Using ergodicity of chaotic systems for improving the global properties of the delayed feedback control method

Kestutis Pyragas[1,2] and Viktoras Pyragas[1]
[1]*Semiconductor Physics Institute, A. Goštauto 11, LT-01108 Vilnius, Lithuania*
[2]*Department of Theoretical Physics, Faculty of Physics, Vilnius University, LT-10222 Vilnius, Lithuania*
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A modified delayed feedback control algorithm with the improved global properties is proposed. The modification is based on the ergodic features of chaotic systems. We do not perturb the system until its state approaches a desired unstable periodic orbit and then we activate the delayed feedback control force. To evaluate the closeness of the system state to the target orbit, a special algorithm is devised. For continuous-time systems, it can be implemented by means of a simple low-pass filter. An additional low-pass filter can be used for selection of the particular orbit from several unstable orbits of the same period.

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Delayed feedback control (DFC) algorithm has been invented in the early 1990s [1] as a simple, robust, and efficient method to stabilize unstable periodic orbits (UPOs) in chaotic systems. Nowadays it has become one of the most popular methods in chaos control research [2]. Successful implementations of the method include quite diverse experimental systems from different fields of science. For the details of experimental implementations as well as various modifications of the DFC algorithm, we refer to the recent review paper [3].

The theory of DFC is rather difficult, since the time-delay dynamics takes place in infinite-dimensional phase spaces. The standard tool for discussing the control performance consists in linear stability analysis [4–7]. But even if such a local analysis predicts stable states, experimental success is not guaranteed because the control performance may strongly depend on initial conditions. The analysis of global properties of DFC systems, such as basins of attraction of stabilized orbits, is much more complicated problem. There exists virtually no systematic investigation of time-delayed feedback control beyond the linear regime.

Numerical analysis of particular systems shows that the DFC algorithm can form incredibly complex basins of attraction [8,9]. Recently, the idea of the transition from sub- to supercritical bifurcations for the estimation of basins of attraction has been proposed [10–12]. However, this approach is not universal and does not guarantee the correct prediction for the system parameters far away from the bifurcation point. The lack of a general theory concerning the global properties of DFC systems represents a serious drawback of the method.

To improve the global properties of the DFC algorithm, several nonlinear modifications have been proposed. A first heuristic idea has been suggested in the original paper [1]. It has been shown that limiting the size of the control force by a simple cutoff increases a basin of attraction of the stabilized orbit. This idea has proved itself in a number of chaotic systems and now it is widely used in experiments. An alternative two-step DFC algorithm has been considered in Ref. [13]. In the first step, this algorithm generates an extraneous stable periodic orbit close to the target orbit and in the second step stabilizes the target. Finally, a nonlinear DFC for systems close to a subcritical Hopf bifurcation has been proposed in Ref. [12]. Here the basin of attraction is enlarged by coupling control forces through the phase of the signal.

Unfortunately, the above-proposed nonlinear DFC schemes are not universal. In this paper, we seek to improve the global properties of the DFC algorithm by invoking an egodicity—the universal feature of chaotic systems. The ergodicity means the fact that a trajectory of any chaotic system visits a neighborhood of each periodic orbit with finite probability. The idea of using ergodicity in chaos control research was first formulated in the seminal paper by Ott, Grebogy, and Yorke (OGY) [14] and has been employed in their OGY control algorithm. However, a straightforward implementation of ergodicity in DFC schemes has not been considered so far. The main motivation of this paper is to fill this gap and adapt the OGY ideas for DFC algorithm. The specific point of our problem is that the DFC force increases the phase dimension of the closed-loop system. As well as in the OGY algorithm, we do not perturb the system until it comes in a small neighborhood of the desired orbit. Using a scalar observable, we develop a technique which allows us to evaluate a moment when the state of the free time-delay system approaches the target orbit. At this moment, we activate the DFC force and stabilize the target. The algorithm does not require a knowledge of location of the orbit and can be easily implemented by means of electronic circuits.

We are going to apply our algorithm for continuous-time chaotic systems. Note that the problem of evaluating the moment when the state of the free system approaches the target orbit has to be considered in an infinite-dimensional phase space, since the DFC force increases the phase dimension of the closed-loop system to infinity. As a first example, we consider the Rössler equations [15] subjected to DFC force

$$\dot{x} = -y - z, \tag{1a}$$

$$\dot{y} = x + ay - kD(t), \tag{1b}$$

$$\dot{z} = b + z(x - c). \tag{1c}$$

Here x, y, z are the dynamic variables of the Rössler system. The parameters $a=0.2$, $b=0.2$, and $c=5.7$ are chosen such

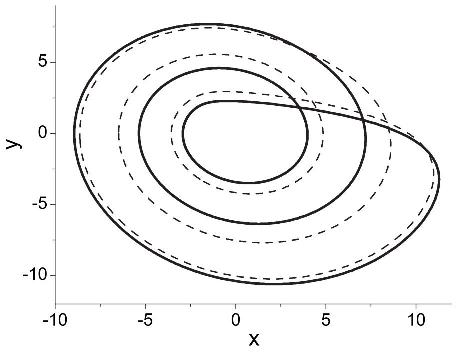

FIG. 1. The (x,y) projection of two period-3 UPOs embedded in chaotic attractor of the free $(k=0)$ Rössler system (1) for $a=0.2$, $b=0.2$, and $c=5.7$. The target UPO with the period $\tau=17.51$ is depicted by solid line. By dashed line is shown another (extraneous) period-3 UPO with the period $\tau=17.60$.

that the system exhibits chaotic behavior. We suppose that $y(t)$ is an observable and the DFC perturbation $kD(t)$ is applied only to the second equation of the Rössler system. Here, k is the feedback gain and $D(t)$ denotes the difference of the observable between the current state and the state delayed by the period τ of UPO

$$D(t) = y(t) - y(t-\tau). \quad (2)$$

Numerical analysis shows that the basins of attraction of the period-1 and period-2 UPOs of the Rössler system are rather large. The straightforward application of the original DFC algorithm (without waiting until the system approaches the desired orbit) is successful for these orbits at any initial conditions placed on the strange attractor. However, this is not the case for the period-3 UPOs. The free $(k=0)$ Rössler system has two period-3 UPOs with slightly different periods $\tau=17.51$ and $\tau=17.60$. The (x,y) projections of their phase portraits are shown in Fig. 1. Here we concentrate on the stabilization of the orbit with the period $\tau=17.51$. In Fig. 1, this target orbit is depicted by solid line. The original DFC algorithm is not able to stabilize this orbit for any initial conditions and thus we need a modification.

The linear analysis of Eqs. (1) with the perturbation (2) shows that the target orbit is stable in the interval of control gain $0.05<k<0.34$. The optimal value of the feedback gain, which leads to the fastest convergence of nearby initial conditions toward the desired orbit, is $k_{op}=0.06$. Although the whole basin of attraction of the stabilized orbit may be very complex, it necessarily occupies some regions in the vicinity of the target orbit. This general feature of the basin of attraction results from linear stability of the orbit. Thus the DFC algorithm should be successful if the initial conditions are in fair proximity to the target. On the other hand, due to the ergodicity, the free system $(k=0)$ should approach the target orbit as close as desired for any initial conditions placed on the strange attractor.

Our strategy is as follows. We do not perturb the system $(k=0)$ when its state is far away from the desired orbit and activate the control in the form of Eq. (2) with $k=k_{op}$ when the state approaches the target UPO. The main problem here is to define the moment when the state of the free system falls in a small neighborhood of the desired orbit. Having a scalar observable $y(t)$, the standard way to reconstruct the phase space of the system is to introduce delay coordinates. Using an $(M+1)$-dimensional delay-coordinate vector $\mathbf{r}(t)=[y(t),y(t-\delta t),\ldots,y(t-M\delta t)]$ with delay δt, one can write inequalities (requirements)

$$|\mathbf{r}_j(t) - \mathbf{r}_j(t-\tau)| \equiv |D(t-j\delta t)| < \varepsilon, \quad j=0,\ldots,M \quad (3)$$

that guarantee that the state of the system at time t is close to the periodic orbit of the period τ in the reconstructed phase space. These inequalities mean that the absolute value of the difference $|D(t)|$ has to be small in $M+1$ equally spaced points in the time interval $[t-\Delta t,t]$, with $\Delta t=M\delta t$. Alternatively, we may replace the above $M+1$ inequalities (3) by one inequality for the averaged difference, $(1/M)\Sigma_{j=0}^{M}|D(t-j\delta t)|<\varepsilon$. For continuous-time systems with an infinite-dimensional phase space, we take the limit $\delta t\to 0$, $M\to\infty$. On the assumption that the product $M\delta t=\Delta t$ is finite, we obtain $(1/\Delta t)\int_{t-\Delta t}^{t}|D(s)|ds<\varepsilon$. The latter condition defines the smallness of a moving average of the difference $|D(t)|$, where Δt is a window of moving average. The main advantage of such a condition is that the moving average can be simply estimated electronically. The simplest moving average filter can be designed as a first-order low-pass filter.

As a result, we come to the following modification of the DFC algorithm. To estimate the moving average of the difference $|D(t)|$, we introduce an auxiliary variable w that satisfies first-order filter equation

$$\tau_w \dot{w} = |D(t)| - w. \quad (4)$$

An asymptotic solution of this equation is

$$w(t) = \frac{1}{\tau_w}\int_{-\infty}^{t} \exp[(s-t)/\tau_w]|D(s)|ds. \quad (5)$$

Thus the variable w represents an exponentially weighted moving average of the difference $|D(t)|$. The characteristic window of the moving average is $\Delta t=\tau_w$. The smallness of this variable indicates the closeness of the system state to the target orbit taking into account an infinite-dimensional phase space.

The control procedure is as follows. We start from initial conditions placed on the strange attractor of the free Rössler system and integrate Eqs. (1) and (4) for $k=0$ as long as $w>\varepsilon$. As soon as the variable w becomes small, $w<\varepsilon$, we set $k=k_{op}$ and continue the integration of the system subjected to DFC. We assume that the control goal is achieved when w decreases to a given value $\varepsilon_F=3\times 10^{-2}\ll\varepsilon$. We repeat this procedure for 10^3 different initial conditions and plot in Fig. 2 histograms of time needed to achieve the control. For suitably chosen filter parameter $\tau_w=\tau/7$ and fairly small $\varepsilon<\varepsilon_c\approx 0.33$, the algorithm produces 100% success rate for any initial conditions. For $\varepsilon=\varepsilon_c$, the mean time of control amounts approximately ten periods of UPO.

We emphasize that our modification is very simple. The closeness of the system state to the target orbit is estimated from the usual DFC signal (2) by means of the standard

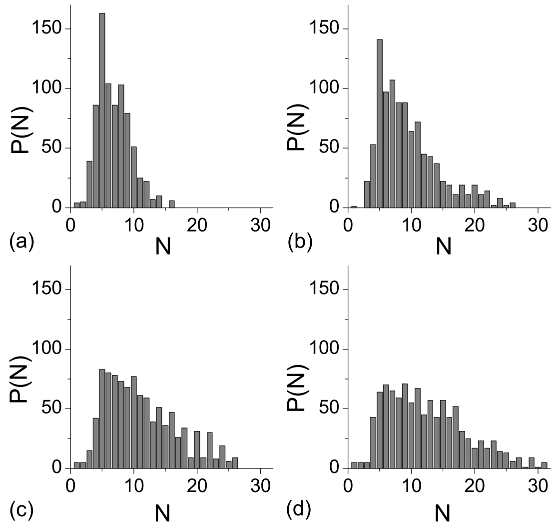

FIG. 2. Histogram of time needed to achieve control in the Rössler system for $k=k_{op}=0.06$, $\tau=17.51$, and $\tau_w=\tau/7$. N is the number of periods τ needed for stabilization of the target UPO and $P(N)$ is the number of successful stabilizations with the given time of control. The parameter ε and the mean number $\langle N\rangle$ of periods needed to achieve control are: (a) $\varepsilon=0.7$, $\langle N\rangle=6.87$; (b) $\varepsilon=\varepsilon_c=0.33$, $\langle N\rangle=9.54$; (c) $\varepsilon=0.25$, $\langle N\rangle=11.25$; (d) $\varepsilon=0.2$, $\langle N\rangle=12.4$. In (a), only 79% of initial conditions are successful, while in (c) and (d) the 100% success rate is obtained.

low-pass filter (4). Thus the modification allows a simple analog implementation and preserves the main advantages of the original DFC technique.

As a second example, we consider a nonautonomous double-well Duffing oscillator subjected to DFC

$$\dot{x} = y - kD(t), \tag{6a}$$

$$\dot{y} = -\beta y + \alpha x - \gamma x^3 + A\cos(\Omega t). \tag{6b}$$

Here, x, y are the dynamic variables, $\alpha=1$, $\gamma=1$ are the parameters of the double-well potential, $\beta=0.16$ is the slope coefficient, and $A=0.27$ and $\Omega=1$ are the amplitude and frequency of the external force, respectively. We suppose that $x(t)$ is an observable and control perturbation $kD(t)$ with

$$D(t) = x(t) - x(t-\tau) \tag{7}$$

is applied to the first Eq. (6a).

For the chosen set of parameters, the free ($k=0$) system exhibits chaotic motion. The chaotic attractor has embedded within it three period-1 UPOs, which are depicted in Fig. 3. All orbits have identical periods coinciding with the period of external force, $\tau=2\pi/\Omega$. One of the orbits is located at the origin. It is symmetrical with respect to both x and y axes and satisfies the so-called odd number limitation [5,6]. This orbit cannot be stabilized by usual DFC technique. Its stabilization with an unstable DFC controller [7] has been considered in the recent paper [13]. Two other nonsymmetric orbits are symmetrically located with respect to the y axis.

Our aim here is to design DFC algorithm capable to select one of two nonsymmetric UPOs, say the right-hand orbit shown in Fig. 3, and stabilize it for any initial conditions

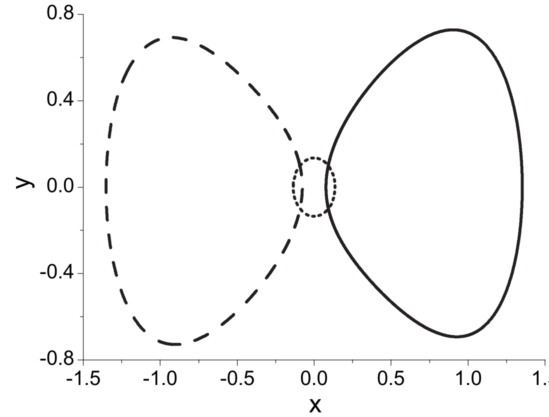

FIG. 3. Three period-1 UPOs embedded in chaotic attractor of the free ($k=0$) double-well oscillator (5) for $\alpha=1$, $\beta=0.16$, $\gamma=1$, $A=0.27$, and $\Omega=1$. All orbits have the same period $\tau=2\pi$. The target orbit is depicted by solid line.

taken on the strange attractor. Now it is not sufficient to require the smallness of the moving average of the delayed difference (7) since it may become small when the system moves along of any of three UPOs. We need a condition, which guarantees that the moving average of $|D(t)|$ becomes small on the particular selected orbit. For this aim, we introduce an additional moving average filter that allows us to estimate the peculiarities of location of the UPO in the phase space. Thus our algorithm here is based on two running average filters

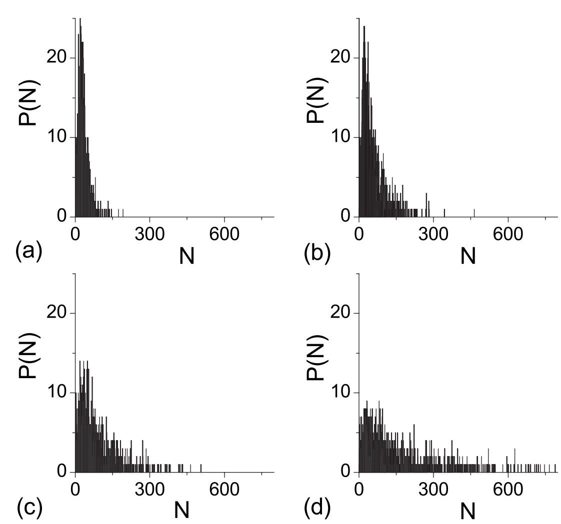

FIG. 4. Histogram of time needed to achieve control in the Duffing system for $k=k_{op}=0.8$, $\tau=2\pi$, $\tau_w=0.1\tau$, $\tau_v=\tau$, $\varepsilon_v=0.2$, and $\varepsilon_F=0.025$. N is the number of periods τ needed for stabilization of the target UPO and $P(N)$ is the number of successful stabilizations with the given time of control. The parameter ε and the mean number $\langle N\rangle$ of periods needed to achieve control are: (a) $\varepsilon=0.2$, $\langle N\rangle=33.58$; (b) $\varepsilon=\varepsilon_c=0.1$, $\langle N\rangle=60.54$; (c) $\varepsilon=0.06$, $\langle N\rangle=86.84$; (d) $\varepsilon=0.04$, $\langle N\rangle=159.27$. In (a), only 77.8% of initial conditions are successful, while in (c) and (d) the 100% success rate is obtained.

$$\tau_w \dot{w} = |D(t)| - w, \tag{8a}$$

$$\tau_v \dot{v} = \mathrm{sgn}(x) - v. \tag{8b}$$

The control procedure is as follows. First we analyze the free running system ($k=0$) by means of the introduced variables w and v. We simultaneously check two conditions $w(t)<\varepsilon$ and $v(t)>1-\varepsilon_v$. As soon as the both conditions are satisfied, we activate the control by setting $k=k_{op}$ and stabilize the target orbit. The first condition $w<\varepsilon$ here is equivalent to that introduced in the previous example. It controls the smallness of the running average of the difference $|D(t)|$ estimated by the filter (8a). The second condition $v>1-\varepsilon_v$ selects the right-hand nonsymmetric UPO. This is achieved by means of the additional running average filter (8b). The function $\mathrm{sgn}(x)$ is the signum function equal to -1 for $x<0$ and equal to 1 for $x>0$. The filter variable $v(t)$ reaches the value close to 1 if $x(t)$ is positive in the characteristic time interval $[t-\tau_v, t]$. Since the target orbit is located in the region $x>0$, the second condition $v>1-\varepsilon_v$ with a small $\varepsilon_v>0$ can be satisfied only for the target orbit.

The linear stability analysis shows that the target orbit is stable for $0.5<k<2.52$. The optimal value of the control gain is $k_{op}=0.8$. In Fig. 4, we show the statistics of successful stabilizations obtained from 10^3 initial conditions randomly chosen on the strange attractor of the free system. For suitably chosen values of the parameters $\tau_w=0.1\tau$, $\tau_v=\tau$, and $\varepsilon_v=0.2$ and sufficiently small $\varepsilon<\varepsilon_c=0.1$, the algorithm successfully selects and stabilizes the target orbit for any initial conditions. Here, the mean time of control is considerably longer than that for the Rössler system. For $\varepsilon=\varepsilon_c$, it is equal approximately to 60 periods of UPO. Such a long transient dynamics is related with the coexistence of three UPOs of the same period and the necessity to select one of them.

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