



## USING PARTIAL DIFFERENTIAL EQUATIONS FOR PRICING OF GOODS AND SERVICES

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### Abstract

*This article is based on the methodology of comparative analysis, using an innovative approach for pricing of various goods and services. Benchmarking is the continuous search to find and adapt better pricing methods that leading to increased profits. We will consider the numerical solution of partial differential equations, based on Black-Scholes model for pricing of goods and services within European option. Also, we will present formulation and numerical behavior of explicit and implicit methods that can be use in pricing for company assets within European option.*

**Keywords:** partial differential equations, economics, pricing, goods, services

**JEL classification:** Y80

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### 1. INTRODUCTION

Every business needs to be permanently in the cash flow conversions and expanding the business. The process begins with a cash infusion, manufacture of products or services to customers, selling and delivery of goods or services, collection of payments and adding of the money received in the treasury of the company. Successful business collects more money from customers than spent to provide service to its products and services. When eventually liquidate the company, profits and cash are equal. But during its existence, the company makes periodic income statements and balance sheets based on accrual, which serves as a measure of performance. It also makes statement of cash flows for measuring the sources and use of actual available amounts (Neftci, 2000; Pliska, 1997; Rich, 1996; Tavella and Randall, 2000).

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To survive a business, it has to have money when you need them. The goal of our study is to investigate the possibility of profit from sale of goods and services, i.e. to show the best possible prices for maximizing profits in some sectors of business. Our survey is based on the Black-Scholes method and use of partial differential equations in pricing for certain goods and services. We will examine two schemes, explicit and implicit, that can use in pricing for essential company assets within the European Option (Tavella and Randall, 2000; Wilmott *et al.*, 1993; Zvan *et al.*, 1997; Zvan *et al.*, 2000).

In addition, the positive buffering money provides a safety net against unforeseen crises in business, failures or management errors and allows the company to take advantage of opportunities that may arise. Availability of sufficient money is needed to survive and grow the business. Businesses do not fail from a lack of growth, lack of profitability or the lack of money to pay bills (Brennan and Schwartz, 1977; Figlewski and Gao, 1999; Hull and White, 1993).

## 2. MATHEMATICAL MODELS

Our research is based on particular differential equations (PDE), which can be used for pricing of some goods on option. The idea is to apply final differential methods for solving the Black-Scholes model (Clemen and Reilly, 1996; Tavella and Randall, 2000; Wilmott, 1999; Zvan *et al.*, 2000).

### 2.1. Final differential methods for Black-Scholes equation

Value of the underlying company asset on Option whose price is  $S(t)$  at a given point in time  $t$ , can be calculated by a function  $f(S, t)$ , which satisfies the following differential equation:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \rho^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad (1)$$

In equation (1) we use appropriate boundary conditions which characterize the type of Option (Pliska, 1997; Taleb, 1996; Brandimarte, 2002). If we change certain assumptions, and if we introduce the dependence on the direction we obtain different equations, where equation (1) is the starting point, and with his help we apply numerical methods based on differential equations to pricing during Option. In this case, to solve the PDE by finite differential methods we must create an unrelated network in terms of time and asset prices. Let  $T$  is the maturity of Option and  $S_{max}$  suitable clear and bright asset prices  $S(t)$ , which cannot be achieved within a given period of time (Topper, 2005; Saaty, 1994; Boyle and Tian, 1998; Black and Scholes, 1973). We need of  $S_{max}$ , because the field of PDE is limitless in terms of asset prices, but we must connect this area with our computing activities:  $S_{max}$  perform  $+\infty$ . The network consists points  $(S, t)$ , such that:

$$\begin{aligned} S &= 0, \varphi S, 2\varphi S, \dots, M\varphi S \equiv S_{max}, \\ t &= 0, 2\varphi t, \dots, N\varphi t \equiv T. \end{aligned}$$

We will use the following annotation for the grid:  $f_{i,j} = f(i \varphi S, j \varphi t)$  (Brandimarte, 2002).

The different ways to approximate the partial derivatives of equation (1) are:

- Progressive difference:

$$\frac{\partial f}{\partial S} = \frac{f_{i+1,j} - f_{i,j}}{\partial S}, \quad \frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j}}{\partial t};$$

- Backward difference:

$$\frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i-1,j}}{\partial S}, \quad \frac{\partial f}{\partial t} = \frac{f_{i,j} - f_{i,j-1}}{\partial t};$$

- Backward difference:

$$\frac{\partial f}{\partial S} = \frac{f_{i+1,j} - f_{i-1,j}}{2\partial S}, \quad \frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j-1}}{2\partial t};$$

- Second derivative:

$$\frac{\partial^2 f}{\partial S^2} = \frac{\left(\frac{f_{i+1,j} - f_{i,j}}{\partial S} - \frac{f_{i,j} - f_{i-1,j}}{\partial S}\right)}{\partial S} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\partial S^2}.$$

Depending on the combination of schemes that we use in the division of the equation, we can make experiment with two different approaches: explicit or implicit (Cvitanic and Zapatero, 2004; Kaas *et al.*, 2008).

Another issue that we have to take into account is the definition of boundary condition. For price  $K$  the boundary condition is:

$$f(S, t) = \max\{S - K, 0\}, \quad \forall S.$$

This problem is not so trivial when we consider boundary conditions in terms of assets prices, since have to solve numerically equation over a limited area, while field of the assets prices is unlimited (Boyle and Tian, 1998; Black and Scholes, 1973).

## 2.2. Pricing for European Option with explicit method

As first attempt to solve the equation (1), we consider the European Option. We found the derivative with respect to  $S$  to the main differential and derivative with respect to time to wobbly differential (Elton and Gruber, 1995; Brandimarte, 2002). Our choice must be compatible with the boundary conditions. As a result we obtain the set of equations:

$$\frac{f_{i,j} - f_{i,j-1}}{\varphi t} + ri\varphi S \frac{f_{i+1,j} - f_{i-1,j}}{2\varphi S} + \frac{1}{2}\rho^2 i^2 \varphi S^2 \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\varphi S^2} = rf_{i,j} \quad (2)$$

which need to be resolved over the following conditions:

$$\begin{aligned} f_{i,N} &= \max[K - i\varphi S, 0], \quad i = 0, 1, \dots, M, \\ f_{0,j} &= Ke^{-r(N-j)\varphi t}, \quad j = 0, 1, \dots, N, \\ f_{M,j} &= 0, \quad j = 0, 1, \dots, N. \end{aligned}$$

Note that since we have a variety of extreme conditions, the equations must be solved to arrears (Brandimarte, 2002; Fabozzi, 1996; Hull, 2003). Let in equation (2)  $j = N$ : the terminal conditions have an unknown value,  $f_{i,N-1}$ , expressed as a function of the three

known values. Going back in time, the same conditions hold for each time layer. For equations (2) we obtain the following explicit scheme:

$$\begin{aligned} f_{i,j-1} &= a_i^* f_{i-1,j} + b_i^* f_{i,j} + c_i^* f_{i+1,j} \\ j &= N-1, N-2, \dots, 1, 0, \\ i &= 1, 2, \dots, M-1 \end{aligned} \quad (3)$$

where:

$$\begin{aligned} a_i^* &= \frac{1}{2} \varphi t (\rho^2 i^2 - ri), \\ b_i^* &= 1 - \varphi t (\rho^2 i^2 - r), \\ c_i^* &= \frac{1}{2} \varphi t (\rho^2 i^2 + ri). \end{aligned}$$

It is very easy to implement this scheme in MATLAB. If the price of the original asset that not lie on the grid must be interpolated between two adjacent points. We use linear interpolation: other good alternative can be complex spline, especially if you are interested in approximations for Option prices (Kemna and Vorst, 1990; Kunitomo and Ikeda, 1992).

### 2.3. Pricing for European Option with implicit method

In the explicit method there is some numerical instability. To overcome this problem, we can use an implicit method (Pliska, 1997; Neftci, 2000; Taleb, 1996). This is achieved by means of a progressive differentiation to make approximations of the partial derivative with respect to time. We obtain the following equation for the grid:

$$\frac{f_{i,j+1} - f_{i,j}}{\varphi t} + ri\varphi S \frac{f_{i+1,j} - f_{i-1,j}}{2\varphi S} + \frac{1}{2} \rho^2 i^2 \varphi S^2 \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\varphi S^2} = r f_{i,j}$$

We can rewrite this equation on the following way (for  $i = 1, 2, \dots, M-1$  and  $j = 0, 1, \dots, N-1$ ):

$$a_i f_{i-1,j} + b_i f_{i,j} + c_i f_{i+1,j} = f_{i,j+1} \quad (4)$$

where, for each  $i$ ,

$$\begin{aligned} a_i &= \frac{1}{2} ri\varphi t - \frac{1}{2} \rho^2 i^2 \varphi t, \\ b_i &= 1 + \rho^2 i^2 \varphi t + r\varphi t, \\ c_i &= -\frac{1}{2} ri\varphi t - \frac{1}{2} \rho^2 i^2 \varphi t. \end{aligned}$$

Here we have three unknown values related with one known value. First, note that for each time layer, we have  $M-1$  equations with  $M-1$  unknowns values: the boundary conditions give two missing values for each time layer, and end conditions give the values for the last time layer (Brandimarte, 2002). As with explicit method we must go back in time by solving sequence of linear systems of equations for  $j = N-1, \dots, 0$ . System for time layer  $j$  is as follows:

$$\begin{bmatrix} b_1 & c_1 & & & & & & & \\ a_2 & b_2 & c_2 & & & & & & \\ & a_3 & b_3 & c_3 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & a_{M-2} & b_{M-2} & c_{M-2} & & & \\ & & & & a_{M-1} & b_{M-1} & & & \end{bmatrix} \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \\ \vdots \\ f_{M-2,j} \\ f_{M-1,j} \end{bmatrix} = \begin{bmatrix} f_{1,j+1} \\ f_{2,j+1} \\ f_{3,j+1} \\ \vdots \\ f_{M-2,j+1} \\ f_{M-1,j+1} \end{bmatrix} - \begin{bmatrix} a_1 f_{2,j} \\ 0 \\ 0 \\ \vdots \\ 0 \\ c_{M-1} f_{M,j} \end{bmatrix}$$

Note that the matrix is a constant for each time layer  $i$ . So you can speed up the calculation by using LU-factorization (Taleb, 1996; Merton, 1973; Rich, 1996).

### 3. RESULTS

We make several computational experiments by MATLAB using explicit and implicit method. For this purpose, we realized the above schemes in individual  $m$ -files in MATLAB. After that we compare our results with the results obtained by *blsprice* tool in MATLAB. The function *blsprice* calculate the pricing of European Option with the right to buy and sell (European put and call Option), using Black-Scholes model. The function is called as follows:

$$[Call, Put] = blsprice(Price, Strike, Rate, Time, Volatility),$$

where:

*Price* – Current price of the underlying asset;

*Strike* – The price of Option;

*Rate* – The annual interest rate, which complicates the risk-free rate of return during Option expressed as positive integer number;

*Time* – Duration of Option in years;

*Volatility* – The annual instability of assets (annualized standard deviation constantly complicate the return on assets), expressed as a positive decimal number.

Using the function *blsprice* with corresponding values for the parameters, we obtain 3.9663 for European Put Option ( $Put = 3.9663$ ):

$$[Call, Put] = blsprice(60, 60, 0.2, 6/12, 0.4).$$

The result that we obtain by the explicit scheme with the same values for corresponding parameters is 3.8993:

$$PricingEurOptExpl(60, 60, 0.2, 6/12, 0.4, 400, 5, 5/1200).$$

We see that the explicit method leads to very good results. We can try to improve these results by using finer grid:

$$[Call, Put] = blsprice(60, 60, 0.2, 6/12, 0.3);$$

$$Put = 2.4963;$$

$$PricingEurOptExpl(60, 60, 0.2, 6/12, 0.3, 400, 5, 5/1200);$$

$$ans = 2.3963.$$

Again, we see that the numerical method gives relatively accurate results. We can still improve the results using the more delicate grid:

$$\begin{aligned} & \text{PricingEurOptExpl}(60, 60, 0.2, 6/12, 0.3, 400, 4.5/1200); \\ & \quad \text{ans} = 2.4344; \\ & \text{PricingEurOptExpl}(60, 60, 0.2, 6/12, 0.3, 400, 1.5/1200); \\ & \quad \text{ans} = -1.5660e+51. \end{aligned}$$

From all examples previously, we conclude that the explicit method leads to numerical instability. One way to avoid this instability is to use implicit methods:

$$\begin{aligned} & [\text{Call}, \text{Put}] = \text{blsprice}(60, 60, 0.2, 6/12, 0.4); \\ & \quad \text{Put} = 3.9663; \\ & \text{PricingEurOptImpl}(60, 60, 0.2, 6/12, 0.4, 400, 5.5/1200); \\ & \quad \text{ans} = 3.8870. \end{aligned}$$

The results of this scheme are approximate and also can be improved using more precise grid, without risk of numerical instability in the implementation.

#### 4. CONCLUSION

In the explicit scheme, the obtained value  $f(S, t)$  is as combination of the values  $f(S + \varphi S, t + \varphi t)$ ,  $f(S, t + \varphi t)$  and  $f(S - \varphi S, t + \varphi t)$ . We can make this interpretation more clearly through the establishment of alternative version on explicit method. We assume that the derivatives of first and second order of  $S$  in point  $(i, j)$  are equal to the derivatives in point  $(i, j + 1)$ :

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{f_{i+1, j+1} - f_{i-1, j+1}}{2\varphi S} \\ \frac{\partial^2 f}{\partial S^2} &= \frac{f_{i+1, j+1} - 2f_{i, j+1} + f_{i-1, j+1}}{2\varphi S} \end{aligned}$$

Alternative way to obtain the same pattern is as replacing the term  $f_{i, j}$  on the right in equation (2) with  $f_{i, j+1}$ . This introduces an error that is bounded and tends to zero; the grid is more accurate (Brandimarte, 2002). After this change, the final differential equation can be shown as:

$$\frac{f_{i, j+1} - f_{i, j}}{\varphi t} + r i \varphi S \frac{f_{i+1, j+1} - f_{i-1, j+1}}{2\varphi S} + \frac{1}{2} \rho^2 i^2 \varphi S^2 \frac{f_{i+1, j+1} - 2f_{i, j+1} + f_{i-1, j+1}}{\varphi S^2} = r f_{i, j}.$$

This equation can be rewritten (for  $i = 1, 2, \dots, M$  and  $j = 0, 1, \dots, N - 1$ ) in the following manner:

$$f_{i, j} = \hat{a}_i f_{i-1, j+1} + \hat{b}_i f_{i, j+1} + \hat{c}_i f_{i+1, j+1},$$

where, for each  $i$ ,

$$\hat{a}_i = \frac{1}{1 + r\varphi t} \left( -\frac{1}{2} r i \varphi t + \frac{1}{2} \rho^2 i^2 \varphi t \right) = \frac{1}{1 + r\varphi t} \pi_a,$$

$$\hat{b}_i = \frac{1}{1+r\varphi t} (1 + \rho^2 i^2 \varphi t) = \frac{1}{1+r\varphi t} \pi_0,$$

$$\hat{c}_i = \frac{1}{1+r\varphi t} \left( \frac{1}{2} r i \varphi t + \frac{1}{2} \rho^2 i^2 \varphi t \right) \frac{1}{1+r\varphi t} \pi_u.$$

Again this scheme is explicit and is subject to numerical instability. However, the coefficients  $\hat{a}_i$ ,  $\hat{b}_i$  and  $\hat{c}_i$  succumb to interpretation. In fact, the coefficients above include terms  $1/(1+r\varphi t)$ , which can be interpreted as reduction factor over time interval of  $\delta t$ . Furthermore, we have:

$$\pi_d + \pi_0 + \pi_u = 1.$$

On this way we suggest interpretation of the coefficients as probability, time discount factor. We need to check the expected value of the increase in asset prices during the time interval  $\varphi t$ :

$$E[\Delta] = -\varphi S \pi_d + 0 \pi_0 + \varphi S \pi_u = r i \varphi S \varphi t = r S \varphi t,$$

which is exactly what we would expect in a risk-neutral world. As regards the deviation of the step, we have:

$$E[\Delta^2] = (-\varphi S)^2 \pi_d + 0 \pi_0 + (\varphi S)^2 \pi_u = \rho^2 i^2 (\varphi S)^2 \varphi t.$$

Therefore, for small  $\varphi t$

$$\text{Var}[\Delta] = E[\Delta^2] - E^2[\Delta] = \rho^2 S^2 \varphi t - r^2 S^2 (\varphi t)^2 \approx \rho^2 S^2 \varphi t,$$

which is also consistent with the geometric Brownian motion in the risk-neutral world. The probabilities  $\pi_d$  and  $\pi_0$  can be negative.

One possibility to avoid the problem described in Hull (2003), is to change the variables. By rewriting Black-Scholes equation in terms of  $Z = \ln S$ , we can derive the conditions for stability. However, the change of variables cannot be good idea for some Option.

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