Using the Matrix Pencil Method to Estimate the Parameters of a Sum of Complex Exponentials

Tapan K. Sarkar and Odilon Pereira
Department of Electrical and Computer Engineering
Syracuse University
Syracuse, New York 13244-1240

1. Abstract

The approximation of a function by a sum of complex exponentials is a problem that is at least two centuries old. Fundamentally, all techniques discussed in this article proceed from using the same sequence of data samples and vary only, but importantly, in how those samples are used in achieving the parameter estimation. All of these techniques, in other words, seek the same quantitative parameters to represent the sampled data, but use different routes to get there. The techniques for estimating the parameters are either linear or nonlinear. The linear techniques are emphasized in this presentation. In particular, the Matrix Pencil Method is described, which is more robust to noise in the sampled data. The Matrix Pencil approach has a lower variance of the estimates of the parameters of interest than a polynomial-type method (Prony's method belongs to this category), and is also computationally more efficient. A bandpass version of the Matrix Pencil can be implemented in hardware, utilizing an AT&T DSP32C chip operating in real time. A copy of the computer program implementing the Matrix Pencil technique is given in Appendix.

2. Introduction

The response of an object to a burst of electromagnetic energy is a problem of interest in many electronic systems. The development of the Singularity-Expansion Method (SEM) by Carl Baum [1] provides a convenient methodology for describing the late-time response of antennas (and other scatterers) in terms of their natural frequencies [2-4]. In essence, this is equivalent to modeling the late-time response of an object irradiated by an electromagnetic pulse as a sum of complex exponentials. Since the complex oscillation frequencies of the electromagnetic response are intrinsic to the geometry of the object, the poles extracted from the late-time portion of the scattered fields' response can be utilized to provide an aspect-independent means for radar-target discrimination [5-7].

However, in recent times, the methodology of approximating a function by a sum of complex exponentials has found applications in other areas of electromagnetics: for example, in the efficient evaluation of the Sommerfeld integrals, where the Sommerfeld kernels are first approximated by sums of complex exponentials [8, 9], and in antenna-pattern synthesis [9]. Other areas of application are in the extraction of the s-parameters of microwave-integrated circuits [10, 11]; in the analysis of propagation of signals over perforated ground planes [12]; in the computation of input impedance of electrically wide slot antennas [13]; in the analysis of complex modes in lossless closed conducting structures [14]; in multiple transient signal processing [15]; in inverse synthetic-aperture radar [16]; in high-resolution imaging of moving targets [17]; and in radio-direction finding [18, 52].

In our discussions of the various approaches, only references which are directly relevant are noted. No attempt has been made to cite the earliest sources. In many cases, additional references may be found in the papers mentioned.

In general, the signal model of the observed late time of electromagnetic-energy-scattered response from an object can be formulated as

$$y(t) = x(t) + n(t) \approx \sum_{i=1}^{M} R_i \exp(s_i t) + n(t); \quad 0 \le t \le T, \qquad (1)$$

where $y(t)$ = observed time response
$n(t)$ = noise in the system
$x(t)$ = signal
R_i = residues or complex amplitudes
$s_i = -\alpha_i + j\omega_i$
α_i = damping factors
ω_i = angular frequencies ($\omega_i = 2\pi f_i$)

After sampling, the time variable, t, is replaced by kT_s, where T_s is the sampling period. The sequence can be rewritten as

$$y(kT_s) = x(kT_s) + n(kT_s) \approx \sum_{i=1}^{M} R_i z_i^k + n(kT_s) \quad \text{for } k = 0,\ldots,N-1 \qquad (2)$$

and

$$z_i = e^{s_i T_s} = e^{(-\alpha_i + j\omega_i)T_s} \quad \text{for } i = 1,2,\ldots,M. \qquad (3)$$

The objective is to find the best estimates of M, R_is, and z_is from the noise-contaminated data, $y(kT_s)$. In general, simultaneous estimation of M, R_is, and z_is is a nonlinear problem. Three of the popular methods of solving the nonlinear problem are described in [19-21]. However, solving the linear problem is interesting and, in many cases, is equivalent to solving the nonlinear problem [22, 23]. In addition, the solution to the linear problem can be used as an initial guess to the non-linear-optimization problems of [19-21].

Two of the popular linear methods are the "polynomial" method and the "matrix pencil" method. The basic difference between the two is that the "polynomial" method is a two-step process in finding the poles, z_i. We will show later that for a polynomial method, one needs to solve a matrix equation for the coefficients of a polynomial, whose roots provide z_i. On the other hand, the "matrix pencil" approach is a one-step process. The poles z_i are found as the solution of a generalized eigenvalue problem. Hence, there is no practical limitation on the number of poles, M, that can be obtained by this method. In contrast, for a polynomial method it is difficult to find roots of a polynomial for, say, M greater than 50. The Matrix Pencil approach is not only more computationally effi-

 1045-9243/93/$03.00©1995 IEEE

cient, but it also has better statistical properties for the estimates of z_i than the "polynomial" method [24-26].

From an historical perspective, the polynomial type of method is much older: at least two centuries old. The method of approximating a function by a sum of complex exponentials in Equation (2) was first developed by Prony in 1795 [27], and further developed in [28, 29]. This method is quite efficient and accurate for extracting poles and residues from given equally spaced transient data. But Prony's method is notorious for its extreme sensitivity to noise [5, 6]. To combat the noise introduced in the observed data, Prony's method was modified, and two schemes for systematically determining the number of poles by using Householder orthogonalization and the dominant eigenvalue/eigen-vector method are given in [30]. This motivated Tufts and Kumaresan to propose the application of the principal-eigenvector method based on singular-value decomposition of the data matrix before applying Prony's method [31, 32].

The Matrix-Pencil technique [24] is relatively new, even though its roots go back to the pencil-of-functions approach, which has been in use for some time [33-41]. The basic difference between the pencil-of-functions method [37] and the Matrix-Pencil (MP) approach (often termed GPOF, or generalized pencil-of-function) [24] is that even though both of them start with the same philosophy, the Matrix-Pencil method is computationally more efficient than the pencil-of-functions method. The original pencil-of-functions method was a variation of the polynomial method, and hence had the same computational bottlenecks as associated with Prony's method. The Matrix-Pencil approach is discussed in the next section. It is shown to be a one-step process, as opposed to the pencil-of-functions method, which provides the solution in two distinct steps, just like Prony's method.

3. Matrix-Pencil (MP) Method

The term "pencil" originated with Gantmacher [42], in 1960. Similar to Gantmacher's definition for matrix pencil, another useful mathematical entity arises when combining two functions defined on a common interval, with a scalar parameter, λ:

$$f(t,\lambda) = g(t) + \lambda h(t). \tag{4}$$

$f(t,\lambda)$ is called a pencil of functions $g(t)$ and $h(t)$, parameterized by λ. To avoid obvious triviality, $g(t)$ is not permitted to be a scalar multiple of $h(t)$. The pencil-of-functions contains very important features about extracting information about z_i, given $y(t)$, when $g(t)$, $h(t)$, and λ are approximately selected. For an historical perspective of the topic, please see [43].

In order to motivate the MP method, we first deal with the noiseless case, and then noise is considered in the latter part of the section.

For noiseless data, we can define two $(N-L)\times L$ matrices, Y_1 and Y_2, defined by

$$[Y_2] = \begin{bmatrix} x(1) & x(2) & \cdots & x(L) \\ x(2) & x(3) & \cdots & x(L+1) \\ \vdots & \vdots & & \vdots \\ x(N-L) & x(N-L+1) & \cdots & x(N-1) \end{bmatrix}_{(N-L)\times L}, \tag{5}$$

$$[Y_1] = \begin{bmatrix} x(0) & x(1) & \cdots & x(L-1) \\ x(1) & x(2) & \cdots & x(L) \\ \vdots & \vdots & & \vdots \\ x(N-L-1) & x(N-L) & \cdots & x(N-2) \end{bmatrix}_{(N-L)\times L}, \tag{6}$$

where L is referred to as the pencil parameter [44-46]. The pencil parameter, L, is very useful in eliminating some effects of noise in the data. The significance of this parameter comes into the picture when we deal with noisy data.

One can write

$$[Y_2] = [Z_1][R][Z_0][Z_2], \tag{7}$$

$$[Y_1] = [Z_1][R][Z_2], \tag{8}$$

where

$$[Z_1] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & & \vdots \\ z_1^{(N-L-1)} & z_2^{(N-L-1)} & \cdots & z_M^{(N-L-1)} \end{bmatrix}_{(N-L)\times M}, \tag{9}$$

$$[Z_2] = \begin{bmatrix} 1 & z_1 & \cdots & z_1^{L-1} \\ 1 & z_2 & \cdots & z_2^{L-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & z_M & \cdots & z_M^{L-1} \end{bmatrix}_{M\times L}, \tag{10}$$

$$[Z_0] = \mathrm{diag}[z_1, z_2, \ldots, z_M], \tag{11}$$

$$[R] = \mathrm{diag}[R_1, R_2, \ldots, R_M], \tag{12}$$

where diag [•] represents a $M \times M$ diagonal matrix.

Now consider the matrix pencil

$$[Y_2] - \lambda[Y_1] = [Z_1][R]\{[Z_0] - \lambda[I]\}[Z_2], \tag{13}$$

where $[I]$ is the $M \times M$ identity matrix. One can demonstrate that, in general, the rank of $\{[Y_2] - \lambda[Y_1]\}$ will be M, provided that $M \le L \le N - M$ [24-26, 45, 46]. However, if $\lambda = z_i$, $i = 1,2,\ldots,M$, the ith row of $\{[Z_0] - \lambda[I]\}$ is zero, and the rank of this matrix is $M-1$. Hence, the parameters z_i may be found as the generalized eigenvalues of the matrix pair $\{[Y_2],[Y_1]\}$. Equivalently, the problem of solving for z_i can be cast as an ordinary eigenvalue problem,

$$\{[Y_1]^+[Y_2] - \lambda[I]\}, \tag{14}$$

where $[Y_1]^+$ is the Moore-Penrose pseudoinverse of $[Y_1]$. This, in turn, is defined as

$$[Y_1]^+ = \{[Y_1]^H[Y_1]\}^{-1}[Y_1]^H, \tag{15}$$

where the superscript "H" denotes the conjugate transpose.

In the presence of noise, some prefiltering needs to be done. To combat noise, the total-least-squares Matrix Pencil has been found to be superior [46-48]. In this implementation, one forms the data matrix $[Y]$ from the noise-contaminated data $y(t)$ by combining $[Y_1]$ and $[Y_2]$ as

$$[Y]=\begin{bmatrix} y(0) & y(1) & \cdots & y(L) \\ y(1) & y(2) & \cdots & y(L+1) \\ \vdots & \vdots & & \vdots \\ y(N-L-1) & y(N-L) & \cdots & y(N-1) \end{bmatrix}_{(N-L)\times(L+1)} \tag{16}$$

Note that $[Y_1]$ is obtained from $[Y]$ by deleting the last column, and $[Y_2]$ is obtained from $[Y]$ by deleting the first column. So, in Equations (5)-(6), the $x(k)$s are replaced by $y(k)$ to obtain $[Y_1]$ and $[Y_2]$. For efficient noise filtering, the parameter L is chosen between $N/3$ to $N/2$ [45-48]. For these values of L, the variance in the parameters z_i, due to noise, has been found to be minimum [47, 48]. Please note that in Equation (16), all the N data samples are utilized, even though L may be considerably less than N.

Next, a singular-value decomposition (SVD) [49] of the matrix $[Y]$ is carried out as

$$[Y]=[U][\Sigma][V]^H. \tag{17}$$

Here, $[U]$ and $[V]$ are unitary matrices, composed of the eigenvectors of $[Y][Y]^H$ and $[Y]^H[Y]$, respectively, and $[\Sigma]$ is a diagonal matrix containing the singular values of $[Y]$, i.e.

$$[U]^H[Y][V]=[\Sigma]. \tag{18}$$

The choice of the parameter M is done at this stage. One looks at the ratio of the various singular values to the largest one. Typically, the singular values beyond M are set equal to zero. The way M is chosen is as follows. Consider the singular value σ_c such that

$$\frac{\sigma_c}{\sigma_{max}} \approx 10^{-p}, \tag{19}$$

where p is the number of significant decimal digits in the data. For example, if the data is accurate up to 3 significant digits, then the singular values for which the ratio in Equation (19) is below 10^{-3} are essentially noise singular values, and they should not be used in the reconstruction of the data. On the other hand, if all the singular values are such that σ_c/σ_{max} does not decrease, then to model the given data by a sum of complex exponentials is not the "correct thing to do." The magnitude of the singular values thus provides a sanity check!

We next consider the "filtered" matrix, $[V']$, constructed so that it contains only M dominant right-singular vectors of $[V]$:

$$[V']=[v_1, v_2, \ldots, v_M]. \tag{20}$$

The right-singular vectors from $M+1$ to L, corresponding to the small singular values, are discarded. Therefore,

$$[Y_1]=[U][\Sigma'][V_1']^H, \tag{21}$$

$$[Y_2]=[U][\Sigma'][V_2']^H, \tag{22}$$

where $[V_1']$ is obtained from $[V']$ with the last row of $[V']$ deleted; $[V_2']$ is obtained by removing the first row of $[V']$; and $[\Sigma']$ is obtained from the M columns of $[\Sigma]$ corresponding to the M dominant singular values.

It can be shown [45-48] that, for the noiseless case, the eigenvalues of the following matrix

$$\{[Y_2]-\lambda[Y_1]\}_{L\times M} \Rightarrow \{[Y_1]^+[Y_2]-\lambda[I]\}_{M\times M}, \tag{23}$$

are equivalent to the eigenvalues of the following matrix

$$\{[V_2']^H-\lambda[V_1']^H\} \Rightarrow \{[V_1']^H\}^+\{[V_2']^H\}^+-\lambda[I]. \tag{24}$$

This methodology of solving for z_is provides minimum variance in the estimate of z_is in the presence of noise [24-26, 46-48]. Typically, up to 20-25 dB of signal-to-noise ratio (SNR) can be handled adequately by this technique [24]. Also, the poles z_i come out much more simply than in a Prony-type method, as outlined in the next section.

Once M and the z_is are known, the residues, R_i, are solved for from the following least-squares problem:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_M^{N-1} \end{bmatrix} \begin{bmatrix} R_1 \\ R_1 \\ \vdots \\ R_M \end{bmatrix}. \tag{25}$$

A real-time implementation of this method is described in Section 6. A listing of a computer program, implementing MP, is included in the Appendix.

4. Polynomial (or a Prony-Type) method

Prony's method was one of the earliest methods used in approximating a function by a sum of complex exponentials. Although the Prony method, the Pisarenko method, and the pencil-of-function method (note the difference between the Matrix Pencil approach and the pencil-of-function method) [22, 23] can be developed from polynomial approaches, we discuss only Prony's method.

It is well known that, given M complex numbers z_i, $i=1,2,\ldots,M$, there exist unique complex numbers a_i, $i=1,2,\ldots,M$, such that

$$1+\sum_{k=1}^{M} a_k z_i^{-k}=0, \quad \text{for } i=1,2,\ldots,M. \tag{26}$$

Thus, finding the signal poles z_i, $i=1,2,\ldots,M$, is equivalent to finding the coefficients a_i, $i=1,2,\ldots,M$, of the Mth-degree polynomial $\sum_{k=0}^{M} a_k z^{-k}$ (with $a_0=1$), which has roots at z_i. This is the essence of the original Prony method. This, however, can be generalized as follows.

Finding the signal-poles z_is is equivalent to finding the coef-

ficients a_i, $i = 1,2,\ldots,L$, of an Lth-degree polynomial $\sum_{k=0}^{L} a_k z^{-k}$ (with $a_0 = 1$ and $L \geq M$), such that of all the L roots of the polynomial, there are M signal roots which are one-to-one functions of z_is, and which are also separable from the other $(L-M)$ (extraneous) roots [49], due to "over modeling," as L is greater than M.

Let

$$p(\lambda) = \sum_{k=0}^{L} a_k \lambda^{-k}, \tag{27}$$

so that $p(z_i) = 0$ for $i = 1,2,\ldots,M$. Then, it can be shown that for $L \leq m \leq N-1$,

$$\sum_{k=0}^{L} y_{m-k} a_k = 0. \tag{28}$$

Therefore, in matrix form, $[Y][\underline{a}] = 0$, where

$$[Y] = [Y_1 \vdots y] = \begin{bmatrix} y(0) & \cdots & y(L-1) & \vdots & y(L) \\ \vdots & & \vdots & \vdots & \\ y(N-L-1) & \cdots & y(N-2) & \vdots & y(N-1) \end{bmatrix}, \tag{29}$$

where $y = [y(L),\ldots,y(N-1)]^{\mathrm{T}}$, and $\underline{a} = [a_L,\ldots,a_0]^{\mathrm{T}}$. Note, also, that

$$[Y] = [Z_1][R][Z_2 \vdots z], \tag{30}$$

with

$$[z] = \left[z_1^L,\ldots,z_M^L\right]^{\mathrm{T}}. \tag{31}$$

Since the roots of polynomial are independent of the uniform scaling of the coefficients a_i, we have let a_0 be one, without any loss of information. Therefore,

$$[Y_1][a] = -[y], \tag{32}$$

where $a = [a_L,\ldots,a_1]^{\mathrm{T}}$. This equation is also referred to as the "forward linear prediction" equation in the digital-signal-processing literature.

Next, we need to find a way to obtain a solution such that the associated polynomial has all the desired signal roots that are also separable from the other, extraneous roots, due to noise (as $L > M$). This special solution is called the minimum-norm solution [31, 32], which is given by

$$[a] = -[Y_1]^{+}[y]. \tag{33}$$

For noisy data, the pseudo inverse $[Y_1]^{+}$ should be replaced by the "truncated rank-M pseudo inverse," which is formed by the first M largest singular values and the corresponding singular vectors. The way to select M has been outlined in the previous section.

5. Relationship between the polynomial method and the matrix pencil method

Although the two methods originated from different approaches, there is a link between the two, as is shown next.

It can be shown [50] that the roots of the polynomial $\sum_{k=0}^{L} a_0 z^{-k}$ (with $a_0 = 1$) are the eigenvalues of the matrix

$$[C_1] = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_L \\ 1 & 0 & \cdots & 0 & -a_{L-1} \\ 0 & 1 & \cdots & 0 & -a_{L-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}_{L\times L} = \left[U_2, U_3,\ldots,U_L, Y_1^{+} y\right], \tag{34}$$

where U_i is the $(L \times 1)$ vector with the ith element equal to 1 and all other elements zero. $[Y_1]^{+}[Y_2]$ can be written as

$$[C_2] = [Y_1]^{+}[Y_2] = \left[Y_1^{+}\underline{y}_1, Y_1^{+}\underline{y}_2,\ldots,Y_1^{+}\underline{y}_L\right], \tag{35}$$

with $\underline{y}_k = [y_k,\ldots,y_{k+N-L-1}]^{\mathrm{T}}$, where $k = 1,\ldots,L$.

As we can see, the ith column of $[C_2]$ is a solution of the following equation:

$$[Y_1][b] = [\underline{y}_i]. \tag{36}$$

But in $[C_1]$, only the last column vector is the minimum-norm solution with $i = L$, while in $[C_2]$, all column vectors are minimum-norm solutions of Equation (36). $[C_1]$ and $[C_2]$ are identical if $L = 1$. Note that $[C_1]$ has M signal eigenvalues at z_i, and $L-M$ extraneous eigenvalues that are nonzero and located inside the unit circle, while $[C_2]$ has M signal eigenvalues at z_i, and $L-M$ extraneous eigenvalues that are zero. Therefore, we see that the polynomial method and the pencil method are different if $L > M$, and identical if $L = M$. For the overdetermined case, $L > M$, the results are different due to the use of different numerical recipes, even though the input data are identical.

The results of the two methods are different under noise. It can be shown that under noise, the statistical variance of the poles z_i for the MP method is always less than that of the polynomial method [45-48], i.e.,

$$\text{varience}\{z_i\}_{polynomial} \geq \text{varience}\{z_i\}_{pencil},$$

with equality when $L = M = 1$. From $M \geq 2$, the two methods yield different variances. The matrix pencil method does deteriorate when the signal-to-noise ratio (SNR) decreases below about 20-26 dB, unless one utilizes the bandpass version [50, 51] of the matrix pencil method, abbreviated as (BPMP).

6. Real-time implementation of the band-pass matrix pencil (BPMP) method

The band-pass matrix pencil (BPMP) method is useful when the signal-spectrum energy is concentrated about a certain region in the frequency domain. By using prior information about the

approximate location and the bandwidth of the signal, the SNR can be enhanced by a prefiltering process. The MP method is then applied to the filtered data, to estimate frequencies and damping factors of sinusoidal signals. However, due to the special requirements of the filtered data by the MP method, the prefiltering process is not trivial [50, 51].

Even though the MP method can filter out part of the noise, by using a SVD of the data matrix and discarding the non-principal singular vectors, some effects of noise still exist in the principal singular vectors. To further combat noise, prefiltering can be used prior to the SVD filtering. The basic idea underlying the BPMP method is first using a digital-filtering technique to enhance the SNR, and then finding the estimate of the poles through the MP algorithm. Thus, intuitively, it can be expected that the BPMP method will have better performance than the MP method, when the effect of noise is considerable. The use of IIR (infinite-impulse response) and FIR (finite-impulse response) filtering can be used for bandpass filters [50, 51]. When IIR filters are used, the data is filtered first using a recursive relation [50], whereas for FIR filters Equation (13) is premultiplied by a circulant matrix $[\mathrm{H}]$, containing the desired filter characteristics. IIR filters are much faster computationally than FIR filters, due to the utilization of the recursive equation. However, they have some special requirements [50, 51]. When the truncation errors due to the finiteness of the filters are made small, FIR filters are preferred.

It has been shown [50, 51] that for the BPMP, the variance of the estimates can come close to the Cramer-Rao bound [51], when the signal-to-noise ratio is greater than 12 dB. Hence, the threshold is extended from 26 dB (for MP) to 12 dB for BPMP [51].

The BPMP method has been implemented on an IBM-AT with a plug-in PC-32C board. This plug-in board consists of an AT&T WE-DSP32C-80 digital-signal processor, 250k bytes of RAM on board, and a 16 M bit/sec buffered bi-directional serial port. The execution program is downloaded via the utility software *D3EMU* [45, 46].

Consider the real-time implementation of the following problem:

$$y(k) = \sum_{i=1}^{2} b_i e^{-\alpha_i k\Delta t} \sin(\omega_i k\Delta t) + n(k). \qquad (37)$$

The number of samples chosen is $k = 0,\ldots,29$. In this case, $M = 4$, $\omega_1 = 0.2\pi$, $\omega_1 = 0.35\pi$, $\alpha_1 = 0.02\pi$, $\alpha_2 = 0.035\pi$, $b_1 = b_2 = 1$. Noise is added to the data so that the SNR = 3.96 dB. The first step is to form the data matrix, $[\mathrm{Y}]$, with the number of columns equal to 11 (i.e., $L = N/3 = 10$). The circulant matrix $[\mathrm{H}]$ [50, 51] is then used to filter the data, and the results are given in Table I. Δ is the relative-error estimate for the variable z_i, and is defined as

$$\Delta = \frac{|x(\mathrm{SNR} = \infty) - x(\mathrm{SNR} = 3.96\ \mathrm{dB})|}{x(\mathrm{SNR} = \infty)} \times 100\% . \qquad (38)$$

Here, $(\mathrm{SNR} = \infty)$ is considered the no-noise case.

The first row of Table I provides the results for the no-noise case, utilizing the MP method, in 50 milliseconds. The second row corresponds to the results when the SNR = 3.96 dB and the method used is BPMP. The third row presents the results due to MP. It is seen that BPMP takes 10 milliseconds more time than MP. It is seen that a significant reduction of the relative errors occurs for the BPMP method, which is attained at a slightly higher computational time.

Table 1: Results for the real-time implementation of the BPMP method.

SNR (dB)	α_1	ω_1	α_2	ω_2	Computational time (ms)
∞	0.0628	0.628	.1099	1.099	50
3.96	0.0627	0.622	0.0992	1.121	60
(BPMP)	Δ = 1.6%	0.96%	9.7%	2.00	
3.96	0.0843	0.602	0.0792	1.109	50
(MP)	Δ = 34%	4.1%	27.9%	0.9%	

7. Conclusion

Out of all linear techniques available to approximate a function by a sum of complex exponentials, the Matrix Pencil method provides smaller variance of the parameters in the presence of noise than a polynomial (or a Prony-type) method. A bandpass version of it can be implemented to perform real-time processing.

8. Acknowledgment

Grateful acknowledgment is made to Drs. D. Dudley, K. Michalski, and E. Miller for providing suggestions for improving the readability of this article.

9. References

1. C. E. Baum, "On the singularity expansion method for the solution of electromagnetic interaction problems", *EMP Interaction Note 8*, Air Force Weapons Laboratory, Kirkland AFB, New Mexico, December 11, 1971.

2. L. Marin and R. W. Latham, "Representation of transient scattered fields in terms of free oscillations of bodies," *Proceedings of the IEEE*, **60**, May 1972, pp. 640-641.

3. F. M. Tesche, "On the analysis of scattering and antenna problems using the singularity expansion technique," *IEEE Transactions on Antennas and Propagation*, **AP-21**, 1, January 1973, pp. 53-62.

4. D. G. Dudley, "Progress in identification of electromagnetic systems," in *IEEE Antennas and Propagation Society Newsletter*, August 1988, pp. 5-11.

5. L. W. Pearson, M. L. VanBlaricum, and R. Mittra, "A new method for radar target recognition based on the singularity expansion for the target," Record of the IEEE 1975 International Radar Conference, Arlington, VA, April 1975, pp. 452-457.

6. D. L. Moffatt and R. K. Mains, "Detection and discrimination of radar target," *IEEE Transactions on Antennas and Propagation*, **AP-23**, May 1975, pp. 358-367.

7. D. G. Dudley and D. M. Goodman, "Transient identification and object classification," in E. K. Miller (ed.), *Time-Domain Measurements in Electromagnetics*, New York, Van Nostrand, 1986.

8. R. M. Shubair and Y. L. Chow, "Efficient computation of the periodic Green's function in Layered Dielectric Media," *IEEE Transactions on Microwave Theory & Techniques*," **MTT-41**, March 1993, pp. 498-502.

9. E. K. Miller and G. J. Burke, "Using model based parameter estimation to increase the physical interpretability and numerical efficiency of computational electromagnetics," *Computer Physics Communications*, **68**, 1991, pp. 43-75.

10. T. K. Sarkar, Z. Maricevic, and M. Kahrizi, "An accurate de-embedding procedure for characterizing discontinuities," *International Journal of Microwave and Millimeter Wave Computer Aided Engineering*," **2**, 2, 1992, pp. 168-178.

11. T. K. Sarkar, Z. Maricevic, M. Kahrizi, and A. R. Djordjevic, "Frequency dependent characterization of radiation from an open end microstrip line," *AEU*, **48**, 2, March 1994, pp. 101-107.

12. M. Kahrizi, T. K. Sarkar, and Z. A. Maricevic, "Dynamic analysis of a microstrip line over a perforated ground plane," *IEEE Transactions on Microwave Theory and Techniques*, **42**, 5, May 1994, pp. 820-825.

13. M. Kahrizi, T. K. Sarkar, and Z. A. Maricevic, "Analysis of a wide radiating slot in the ground plane of a microstrip line," *IEEE Transactions on Microwave Theory and Techniques*, January 1993, pp. 29-37.

14. T. K. Sarkar and M. Salazar, "An alternate interpretation of complex modes in closed perfectly conducting (lossless) structures," *AEU*, **48**, May 1994, pp. 123-129.

15. Y. Hua and T. K. Sarkar, "Parameter estimation of multiple transient signals," *Signal Processing*, **28**, 1992, pp. 109-115.

16. Y. Hua, F. Baqai, Y. Zhu, and D. Heilbronn, "Imaging of point scatterers from step frequency ISAR data," *IEEE Transactions on Aerospace and Electronic Systems*, **AES-29**, January 1993, pp. 195-205.

17. Y. Hua, "High resolution imaging of continuously moving object using stepped frequency radar," *Signal Processing*, **35**, 1, January 1994, pp. 33-40.

18. H. Ouibrahim, D. D. Weiner, and T. K. Sarkar, "Matrix pencil approach to direction finding," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-36**, April 1988, pp. 610-612.

19. A. G. Evans, "Least squares systems parameter identification and time domain approximation," Ph.D. thesis, Drexel University, 1972.

20. K. Stieglitz, "On simultaneous estimation of poles and zeros in speech analysis," *IEEE Transactions Acoustics, Speech and Signal Processing*, **ASSP-25**, June 1977, pp. 229-234.

21. D. M. Goodman and D. G. Dudley, "An output error model and algorithm for electromagnetic system identification," *Circuits, Systems and Signal Processing*, **6**, 4, 1987, pp. 471-505.

22. T. K. Sarkar, S. A. Dianat, and D. D. Weiner, "A discussion of various approaches to the linear system identification problem," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-32**, 3, June 1984, pp. 654-656.

23. T. K. Sarkar, J. Nebat, D. D. Weiner, and V. K. Jain, "A discussion of various approaches to the identification/ approximation problem," *IEEE Transactions on Antennas and Propagation*," January 1982, pp. 89-98.

24. Y. Hua, "On techniques for estimating parameters of exponentially damped/undamped sinusoids in noise," PhD dissertation, Syracuse University, Syracuse, NY, August 1988.

25. Y. Hua and T. K. Sarkar, "Perturbation analysis of TK method for harmonic retrieval problems," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-36**, 2, February 1988, pp. 228-240.

26. Y. Hua and T. K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-38**, 5, May 1990, pp. 814-824.

27. R. Prony, "Essai experimental et analytique ....," *Paris J. l'Ecola Polytechnique*, **1**, Calvier 2, 1795, pp. 24-76.

28. F. B. Hildebrand, *Introduction to Numerical Analysis*, New York, McGraw Hill, pp. 378-382, 1956.

29. M. L. VanBlaricum and R. Mittra, "A technique for extracting the poles and residues of a system directly from its transient response," *IEEE Transactions on Antennas and Propagation*," **AP-23**, November 1975, pp. 777-781.

30. M. L. VanBlaricum and R. Mittra, "Problems and solutions associated with Prony's method for processing transient data," *IEEE Transactions on Antennas and Propagation*," **AP-26**, January 1978, pp. 174-182.

31. D. W. Tufts and R. Kumaresan, "Frequency estimation of multiple sinusoids: making linear prediction perform like maximum likelihood," *Proceedings of the IEEE*, **70**, September 1982, pp. 975-989.

32. R. Kumaresan and D. W. Tufts, "Estimating the parameters of exponentially damped sinusoids and pole-zero modeling in noise," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-30**, December 1982, pp. 833-840.

33. V. K. Jain and R. D. Gupta, "Identification of linear systems through a Grammian technique," *International Journal of Control*, **12**, 3, 1970, pp. 421-431.

34. V. K. Jain, "Filter analysis by grammian method," *IEEE Transactions on Audio Electrocoustics*, **AU-21**, 1973, pp. 120-123.

35. V. K. Jain, "Filter analysis by use of pencil-of-functions: Part I," *IEEE Transactions on Circuits and Systems*, **CAS-21**, September 1974, pp. 574-579.

36. V. K. Jain, "Filter analysis by use of pencil-of-functions: Part II," *IEEE Transactions on Circuits and Systems*, **CAS-21**, September 1974, p.580-583.

37. T. K. Sarkar, J. N. Nebat, D. D. Weiner, and V. K. Jain, "Suboptimal approximation/identification of transient waveforms from electromagnetic systems by pencil-of-function method," *IEEE Transactions on Antennas and Propagation*, **AP-37**, 2, February 19, 1979, pp. 229-234.

38. V. K. Jain, T. K. Sarkar, and D. D. Weiner, "Rational modeling by the pencil-of-functions method," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-31**, June 1983, pp. 564-573.

39. J. A. Mackay and A. McCowen, "An improved pencil-of-functions method and comparisons with traditional methods of pole extraction," *IEEE Transactions Antennas and Propagation*, **AP-35**, April 1987, pp. 435-441.

40. V. K. Jain and T. K. Sarkar, "High performance signal modeling by pencil-of-function method," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-34**, August 1986, pp. 997-1000.

41. V. K. Jain and T. K. Sarkar, "Multivariable system identification by pencil-of-functions method," *IEEE Transactions on Instruments and Measurements*, **IM-34**, December 1985, pp. 550-557.

42. F. R. Gantmacher, *Theory of Matrices, Volume 1*, New York, Chelsea, 1960.

43. T. K. Sarkar, F. Hu, Y. Hua, and M. Wicks, "A real time signal processing technique for approximating a function by a sum of complex exponentials utilizing the Matrix Pencil approach," *Signal Processing - A Review Journal*, **4**, 2, April 1994, pp. 127-140.

44. Y. Hua and T. K. Sarkar, "Generalized pencil-of-function method for extracting poles of an EM system from its transient response," *IEEE Transactions on Antennas and Propagation*, **AP-37**, 2, February 1989, pp. 229-234.

45. Y. Hua and T. K. Sarkar, "Matrix pencil and system poles," *Signal Processing*, **21**, 2, October 1990, pp. 195-198.

46. Y. Hua and T. K. Sarkar, "On SVD for estimating generalized eigenvalues of singular matrix pencil in noise," *IEEE Transactions on Signal Processing*, **SP-39**, 4, April 1991, pp. 892-900.

47. Y. Hua and T. K. Sarkar, "A perturbation property of the TLS-LP method," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-38**, November 1990, pp. 2004-2005.

48. Y. Hua and T. K. Sarkar, "On the total least squares linear prediction method for frequency estimation," *IEEE Transactions on Acoustics, Speech and Signal Processing*, **ASSP-38**, December 1990, pp. 2186-2189.

49. G. H. Golub and C. F. Van Loan, *Matrix Computations*, Baltimore, Maryland, John Hopkins University, 1983.

50. F. Hu, "The band pass matrix pencil method for parameter estimation of exponentially damped/undamped sinusoidal signals in noise," Ph.D. Dissertation, Syracuse University, Syracuse, NY, Dec. 1990.

51. F. Hu, T. K. Sarkar, and Y. Hua, "Utilization of bandpass filtering from the matrix pencil method," *IEEE Transactions on Signal Processing*, **SP-41**, 1, Jan 1993, pp. 442-446.

52. B. Himed, "Application of the Matrix Pencil Approach to Direction Finding," Ph.D. Thesis, Syracuse University, May 1990.

Appendix: A Matrix Pencil Method Subroutine

```
C
      Subroutine PENCIL(N,X,DT,M,R,S)
C-----------------------------------------------------------------------
C Purpose:
C   Complex exponential fit of a sampled complex waveform by the
C    Matrix Pencil Method.
C Authors:
C   Z. A. Maricevic, T. K. Sarkar, and Y. Hua, Copyright 1992.
C   (This version by K. A. Michalski, October 1994.)
C Library:
C   Calls DLSVCR, DEVCHF, and DGVLCG routines from IMSL, v. 2.0.
C Arguments:
C   N    - Number of data samples. (Input)
C   X(I) - Complex*16 array containing sampled waveform. (Input)
C   DT   - Real*8 sample interval. (Input)
C   M    - Integer used to control the model order. (Input/Output)
C          If M>0, the model order (number of poles) is set to M.
C          If M<0, then -M is interpreted as the number of significant
C          digits in the input data, and is used to automatically
C          select the model number. This estimate is returned in M.
C   R(I) - Complex*16 array containing residues (amplitudes). (Output)
C   S(I) - Complex*16 array containing exponents (poles). (Output)
C-----------------------------------------------------------------------
      Implicit None
      Integer NX,NY,MX
C NX - maximum number of samples; MX - maximum model order
      Parameter (NX = 150,NY = NX/2,MX = 10)
      Integer N,M,L,L1,NL,I,J,K,IPATH,IRANK
      Real*8 DT,TOL,S2(NY)
      Complex*16 X(NX),Y(NX,NY),V(NY,NY),Y1(MX,MX),Y2(MX,MX)
      Complex*16 Z(NY),R(MX),S(MX),U(NX,NX)
      Intrinsic DCONJG,MINO,CDABS,CDLOG,DLOG10
      If (N.GT.NX.OR.M.GT.MX) Then
        Stop ' PENCIL: Exceeded maximum N or M!'
      End If
C set the pencil parameter L
      L = N/2-1
      If (L.LE.M) Then
        Stop ' PENCIL: N too small for requested M!'
```

```
      End If
C fill the Hankel data matrix
      L1 = L+1
      NL = N-L
      Do J = 1,L1
        Do I = 1,NL
          Y(I,J) = X(I+J-1)
        End Do
      End Do
C compute singular values and right singular vectors of Y (call IMSL)
      IPATH = 01
      TOL = 1.D-10
      Call DLSVCR(NL,L1,Y,NX,IPATH,TOL,IRANK,Z,U,NX,V,NY)
C determine the model order
      If (M.LT.0) Then
        Do I = 1,L
          If (DLOG10(CDABS(Z(I+1)/Z(1))).LT.M) Exit
        End Do
        M = MIN0(I,L-1,MX)
      End If
C set up the generalized eigensystem for pole computation
      Do I = 1,M
        Do J = 1,M
          Y1(I,J) = (0.D0,0.D0)
          Y2(I,J) = (0.D0,0.D0)
          Do K = 1,L
            Y1(I,J) = Y1(I,J)+DCONJG(V(K+1,I))*V(K,J)
            Y2(I,J) = Y2(I,J)+DCONJG(V(K,I))*V(K,J)
          End Do
        End Do
      End Do
C compute eigenvalues of Y1-LAMBDA*Y2 (call IMSL)
      Call DGVLCG(M,Y1,MX,Y2,MX,S,R)
      Do I = 1,M
        If (CDABS(R(I)).GT.0.D0) Then
          S(I) = S(I)/R(I)
        Else
          Stop ' PENCIL: Infinite eigenvalue???'
        End If
      End Do
C find residues using least squares
C compute Z¬H*Z, where Z is the Vandermonde matrix
C  (upper triangular part only)
      Do I = 1,M
        Do J = I,M
          Y(I,J) = (0.D0,0.D0)
          Do K = 1,N
            Y(I,J) = Y(I,J)+(DCONJG(S(I))*S(J))**(K-1)
          End Do
        End Do
      End Do
C compute eigenvalues and eigenvectors of Z¬H*Z (call IMSL)
      Call DEVCHF(M,Y,NX,S2,V,NY)
      Do I = 1,M
        R(I) = (0.D0,0.D0)
        Do K = 1,N
          R(I) = R(I)+DCONJG(S(I))**(K-1)*X(K)
        End Do
      End Do
      Do I = 1,M
        Z(I) = (0.D0,0.D0)
        Do K = 1,M
          Z(I) = Z(I)+DCONJG(V(K,I))*R(K)
        End Do
        Z(I) = Z(I)/S2(I)
      End Do
      Do I = 1,M
        R(I) = (0.D0,0.D0)
        Do K = 1,M
          R(I) = R(I)+V(I,K)*Z(K)
        End Do
      End Do
      Do I = 1,M
        S(I) = CDLOG(S(I))/DT
      End Do
      End
```