Using the Universal Modality: Gains and Questions*

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Abstract

The paper investigates a simple and natural enrichment of the usual modal language $\mathscr{L} = \mathscr{L}(\Box)$ with an auxiliary 'universal' modality $\boxed{u}$ having Kripke semantics: $\boxed{u}\phi$ is true at a world of a model iff ϕ is valid in the model.

The enriched language $\mathscr{L}_{\boxed{u}} = \mathscr{L}(\Box, \boxed{u})$ turns out to be fairly different from the classical one. In particular the notions of satisfiability, validity and consequence in models become interreducible.

Section 2 is devoted to *modal definability* in $\mathscr{L}_{\boxed{u}}$. Model-theoretic characterizations of this definability are obtained. $\mathscr{L}_{\boxed{u}}$-definability is proved to be equivalent with *sequential definability* in $\mathscr{L}$ introduced by Kapron. In section 3 the minimal normal $\mathscr{L}_{\boxed{u}}$-logic $K_{\boxed{u}}$ is axiomatized and a general model-completeness theorem for the family of normal extensions of $K_{\boxed{u}}$ is proved. Section 4 deals with *minimal extensions*—$\mathscr{L}_{\boxed{u}}$-logics axiomatized with schemata of $\mathscr{L}$ over $K_{\boxed{u}}$. A general study on *transfer of properties* of $\mathscr{L}$-logics to their minimal extensions is initiated. Transfer of incompleteness, strong completeness, compactness and filtration is proved. The problems of transferring completeness, finite completeness and decidability are investigated and several general results are obtained. Uniform reductions of these properties of $\mathscr{L}_{\boxed{u}}$-logics to corresponding natural properties in their classical fragments are established. For a large class of $\mathscr{L}$-logics, completeness is shown to be inherited in their minimal extensions. However, the general transferring problems remain still open. In section 5 several concrete completeness and decidability results for logics with essentially $\mathscr{L}_{\boxed{u}}$-axiomatics are stated and some other applications of $\boxed{u}$ are sketched. In an appendix *independent join* of $\mathscr{L}_{\boxed{u}}$-logics is introduced and proved to preserve completeness when applied to minimal extensions.

Besides the technical results, the paper pursues two main purposes: first, to advertise the universal modality as a natural and helpful tool, providing a better medium for the mission of modality; and second, to illustrate the typical problems arising when enrichments of modal languages are investigated.

Keywords: Modal logic; universal modality; modal definability; minimal extensions; transfer of properties; independent join.

1. Introduction

Nowadays the propositional languages enriched with intensional operators (modalities in a broad sense) are enjoying an increasing recognition as expedient and useful environments for uniform formalizing of seemingly

* Research partly supported by Contract No.56 with the Ministry of Science and Education of Bulgaria.

disintegrated situations and ideas arising in various scientific fields, such as computer science, artificial intelligence, linguistics, philosophy, cognitive science, etc. The expanding areas of application of (poly-) modal logics (including modal, temporal, dynamic, epistemic ones, etc.) cut across both the syntactic and semantic canons of the traditional approach to modality, featured in the classical works of Lewis, Kripke, Prior and others. On one hand, having fixed the usual syntax of the modality as a unary propositional operator, its Kripke-style semantics naturally leaves the classical paradigm reflecting the Leibnizian view on the necessity and varies over arbitrary first- or higher-order quantificational schemata involving the truth at the actual world of valuation and the 'accessibility' relation. An analogy can be made with the process of transition from Fregean first-order quantifiers to generalized quantifiers in the contemporary researches on natural language semantics. On the other hand, the syntactic appearance of the modality can vary, too. Familiar examples are the 0-ary modalities *loop* and *repeat* in dynamic logic as well as the relevant implication which can be regarded as a binary modality or Kamp's operators *Since* and *Until* in temporal logic. For a more general approach towards modality in temporal setting see also [8].

The syntactic and semantic deviations from the classical modality yield a large variety of modal operators, increasing the expressive power of the language in one or other direction, and thus making it more adequate to the envisaged purposes. Among this variety of modalities there are several basic patterns of fundamental importance both for the expressiveness of the language and for axiomatization of the corresponding deductive machine. The most natural and simplest one, and at the same time the most useful as an auxiliary tool, is the *universal modality* $\boxed{u}$ which is the focus of the present paper. The universal modality is interpreted in the Kripke semantics on a frame $\langle W, R\rangle$ by the Cartesian square W^2 of the universe W, i.e. $\boxed{u}\phi$ is true at a point of the model iff ϕ is valid in the model. Thus the main technical effect of the introduction of $\boxed{u}$ in the language is that it overcomes the restriction on the power of local (pointwise) truth imposed by the accessibility relation, and allows an expression of statements valid in the whole universe standing at any world of this universe. This ability of the universal modality has its undoubted value in all traditional interpretations of the modal language, i.e. it makes it possible to express eternal truths in temporal logic or invariance properties and partial correctness statements in logics of programs. That is why $\boxed{u}$ has been introduced, explicitly or not, many times by a number of authors, under different names and in different contexts, e.g. in temporal logic, [5], [6], [11]; in dynamic logic, [22], [23]; in logic of programs, [1]; and just technically, [4], [19], [24], etc. These are only a few references among many. Indeed, taken in isolation, $\boxed{u}$ is nothing more than the well-known old S5-modality. The point of the present paper, however, is to regard $\boxed{u}$ just as an *auxiliary* modality. We aim at a systematic extraction and investigation of the effects generated by the universal modality and that is why we consider it in the classical modal environment. Nevertheless, as a rule, the results obtained in the paper readily spread over arbitrary polymodal languages with $\boxed{u}$. Our other goal is to illustrate the typical difficulties and the challenging problems arising when enrichments, even so simple as this one, of modal languages are put to a systematic investigation.

2. Preliminaries

Throughout the paper we fix a propositional modal language $\mathcal{L} = \mathcal{L}(\Box)$ of one modality $\Box$ and its dual $\Diamond =_{DF} \neg\Box\neg$. The set of formulae of $\mathcal{L}$ is denoted by FOR. We assume familiar the notions of *frame, model, general frame, modal algebra* and *validity* in them as well as the basic *frame constructions—subframe* (this will mean *generated* subframe). *p-morphic image, disjoint union* and the basic *algebraic constructions—subalgebra, homomorphic image, direct product* (for exact definitions see [10], [16], [3], [15]. Some notation: $\mathcal{M} \models \phi[x]$ means that the formula ϕ is true at the world x of the model $\mathcal{M}$; $\mathcal{M} \models \phi$ means that ϕ is valid in $\mathcal{M}$. Notation for truth and validity in frames, general frames and modal algebras is analogous.

Also we use the categorical connections between general frames and modal algebras [10]: to each general frame $\mathcal{F}$ there corresponds a modal algebra $\mathcal{F}^+$ and to each modal algebra $\mathcal{U}$ the general frame $\mathcal{U}_+$ which is its *Stone representation.* (Here we stick to the notation of [10].) Hereafter, frames will be freely identified, when necessary, with the full general frames based on them.

If $\mathcal{F}$ is a general frame then $(\mathcal{F}^+)_+$ is called a *Stone representation* of $\mathcal{F}$, denoted also by $\mathrm{Sr}(\mathcal{F})$. If F is a frame, then the underlying frame of $(F^+)_+$ is called an *ultrafilter extension* of F, denoted by $\mathrm{ue}(F)$. F is called an *ultrafilter contraction* of $\mathrm{ue}(F)$.

Another construction to be used is the *ultraproduct of general frames* (see [10] or [3]). Note that this construction, applied to a family of frames, yields a general frame (unlike the ordinary ultraproduct of frames) since it regards frames as full general frames. That is why we call it a *general ultraproduct.*

Denote the language obtained from $\mathcal{L}$ adding the *universal modality* $\boxed{u}$ (and its dual $\langle u \rangle$) by $\mathcal{L}_{\boxed{u}}$ and the set of formulae of $\mathcal{L}_{\boxed{u}}$ by $\mathrm{FOR}_{\boxed{u}}$. Here are some basic notions for the new language.

A *frame* for $\mathcal{L}_{\boxed{u}}$ (*$\mathcal{L}_{\boxed{u}}$-frame*) is a frame $\langle W, R, W^2 \rangle$ which will be identified with $\langle W, R \rangle$. Operators $\Box$ and $\boxed{u}$ on subsets of the universe are defined in a frame $F = \langle W, R \rangle$ as follows:

$$\Box X = \{x \in W / R(x) \subseteq X\} \quad \text{and} \quad \boxed{u} X = \begin{cases} W & \text{if } X = W \\ \varnothing & \text{otherwise} \end{cases}.$$

$\mathscr{L}_{\boxed{u}}$*-model* is a pair $\langle F, V\rangle$ where F is a frame and V is a valuation of the variables, extended to $\text{FOR}_{\boxed{u}}$ via the usual clauses for $\mathscr{L}$ and $V(\boxed{u}\phi) = \boxed{u}V(\phi)$.

The notion of *general* $\mathscr{L}_{\boxed{u}}$*-frame* is also defined in due standard way as a pair $\langle F, \mathbb{W}\rangle$ where $F = \langle W, R\rangle$ is a frame and $\mathbb{W} \subseteq P(W)$ is closed under the Boolean operations, $\Box$ and $\boxed{u}$. Clearly, the operator $\boxed{u}$ does not impose extra closure conditions and so we can identify (general) $\mathscr{L}$-frames with (general) $\mathscr{L}_{\boxed{u}}$-frames. An $\mathscr{L}_{\boxed{u}}$*-algebra* is a non-trivial modal algebra with an additional unary operator $\boxed{u}$, satisfying the condition:

$$\boxed{u}a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

for each element a. Let $\mathbb{M}_{\boxed{u}}$ be the class of all $\mathscr{L}_{\boxed{u}}$-algebras. It is easy to see that the $\mathscr{L}_{\boxed{u}}$-algebras are exactly those bimodal algebras which are isomorphic to $\mathscr{F}^+$ where $\mathscr{F}$ is a general $\mathscr{L}_{\boxed{u}}$-frame. The notions of *validity* ($\models$) in $\mathscr{L}_{\boxed{u}}$-models, general $\mathscr{L}_{\boxed{u}}$-frames, $\mathscr{L}_{\boxed{u}}$-frames and $\mathscr{L}_{\boxed{u}}$-algebras are defined in the standard way. *Closed formulae in* $\mathscr{L}_{\boxed{u}}$ are the Boolean combinations of formulae, beginning with $\boxed{u}$.

Here we sketch several specific effects of the enriched language. The universal modality makes it possible to express global properties (for the whole model or frame) by means of local (pointwise) ones. This is grounded on the obvious fact that truth of a closed formula at a point (local validity) is equivalent to validity of this formula in the whole model (global validity). Here are some issues of this effect:

PROPOSITION 2.1

(1) Global validity of any $\mathscr{L}_{\boxed{u}}$-formula ϕ is equivalent to local validity of $\boxed{u}\phi$;

(2) Global consequence $\Gamma \models \phi$ (meaning that for every model $\mathscr{M} \models \Gamma$ implies $\mathscr{M} \models \phi$) equivalent to local consequence $\boxed{u}(\Gamma) \models_1 \phi$ (meaning that for every model $\mathscr{M}$ and point x in it, $\mathscr{M} \models \boxed{u}(\Gamma)[x]$ implies $\mathscr{M} \models \phi[x]$), where $\boxed{u}(\Gamma) = \{\boxed{u}\gamma/\gamma \in \Gamma\}$; this is equivalent to the validity $\models \boxed{u}(\Gamma) \rightarrow \phi$ when Γ is finite.

Analogous effect appears in *first-order definability* [3]: an $\mathscr{L}_{\boxed{u}}$-formula ϕ is (globally) first-order definable iff $\boxed{u}\phi$ is locally first-order definable.

3. Modal definability in $\mathscr{L}_{\boxed{u}}$

3.1. Classes of algebras and frames, modally definable in $\mathscr{L}_{\boxed{u}}$

If C is a class of $\mathscr{L}_{\boxed{u}}$-frames then the *modal theory of C*, $\text{MT}_{\boxed{u}}(C)$, is the set of all $\mathscr{L}_{\boxed{u}}$-formulae valid in C. If Γ is a set of $\mathscr{L}_{\boxed{u}}$-formulae then $\text{FR}(\Gamma)$ is the class of frames in which the formulae of Γ are valid.

Definition

A class of frames C is *modally definable* in $\mathscr{L}_{\boxed{u}}$ (*$\mathscr{L}_{\boxed{u}}$-definable*) if there exists a set $\Gamma \subseteq \mathrm{FOR}_{\boxed{u}}$ such that for each frame $F: F \in C$ iff $F \models \Gamma$.

The class of the modally definable classes of frames in $\mathscr{L}_{\boxed{u}}$ will be denoted by $\mathrm{MD}(\mathscr{L}_{\boxed{u}})$. We will describe the $\mathscr{L}_{\boxed{u}}$-definability in a model-theoretic fashion, by means of closure under certain constructions. For this purpose we will define some operators on classes of algebras and frames. Let A be a class of algebras of some signature Ω. Then we denote by $\mathbf{I}(A)$ ($\mathbf{S}(A)$, $\mathbf{H}(A)$, $\mathbf{P}(A)$, $\mathbf{U}(A)$) the class of all isomorphic copies (subalgebras, homomorphic images, direct products, ultraproducts) of algebras from A. Analogously, let C be a class of frames. Then we denote by $\mathbf{I}_f(C)$ ($\mathbf{H}_f(C)$, $\mathbf{U}_f(C)$, $\mathbf{SR}(C)$, $\mathbf{E}_u(C)$, $\mathbf{C}_u(C)$) the class C extended with all isomorphic copies (p-morphic images, ultraproducts, Stone representations, ultrafilter extensions, ultrafilter contractions) of frames from C.

The same notation will be used for classes of general frames.

Fact 3.0

All of the operators defined above preserve the validity of modal formulae. (See [10].)

Definition

Modally definable closure of a class C of frames in $\mathscr{L}_{\boxed{u}}$ is the smallest $\mathscr{L}_{\boxed{u}}$-definable class $[C]$ containing C. Explicitly: $[C] = \mathrm{FR}(\mathrm{MT}_{\boxed{u}}(C))$.

The definitions and notations for modally definable classes and modally definable closures of classes of general frames, models and modal algebras are in the same spirit.

The following results in this section are obtained as close analogues to those in [13] where definability in the bimodal language $\mathscr{L}(R, -R)$ (with modalities both over a relation and its complement) is studied. That is why the proofs will be omitted or just sketched.

Lemma 3.1

$\mathbb{M}_{\boxed{u}}$ consists of simple algebras (without proper congruences).

Proof. Let $\mathfrak{U} \in \mathbb{M}_{\boxed{u}}$, $a, b \in \mathfrak{U}$, $a \neq b$ and for some congruence $\varkappa$ in $\mathfrak{U}$, $a\varkappa b$. Then $0 = \boxed{u}(a \leftrightarrow b)\varkappa\boxed{u}(a \leftrightarrow a) = 1$. #

Lemma 3.2

If $K \subseteq \mathbb{M}_{\boxed{u}}$ then $[K] = \mathbf{HSP}(K) \cap \mathbb{M}_{\boxed{u}} = \mathbf{ISU}(K)$.

Proof. By Lemma 3.1 and Jonsson's result that every subdirectly irreducible algebra in a variety generated by a class K of algebras with distributive lattices of congruences, belongs to $\mathbf{HSU}(K)$. #

As a consequence, a class of $\mathscr{L}_{\boxed{u}}$-algebras is modally definable iff it is closed under isomorphisms, subalgebras and ultraproducts. A well-known model-theoretic result implies that the $\mathcal{L}_{\boxed{u}}$-definable classes of algebras are exactly the universal classes.

Now we shall define specifically for $\mathcal{L}_{\boxed{u}}$, a simpler version of the notion of SA-construction introduced by Goldblatt and Thomason [12] and used to characterize $\mathcal{L}$-definability.

DEFINITION
Let $\mathcal{F} = \langle W, R, \mathbb{W}\rangle$ and $F' = \langle W', R'\rangle$. F' is said to be a $\boxed{u}$*-collapse* of $\mathcal{F}$ iff F'^{+} is subalgebra of $\mathcal{F}^{+}$.

It can be shown (similarly to Lemma 3.10 from [13]) that the above definition means that there exists a complete atomic subalgebra of $\mathcal{F}^{+}$, $\mathcal{F}_1^{+} = \langle W, R, \mathbb{W}_1\rangle^{+}$ such that W' is the set of atoms of $\mathcal{F}_1'^{+}$, for each $a, b \in W'$: $R'ab$ iff $a \subseteq \Diamond b$ (i.e. $\forall x \in a\, \exists y \in b Rxy$) and the following condition holds:

$$\forall a \in W'\ \forall X \in \mathbb{W}_1 (\forall b \in W'(R'ab \rightarrow b \subseteq X) \rightarrow R(a) \subseteq X)$$

Let C be a class of general frames. The class of all $\boxed{u}$-collapses of C will be denoted by $\mathbf{C}_{\boxed{u}}(C)$.

From Lemma 2.2 it follows:

THEOREM 3.3
(1) If C is a class of frames then $[C] = \mathbf{I}_f \mathbf{C}_{\boxed{u}} \mathbf{U}_f(C)$.
(2) A class of frames C is in MD($\mathcal{L}_{\boxed{u}}$) iff it is closed under isomorphisms and $\boxed{u}$-collapses of general ultraproducts.

The essential difference between this characterization and the classical result of Goldblatt and Thomason [12] is due to the fact that in the enriched language the notions of (generated) subframe and disjoint union of frames are trivialized.

DEFINITION
A class of frames C is Δ*-elementary* iff there is a set Σ of formulae of the first-order language with equality and a binary predicate symbol R, such that for each frame F, $F \in C$ iff $F \models \Sigma$.

COROLLARY 3.4
(i) If C is a class of frames closed under ultraproducts then $[C] = \mathbf{I}_f \mathbf{C}_{\boxed{u}}(C)$.
(ii) A Δ-elementary class is $\mathcal{L}_{\boxed{u}}$-definable iff it is closed under $\boxed{u}$-collapses.

3.2. Definability in $\mathcal{L}_{\boxed{u}}$ and sequential definability

Kapron [18] considers definability by means of sequents in the usual modal language as follows:

DEFINITION

(1) A *modal sequent* in $\mathcal{L}$ (*$\mathcal{L}$-sequent*) is a pair $\langle \Gamma, \Delta \rangle$ of finite sets of formulae of $\mathcal{L}$;

(2) An $\mathcal{L}$-sequent $\langle \Gamma, \Delta \rangle$ is *valid in a model $\mathcal{M}$*, notation $\mathcal{M} \models \langle \Gamma, \Delta \rangle$, if $(\forall \phi \in \Gamma)(\mathcal{M} \models \phi)$ implies $(\exists \psi \in \Delta)(\mathcal{M} \models \psi)$;

(3) $\langle \Gamma, \Delta \rangle$ is *valid in a frame F*, notation $F \models \langle \Gamma, \Delta \rangle$, if $\langle \Gamma, \Delta \rangle$ is valid in each model on F;

(4) A *set of modal sequents Ξ is valid in a frame F, $F \models \Xi$*, if each member of Ξ is.

(5) A *class of frames C is modally sequentially-definable* if there exists a set Ξ of modal sequents such that for each frame $F: F \models \Xi$ iff $F \in C$. (We do not use the notions of 'axiomatic' and 'sequent-axiomatic' class (cf. [12] and [18]) because a class of frames can be defined by a set of formulae or sequents but not axiomatized by this set.)

The class of modally sequent-definable classes in $\mathcal{L}$ will be denoted by MSD($\mathcal{L}$).

LEMMA 3.5
MSD($\mathcal{L}$) $\subseteq$ MD($\mathcal{L}_{\boxed{u}}$).

PROOF. Let a class C be defined by a set of sequents Ξ. For each sequent $\sigma = \langle \Gamma, \Delta \rangle \in \Xi$ we define $\phi_\sigma \in \mathrm{FOR}_{\boxed{u}}$:

$$\phi_\sigma =_{\mathrm{DF}} \bigwedge_{\psi \in \Gamma} \boxed{u}\psi \rightarrow \bigvee_{\theta \in \Delta} \boxed{u}\theta$$

It is easy to see, using 1.1, that for each model $\mathcal{M}: \mathcal{M} \models \sigma$ iff $\mathcal{M} \models \phi_\sigma$. So C is defined by the set of formulae $\{\phi_\sigma / \sigma \in \Xi\}$. #

The opposite inclusion will be proved using a kind of normal form of the formulae of $\mathcal{L}_{\boxed{u}}$.

DEFINITION

(1) *An elementary conjunction* (*disjunction*) is any formula of the type $\chi \wedge \boxed{u}\chi_0 \wedge \langle u \rangle\chi_1 \wedge \ldots \wedge \langle u \rangle\chi_s (\chi \vee \langle u \rangle\chi_0 \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s)$, where $\chi, \chi_i \in \mathcal{L}(\Box)$;

(2) *A conjunctive form*, CF *for short* (*disjunctive form*, DF), is any conjunction (disjunction) of elementary disjunctions (conjunctions).

By a standard argument, each CF is equivalent to a DF and vice versa. So, by *form* we will mean either CF or DF.

PROPOSITION 3.6
For each formula ϕ and closed formula ψ,
(a) $\models\Box(\phi \vee \psi) \leftrightarrow (\Box\phi \vee \psi)$;
(b) $\models\boxed{u}(\phi \vee \psi) \leftrightarrow (\boxed{u}\phi \vee \psi)$;

PROOF. Standard semantic arguments, using 1.1. #

THEOREM 3.7
For each $\phi \in FOR_{\boxed{u}}$ there is a form equivalent to ϕ.

PROOF. By induction on ϕ. The Boolean steps are standard. Let $\phi = \Box\psi$ and $\psi' = \psi_1 \wedge \ldots \wedge \psi_n$ be a CF of ψ. Then $\phi \equiv \Box\psi_1 \wedge \ldots \wedge \Box\psi_n$ and Proposition 3.6(a) guarantees that all $\Box\psi_i$s have equivalent elementary disjunctions and so ϕ has an equivalent CF. For $\phi = \boxed{u}\psi$, the proof is the same, using Proposition 3.6(b). #

DEFINITION
Let $\phi \in FOR_{\boxed{u}}$ and ϕ' be a CF equivalent to ϕ, $\phi' = \psi_1' \wedge \ldots \wedge \psi_k'$. For every elementary disjunction $\psi = \chi \vee \diamondsuit\chi_0 \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s$ in ϕ' we denote $\tau(\psi) = \boxed{u}\neg\chi_0 \rightarrow (\boxed{u}\chi \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s)$ and finally we put $\tau(\phi') = \tau(\psi_1') \wedge \ldots \wedge \tau(\psi_k')$ and $\tau(\phi) = \tau(\phi')$.

$\tau(\phi)$ will be called a *sequential closure of* ϕ. Note that $\tau(\phi)$ is a formula without nested occurrences of $\boxed{u}$. The circumstance that a formula can have many equivalent sequential closures will be harmless.

For every $\mathcal{L}_{\boxed{u}}$-model $\mathcal{M}$ and an elementary disjunction $\chi \vee \diamondsuit\chi_0 \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s$ the following hold:

$\mathcal{M} \models \chi \vee \diamondsuit\chi_0 \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s$ iff $\mathfrak{M} \models \boxed{u}(\chi \vee \diamondsuit\chi_0 \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s)$
iff $\mathcal{M} \models \boxed{u}\chi \vee \diamondsuit\chi_0 \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s$
iff $\mathcal{M} \models \boxed{u}\neg\chi_0 \rightarrow (\boxed{u}\chi \vee \boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s)$

Thus, every $\mathcal{L}_{\boxed{u}}$-formula is equivalent to its sequential closure with respect to validity in models. Obviously, every conjunctive member of $\tau(\phi)$ equivalent, with respect to validity in an $\mathcal{L}_{\boxed{u}}$-model (hence in a frame), to the corresponding sequent $\langle \neg\chi_0, \{\chi, \chi_1, \ldots, \chi_s\}\rangle$. This observation and Theorem 3.7 yield an equivalence between sequential definability in $\mathcal{L}$ and definability in $\mathcal{L}_{\boxed{u}}$:

THEOREM 3.8
$MSD(\mathcal{L}) = MD(\mathcal{L}_{\boxed{u}})$.

Moreover, definability and sequential definability coincide in each polymodal language having $\boxed{u}$ explicitly definable.

Now, Theorem 7 from [18] is directly translated into $\mathcal{L}_{\boxed{u}}$:

COROLLARY 3.9
A Δ–elementary class of frames is MD in $\mathcal{L}_{\boxed{u}}$ iff it is closed under p-morphisms and ultrafilter contractions.

Thus, for Δ-elementary classes of frames, closedness under $\boxed{u}$-collapses turns out to be equivalent to closedness under p-morphisms and ultrafilter contractions—a result which can be proved directly, too.

In particular, a first-order condition is definable in $\mathscr{L}_{\boxed{u}}$ iff it is preserved under p-morphisms and ultrafilter contractions. For instance, $\forall x \neg Rxx$ and $\exists xRxx$ are not definable in $\mathscr{L}_{\boxed{u}}$ since the former fails after an appropriate p-morphism (e.g. the only mapping from $\langle \{x, y\}, \{\langle x, y\rangle, \langle y, x\rangle\}\rangle$ onto $\langle \{u\}, \{\langle u, u\rangle\}\rangle$) and the latter fails after an appropriate ultrafilter contraction (e.g. if $\mathbb{N}$ is the set of natural number then $\langle \mathbb{N}, <\rangle \nvDash \exists xRxx$ but $ue(\langle \mathbb{N}, <\rangle) \vDash \exists xRxx$).

Corollary 3.10
If C is a Δ-elementary class of frames then $[C] = \mathbf{C}_u\mathbf{H}_f(C)$.

This result is carried over into multimodal enrichments of $\mathscr{L}$ which have $\boxed{u}$ explicitly definable. It has some methodological value: frequently such enrichments have non-standard semantics extending the standard ones because the semantic connections between the different modalities may not be syntactically expressible. So, let L be a first-order definable logic in such an enrichment of $\mathscr{L}$ which is proved to be complete with respect to the class NS(L) of non-standard L-frames. Now the problem arises how to prove completeness with respect to the class of standard L-frames S(L), if it holds. Extending a well-known result from Fine [7] we can conclude that L is canonical, hence complete with respect to the class DNS(L) of (non-standard) L-frames which carry *descriptive* (see [10]) general frames. If L is complete with respect to S(L) then $\mathrm{NS}(L) = [\mathrm{S}(L)] = \mathbf{C}_u\mathbf{H}_f(\mathrm{S}(L))$, hence $\mathrm{DNS}(L) \subseteq \mathbf{C}_u\mathbf{H}_f(\mathrm{S}(L))$, so $\mathbf{E}_u(\mathrm{DNS}(L)) \subseteq \mathbf{H}_f(\mathrm{S}(L))$. But it is easy to see that every frame F which carry a descriptive general frame is a p-morphic image of its ultrafilter extension. Indeed, if $\mathscr{F} = \langle F, \mathbb{W}\rangle$ then $\mathscr{F}^+$ is a subalgebra of F^+ hence if $\mathscr{F}$ is descriptive then $\mathscr{F} \cong (\mathscr{F}^+)_+$ is a p-morphic image of $(F^+)_+$ hence F is a p-morphic image of ue(F). Thus we obtain $\mathrm{DNS}(L) \subseteq \mathbf{H}_f\mathbf{E}_u(\mathrm{DNS}(L)) \subseteq \mathbf{H}_f\mathbf{H}_f(\mathrm{S}(L)) = \mathbf{H}_f(\mathrm{S}(L))$. This calculation means that if standard completeness hold for L one must be able to prove it as follows: for any L-consistent formula ϕ take a canonical L-frame F (non-standard in general) satisfying ϕ and then construct a p-morphic counter-image of F which is a standard L-frame (using, for example, the 'copying' technique, [9], [14], [27]). If one succeeds in this, the standard completeness of L is proved.

Corollary 3.11
A class of general frames C is MD in $\mathscr{L}_{\boxed{u}}$ iff C is closed under p-morphisms, ultraproducts and Stone representations, and the complement of C is closed under Stone representations. (Kapron [18], Theorem 6).

Actually, the proof of Kapron [18], Theorem 6 gives something more:

Corollary 3.12
If C is a class of general frames then $[C] = \mathbf{S_R}^{-1}\mathbf{H_f S_R U_f}(C)$.

Here are some examples of conditions that are $\mathscr{L}_{\boxed{u}}$-definable but not $\mathscr{L}$-definable.

Semantic condition	Modal formula
$R = W^2$	$\Box p \rightarrow \boxed{u} p$
$\exists x\, \forall y \neg Rxy$	$\langle u \rangle \Box \bot$
$\exists x\, \exists y Rxy / R \neq \varnothing /$	$\langle u \rangle \Diamond \top$
$\lvert W \rvert = 1$	$p \rightarrow \boxed{u} p$
$\lvert W \rvert \leq n$	$\bigwedge_{i=1}^{n+1} \langle u \rangle p_i \rightarrow \bigvee_{i \neq j} \langle u \rangle (p_i \wedge p_j)$
$\forall x\, \forall y\, \forall z (Rxz \rightarrow Ryz)$	$\Box p \rightarrow \boxed{u} \Box p$
$\forall x\, \forall y\, \forall z (Rxz\, \&\, x \neq z \rightarrow Ryz)$	$(p \wedge \langle u \rangle \Box p) \rightarrow \Box p$
$\forall x\, \exists y Ryx$	$\boxed{u} \Box p \rightarrow p$
R^{-1} is well-founded	$\boxed{u}(\Box p \rightarrow p) \rightarrow p$

One could prove non-definability in $\mathscr{L}$ of these conditions, using the criteria of Goldblatt and Thomason [12] Theorems 3 and 8. See also [2].

4. $\mathscr{L}_{\boxed{u}}$-logics

In this section we consider normal modal logics in $\mathscr{L}_{\boxed{u}}$. First we have to define the minimal normal $\mathscr{L}_{\boxed{u}}$-logic, i.e. the analogue of K. Here are several obviously valid schemata:

$(\boxed{u})$	$\boxed{u}(p \rightarrow q) \rightarrow (\boxed{u} p \rightarrow \boxed{u} q)$
(ref$_{\boxed{u}}$)	$\boxed{u} p \rightarrow p$
(sym$_{\boxed{u}}$)	$p \rightarrow \boxed{u} \langle u \rangle p$
(trans$_{\boxed{u}}$)	$\boxed{u} p \rightarrow \boxed{u}\boxed{u} p$
(incl)	$\boxed{u} p \rightarrow \Box p$

These schemata determine that $\boxed{u}$ is an S5-modality with corresponding equivalence relation U containing the relation R corresponding to $\Box$. This does not guarantee that U is a universal relation but this property cannot be expressed by means of model formulae since it is not preserved in disjoint unions. Indeed, we shall see (as a consequence of the completeness theorem) that the above schemata are all we can say about $\boxed{u}$. The extension of the minimal normal modal logic K with these schemata and the rule

$$(\text{NEC}_{\boxed{u}}): \frac{\phi}{\boxed{u}\phi}$$

will be called $K_{\boxed{u}}$.

Note that the rule ($\text{NEC}_{\boxed{u}}$), combined with (incl), makes the rule

$$(\text{NEC}_{\Box}): \frac{\phi}{\Box\phi}$$

redundant.

So we have another semantics, larger than the one envisaged thus far, namely, models over frames $\langle W, R, U\rangle$ where U is an equivalence relation containing R. These frames, when $U \neq W^2$, will be called *non-standard frames for* $\mathscr{L}_{\boxed{u}}$ and the frames $\langle W, R, W^2\rangle$ will be *standard* ones. Analogous terminology will be accepted for general frames and models over standard and non-standard frames.

DEFINITION
A simple extension of $K_{\boxed{u}}$, or $\mathscr{L}_{\boxed{u}}$*-logic,* is any extension of $K_{\boxed{u}}$ by means of schemata of $\mathscr{L}_{\boxed{u}}$.

Now two general notions of completeness arise: completeness with respect to the general semantics and completeness with respect to the standard one. Of course we are interested in the latter, but, with S5 in mind, it is clear that these two notions are equivalent since each generated subframe (as a bi-relational frame) of an $\mathscr{L}_{\boxed{u}}$-frame is a standard $\mathscr{L}_{\boxed{u}}$-frame and each formula refuted in a frame is refuted in some of its generated subframes. Combining the above observations with the usual canonical model technique we obtain the *general model-completeness theorem for* $\mathscr{L}_{\boxed{u}}$*-logics*:

THEOREM 4.1
All $\mathscr{L}_{\boxed{u}}$-logics are complete with respect to the class of their standard $\mathscr{L}_{\boxed{u}}$-models.

In particular $K_{\boxed{u}}$ is complete with respect to the standard $\mathscr{L}_{\boxed{u}}$-frames, i.e. it is actually the minimal normal $\mathscr{L}_{\boxed{u}}$-logic.

We shall investigate in detail a special class of $\mathscr{L}_{\boxed{u}}$-logics, axiomatized by classical schemata over $K_{\boxed{u}}$.

DEFINITION
Given an $\mathscr{L}$-logic L, the *minimal extension* of L in $\mathscr{L}_{\boxed{u}}$ is the simple extension $L_{\boxed{u}}$ of $K_{\boxed{u}}$ with the schemata of L, spread over $\mathscr{L}_{\boxed{u}}$.

Let us make a simple observation. If V is a valuation on a frame and Γ a set of formulae, we denote $V[\Gamma] = \{V(\phi)/\phi \in \Gamma\}$. Clearly, for any $\mathscr{L}_{\boxed{u}}$-model $V[\text{FOR}] = V[\text{FOR}_{\boxed{u}}]$ since every formula beginning with $\boxed{u}$ is equivalent in the model either to ⊤ or to ⊥. This implies that for every $\mathscr{L}$-logic L, if $\mathcal{M}$ is an L-model then $\mathcal{M}$ validates all schemata of $L_{\boxed{u}}$ and can be regarded as a standard $L_{\boxed{u}}$-model. As a consequence, every minimal extension is conservative.

5. Minimal extensions and transfer of properties

The notion of minimal extension appears every time an enrichment of a propositional language is considered. A general question of *transfer of properties* is connected with this notion:

Let $\mathscr{P}$ be some property of logics. Is it the case that if an $\mathscr{L}$-logic L enjoys the property $\mathscr{P}$ then so does its minimal extension in the enriched language?

As a rule it is not difficult to prove such results for particular logics but the general problems seem hard.

5.1. Transfer of completeness. Strong completness of $\mathscr{L}$-logics

An $\mathscr{L}_{\boxed{u}}$-logic L is *complete* if for each $\phi \in \mathrm{FOR}_{\boxed{u}}$ such that $L \nvdash \phi$ there exists a frame F such that $F \vDash L$ and $F \nvDash \phi$.

Let us observe that, due to conservativeness of the minimal extensions, the transfer of incompleteness is obvious.

The problems to be overcome while proving completeness of $\mathscr{L}_{\boxed{u}}$-logics seem to be the same as those for $\mathscr{L}$-logics (the universal modality is not expected to introduce new difficulties) so the methods will be the same too. Anyway, one should surely prefer not to re-create here all familiar completeness achievements in the usual modal logics but to effortlessly transfer as many of them as possible to the enriched language. So, the following seems quite plausible and desirable:

Conjecture 1

If an $\mathscr{L}$-logic L is complete, then its minimal extension $L_{\boxed{u}}$ is complete too.

We will make a digression from $\mathscr{L}_{\boxed{u}}$ in order to translate the problem into an equivalent one in the classical language $\mathscr{L}$.

Definition

Let L be an $\mathscr{L}$-logic and ϕ, $\psi \in \mathrm{FOR}$.

(1) A *normal ϕ-theory over L*, denoted by $\mathrm{Th}_L(\phi)$, is the set of formulae derivable from $L \cup \{\phi\}$ using MP and NEC; every $\psi \in \mathrm{Th}_L(\phi)$ is said to be *normally derivable from ϕ over L*, denoted $\varphi \vdash_L \psi$;

(2) *ψ is a model consequence of ϕ over L*, notation $\phi \vDash_L \psi$, if for each L-model $\mathcal{M}$: $\mathcal{M} \vDash \phi$ implies $\mathcal{M} \vDash \psi$;

(3) *ψ is a normal model consequence of ϕ over L*, notation $\phi \vDash_{nL} \psi$, if for each *normal* (i.e. based on an L-frame) L-model $\mathcal{M}$: $\mathcal{M} \vDash \phi$ implies $\mathcal{M} \vDash \psi$.

Denote $\Box^{(k)}\phi = \phi \wedge \Box\phi \wedge \ldots \wedge \Box^k\phi$ and $\Box^{(\omega)}\phi = \{\phi, \Box\phi, \ldots, \Box^n\phi, \ldots\}$.

LEMMA 5.1 (Deduction lemma for normal ϕ-theories)
Let L be an $\mathscr{L}$-logic and $\phi, \psi \in$ FOR. Then $\phi \vdash_L \psi$ iff for some $k \geq 0$, $L \vdash \Box^{(k)}\phi \rightarrow \psi$.

PROOF. An easy induction on the inference $\phi \vdash_L \psi$. #

PROPOSITION 5.2 (General model-completeness theorem—sequential version)

Let L be an $\mathscr{L}$-logic and $\phi, \psi \in$ FOR. Then $\phi \vdash_L \psi$ iff $\phi \vDash_L \psi$.

PROOF. Since validity in a model is preserved under MP and NEC, we obtain the soundness-direction. Vice versa, suppose $\phi \vdash_L \psi$ does not hold. Then the set $X = \Box^{(\omega)}\phi \cup \{\neg\psi\}$ is L-consistent: otherwise some finite subset should be L-inconsistent hence $\phi \vdash_L \psi$ by 5.1. So there exists a maximal L-theory x containing X. Then ϕ is valid in the x-generated submodel $\mathscr{M}_x^L$ of the canonical L-model while ψ is refuted in the root, whence $\phi \vDash_L \psi$ fails. #

COROLLARY 5.3
For every $\phi, \psi \in$ FOR and $\mathscr{L}$-logic L, $\phi \vdash_L \psi$ iff $L_{\boxed{u}} \vdash \boxed{u}\phi \rightarrow \boxed{u}\psi$.

Now a question arises: what will be the sequential version of the completeness theorem with respect to frames? A natural candidate for an answer is the following:

CONJECTURE 2
An $\mathscr{L}$-logic L is complete iff it satisfies the condition: (*) for each $\phi, \psi \in$ FOR: $\phi \vdash_L \psi$ iff $\phi \vDash_{nL} \psi$.
Note that

(1) (*), when ϕ is $\top$, expresses frame-completeness of L;

(2) $\phi \vdash_L \psi$ implies $\phi \vDash_{nL} \psi$ by virtue of the soundness of L and the preservation of validity in a model under MP and NEC.

DEFINITION
An $\mathscr{L}$-logic is called *sequentially complete* if it satisfies the condition (*).

Let us note that the reader should not consider the adjective 'sequential', used above and henceforth, with its traditionally accepted logical etymology, although there are certain reasons for that. It is just a more or less appropriate name for a (at least definitively) new kind of completeness, decidability, etc.

So Conjecture 2 states that the sequential completeness is not stronger than the ordinary one. So far the Conjectures 1 and 2 are open but we are going to prove that they actually claim the same.

THEOREM 5.4
An $\mathscr{L}$-logic L is sequentially complete iff its minimal extension $L_{\boxed{u}}$ is complete.

PROOF. (1) Let $L_{\boxed{u}}$ be complete, $\phi, \psi \in$ FOR and $\phi \nVdash_L \psi$. Then there exists an $\mathscr{L}$-model $\mathcal{M}$ such that $\mathcal{M} \vDash L$, $\mathcal{M} \vDash \phi$ and $\mathcal{M} \nvDash \psi$. Now regarding $\mathcal{M}$ as an $\mathscr{L}_{\boxed{u}}$-model we have $\mathcal{M} \vDash L_{\boxed{u}}$ and $\mathcal{M} \nvDash \boxed{u}\phi \to \boxed{u}\psi$, so $L_{\boxed{u}} \nvdash \boxed{u}\phi \to \boxed{u}\psi$, and hence there exists a normal $L_{\boxed{u}}$-model $\mathcal{M}'$ such that $\mathcal{M}' \nvDash \boxed{u}\phi \to \boxed{u}\psi$, hence $\mathcal{M}' \vDash \phi$ and $\mathcal{M}' \nvDash \psi$, i.e. $\phi \nvDash_{nL} \psi$.

(2) Let L be sequentially complete, $\phi \in \mathrm{FOR}_{\boxed{u}}$ and $L_{\boxed{u}} \nvdash \phi$. Then $L_{\boxed{u}} \nvdash \tau(\phi)$ (recall the definition of $\tau(\phi)$ from section 3), so there exists a conjunctive member $\boxed{u}\chi \to (\boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s)$ of $\tau(\phi)$ for some $\chi, \chi_1, \ldots, \chi_s \in$ FOR, which is not derivable in $L_{\boxed{u}}$. We shall find normal L-models $\mathcal{M}_i$ such that $\mathcal{M}_i \vDash \chi$ and $\mathcal{M}_i \nvDash \chi_i$ for $i = 1, \ldots, s$. Assume that for some i no such a model exists, i.e. $\chi \vDash_{nL} \chi_i$. Then, by the sequential completeness of L, $\chi \vdash_L \chi_i$, whence by Lemma 5.1. $L \vdash \Box^{(k)}\chi \to \chi_i$ for some k. Therefore $L_{\boxed{u}} \vdash \Box^{(k)}\chi \to \chi_i$; also $L_{\boxed{u}} \vdash \boxed{u}\chi \to \Box^{(k)}\chi$ hence $L_{\boxed{u}} \vdash \boxed{u}\chi \to \chi_i$ so $L_{\boxed{u}} \vdash \boxed{u}\chi \to \boxed{u}\chi_i$ and $L_{\boxed{u}} \vdash \boxed{u}\chi \to (\boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s)$—a contradiction. So, let $\mathcal{M}_1, \ldots, \mathcal{M}_s$ be the normal L-models with the desired property. Let $\mathcal{M}$ be their disjoint union. Considered as an $\mathscr{L}_{\boxed{u}}$-model $\mathcal{M}$ is a normal *$L_{\boxed{u}}$-model such that* $\mathcal{M} \vDash \boxed{u}\chi$ and $\mathcal{M} \vDash \langle u\rangle\neg\chi_i$ for $i = 1, \ldots, s$, so $\mathcal{M} \nvDash \boxed{u}\chi \to (\boxed{u}\chi_1 \vee \ldots \vee \boxed{u}\chi_s)$, therefore $\mathcal{M} \nvDash \tau(\phi)$, and so $\mathcal{M} \nvDash \phi$. This shows that $L_{\boxed{u}}$ is complete. #

DEFINITION

(1) A set of formulae Γ is *normally satisfiable in a logic* L if Γ is satisfiable in a normal L-model;

(2) Γ is *locally normally satisfiable in* L if every finite subset of Γ is normally satisfiable in L;

(3) A logic L is *compact* if every locally normally satisfiable in L set is normally satisfiable in L;

(4) A logic L is *strongly complete* if every L-consistent set is normally satisfiable in L.

Actually, strong completeness = frame-completeness + compactness.

Let L be an $\mathscr{L}$-logic and $\phi, \psi \in$ FOR. Let us note that, as a consequence of 5.1, $\phi \nvdash \psi$ is equivalent to L-consistency of the set $\{\neg\psi\} \cup \Box^{(\omega)}\phi$. So, an equivalent definition of sequential completeness is: L is sequentially complete if every L-consistent set of formulae of the kind $\{\psi\} \cup \Box^{(\omega)}\phi$ is normally satisfiable in L. This condition corresponds to a particular case of compactness: call a logic L *sequentially compact* if it is compact with respect to all sets of the kind $\{\psi\} \cup \Box^{(\omega)}\phi$. Thus: an $\mathscr{L}$-logic L is sequentially complete iff it is complete and sequentially compact.

THEOREM 5.5
Let L be an $\mathscr{L}$-logic.
(1) L is compact iff $L_{\boxed{u}}$ is compact;
(2) L is strongly complete iff $L_{\boxed{u}}$ is strongly complete.

PROOF. (1) If $L_{\boxed{u}}$ is compact then L is compact too, since $L_{\boxed{u}}$ is conservative over L and every normal L-model can be extended to a normal $L_{\boxed{u}}$-model. Vice versa, let L be compact and Γ be a locally normally satisfiable in $L_{\boxed{u}}$ set. By virtue of 3.7 one can regard every $\phi \in \Gamma$ of the type $\chi \vee \langle u \rangle \chi_0 \vee \boxed{u} \chi_1 \vee \ldots \vee \boxed{u}\chi_n$. Then we can successively choose disjunctive items from every such formula of Γ and thus form a new set Δ which is locally normally satisfiable in $L_{\boxed{u}}$, too. Partition Δ to 3 sets Δ_1, Δ_2 and Δ_3 as follows:

$\Delta_1 = \mathrm{FOR} \cap \Delta$, $\Delta_2 = \{\langle u \rangle\chi / \langle u \rangle\chi \in \Delta\}$ and $\Delta_3 = \{\boxed{u}\chi / \boxed{u}\chi \in \Delta\}$. Put $\Delta_3' = \bigcup \{\chi, \Box\chi, \ldots, \Box^n\chi, \ldots / \boxed{u}\chi \in \Delta_3\}$. $\Delta_1 \cup \Delta_3'$ is locally normally satisfiable in $L_{\boxed{u}}$ (because $\Delta_1 \cup \Delta_3$ is) hence in L. Then $\Delta_1 \cup \Delta_3'$ is satisfied at a point x of a normal L-model $\mathcal{M}_0$. We can choose this model to be generated by x. Further, for every $\langle u \rangle\chi \in \Delta_2$, the set $\Delta_3' \cup \{\chi\}$ is locally normally satisfiable in $L_{\boxed{u}}$ hence satisfiable at the root of some normal L-model $\mathcal{M}_\chi$. Finally, take $\mathcal{M}$ to be the disjoint union of $\mathcal{M}_0$ and all $\mathcal{M}_\chi$ for $\langle u \rangle\chi \in \Delta_2$. Then $\mathcal{M}$ is a normal L-model (hence $L_{\boxed{u}}$-model) which satisfies Δ, hence Γ, at x.

(2) is proved in the same way since completeness implies (consistency = local normal satisfiability). #

Note that, by virtue of a result in Fine [7], all first-order definable complete logics are strongly complete (being canonical) hence sequentially complete and their minimal extensions in $\mathscr{L}_{\boxed{u}}$ are strongly complete too. In particular, for every strongly complete (e.g. canonical) logic, Conjectures 1 and 2 hold.

Here is a sufficient condition for sequential compactness.

PROPOSITION 5.6
If an $\mathscr{L}$-logic L contains a theorem of the type $\Box^k p \to \Box^m p$ for some m, k such that $m > k$, then L is sequentially compact.

PROOF. Let $L \vdash \Box^k p \to \Box^m p$, $m > k$. Then a set $\{\chi\} \cup \Box^{(\omega)}\phi$ is normally satisfiable in L iff $\{\chi\} \cup \Box^{(m-1)}\phi$ is such, since for each $n \geq m$ a formula $\Box^r p \to \Box^n p$ is a theorem of L for some integer r, such that $k \leq r < m$. (More exactly, we can choose $r =_{\mathrm{DF}} k + r_0$ where r_0 is the remainder of $n - k$ modulo $m - k$.). #

Thus, for example, every complete extension of K4 is sequentially complete. As one of the referees hints, the above statement can be obviously strengthened: if $L \vdash (p \wedge \Box p \wedge \ldots \wedge \Box^n p) \to \Box^{n+1} p$ for some n, then L is sequentially compact.

Now we can show that strong completeness is stronger than sequential completeness. For example,

$$\mathrm{K4.3W} = K + \Box(\Box p \to p) \to \Box p + \Box((\Box p \wedge p) \to q) \vee \Box((\Box q \wedge q) \to p)$$

which is sequentially complete (being a complete extension of K4) but not compact [16].

By the way, here is a sufficient model-theoretic condition for compactness.

Proposition 5.7
If FR(L) is closed under ultrapowers, then L is compact.

Proof. FR(L) is closed under generated subframes, disjoint unions and isomorphisms, therefore closedness under ultrapowers implies closedness under ultraproducts (see [3], 8.2). Now, let S be a locally normally satisfiable in L set and S_f be the set of all finite subsets of S. For each $\Gamma \in S_f$ there exists a normal L-model $\mathcal{M}_\Gamma = \langle W_\Gamma, R_\Gamma, V_\Gamma \rangle$ and $x_\Gamma \in W_\Gamma$, such that $\mathcal{M}_\Gamma \models (\bigwedge_{\phi \in \Gamma} \phi)[x_\Gamma]$, i.e. $\mathcal{M}_\Gamma \models (\bigwedge_{\phi \in \Gamma} \mathrm{ST}(\phi))[x_\Gamma]$ where $\mathcal{M}_\Gamma$ is considered as a model for the first-order language L_1 having a binary predicate symbol R and unary predicate symbols (corresponding to the propositional variables) $P_1, P_2, \ldots$ and ST(ϕ) is the standard translation of ϕ in L_1 (cf. [3]). Let for each $\Gamma \in S_f$ $X_\Gamma = \{\Delta \in S_f / \Gamma \subseteq \Delta\}$. The family $X = \{X_\Gamma / \Gamma \in S_f\}$ is centered, hence it is included in a ultrafilter D. Let $\langle \mathcal{M}, x \rangle = (\prod_{\Gamma \in S_f} \langle \mathcal{M}_\Gamma, x_\Gamma \rangle)/D$. $\mathcal{M}$ is a normal L-model (the underlying frame for $\mathcal{M}$ being an L-frame) and $\mathcal{M} \models S[x]$. #

The above results show that, if our Conjectures 1 and 2 are not true, a counter-example should be a relatively weak, complete but not compact extension of K.

Warning (Vakarelov, personal communication). The results about transfer of completeness do not carry over to completeness results with respect to classes of frames, defined through additional semantic conditions, inexpressible syntactically. For instance, the logic S4.3 is complete w.r.t. the class of all linear orderings LO [25] but is characterized by the class of weak linear orderings WLO. However, S4.3$_{[u]}$ (which is characterized by WLO too, thanks to 5.2, the canonicity of S4.3 and 4.5) is not complete w.r.t. LO since the formula $[u](\Box p \to q) \vee [u](\Box q \to p)$ is true in LO and not true in WLO, hence not a theorem of S4.3$_{[u]}$.

Let us notice that another, easily achieved transfer (as pointed out by one of the referees) concerns *canonicity* in the classically adopted sense: ([7]) a logic L is called canonical if every frame carrying a descriptive general L-frame is an L-frame itself. Now, if L is an $\mathcal{L}$-logic and $\mathcal{F}$ is descriptive regarded as a standard general $L_{[u]}$-frame then it is descriptive as an L-frame, too. This obviously leads to transfer of canonicity.

5.2. Transfer of finite completeness in $\mathscr{L}_{\boxed{u}}$

Now we shall consider transfer of finite completeness. Of course, we can confine ourselves to the class of standard models. We can translate the problem into $\mathscr{L}$, too:

DEFINITION
An $\mathscr{L}$-logic L is *sequentially finitely complete* if for each $\phi, \psi \in$ FOR such that $\phi \nVdash_L \psi$ there exists a finite normal L-model $\mathcal{M}$ such that $\mathcal{M} \vDash \phi$ and $\mathcal{M} \nvDash \psi$.

In fact, the requirement of normality of the refuting model can be dropped by virtue of the analogue of Segerberg's theorem [25] in $\mathscr{L}_{\boxed{u}}$ about an equivalence between finite model property (FMP) and finite frame property.

THEOREM 5.8
An $\mathscr{L}$-logic L is sequentially finitely complete iff the minimal extension $L_{\boxed{u}}$ is finitely complete.

PROOF. The same as the proof of 5.4, since a finite disjoint union of finite models is a finite model, too. #

Here we set a series of open questions: Do canonicity and finite completeness in $\mathscr{L}$ imply sequential finite completeness? Do finite completeness and sequential completeness imply sequential finite completeness? Does finite completeness imply sequential completeness? Does finite completeness imply sequential finite completeness?

Still, we can easily ascertain the transferring of the most frequently used technique for proving FMP, namely, filtration. There could be a number of different definitions of what it means for a logic to admit filtration. We shall adopt here an acceptable and general enough version of this property. We say that a logic L admits a filtration if there exists a class of L-models C such that L is complete w.r.t. C and for every model $\mathcal{M} \in C$ and a finite set of formulae Γ there exists a finite set Γ' containing Γ and closed under subformulae, such that an L-model can be obtained from $\mathcal{M}$ by filtration over Γ' (in the usual sense, see for example [16]).

THEOREM 5.9
If an $\mathscr{L}$-logic L admits filtration then $L_{\boxed{u}}$ does, too.

PROOF. Let $\Gamma \subseteq \mathrm{FOR}_{\boxed{u}}$ be closed under subformulae and $\mathcal{M} = \langle F, V \rangle$ be an $L_{\boxed{u}}$-model. For each $\phi \in \Gamma$ we take a formula $\phi' \in$ FOR obtained from ϕ by replacing all occurrences of subformulae of the sort $\boxed{u}\psi$ by $\top$ or $\bot$ in accordance with $V(\boxed{u}\psi)$. Obviously $V(\phi) = V(\phi')$. Thus we obtain a set $\Gamma' \subseteq$ FOR which is closed under subformulas, too. The sets Γ and Γ' will lead to the same filtrations since $\boxed{u}$ does not add new conditions. We can obtain by filtration on Γ (hence on Γ') an L-model hence a standard $L_{\boxed{u}}$-model. #

5.3. Transfer of decidability in $\mathscr{L}_{\boxed{u}}$

The next general question is: Does decidability of an $\mathscr{L}$-logic L imply decidability of $L_{\boxed{u}}$?

PROPOSITION 5.10
The disjunction property

$$\frac{\vdash \boxed{u}\phi \to (\boxed{u}\psi \vee \boxed{u}\chi)}{\vdash \boxed{u}\phi \to \boxed{u}\psi \text{ or } \vdash \boxed{u}\phi \to \boxed{u}\chi}, \quad \phi, \psi, \chi \in \mathrm{FOR},$$

holds in $L_{\boxed{u}}$, for each $\mathscr{L}$-logic L.

PROOF. Let us assume $L_{\boxed{u}} \nvdash \boxed{u}\phi \to \boxed{u}\psi$ and $L_{\boxed{u}} \nvdash \boxed{u}\phi \to \boxed{u}\chi$. Then there exist $L_{\boxed{u}}$-models $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$ and points $x_1 \in W_1$ and $x_2 \in W_2$ such that $\mathcal{M}_1 \models \phi$, $\mathcal{M}_1 \nvDash \psi[x_1]$, $\mathcal{M}_2 \models \phi$ and $\mathcal{M}_2 \nvDash \chi[x_2]$. Let $\mathcal{M}$ be the disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$, considered as L-models. $\mathcal{M} \models L \Rightarrow \mathcal{M} \models L_{\boxed{u}}$. Moreover $\mathcal{M} \models \phi$ hence $\mathcal{M} \models \boxed{u}\phi$ but $\mathcal{M} \nvDash \psi[x_1] \Rightarrow \mathcal{M} \nvDash \boxed{u}\psi$ and $\mathcal{M} \nvDash \chi[x_2] \Rightarrow \mathcal{M} \nvDash \boxed{u}\chi$ hence $\mathcal{M} \nvDash \boxed{u}\phi \to (\boxed{u}\psi \vee \boxed{u}\chi)$. Therefore $L_{\boxed{u}} \nvdash \boxed{u}\phi \to (\boxed{u}\psi \vee \boxed{u}\chi)$.
#

Through the translation τ, Proposition 5.10 reduces the decision problem for $L_{\boxed{u}}$ to the problem of deciding provability in $L_{\boxed{u}}$ of formulae of the form $\boxed{u}\phi \to \boxed{u}\psi$ where $\phi, \psi \in \mathrm{FOR}$.

We can uniformly transfer the decidability problem from $\mathscr{L}_{\boxed{u}}$ to $\mathscr{L}$.

DEFINITION
An $\mathscr{L}$-logic L is *sequentially decidable* if the set of valid consequences $\{\phi \vdash_L \psi\}$ is decidable.

From 5.3 and 5.10 immediately follows

PROPOSITION 5.11
An $\mathscr{L}$-logic L is sequentially decidable iff $L_{\boxed{u}}$ is decidable.

Here we hazard a positive answer of the last question, raising

CONJECTURE 3
For each [decidable] $\mathscr{L}$-logic L there exists an effective function $f_L : \mathrm{FOR} \times \mathrm{FOR} \to \mathbb{N}$ such that for each $\phi, \psi \in \mathrm{FOR}$ $\phi \vdash_L \psi$ iff $\vdash_L \Box^{(n)}\phi \to \psi$ where $n = f_L(\phi, \psi)$.

We finish this section with two more open questions. Do minimal extensions preserve complexity? Do they preserve the interpolation property?

6. Some uses of the universal modality

The universal modality can be a fairly useful tool for axiomatization. Here we sketch some examples demonstrating its merits.

Let us first mention that the standard techniques for proving completeness and finite model property in $\mathcal{L}$ (canonical model, filtrations, etc.) work as well in $\mathcal{L}_{\boxed{u}}$. As we have already noticed, the canonical model technique will cause no additional complications, connected with the non-standard models, since all $\boxed{u}$-rooted models are standard, which is sufficient for the purposes of the completeness. For instance it is a routine task to prove that all conditions, listed in section 3 are *axiomatized* by the corresponding formulae, added to $K_{\boxed{u}}$. Indeed, all of them but the last are canonical (note that $\boxed{u}\Box$ corresponds to the composition $W^2 \circ R$ and $\boxed{u}\Box p \rightarrow p$ says that this relation is reflexive). All these examples axiomatize logics which admit filtration and therefore have the finite model property and are decidable. (The proof for the logic of finite paths $K_{\boxed{u}} + \boxed{u}(\Box p \rightarrow p) \rightarrow p$ goes through a minimal filtration and is a slight modification of the well-known proof of completeness for GL.)

Another curious example is due to Dimiter Vakarelov. The condition $\exists x Rxx$ is definable neither in $\mathcal{L}$ nor in $\mathcal{L}_{\boxed{u}}$ as we have already known. This condition is axiomatized in $\mathcal{L}$ by K, i.e. no part of it can be expressed there. In $\mathcal{L}_{\boxed{u}}$ however, it is axiomatized over $K_{\boxed{u}}$ by the infinite set of axioms $\{\theta_n\}_{n \in \mathbb{N}}$ where $\theta_n = \langle u \rangle((p_1 \rightarrow \Diamond p_1) \wedge \ldots \wedge (p_n \rightarrow \Diamond p_n))$ (without being defined by them). First, all frames with a reflexive world satisfy all θ_n. Actually, validity of θ_n in $F = \langle W, R \rangle$ means that for every n subsets $P_1, \ldots, P_n$ of W there exists a world x which has R-successors in all Ps containing x. In particular, if F is finite and $W = \{x_1, \ldots, x_n\}$ then $F \models \theta_n$ implies (taking the subsets $\{x_1\}, \ldots, \{x_n\}$) that $F \models \exists x Rxx$. So, the axioms $\{\theta_n\}_{n \in \mathbb{N}}$ guarantee existence of an R-reflexive world in all finite frames which satisfy them, though not in all such infinite frames. The proof of completeness uses the standard canonical model technique: observe that if $L = K_{\boxed{u}} + \{\theta_n\}_{n \in \mathbb{N}}$ and $L \nvdash \neg\phi$ then $\{\langle u \rangle \phi\} \cup \{\Box\alpha \rightarrow \alpha / \alpha \in \mathrm{FOR}_{\boxed{u}}\}$ is consistent and hence included in a maximal L-consistent set which is reflexive.

The finitely axiomatized $\mathcal{L}_{\boxed{u}}$-logics form a lattice (unlike the finitely axiomatized $\mathcal{L}$-logics, [3] Ch. 5) as follows from the next proposition.

PROPOSITION 6.1
If $L_1 = K_{\boxed{u}} + \phi_1$ and $L_2 = K_{\boxed{u}} + \phi_2$ are $\mathcal{L}_{\boxed{u}}$-logics and ϕ_1 and ϕ_2 do not share common variables then $L_1 \cap L_2 = K_{\boxed{u}} + \boxed{u}\phi_1 \vee \boxed{u}\phi_2$.

PROOF. It is clear that $L_1 \cap L_2 \vdash \boxed{u}\phi_1 \vee \boxed{u}\phi_2$, hence $K_{\boxed{u}} + \boxed{u}\phi_1 \vee \boxed{u}\phi_2 \subseteq L_1 \cap L_2$. Vice versa, a standard deduction lemma for $\mathcal{L}_{\boxed{u}}$-logics shows that $L_1 \vdash \psi$ iff $K_{\boxed{u}} \vdash \boxed{u}\phi_1^1 \wedge \ldots \wedge \boxed{u}\phi_1^k \rightarrow \psi$ for certain substitution instances $\phi_1^1, \ldots, \phi_1^k$ of ϕ_1; analogously $L_2 \vdash \psi$ iff $K_{\boxed{u}} \vdash \boxed{u}\phi_2^1 \wedge \ldots \wedge \boxed{u}\phi_2^m \rightarrow \psi$ for some $\phi_2^1, \ldots, \phi_2^m$. But $K_{\boxed{u}} + \boxed{u}\phi_1 \vee \boxed{u}\phi_2 \vdash (\boxed{u}\phi_1^1 \wedge \ldots \wedge \boxed{u}\phi_1^k) \vee (\boxed{u}\phi_2^1 \wedge \ldots \wedge \boxed{u}\phi_2^m)$, whence $L_1 \cap L_2 \subseteq K_{\boxed{u}} + \boxed{u}\phi_1 \vee \boxed{u}\phi_2$. #

The above fact is certainly not surprising; an analogous property is proved by analogous arguments, for the normal extensions of S4 in Maksimova and Rybakov [21].

The prime stimulus for considering the universal modality has come up in the context of the *proper names for the possible worlds* (see Passy and Tinchev [22, 23]). They are special kinds of propositional variables evaluated in the Kripke semantics in single worlds which, added to modal and dynamic languages, strongly increase their expressiveness and deductive power. A complete axiomatization of the minimal normal logic K_N in the modal language with names is given in [9] using special kinds of axiom schemata, called in [11] *admissible forms*. The names are axiomatized by the scheme $M(c \wedge A) \rightarrow L(c \rightarrow A)$, where c is a name, A is a formula, M is a possibility form and L is a necessity form.

After adding the universal modality to the language, the need of admissible forms disappears because the form scheme is replaced by the axiom scheme $\langle u \rangle(c \wedge A) \rightarrow [u](c \rightarrow A)$. In addition, we can already say that each name has a denotation by means of the schema $\langle u \rangle c$ and thus to give a complete axiomatization of the names over $K_{[u]}$.

Using $[u]$ one could elegantly axiomatize puzzling non-classical modalities. As an example let us consider a modality $\boxtimes$ with the following semantics in an ordinary Kripke model $\mathcal{M} = \langle W, R, V \rangle$:

(*) $\mathcal{M} \models \boxtimes\phi[x]$ iff $\forall y(Rxy \leftrightarrow \mathcal{M} \models \phi[y])$ i.e. $R(x) = V(\phi)$

We shall call $\boxtimes$ the 'iff-modality' having in mind a natural interpretation as 'necessary and sufficient' (see [9]) or 'all and only' [17]. This is a fairly strange modality: neither monotonic, nor anti-monotonic, but extensional; no formula of the kind $\boxtimes\phi$ or its negation is universally valid.

Humberstone [17] has axiomatized $\boxtimes$ by means of an infinite set of schemata and an infinite set of rules. Adding the universal modality to the language we can replace that really ingenious axiomatics by the following transparent one in the language $\mathcal{L}(\boxtimes, [u])$:

Axiom schemata of the logic IFF:

(1) all propositional tautologies;
(2) S5 axioms for $[u]$;
(3) $(\boxtimes_1) : (\boxtimes p \wedge \boxtimes q) \rightarrow [u](p \leftrightarrow q)$;
(4) $(\boxtimes_2) : [u](p \leftrightarrow q) \rightarrow (\boxtimes p \leftrightarrow \boxtimes q)$;
Rules: MP and $NEC_{[u]}$.

THEOREM 6.2
The logic IFF is sound and complete.

PROOF. Soundness is straightforward. Completeness: let ϕ be an IFF-consistent formula and w be a maximal IFF-consistent set containing ϕ. (There are no problems in the Lindenbaum lemma.) Denote $W_0 = \{y / y$ is a maximal IFF-consistent set and $[u]w \subseteq y\}$ where $[u]w = \{\phi / [u]\phi \in w\}$. Let w'

be a copy of w and $W = W_0 \cup \{w'\}$. It is clear that for every $x, y \in W$, $\boxed{u}x \subseteq y$. Now we define a relation R in W:

$$Rxy \text{ iff } ((\boxtimes\psi \in x \text{ and } \psi \in y \text{ for some } \psi) \text{ or }$$
$$(y = w' \text{ and } \boxtimes\psi \notin x \text{ for every } \psi))$$

Obviously $R(w) = R(w')$. Consider the model $\langle W, R, V\rangle$ with the canonical valuation $V: V(p) = \{x \in W / \psi \in x\}$ for each propositional variable p. Extend V to a valuation on all formulae through the standard semantics of $\boxed{u}$ and (*). Now we shall prove the *truth lemma*: #

For each formula ψ, $V(\psi) = \{x \in W / \psi \in x\}$. The only non-trivial case in the induction on ψ is that $V(\boxtimes\psi) = \{x / \boxtimes\psi \in x\}$.

Let us first observe that if Rxy, then for every θ, $\boxtimes\theta \in x$ implies $\theta \in y$. Indeed, Rxy and $\boxtimes\theta \in x$ imply $\boxtimes\chi \in x$ and $\chi \in y$ for some χ. Then $\boxed{u}(\chi \leftrightarrow \theta) \in x$ by $(\boxtimes_1)$, so $\chi \leftrightarrow \theta \in y$, hence $\theta \in y$.

(1) Let $\boxtimes\psi \in x$. If $y \in V(\psi)$ then by IH $\psi \in y$ and Rxy by definition; if Rxy then $\psi \in y$ by the above observation. Thus $R(x) = V(\psi)$, so $x \in V(\boxtimes\psi)$.

(2) Let $\boxtimes\psi \notin x$. Two cases are possible:

(a) for each χ, $\boxtimes\chi \notin x$. Therefore Rxw' but not Rxw. If $\psi \notin w'$ then $w' \in R(x)\backslash V(\psi)$; if $\psi \in w'$ then $\psi \in w$ and so $w \in V(\psi)\backslash R(x)$ and $x \notin V(\boxtimes\psi)$.

(b) $\boxtimes\chi \in x$ for some χ. Then $\boxtimes\chi \to \boxtimes\psi \notin x$ hence by $(\boxtimes_2)\boxed{u}(\chi \leftrightarrow \psi) \notin x$ so $\langle u\rangle((\psi \wedge \neg\chi) \vee (\chi \wedge \neg\psi)) \in x$, i.e. for some y, $(\psi \wedge \neg\chi) \in y$ or $(\chi \wedge \neg\psi) \in y$.

(b1) $(\psi \wedge \neg\chi) \in y$. Then $\neg Rxy$ since otherwise $\chi \in y$; so $y \in V(\psi)\backslash R(x)$.

(b2) $(\neg\psi \wedge \chi) \in y$. Then $\boxtimes\chi \in x$ and $\chi \in y$ imply Rxy and so $y \in R(x)\backslash V(\psi)$.

The proof of the truth lemma is finished. Thus the theorem is proved.

Let us note that both modalities $\boxed{u}$ and $\boxtimes$ are expressible in the bimodal language $\mathscr{L}(R, -R)$: $\boxed{u}p = \boxplus p \wedge \boxminus p$ and $\boxtimes p = \boxplus p \wedge \boxminus\neg p$ where $\boxplus$ and $\boxminus$ are the modalities corresponding to R and $-R$. The minimal logic of $\mathscr{L}(R, -R)$, $K^{\sim}$ is axiomatized (see [9]) just by S5-axioms for $\boxed{u}$ thus expressed and is proved to have the FMP and hence to be decidable. Since $K^{\sim}$ is conservative over IFF (by an easy semantic argument) we have the FMP and the decidability of IFF.

The last two examples suggest that the universal modality can fairly well play the role of the admissible forms and more precisely, that the admissible forms are devised as rough approximations of $\boxed{u}$.

7. Appendix: Independent joint of $\mathscr{L}_{\boxed{u}}$-logics

This appendix contains a first author's result concerning polymodal logics with $\boxed{u}$. The picture in the polymodal case is much more complicated because of the interaction between the basic modalities but there exists an important particular case—an independent join of modal logics (cf. [26]), without interacting axioms. A natural question is which virtues of the compounding logics are inherited in the independent join. Our result concerns join of $\mathscr{L}_{\boxed{u}}$-logics but after some modifications the technique can be applied to classical modal logics as well. Such results about transfer of basic logical properties as completeness, finite model property, decidability, compactness, etc., are independently obtained in a recent work of Kracht and Wolter [20].

We shall consider join of two logics but the results are readily generalized to the polymodal case (even with infinitely many modalities). Let us start with the exact definitions.

Definition

Let $\mathscr{L}_1 = \mathscr{L}(\boxed{1})$ and $\mathscr{L}_2 = \mathscr{L}(\boxed{2})$ be two modal languages and L_1, L_2 be normal logics in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. We call the logic $L_1 \oplus L_2$ in $\mathscr{L}(\boxed{1}, \boxed{2})$ axiomatized by the schemata both of L_1 and L_2 spread over the joint language an *independent join of L_1 and L_2*.

When consider languages with $\boxed{u}$ we modify the above definition in accordance with the particular role of the universal modality.

Definition

Let $\mathscr{L}_1 = \mathscr{L}(\boxed{1}, \boxed{u})$ and $\mathscr{L}_2 = \mathscr{L}(\boxed{2}, \boxed{u})$ be two modal languages with $\boxed{u}$ (note that both languages share the same universal modality) and L_1, L_2 be normal logics in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. We call the logic $L_1 \oplus L_2$ in $\mathscr{L}(\boxed{1}, \boxed{2}, \boxed{u})$ axiomatized by the schemata both of L_1 and L_2 spread over the joint language an *independent join of L_1 and L_2 over $\mathscr{L}(\boxed{u})$*.

Theorem 7.1

An independent join of sequentially complete normal $\mathscr{L}$-logics is sequentially complete, too.

Proof. Let $\mathscr{L}_1 = \mathscr{L}(\boxed{1})$ and $\mathscr{L}_2 = \mathscr{L}(\boxed{2})$ be two ordinary modal languages and L_1 and L_2 be sequentially complete normal logics in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Denote $\mathscr{L}_{1,2} = \mathscr{L}(\boxed{1}, \boxed{2})$.

We need some preparation and preliminaries.

Definition

An *$\mathscr{L}_i$-approximation of an $\mathscr{L}_{1,2}$-formula ϕ* is the result of the replacement of all maximal occurrences of subformulae beginning by $[j]$, for $j \neq i$, by different propositional variables not occurring in ϕ.

Definition

i-cactus with a set of j-thorns X (for $i = 1, 2, j \neq i$). Inductively:

(1) Every normal L_1-model $\mathcal{M} = \langle W, R_1, V \rangle$ is a 2-cactus $\langle W, R_1, \varnothing, V \rangle$ with a set of 1-thorns W. The same with 1 and 2 exchanged.

(2) Let $\mathcal{M}' = \langle W', R_1, R_2, V' \rangle$ be a 2-cactus with a set of 1-thorns X. Now let $\{\mathcal{M}_x = \langle W_x, R_x^2, V_x \rangle / x \in X\}$ be a set of disjoint normal L_2-models such that:

(i) $\mathcal{M}_x$ is generated by x for every $x \in X$.
(ii) $W_x \cap W' = \{x\}$.
(iii) V_x agrees with V' over x.

Then the structure

$$\mathcal{M} = \langle W' \cup \bigcup \{W_x / x \in X\}, R_1, R_2 \cup \bigcup \{R_x^2 / x \in X\}, V \rangle \quad \text{where}$$
$$V_{|W_x} = V_x \quad \text{and} \quad V_{|W'} = V'$$

is a 1-cactus with a set of 2-thorns $\bigcup \{W_x \backslash \{x\} / x \in X\}$. Analogously, starting from a 1-cactus we obtain a 2-cactus.

Now let $\mathcal{M}_1, \ldots, \mathcal{M}_n, \ldots$, by an infinite sequence of cactuses, successively constructed as above. They form an increasing chain by inclusion. So we can consider the 'limit' of the above construction: $\mathcal{M} = \langle \bigcup W_k, \bigcup R_1^k, \bigcup R_2^k, \bigcup V_k \rangle$. It will be called an *infinitary cactus.*

FACT
Every infinitary cactus is a normal $L_1 \oplus L_2$-model. Indeed, erasing all R_1-arrows in the cactus we obtain a disjoint union of normal L_2-models and vice versa.

Now, we are ready to start the proof. Suppose not $\phi \vdash_{L_1 \oplus L_2} \neg\psi$. We will construct an infinitary cactus $\mathcal{M}$, generated by some w, such that $\mathcal{M} \models \phi$ and $\mathcal{M} \models \psi[w]$.

We start constructing the series of 1-cactuses and 2-cactuses. Let $\gamma = \phi \wedge \psi$ and $p_1, \ldots, p_k$ be all propositional variables occurring in γ, $\chi_1, \ldots, \chi_s$ be all subformulae of γ beginning with $\boxed{1}$ and $\chi_{s+1}, \ldots, \chi_m$ be those subformulae which begin with $\boxed{2}$. Now, let $p_{k+1}, \ldots, p_{k+m}$ be new, different, propositional variables which will represent $\chi_1, \ldots, \chi_m$ in the approximations. Hereafter we shall take care only for the variables $p_1, \ldots, p_{k+m}$; all others will be irrelevant for what we shall do.

Let $\gamma^1 = \phi^1 \wedge \psi^1$ be the corresponding $\mathcal{L}_1$-approximation of γ.

Denote by $\theta_1, \ldots, \theta_r$ all formulae of the type:

$$\theta_j = \varepsilon_1^j p_1 \wedge \ldots \wedge \varepsilon_k^j p_k \wedge \varepsilon_{k+1}^j \chi_1 \wedge \ldots \wedge \varepsilon_{k+m}^j \chi_m$$

where each ε is either empty or negation, and such that not $\phi \vdash_{L_1 \oplus L_2} \neg\theta_j$.

Let $\theta_j^i = \varepsilon_1^j p_1 \wedge \ldots \wedge \varepsilon_k^j p_k \wedge \varepsilon_{k+1}^j \chi_1^i \wedge \ldots \wedge \varepsilon_{k+m}^j \chi_m^i$, $i = 1, 2$, be their approximations. Denote $\theta = \theta_1 \vee \ldots \vee \theta_r$ and $\theta^i = \theta_1^i \vee \ldots \vee \theta_r^i$, $i = 1, 2$. We call θ a *consistency description for* ϕ *and* ψ and denote it by CD(ϕ, ψ). By propositional reasoning, $\phi \vdash_{L_1 \oplus L_2} \theta$. Therefore not $\phi \wedge \theta \vdash_{L_1 \oplus L_2} \neg\psi$ hence not $\phi^1 \wedge \theta^1 \vdash_{L_1} \neg\psi^1$. Then, by sequential completeness of L_1, there exists a normal L_1-model $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ generated by some w, and such that $\mathcal{M}_1 \models \phi^1 \wedge \theta^1$ and $\mathcal{M}_1 \models \psi^1[w]$. We may assume that V_1 evaluates only the variables which may appear in $\mathcal{L}_1$-approximations, i.e. $\{p_1, \ldots, p_k\} \cup \{p_{k+s+1}, \ldots, p_{k+m}\}$. Then we extend V_1 over $p_{k+1}, \ldots, p_{k+s}$ in accordance with their destination: $V_1(p_{k+i}) =_{DF} V_1(\chi_i^1)$, $i = 1, \ldots, s$. Consider $\mathcal{M}_1$ as a 2-cactus with a set of 1-thorns W_1. Now for each 1-thorn x there exists exactly one θ_j such that $\mathcal{M}_1 \models \theta_j^1[x]$. Since not $\phi \vdash_{L_1 \oplus L_2} \neg\theta_j$ then not $\phi^2 \vdash_{L_2} \neg\theta_j^2$. Let $\mathcal{M}_x = \langle W_x, R_x^2, V_x\rangle$ be a normal L_2-model, generated by x and such that $W_x \cap W_1 = \{x\}$, $\mathcal{M}_x \models \theta_j^2[x]$ and $\mathcal{M}_x \models \phi^2 \wedge (\mathrm{CD}(\phi, \theta_j))^2$ (repeat the above reasoning with θ_j instead of ψ). Again we may assume that V_x evaluates only those variables which may appear in $\mathcal{L}_2$-approximations, i.e. $\{p_1, \ldots, p_k, p_{k+1}, \ldots, p_{k+s}\}$. Extend V_x over $p_{k+s+1}, \ldots, p_{k+m}$ accordingly: $V_x(p_{k+i}) =_{DF} V_1(\chi_i^2)$, $i = s+1, \ldots, m$. Then clearly V_1 and V_x agree on x.

Taking such disjoint normal L_2-models for every $x \in W_1$ and proceeding in the same way we construct a 1-cactus as in the definition. Repeating this procedure infinitely many times we obtain a sequence of cactuses and finally an infinitary cactus $\mathcal{M} = \langle W, R_1, R_2, V\rangle$.

We shall prove by a structural induction that for every subformula α of $\phi \wedge \psi$, $\mathcal{M} \models \alpha[x]$ iff $\mathcal{M}^1 \models \alpha^1[x]$ iff $\mathcal{M}^2 \models \alpha^2[x]$ where $\mathcal{M}^i$ is $\mathcal{M}$ considered as an $\mathcal{L}_i$-model, for $i = 1, 2$.

When α is a propositional variable there is nothing to prove. The Boolean steps are trival as usual.

Let $\alpha = \boxed{1}\beta$. Then $\alpha^1 = \boxed{1}\beta^1$ and $\mathcal{M} \models \boxed{1}\beta[x]$ iff $\forall y \in \mathcal{M}(R_1xy \Rightarrow \mathcal{M} \models \beta[y])$ iff (by IH) $\forall y \in \mathcal{M}(R_1xy \Rightarrow \mathcal{M}^1 \models \beta^1[y])$ iff $\mathcal{M}^1 \models \boxed{1}\beta^1[x]$.

Now let $\boxed{1}\beta = \chi_i$ for some $i \in \{1, \ldots, s\}$. Then $\alpha^2 = p_{k+i}$. We shall prove $\mathcal{M}^1 \models \boxed{1}\beta^1[x]$ iff $\mathcal{M}^2 \models p_{k+i}[x]$. Let $\mathcal{M}_k$ be the cactus from the sequence constructed above, where x appears as a thorn and let $\mathcal{M}_k^1$ and $\mathcal{M}_k^2$ be its restrictions to $\mathcal{L}_1$ and $\mathcal{L}_2$. Also, let $\mathcal{M}_x$ be the model grafted at x. We consider two cases:

(a) x is a 1-thorn. Then $\mathcal{M}^1 \models \boxed{1}\beta^1[x]$ iff $\mathcal{M}_k^1 \models \boxed{1}\beta^1[x]$ and $\mathcal{M}^2 \models p_{k+i}[x]$ iff $\mathcal{M}_x \models p_{k+i}[x]$ ($\mathcal{M}_x$ is an L_2-model). Let $j \in \{1, \ldots, r\}$ be such that $\mathcal{M}_k^1 \models \theta_j^1[x]$ (j exists by the construction). Now, $\mathcal{M}_k^1 \models \boxed{1}\beta^1[x]$ iff $\boxed{1}\beta$ has a positive occurrence in θ_j iff p_{k+i} has a positive occurrence in θ_j^2 iff $\mathcal{M}_x \models p_{k+i}[x]$ because $\mathcal{M}_x \models \theta_j^2[x]$.

(b) x is a 2-thorn. Then $\mathcal{M}^1 \models \boxed{1}\beta^1[x]$ iff $\mathcal{M}_x \models \boxed{1}\beta^1[x]$ ($\mathcal{M}_x$ is an L_1-model) and $\mathcal{M}^2 \models p_{k+i}[x]$ iff $\mathcal{M}_k^2 \models p_{k+i}[x]$. Let $j \in \{1, \ldots, r\}$ be such that $\mathcal{M}_k^2 \models \theta_j^2[x]$. Now, $\mathcal{M}_k^2 \models p_{k+i}[x]$ iff p_{k+i} has a positive occurrence in θ_j^2 iff $\boxed{1}\beta$ has a positive occurrence in θ_j iff $\mathcal{M}_x \models \boxed{1}\beta^1[x]$ because $\mathcal{M}_x \models \theta_j^1[x]$.

The case $\alpha = \boxed{2}\beta$ is analogous. The induction is completed. As immediate consequences $\mathcal{M} \models \phi$ and $\mathcal{M} \models \psi[w]$, what we need. #

As a corollary of this proof we can uniformly translate derivability in $L_1 \oplus L_2$ to derivability in the components:

$$\phi \underset{L_1 \oplus L_2}{\vdash} \psi \quad \text{iff} \quad \phi^1 \wedge (\mathrm{CD}(\phi, \psi))^1 \underset{L_1}{\vdash} \psi^1 \quad \text{iff} \quad \phi^2 \wedge (\mathrm{CD}(\phi, \psi))^2 \underset{L_2}{\vdash} \psi^2$$

However, this translation seems not to be really effective since it involves again derivability in $L_1 \oplus L_2$ in order to define $\mathrm{CD}(\phi, \psi)$.

Now, let us note that the results of section 5 are easily generalized for polymodal logics, i.e. a polymodal logic is (finitely) sequentially complete iff its minimal extension in $\mathscr{L}_{\boxed{u}}$ is (finitely) complete. Since the constructions join and minimal extension commute, we obtain:

PROPOSITION 7.2
The independent join over $\mathscr{L}(\boxed{u})$ of the minimal extensions of normal $\mathscr{L}$-logics L_1 and L_2 coincides with the minimal extension of the independent join of L_1 and L_2.

COROLLARY 7.3
The independent join over $\mathscr{L}(\boxed{u})$ of complete minimal extensions of normal $\mathscr{L}$-logics L_1 and L_2 is complete, too.

Modifying the technique used in the proof of Theorem 7.1 one can obtain similar results concerning strong completeness and compactness of independent join of minimal extensions. Then, as a corollary of Theorem 5.5 these results can be transferred to usual $\mathscr{L}$-logics.

As for independent join over $\mathscr{L}(\boxed{u})$ of arbitrary $\mathscr{L}_{\boxed{u}}$-logics, the generalization of Theorem 7.1 is another open problem.

Acknowledgements

We are indebted to Tinko Tinchev, who—introducing the universal modality in combinatory dynamic logic—introduced us to this modality, too. Also we thank Dimiter Vakarelov, who made some comments and communicated several useful examples to us, that have been included in the text. Finally, we are grateful to the referees for the careful reading and the valuable remarks which have improved both the style and the content of the paper.

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