Using symbolic computation to prove nonexistence of distance-regular graphs

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Abstract

A package for the Sage computer algebra system is developed for checking feasibility of a given intersection array for a distance-regular graph. We use this tool to show that there is no distance-regular graph with intersection array

$$\{(2r+1)(4r+1)(4t-1), 8r(4rt-r+2t), (r+t)(4r+1); \\ 1, (r+t)(4r+1), 4r(2r+1)(4t-1)\} \quad (r, t \geqslant 1),$$

$\{135,128,16;1,16,120\}$, $\{234,165,12;1,30,198\}$ or $\{55,54,50,35,10;1,5,20,45,55\}$. In all cases, the proofs rely on equality in the Krein condition, from which triple intersection numbers are determined. Further combinatorial arguments are then used to derive nonexistence.

Mathematics Subject Classifications: 05E30

1 Introduction

Distance-regular graphs were introduced around 1970 by N. Biggs [1]. As they are intimately linked to many other combinatorial objects, such as finite simple groups, finite geometries, and codes, a natural goal is trying to classify them.

Many distance-regular graphs are known, however constructing new ones has proved to be a difficult task. A number of feasibility conditions for distance-regular graphs have been found, which allows us to compile a list of feasible intersection arrays for small distance-regular graphs (or related structures, such as Q-polynomial association schemes), see Brouwer et al. [2, 3, 4] and Williford [22]. However, feasibility is no guarantee for

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existence, so proofs of nonexistence of distance-regular graphs with feasible intersection arrays are also a contribution to the classification. In certain cases, single intersection arrays have been ruled out [12, 13], while other proofs may show nonexistence for a whole infinite family of feasible intersection arrays [6, 9, 19]. In this paper we give proofs of nonexistence for distance-regular graphs belonging to a two-parameter infinite family, as well as for graphs with intersection arrays

$$\{135, 128, 16;\ 1, 16, 120\},$$
$$\{234, 165, 12;\ 1, 30, 198\},$$
$$\{55, 54, 50, 35, 10;\ 1, 5, 20, 45, 55\}.$$

We develop a package called `sage-drg` [21] for the Sage computer algebra system [18]. Sage is free open-source software written in the Python programming language [17], with many functionalities deriving from other free open-source software, such as Maxima [16], which Sage uses for symbolic computation. The `sage-drg` package is thus also free open-source software available under the MIT license, written in the Python programming language, making use of the Sage library. The package can be used to check for feasibility of a given intersection array against known feasibility conditions, see Van Dam, Koolen and Tanaka for an up-to-date survey [7]. Furthermore, using equality in the Krein condition (see Theorem 1), restrictions on triple intersection numbers can be derived. In this paper, we use them to derive some nonexistence results. The `sage-drg` package also includes Jupyter notebooks demonstrating its use to obtain these results, as well as the notebook `jupyter/Demo.ipynb` giving some general examples of use of the package. A more detailed description of the `sage-drg` package is given in the supplementary files[1].

The results from Sections 3, 4 and 6 appeared in the author's PhD thesis [20], where computation was done using a Mathematica [23] notebook originally developed by M. Urlep. Thus, the `sage-drg` package can be seen as a move from closed-source proprietary software to free open-source software, which allows one to check all code for correctness, thus making the results verifiable.

2 Preliminaries

In this section we review some basic definitions and concepts. See Brouwer, Cohen and Neumaier [3] for further details.

Let Γ be a connected graph with diameter d and n vertices, and let $\partial(u, v)$ denote the distance between the vertices u and v of Γ. The graph Γ is *distance-regular* if there exist constants p^h_{ij} $(0 \leqslant h, i, j \leqslant d)$, called the *intersection numbers*, such that for any pair of vertices u, v at distance h, there are precisely p^h_{ij} vertices at distances i and j from u and v, respectively. In fact, all intersection numbers can be computed given only the intersection numbers $b_i = p^i_{1,i+1}$ and $c_{i+1} = p^{i+1}_{1,i}$ $(0 \leqslant i \leqslant d-1)$ [3, §4.1A]. These intersection numbers are usually gathered in the *intersection array* $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$. We also

[1]Available in the 'Appendix' at `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v25i4p21`

define the *valency* $k = b_0$ and $a_i = k - b_i - c_i$ $(0 \leqslant i \leqslant d)$, where $b_d = c_0 = 0$. A connected noncomplete *strongly regular* graph with parameters (v, k, λ, μ) is a distance-regular graph of diameter 2 with v vertices, valency k and intersection numbers $a_1 = \lambda$, $c_2 = \mu$.

Let A_i $(0 \leqslant i \leqslant d)$ be a binary square matrix indexed with the vertices of a graph Γ of diameter d, with entry corresponding to vertices u and v equal to 1 precisely when $\partial(u, v) = i$. The matrix $A = A_1$ is the *adjacency matrix* of Γ. The graph Γ is called *primitive* if all A_i $(1 \leqslant i \leqslant d)$ are adjacency matrices of connected graphs. A distance-regular graph of valency $k \geqslant 3$ that is not primitive is bipartite or antipodal (or both) [3, Thm. 4.2.1]. The spectrum of Γ is defined to be the spectrum of A (i.e., eigenvalues with multiplicities) and can be computed directly from the intersection array of Γ [3, §4.1B].

Suppose that Γ is distance-regular. Let $\mathcal{M}$ be the *Bose-Mesner* algebra, i.e., the algebra generated by A. The matrices $\{A_i\}_{i=0}^d$ form a basis of $\mathcal{M}$, which also has a second basis $\{E_i\}_{i=0}^d$ consisting of projectors to the eigenspaces of A [3, §2.2]. Note that the indexing in this second basis depends on the ordering of eigenvalues. The descending ordering of eigenvalues is known as the *natural ordering*. We define the *eigenmatrix* P and *dual eigenmatrix* Q as $(d + 1) \times (d + 1)$ matrices such that $A_j = \sum_{i=0}^d P_{ij} E_i$ and $E_j = n^{-1} \sum_{i=0}^d Q_{ij} A_i$. The graph Γ is called *formally self-dual* [3, p. 49] if $P = Q$ holds for some ordering of eigenvalues. The graph Γ is called *Q-polynomial* [3, §2.7] with respect to some ordering of eigenvalues if there exist real numbers $z_0, \dots, z_d$ and polynomials q_j of degree j such that $Q_{ij} = q_j(z_i)$ $(0 \leqslant i, j \leqslant d)$. Finally, we define the *Krein parameters* q_{ij}^h [3, §2.3] as such numbers that $E_i \circ E_j = n^{-1} \sum_{h=0}^d q_{ij}^h E_h$, where $\circ$ represents entrywise multiplication of matrices. A formally self-dual distance-regular graph is also Q-polynomial with respect to the corresponding ordering of eigenvalues and has $p_{ij}^h = q_{ij}^h$ $(0 \leqslant i, j, h \leqslant d)$. In this paper, we will use the natural ordering for indexing, noting when a graph is Q-polynomial or formally self-dual for some other ordering.

For vertices u, v, w of the distance-regular graph Γ and integers i, j, h $(0 \leqslant i, j, h \leqslant d)$ we denote by $\begin{bmatrix} u & v & w \\ i & j & h \end{bmatrix}$ (or simply $[i\ j\ h]$ when it is clear which triple (u, v, w) we have in mind) the number of vertices of Γ that are at distances i, j, h from u, v, w, respectively. We call these numbers *triple intersection numbers*. They have first been studied in the case of strongly regular graphs [5], and later also for distance-regular graphs, see for example [6, 8, 9, 10, 19]. Unlike the intersection numbers, these numbers may depend on the particular choice of vertices u, v, w and not only on their pairwise distances. We may however write down a system of $3d^2$ linear Diophantine equations with d^3 triple intersection numbers as variables, thus relating them to the intersection numbers, cf. [9]:

$$\sum_{\ell=1}^{d} [\ell\ j\ h] = p_{jh}^U - [0\ j\ h], \qquad \sum_{\ell=1}^{d} [i\ \ell\ h] = p_{ih}^V - [i\ 0\ h], \qquad \sum_{\ell=1}^{d} [i\ j\ \ell] = p_{ij}^W - [i\ j\ 0], \qquad (1)$$

where $U = \partial(v, w)$, $V = \partial(u, w)$, $W = \partial(u, v)$, and

$$[0\ j\ h] = \delta_{jW} \delta_{hV}, \qquad [i\ 0\ h] = \delta_{iW} \delta_{hU}, \qquad [i\ j\ 0] = \delta_{iV} \delta_{jU}.$$

Furthermore, we can use the triangle inequality to conclude that certain triple intersection numbers must be zero. Moreover, the following theorem sometimes gives additional equations.

Theorem 1. ([6, Theorem 3], cf. [3, Theorem 2.3.2]) *Let Γ be a distance-regular graph with diameter d, dual eigenmatrix Q and Krein parameters q_{ij}^h $(0 \leqslant i, j, h \leqslant d)$. Then,*

$$q_{ij}^h = 0 \quad \Longleftrightarrow \quad \sum_{r,s,t=0}^{d} Q_{ri} Q_{sj} Q_{th} \begin{bmatrix} u & v & w \\ r & s & t \end{bmatrix} = 0 \quad \textit{for all} \;\; u, v, w \in V\Gamma. \qquad \square$$

Together with integrality and nonnegativity of triple intersection numbers, we can use all of the above to either derive that the system of equations has no solution, or arrive at a small number of solutions, which gives us new information on the structure of the graph and may lead to proving its nonexistence.

3 A two-parameter family of primitive graphs of diameter 3

In [9], graphs meeting necessary conditions for the existence of extremal codes were studied. One of the families of primitive graphs of diameter 3 for which these conditions were met was

$$\{a(p+1), (a+1)p, c; 1, c, ap\}, \tag{2}$$

where $a = a_3$, $c = c_2$ and $p = p_{33}^3$. Graphs belonging to this family are Q-polynomial with respect to the natural ordering of eigenvalues precisely when the Krein parameter q_{11}^3 is zero, which is equivalent to

$$c = \frac{1}{4}\left((p+1)^2 + \frac{2a(p+1)}{p+2}\right). \tag{3}$$

Hence, $p+2$ must divide $2a$ for c to be integral. If $p = 2r-1$, then $a = t(2r+1)$ and $c = r(r+t)$ for some positive integers r, t, which gives us the two-parameter family

$$\{2rt(2r+1), (2r-1)(2rt+t+1), r(r+t); 1, r(r+t), t(4r^2-1)\}.$$

In [9], nonexistence was shown for a feasible subfamily with $r = t \geqslant 2$. If, on the other hand, p is even, integrality of the multiplicity of the second largest eigenvalue implies that we must have $p = 4r$, $a = (2r+1)(4t-1)$ and $c = (r+t)(4r+1)$ for some positive integers r, t, giving the family

$$\begin{aligned}\{(2r+1)(4r+1)(4t-1), 8r(4rt-r+2t), (r+t)(4r+1);& \\ 1, (r+t)(4r+1), 4r(2r+1)(4t-1)\}.&\end{aligned} \tag{4}$$

We find two one-parameter infinite subfamilies of feasible intersection arrays by setting $t = 4r^2$ or $t = 4r^2 + 2r$:

$$\begin{aligned}\{(2r+1)(4r+1)(16r^2-1), 8r^2(16r^2+8r-1), r(4r+1)^2;& \\ 1, r(4r+1)^2, 4r(2r+1)(16r^2-1)\},& \\ \{(2r+1)(4r+1)(16r^2+8r-1), 8r^2(4r+1)(4r+3), r(4r+1)(4r+3);& \\ 1, r(4r+1)(4r+3), 4r(2r+1)(16r^2+8r-1)\}.&\end{aligned}$$

There are also other feasible cases – for instance, when $r = 2$, we have, besides the cases from the two subfamilies above, feasible examples when $t \in \{4, 7, 196\}$. The case with $r = 1$ and $t = 4$ belonging to the first subfamily above is also listed in the list of feasible parameter sets for 3-class Q-polynomial association schemes by J. S. Williford [22].

We now prove that a graph Δ with intersection array (4) does not exist. The proof parallels that of [9, Lems. 1, 3] – in fact, a significant part of the proof may be extended to the entire family (3), as it has been done in [20]. The computation needed to obtain the results in this section is illustrated in the `jupyter/DRG-d3-2param.ipynb` notebook included in the `sage-drg` package [21].

Lemma 2. *Let Δ be a distance-regular graph with intersection array (4), and u', v, w be vertices of Δ with $\partial(u', v) = 1$, $\partial(u', w) = 2$ and $\partial(v, w) = 3$. Then $\begin{bmatrix} u' & v & w \\ 3 & 3 & 3 \end{bmatrix} = 1$.*

Proof. Let u be a vertex of Δ at distance 3 from both v and w (such a vertex exists since $p^3_{33} = 4r > 0$). We consider the triple intersection numbers $[i\ j\ h]$ that correspond to (u, v, w). As $q^3_{11} = q^1_{13} = q^1_{31} = 0$, Theorem 1 gives three additional equations to the system (1), allowing us to express its solution in terms of a single parameter $\alpha = [3\ 3\ 3]$. Let us express the counts of vertices at distance 1 or 2 from one of u, v, w and at distance 3 from the other two vertices:

$$[3\ 3\ 1] = [3\ 1\ 3] = [1\ 3\ 3] = \frac{(\alpha - 4r + 1)(4r + 1)}{4r - 1},$$
$$[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] = \frac{8r(4r - 1 - \alpha)}{4r - 1}.$$

For the values above to be nonnegative, we must have $\alpha = 4r - 1$, which means that they are all zero. As the choice of u, v, w was arbitrary, this implies that any pair of vertices at distance 3 induces a set of $4r + 2$ vertices pairwise at distance 3 – in the terminology of [9], this is a maximal 1-code in Δ. Since we have $a_3 p^3_{33} = 4r(2r + 1)(4t - 1) = c_3$, it follows by [9, Prop. 2] that $\begin{bmatrix} u' & v & w \\ 3 & 3 & 3 \end{bmatrix} = 1$ holds. □

Theorem 3. *A distance-regular graph Δ with intersection array (4) does not exist.*

Proof. Let u', v, w be vertices of Δ with $\partial(u', v) = 1$, $\partial(u', w) = 2$ and $\partial(v, w) = 3$ (such vertices exist, since we have $p^2_{13} = b_2 = (r + t)(4r + 1) > 0$). We consider the triple intersection numbers $[i\ j\ h]$ that correspond to (u', v, w). By Lemma 2, we have $[3\ 3\ 3] = 1$. Using $q^3_{11} = 0$, Theorem 1 gives an additional equation which allows us to obtain a unique solution to the system (1). However, we obtain $[1\ 1\ 3] = 2t - 1/2$, which is nonintegral for all integers t. Therefore, the graph Δ does not exist. □

4 A primitive graph with diameter 3 and 1360 vertices

Let Λ be a distance-regular graph with intersection array

$$\{135, 128, 16; 1, 16, 120\}. \qquad (5)$$

This intersection array can be obtained from (2) by setting $a = 15$, $c = 16$ and $p = 8$. The graph Λ has diameter 3 and 1360 vertices. It is not Q-polynomial, however its Krein parameter q^3_{33} is zero. We show that such a graph does not exist. The computation needed to prove Theorem 4 is illustrated in the `jupyter/DRG-135-128-16-1-16-120.ipynb` notebook included in the `sage-drg` package [21].

Theorem 4. *A distance-regular graph Λ with intersection array* (5) *does not exist.*

Proof. Let u, v, w be three pairwise adjacent vertices of Λ (such vertices exist, since we have $p^1_{11} = 6 > 0$). We consider triple intersection numbers $[i\ j\ h]$ that correspond to (u, v, w). As $q^3_{33} = 0$, Theorem 1 gives an additional equation to the system (1), allowing us to express its solution in terms of a single parameter $\alpha = [1\ 1\ 1]$. In particular, we obtain

$$[3\ 3\ 3] = \frac{71 - 27\alpha}{8}.$$

Clearly, α must be a nonnegative integer. For $[3\ 3\ 3]$ to be nonnegative, we must have $\alpha \in \{0, 1, 2\}$. However, $[3\ 3\ 3]$ is still nonintegral in these cases, showing that the graph Λ does not exist. □

5 A primitive graph with diameter 3 and 1600 vertices

Let Ξ be a distance-regular graph with intersection array

$$\{234, 165, 12; 1, 30, 198\}. \tag{6}$$

The graph Ξ has diameter 3 and 1600 vertices. The intersection array (6) has been found as an example of a feasible parameter set for a distance-regular graph which is formally self-dual for an ordering of eigenvalues distinct from the natural ordering – in fact, Ξ is Q-polynomial for the ordering $0, 2, 3, 1$, so its Krein parameters q^1_{22}, q^2_{12} and q^2_{21} are zero. The intersection array (6) is also listed in the list of feasible parameter sets for 3-class Q-polynomial association schemes by J. S. Williford [22]. We show that such a graph does not exist. The computation needed to prove Theorem 5 is illustrated in the `jupyter/DRG-234-165-12-1-30-198.ipynb` notebook included in the `sage-drg` package [21].

Theorem 5. *A distance-regular graph Ξ with intersection array* (6) *does not exist.*

Proof. Let u, v, w be three vertices of Ξ that are pairwise at distance 3 (such vertices exist, since we have $p^3_{33} = 8 > 0$). We consider triple intersection numbers $[i\ j\ h]$ that correspond to (u, v, w). As $q^1_{22} = q^2_{12} = q^2_{21} = 0$, Theorem 1 gives three additional equations to the system (1), allowing us to express its solution in terms of a single parameter $\alpha = [3\ 3\ 3]$. In particular, we obtain

$$[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] = -17 - 4\alpha.$$

Clearly, α must be nonnegative, but then we have $[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] < 0$, a contradiction. We conclude that the graph Ξ does not exist. □

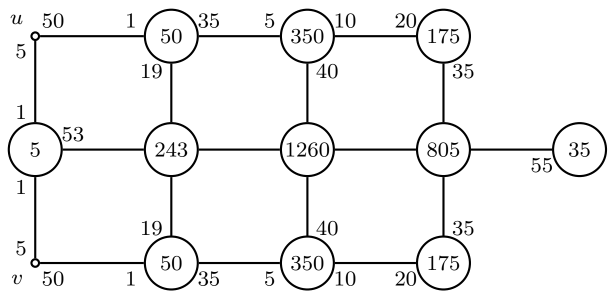

Figure 1: The partition of vertices of Σ by distance from a pair of vertices u, v at distance 2. The part that is at distance i from u and distance j from v has size p^2_{ij}. As the graph is bipartite, the intersection number p^2_{ij} is nonzero only when $i + j$ is even. Moreover, there are no edges within each part. It is natural to consider [1 1 1] for w at distance 2 from both u and v, see Lemma 6.

6 A bipartite graph with diameter 5

Let Σ be a distance-regular graph with intersection array

$$\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}. \tag{7}$$

This intersection array appears in the list of feasible intersection arrays for bipartite non-antipodal distance-regular graphs of diameter 5 by Brouwer et. al. [3, p. 418] as an open case. The existence of such a graph would give a counterexample to a conjecture by MacLean and Terwilliger [15], cf. Lang [14]. The computation needed to obtain the results in this section is illustrated in the `jupyter/DRG-55-54-50-35-10-bipartite.ipynb` notebook included in the `sage-drg` package [21].

The graph Σ has diameter 5 and 3500 vertices. The partition of Σ corresponding to two vertices at distance 2 is shown in Figure 1. The graph is Q-polynomial for the natural ordering of eigenvalues, see for example [3, p. 418]. Moreover, as the graph is bipartite, it is also Q-antipodal [3, Thm. 8.2.1]. Many Krein parameters are zero, in particular q^3_{11} and q^4_{11} due to the triangle inequality. We use this fact in the proof of the following statement.

Lemma 6. *Let Σ be a distance-regular graph with intersection array* (7)*, and u, v, w be vertices of Σ that are pairwise at distance* 2*. Then* $\begin{bmatrix} u & v & w \\ 1 & 1 & 1 \end{bmatrix} \leqslant 1$.

Proof. We consider the triple intersection numbers $[i\ j\ h]$ that correspond to (u, v, w). Since the graph Σ is bipartite, we have $[i\ j\ h] = 0$ whenever any of the sums $i + j$, $j + h$, $h + i$ is odd. As $q^3_{11} = q^4_{11} = 0$, Theorem 1 gives us two additional equations to the system (1), thus allowing us to express the solution of the system in terms of a single parameter $\alpha = [1\ 1\ 1]$. In particular, we obtain

$$[5\ 5\ 5] = 20 - 12\alpha.$$

The integrality and nonnegativity of [5 5 5] now gives $\alpha \leqslant \lfloor 5/3 \rfloor = 1$. ☐

Note. It can also be shown with a method similar to the one used in Lemma 6 that the graph $[\Sigma_5(u)]_2$ for a vertex $u \in V\Sigma$ (i.e., the graph of vertices at distance 5 from a vertex u, with adjacency corresponding to distance 2 in Σ) is strongly regular with parameters $(v, k, \lambda, \mu) = (210, 99, 48, 45)$. A strongly regular graph with such parameters has been constructed by M. Klin [11].

Theorem 7. *A distance-regular graph* Σ *with intersection array* (7) *does not exist.*

Proof. Let u and v be vertices of Σ at distance 2, see Figure 1, and let $\{i\ j\}$ denote the set of vertices at distances i and j from u and v, respectively. There are $p^2_{11}(k-2) = 5 \cdot 53 = 265$ edges between the sets $\{1\ 1\}$ and $\{2\ 2\}$. However, the cardinality of the latter set is $p^2_{22} = 243 < 265$, so there is a vertex $w \in \{2\ 2\}$ that has at least two neighbours in $\{1\ 1\}$, i.e., $\begin{bmatrix} u & v & w \\ 1 & 1 & 1 \end{bmatrix} \geqslant 2$, which is in contradiction with Lemma 6. Hence, the graph Σ does not exist. $\square$

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