} is the "shut-off pressure rise." We translate $\Psi_c(\Phi)$ to Φ_0

$$\psi_c(\phi) = \Psi_c(\Phi) - \Psi_c(\Phi_0) = \frac{9}{8}\phi - \frac{1}{2}\left(\phi + \frac{3}{2}\right)^2\phi \tag{11}$$

and consider its maximum at $\Phi_0 = 1$, that is $\phi = 0$ as the desired equilibrium. To stabilize this equilibrium for (9), we need $\psi_{\rm des}(\phi)$ such that $[-\psi_{\rm des}(\phi) + \psi_c(\phi)]\phi \leq 0$, that is, the function $-\psi_{\rm des}(\phi) + \psi_c(\phi)$ should lie only in the second and the fourth quadrants. The feedback $\psi_{\rm des}(\phi) = \frac{9}{8}\phi$ guarantees global stability of $\dot{\phi} = -\frac{1}{2}(\phi + \frac{3}{2})^2\phi$. For global *asymptotic* stability, we pick $\psi_{\rm des}(\phi) = (c_1 + \frac{9}{8})\phi,\ c_1 > 0$. To stabilize another equilibrium, we need a different amount of linear gain to move $-\psi_{\rm des}(\phi) + \psi_c(\phi)$ out of the first and the third quadrants. However, it is clear by inspection of Fig. 1 that the gain equal to the maximum positive slope of $\Psi_c(\Phi)$ [which is the same as $\psi_c(\phi)$] is sufficient for any equilibrium on the characteristic. This maximum *positive* slope of (10) is $a = 3/2$. Hence, if we select

$$\psi_{\rm des}(\phi) = (c_1 + a)\phi = c_0\phi, \qquad c_1 > 0 \tag{12}$$

the resulting system (9) becomes

$$\dot{\phi} = -c_1\phi - \rho(\phi)\phi, \qquad \rho(\phi) \triangleq -\frac{\psi_c(\phi) - a\phi}{\phi} \geq 0, \qquad \forall\phi. \tag{13}$$

The nonlinearity $-\rho(\phi)\phi$ is useful because it provides damping to the ϕ-equation. Denoting $\tilde{\psi} = \psi - \psi_{\rm des}(\phi)$ we rewrite (13) as

$$\dot{\phi} = -c_1\phi - \rho(\phi)\phi - \tilde{\psi}. \tag{14}$$

Step 2: At this step we benefit from preserving a useful nonlinearity at the first step. Differentiating $\tilde{\psi} = \psi - c_0\phi$, we get

$$\dot{\tilde{\psi}} = \frac{1}{\beta^2}(\phi + \Phi_0 + 1 - \gamma\sqrt{\Psi}) + c_0(\tilde{\psi} + c_1\phi + \rho(\phi)\phi). \tag{15}$$

Denoting $u = \frac{1}{\beta^2}(\Phi_0 + 1 - \gamma\sqrt{\Psi})$, the system (14) and (15) is rewritten as

$$\dot{\phi} = -c_1\phi - \rho(\phi)\phi - \tilde{\psi} \tag{16}$$

$$\dot{\tilde{\psi}} = u + c_0\tilde{\psi} + \left(c_0c_1 + \frac{1}{\beta^2}\right)\phi + c_0\underbrace{\rho(\phi)\phi}. \tag{17}$$

Whether $\rho(\phi)\phi$ is useful or not is what we now determine with the help of a Lyapunov function. For $V = \frac{1}{2}(c_0c_1 + \frac{1}{\beta^2})\phi^2 + \frac{1}{2}\tilde{\psi}^2$, the nonlinearity $\rho(\phi)\phi$ in (17) does not appear to be useful and would have to be cancelled. As a refinement of backstepping we consider a more flexible Lyapunov function

$$V = \frac{1}{2}\left(c_0c_1 + \frac{1}{\beta^2}\right)\phi^2 + U(\phi) + \frac{1}{2}\tilde{\psi}^2 \tag{18}$$

in which $U(\phi)$ is a positive definite function to be chosen. The derivative of V for (16) and (17) is

$$\dot{V} = -\left(c_0c_1 + \frac{1}{\beta^2}\right)[c_1 + \rho(\phi)]\phi^2 + \frac{dU(\phi)}{d\phi}[-c_1\phi - \rho(\phi)\phi] + \tilde{\psi}(u + c_0\tilde{\psi}) + \tilde{\psi}\left[-\frac{dU(\phi)}{d\phi} + c_0\underbrace{\rho(\phi)\phi}\right]. \tag{19}$$

We now pick $U(\phi)$ to eliminate the indefinite term in the last brackets

$$U(\phi) = c_0\int \rho(\phi)\phi\, d\phi. \tag{20}$$

This function is positive definite, and moreover, $\phi\frac{dU(\phi)}{d\phi} = c_0\rho(\phi)\phi^2 \geq 0$, for all ϕ. Thus, (19) becomes

$$\dot{V} = -c_0\left[\frac{1}{c_0\beta^2} + c_1 + \rho(\phi)\right][c_1 + \rho(\phi)]\phi^2 + \tilde{\psi}(u + c_0\tilde{\psi}). \tag{21}$$

The control u is now selected to achieve

$$\dot{V} = -c_0\left[\frac{1}{c_0\beta^2} + c_1 + \rho(\phi)\right][c_1 + \rho(\phi)]\phi^2 - c_2\tilde{\psi}^2 \tag{22}$$

which yields

$$u = -(c_0 + c_2)\tilde{\psi}, \qquad c_2 > 0 \tag{23}$$

and establishes global asymptotic stability of the equilibrium $(\Phi, \Psi) = (\Phi_0, \Psi_C(\Phi_0))$.

For the throttle opening γ, the control law (22) is

$$\gamma = \frac{\Gamma + \bar{\beta}^2 k(\Psi - c_0\Phi)}{\sqrt{\Psi}}, \qquad k > c_0 > a \tag{24}$$

where $\bar{\beta}$ is an upper bound on β (which we consider unknown), and

$$\Gamma = 1 + \Phi_0 + \beta^2(c_2 + c_0)[c_0\Phi_0 - \Psi_C(\Phi_0)]. \tag{25}$$

By design we have achieved that the control law (24) depends on the compressor characteristic $\Psi_C(\Phi)$ and the Greitzer parameter β only through the "set-point" parameter $\Gamma = \Gamma(\Phi_0)$ which is an invertible function of Φ_0.

Theorem 2.1: For each value of Γ, the system (5) and (6) with the control law (24) has a unique equilibrium, and this equilibrium is globally asymptotically stable.

For comparison, we also derive a feedback linearizing controller for (5) and (6)

$$\gamma = \frac{1}{\sqrt{\Psi}}\left\{\Phi + 1 + \beta^2\left[k_1(\Phi_0 - \Phi) + \left(\frac{3}{2} - \frac{3}{2}\underbrace{\Phi^2} + k_2\right) \times \left(\Psi - \Psi_{C0} - 1 - \frac{3}{2}\Phi + \frac{1}{2}\underbrace{\Phi^3}\right)\right]\right\}. \tag{26}$$

The simplification achieved with backstepping is striking. While the backstepping controller (24) is linear, the linearizing control (26) grows as Φ^5 and requires much higher control effort for large signals. Furthermore, while the linearizing controller is based on the exact knowledge of $\Psi_C(\Phi)$ and β, backstepping needs only a and $\bar{\beta}$.

The avoidance of cancellation endows the backstepping controller with another significant property not possessed by the feedback linearizing controller—inverse optimality [8]. Using the results in [18], we can prove that the control law (24) with $k > lc_0$, $l \geq 2$, minimizes the cost functional

$$J = \int_0^\infty \left\{2lc_0\left[\frac{1}{c_0\beta^2} + c_1 + \rho(\phi)\right][c_1 + \rho(\phi)]\phi^2 + l(lc_2 + (l-2)c_0)(\psi - c_0\phi)^2 + \frac{1}{(c_2 + c_0)\beta^4}[\sqrt{\Psi}\gamma - (\Phi_0 + 1)]^2\right\}dt. \tag{27}$$

Instead of a quadratic form $\int_0^\infty x^TQx + ru^2$, this cost functional has a higher order nonlinear form in ϕ. For the problem at hand, this is a more meaningful cost because the rapidly growing nonlinearities in ϕ are beneficial. One of the main benefits of inverse optimality is an infinite gain margin reflected in the property that $l \geq 2$ can be arbitrarily large [18].

III. Stall/Surge Stabilization Without Stall Measurement

With $\Psi_C(\Phi)$ in (10) and $R = A^2/4$, the model (1)–(3) becomes

$$\dot{\Phi} = -\Psi + \Psi_C(\Phi) - 3\Phi R \tag{28}$$

$$\dot{\Psi} = \frac{1}{\beta^2}(\Phi - \Phi_T) \tag{29}$$

$$\dot{R} = \sigma R(1 - \Phi^2 - R), \qquad R(0) \geq 0. \tag{30}$$

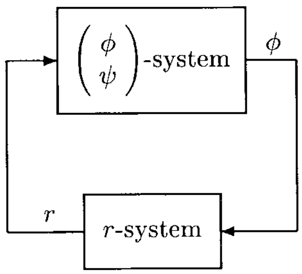

Fig. 2. Feedback connection created by a partial-state feedback controller.

Denoting $\Psi_S(\Phi) = \Psi_{C0} + 1 - \frac{3}{2}\Phi + \frac{5}{2}\Phi^3$, the equilibria are

$$(\Phi, R, \Psi)_e = \begin{cases} (\Phi_0, 0, \Psi_C(\Phi_0)), & \forall \Phi_0 \\ \big(\Phi_0, 1-\Phi_0^2, \Psi_S(\Phi_0)\big), & |\Phi_0| \le 1. \end{cases} \tag{31}$$

The no-stall equilibria for $|\Phi_0| \le 1$ cannot be stabilized by throttle feedback. We will stabilize the no-stall equilibria for $|\Phi_0| > 1$ and the stall equilibria for $|\Phi_0| \le 1$. In the error coordinates $\phi = \Phi - \Phi_0$, $\psi = \Psi - \Psi_S(\Phi_0)$, $r = R - 1 + \Phi_0^2$, the model (28)–(30) is

$$\dot\phi = -\psi + \frac{3}{2}\left(1 - \frac{1}{4}\Phi_0^2\right)\phi - \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi - 3R\phi - 3\Phi_0 r \tag{32}$$

$$\dot\psi = \frac{1}{\beta^2}(\phi + \Phi_0 + 1 - \gamma\sqrt{\Psi}) \tag{33}$$

$$\dot r = \sigma\big(r + 1 - \Phi_0^2\big)(-r - 2\Phi_0\phi - \phi^2), \qquad r(0) + 1 - \Phi_0^2 \ge 0. \tag{34}$$

Our goal is to design a feedback law $\gamma(\phi,\psi)$ that does not depend on r. To this end, we observe that the model (32)–(34) is a feedback system as in Fig. 2. The coupling between the ϕ- and r-equations can be destabilizing. Our two-step Lyapunov design of $\gamma(\phi,\psi)$ employs lengthy calculations, but it reduces the conservativeness of input–output design that would be applicable to this problem using the "nonlinear small gain" theorem of Jiang *et al.* [11].

Step 1: To find a partial-state feedback $\psi = \psi_{\rm des}(\phi)$ to stabilize the equilibrium $(\phi, r) = 0$ of the system (32) and (34) rewritten as

$$\begin{aligned}\dot\phi &= -\psi_{\rm des}(\phi) + \frac{3}{2}\left(1 - \frac{1}{4}\Phi_0^2\right)\phi \\ &\quad - \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi - 3R\phi - 3\Phi_0 r \end{aligned}\tag{35}$$

$$\dot r = \sigma\big(r + 1 - \Phi_0^2\big)(-r - 2\Phi_0\phi - \phi^2) \tag{36}$$

we use the Lyapunov function

$$V_1 = \frac{1}{2}\left(\phi^2 + \frac{5}{\sigma}V_r(r)\right) \tag{37}$$

where $V_r(r)$ is a positive definite radially unbounded function on the interval $[-1 + \Phi_0^2, +\infty)$ and given by

$$V_r(r) = r - \big(1 - \Phi_0^2\big)\log\frac{r + 1 - \Phi_0^2}{1 - \Phi_0^2}. \tag{38}$$

$V_r(r)$ is quadratic around $r = 0$ and linear for large r. It "blows up" at $r = \Phi_0^2 - 1$, that is, at the invariant manifold $R = 0$, except for $\Phi_0 = 1$ where $V_r(R) = R$. An important property of $V(r)$ is

$$\frac{dV_r(r)}{dr} = \frac{r}{r + 1 - \Phi_0^2} \tag{39}$$

so that

$$\dot V_r = \sigma(-r^2 - 2\Phi_0\phi r - \phi^2 r) \tag{40}$$

and in view of $-r = -R + 1 - \Phi_0^2 \le 1 - \Phi_0^2$, we get

$$\dot V_r \le \sigma\big[-r^2 - 2\Phi_0\phi r + \big(1 - \Phi_0^2\big)\phi^2\big]. \tag{41}$$

The derivative of V_1 along the trajectories of (35) and (36) is

$$\begin{aligned}\dot V_1 &\le -\psi_{\rm des}(\phi)\phi + \frac{3}{2}\left(1 - \frac{1}{4}\Phi_0^2\right)\phi^2 - \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi^2 \\ &\quad - 3R\phi^2 - 3\Phi_0 r\phi - \frac{5}{2}r^2 - 5\Phi_0 r\phi + \frac{5}{2}\big(1 - \Phi_0^2\big)\phi^2. \end{aligned}\tag{42}$$

By completing squares, $-2r^2 - 8\Phi_0 r\phi \le 8\Phi_0^2\phi^2$, we get

$$\begin{aligned}\dot V_1 &\le -\psi_{\rm des}(\phi)\phi + \left[\frac{3}{2}\left(1 - \frac{1}{4}\Phi_0^2\right) + 8\Phi_0^2 + \frac{5}{2}\big(1 - \Phi_0^2\big)\right]\phi^2 \\ &\quad - \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi^2 - 3R\phi^2 - \frac{1}{2}r^2. \end{aligned}\tag{43}$$

For $\Phi_0 \in [-1, 1]$, the bracketed term achieves the maximum at the ends of the interval, so that

$$\begin{aligned}\dot V_1 &\le -\psi_{\rm des}(\phi)\phi + \left(\frac{3}{2} + 7\frac{5}{8}\right)\phi^2 \\ &\quad - \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi^2 - 3R\phi^2 - \frac{1}{2}r^2. \end{aligned}\tag{44}$$

Hence, we can select

$$\psi_{\rm des}(\phi) = \left(c_1 + \frac{3}{2}\right)\phi = c_0\phi, \qquad c_1 > 7\frac{5}{8}. \tag{45}$$

Denoting $\tilde\psi = \psi - \psi_{\rm des}(\phi)$, the derivative of V_1 for (32) and (34) becomes

$$\dot V_1 \le -\left(c_1 - 7\frac{5}{8}\right)\phi^2 - \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi^2 - 3R\phi^2 - \frac{1}{2}r^2 - \phi\tilde\psi. \tag{46}$$

The resulting form of the ϕ-equation is

$$\dot\phi = -\left(c_1 + \frac{3}{8}\Phi_0^2\right)\phi - \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi - 3R\phi - 3\Phi_0 r - \tilde\psi. \tag{47}$$

Step 2: Differentiating $\tilde\psi$, we get

$$\begin{aligned}\dot{\tilde\psi} &= \frac{1}{\beta^2}(\phi + \Phi_0 + 1 - \gamma\sqrt{\Psi}) \\ &\quad + c_0\left[\tilde\psi + \left(c_1 + \frac{3}{8}\Phi_0^2\right)\phi + \frac{1}{2}\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi + 3R\phi + 3\Phi_0 r\right]. \end{aligned}\tag{48}$$

Now consider the Lyapunov function

$$\begin{aligned}V_2(\phi,\psi,r) &= \frac{c_0}{2}\left[2\left(c_1 + \frac{3}{8}\Phi_0^2 + \frac{1}{c_0\beta^2}\right)V_1(\phi, r)\right. \\ &\quad \left. + \int\left(\phi + \frac{3}{2}\Phi_0\right)^2\phi\, d\phi + 3R\phi^2\right] + \frac{1}{2}\tilde\psi^2. \end{aligned}\tag{49}$$

In addition to the (ϕ,ψ)-part and the r-part, the Lyapunov function $V_2(\phi,\psi,r)$ also has the cross-term $R\phi^2$ which accounts for the beneficial coupling between the two subsystems. The derivative of V_2 for (32)–(34) is

$$\begin{aligned}\dot{V}_2 \le &-c_0\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}\right)\\ &\left[\left(c_1-7\frac{5}{8}\right)\phi^2+\frac{1}{2}\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi^2+3R\phi^2+\frac{1}{2}r^2+\phi\tilde{\psi}\right]\\ &+c_0\left[3R+\frac{1}{2}\left(\phi+\frac{3}{2}\Phi_0\right)^2\right]\\ &\left[-\left(c_1+\frac{3}{8}\Phi_0^2\right)\phi^2-\frac{1}{2}\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi^2-3R\phi^2\right.\\ &\left.-3\Phi_0 r\phi-\phi\tilde{\psi}\right]\\ &+\frac{3}{2}c_0\sigma\phi^2R(-R+1-\Phi^2)+\tilde{\psi}\\ &\left[\frac{1}{\beta^2}(\Phi_0+1-\gamma\sqrt{\Psi})+c_0\tilde{\psi}+c_0\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}\right)\phi\right.\\ &\left.+\frac{c_0}{2}\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi+3c_0R\phi+3c_0\Phi_0 r\right] \end{aligned}\tag{50}$$

where we have substituted $\dot{V}_1$ from (46). After some manipulations which eliminate $c_0(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2})\phi+\frac{c_0}{2}(\phi+\frac{3}{2}\Phi_0)^2\phi+3c_0R\phi$ from the last bracket, we obtain

$$\begin{aligned}\dot{V}_2 \le &-c_0\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}\right)\left(c_1-7\frac{5}{8}\right)\phi^2\\ &-\frac{c_0}{2}\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}\right)r^2\\ &+c_0\left[\left(c_1-3\left(\frac{3}{8}\Phi_0^2+\frac{1}{2c_0\beta^2}\right)-\frac{\sigma}{2}\right)R\phi^2\right.\\ &-3R\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi^2\\ &-\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{2c_0\beta^2}\right)\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi^2\\ &\left.-\frac{3}{2}\Phi_0 r\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi-9R^2\phi^2-9R\Phi_0 r\phi\right]\\ &+\tilde{\psi}\left[\frac{1}{\beta^2}(\Phi_0+1-\gamma\sqrt{\Psi})+c_0\tilde{\psi}+3c_0\Phi_0 r\right].\end{aligned}\tag{51}$$

It is not hard to show with three completions of squares that

$$\begin{aligned}&-3R\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi^2-\left(c_1+\frac{3}{8}\Phi_0^2\right)\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi^2\\ &-\frac{3}{2}\Phi_0 r\left(\phi+\frac{3}{2}\Phi_0\right)^2\phi\\ &\le\frac{3}{16}\Phi_0^2R\phi^2+\frac{1}{c_1}\left(\frac{9}{8}\right)^2\Phi_0^4r^2+\frac{3}{2}\left(1-\Phi_0^2\right)^2\phi^2.\end{aligned}\tag{52}$$

We also note that

$$-9R^2\phi^2-9R\Phi_0 r\phi\le\frac{9}{4}\Phi_0^2r^2\quad\text{and}\quad 3\Phi_0 r\tilde{\psi}\le\tilde{\psi}^2+\frac{9}{4}\Phi_0^2r^2.\tag{53}$$

Substituting (52) and (53) into (51), we finally get

$$\begin{aligned}\dot{V}_2 \le &-c_0\left[\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}\right)\left(c_1-7\frac{5}{8}\right)-\frac{3}{2}\left(1-\Phi_0^2\right)^2\right]\phi^2\\ &-\frac{c_0}{2}\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}-9-\frac{81}{32c_1}\Phi_0^4\right)r^2\\ &-6c_0\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{2c_0\beta^2}-\frac{\sigma}{4}-\frac{1}{32}\Phi_0^2\right)R\phi^2\\ &+\tilde{\psi}\left[\frac{1}{\beta^2}(\Phi_0+1-\gamma\sqrt{\Psi})+2c_0\tilde{\psi}\right].\end{aligned}\tag{54}$$

The bracketed term in front of ϕ^2 is positive provided $c_1 > 7.817$. The term in front of r^2 is positive provided that $c_1 > 9.273$. The term in front of $R\phi^2$ is positive if $c_1 > \frac{\sigma}{4}+\frac{1}{32}$. The control law is selected as

$$\gamma(\phi,\psi)=\frac{\Phi_0+1+\bar{\beta}^2(c_2+2c_0)\tilde{\psi}}{\sqrt{\Psi}},\qquad c_2>0.\tag{55}$$

This control law guarantees global asymptotic stability provided $c_1 > \max\{9.273, \frac{\sigma}{4}+\frac{1}{32}\}$

$$\begin{aligned}\dot{V}_2 \le &-c_0\left[\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}\right)\left(c_1-7\frac{5}{8}\right)-\frac{3}{2}\left(1-\Phi_0^2\right)^2\right]\phi^2\\ &-\frac{c_0}{2}\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{c_0\beta^2}-9-\frac{81}{32c_1}\Phi_0^4\right)r^2\\ &-6c_0\left(c_1+\frac{3}{8}\Phi_0^2+\frac{1}{2c_0\beta^2}-\frac{\sigma}{4}-\frac{1}{32}\Phi_0^2\right)R\phi^2-c_2\tilde{\psi}^2.\end{aligned}\tag{56}$$

The control law (55) can be expressed also as

$$\gamma=\frac{\Gamma+\bar{\beta}^2k(\Psi-c_0\Phi)}{\sqrt{\Psi}}\tag{57}$$

where $k > 2c_0, c_0 > \max\{10.773, \frac{\sigma}{4}+1.53\}$ (to our knowledge, σ on real compressors is seldom larger than four) and

$$\Gamma=1+\Phi_0+\bar{\beta}^2k[c_0\Phi_0-\Psi_S(\Phi_0)].\tag{58}$$

Theorem 3.1: Consider the system (28)–(30) evolving in the set $\mathcal{G}=\{(\Phi,\Psi,R)\in\mathbb{R}^3 \mid R\ge 0\}$ with the control law (57). For each value of Γ, the system has

- either a unique asymptotically stable equilibrium on the axisymmetric compressor characteristic, with a region of attraction equal to the entire set $\mathcal{G}$;
- or two equilibria:

—one equilibrium on the axisymmetric characteristic which is unstable but attracts all solutions starting in $\partial\mathcal{G} = \{(\Phi,\Psi,R)\in\mathbb{R}^3 \mid R=0\}$;

—one equilibrium on the stall characteristic which is asymptotically stable with a region of attraction equal to $\mathcal{G}\setminus\partial\mathcal{G} = \{(\Phi,\Psi,R)\in\mathbb{R}^3 \mid R>0\}$.

IV. Bifurcation Diagrams

Open-loop bifurcation diagrams generated by varying the throttle opening γ are given in Fig. 3 for a low-speed few-stage compressor studied in [7], for which $\Psi_{C0}=0.72$, $\sigma=4$, and $\beta=0.71$. The equilibria with low-stall amplitude are unstable and we observe

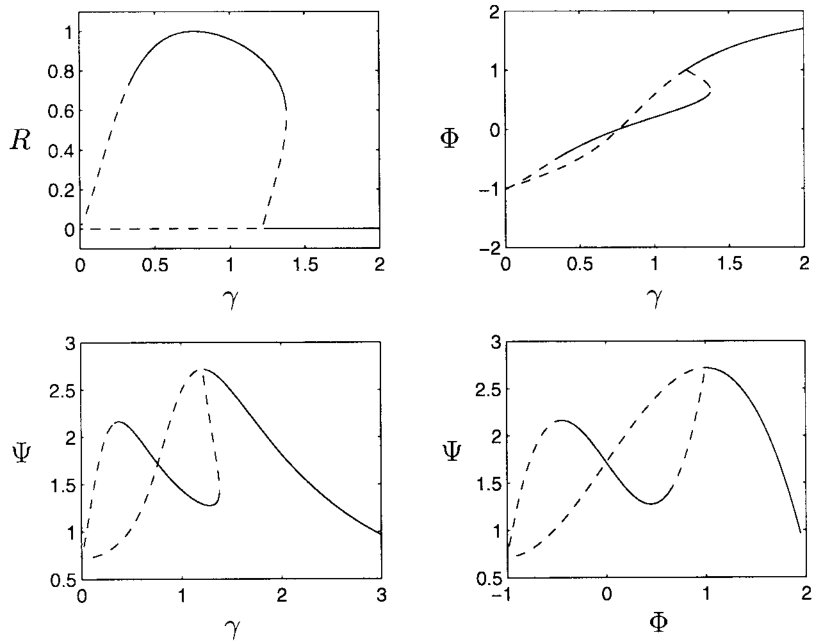

Fig. 3. Bifurcations in the uncontrolled MG3 with γ as the bifurcation parameter.

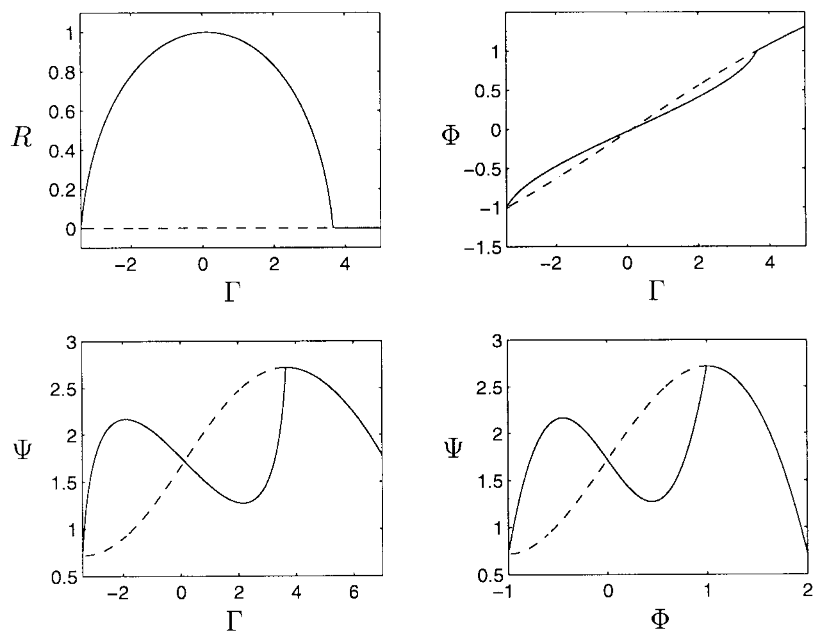

Fig. 4. Bifurcations in MG3 model controlled by the *backstepping* controller (57). The "set-point" parameter Γ is the bifurcation parameter.

TABLE I
CATALOG OF CONTROLLERS FOR THE MOORE–GREITZER MODEL. SYMBOLS: k, k_1, k_2, c_1, c_2 ARE CONSTANT GAINS AND γ_0 AND Γ ARE BIFURCATION PARAMETERS

Liaw and Abed [14]	$\gamma(A) = \gamma_0 + kA^2$	2D sensing	local
Eveker et al. [7]	$\gamma(A, \dot{\Phi}) = \gamma_0 + k_1 A^2 - k_2 \dot{\Phi}$	2D sensing + "differentiation"	large operating region
Backstepping	$\gamma(\Phi, \Psi) = \dfrac{\Gamma + c_1 \Psi - c_2 \Phi}{\sqrt{\Psi}}$	1D sensing	global

"hysteresis," caused by the subcritical bifurcation. The bifurcation diagrams for the backstepping controller (57) are shown in Fig. 4 for $c_0 = 6$ and $\bar{\beta}^2 k = 0.5$. This controller "softens" the bifurcation from subcritical to supercritical and eliminates the hysteresis. In addition, it stabilizes all stall equilibria and prevents surge for all values of Γ.

While all three designs in Table I soften the bifurcation, the global design achieved with backstepping is due to a methodological difference. The bifurcation designs in [14] and [7] are based on local stability properties established by the center manifold theorem because the maximum of the compressor characteristic is a bifurcation point that is not linearly controllable. Hence stabilization is inherently nonlinear and results in asymptotic but not exponential stability. Our Lyapunov-based design incorporates the good features of a bifurcation-based design. For $\Phi_0 = 1$, the term $V_r(r)$ in the Lyapunov function (49), becomes $V_r(R) = R$, so that the Lyapunov function

$$V_2 = \frac{c_0}{2}\left[\left(c_1 + \frac{3}{8} + \frac{1}{c_0\beta^2}\right)\left(\phi^2 + \frac{5}{\sigma}R\right) + \frac{9}{8}\phi^2 + \phi^2 + \frac{1}{4}\phi^4 + 3R\phi^2\right] + \frac{1}{2}(\psi - c_0\phi)^2 \tag{59}$$

is (locally) quadratic in ϕ and ψ but only linear in R. Since the derivative of V_2 is quadratic in all three variables, $\dot{V}_2 \leq -a_1\phi^2 - a_2R^2 - a_3(\psi - c_0\phi)^2$ [see (56)], this clearly indicates that the achieved type of stability is asymptotic but not exponential. However, to satisfy the requirements not only for local but also for global stability, our analysis is considerably more complicated.

V. Conclusion

Experimental validation of the controller presented here is planned but is beyond the scope of this paper. Measurement of Φ represents a challenge but it is not expected to be insurmountable considering that a controller that employs the *derivative* of Φ has been successfully implemented [7].

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A Deterministic Analysis of Stochastic Approximation with Randomized Directions

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***Abstract*— We study the convergence of two stochastic approximation algorithms with randomized directions: the simultaneous perturbation stochastic approximation algorithm and the random direction Kiefer–Wolfowitz algorithm. We establish deterministic necessary and sufficient conditions on the random directions and noise sequences for both algorithms, and these conditions demonstrate the effect of the "random" directions on the "sample-path" behavior of the studied algorithms. We discuss ideas for further research in analysis and design of these algorithms.**

***Index Terms*—Deterministic analysis, random directions, simultaneous perturbation, stochastic approximation.**

I. Introduction

One of the most important applications of stochastic approximation algorithms is in solving local optimization problems. If an estimator of the gradient of the criterion function is available, the Robbins–Monro algorithm [9] can be directly applied. In [7], Kiefer

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