Progress of Theoretical Physics, Vol. 99, No. 1, January 1998

$USp(2k)$ Matrix Model: F Theory Connection

Hiroshi ITOYAMA and Akira TOKURA

Department of Physics, Graduate School of Science
Osaka University, Toyonaka 560, Japan

(Received August 26, 1997)

We present a zero-dimensional matrix model based on $USp(2k)$ with supermultiplets in symmetric, antisymmetric and fundamental representations. The four-dimensional compactification of this model naturally captures the exact results of Sen [1] in F theory. Eight dynamical and eight kinematical supercharges are found, as required for critical string interpretation. The classical vacuum has ten coordinates and is equipped with orbifold structure. We clarify the issue of spacetime dimensions which F theory represented by this matrix model produces.

§1. Introduction

Recently there have been a few interesting directions emerging on the formulation of strings. On the one hand, a nonperturbative formulation of string theory as matrix models via the notion of noncommuting coordinates [2] is developing. This includes vigorous activities on the large N quantum mechanical model [3] which formulates M theory [4] as well as the zero-dimensional model [5),6] of type-IIB superstrings. Notions of string compactification and attendant counting of degrees of freedom appear to be very different from what we had thought from unification based on perturbative strings. So far we have been able to treat only toroidal compactifications with/without discrete projection [3),7)-9] through a specific procedure. [9]

Another interesting direction includes the developments centered around F theory. [10] This provides a new perspective on treating type-IIB strings on exact quantum backgrounds through a purely geometrical framework. F theory captures the intriguing phenomenon of string coupling depending on internal space*) beyond perturbative considerations. One way of viewing this F theory is that it provides us with a new scheme of compactifications of string theory which are defined beyond perturbation theory. This is the point of view we wish to adopt in the present paper. A series of compactifications whose perturbative limits are those of orientifold are prototypical examples.

In this paper, we wish to give this scheme of F theory a constructive framework as a matrix model. We present a $USp(2k)$ matrix model in zero dimensions and discuss several properties. We argue that our model in the particular large N limit produces an exact F theory compactification. The model consists of matrices belonging to symmetric (adjoint) and antisymmetric representations and of n_f $2k$-dimensional vectors. The $n_f = 4$ and $n_f = 16$ cases are of special significance. The

*) In the conventional approach of the first quantized strings, this is physics related to the orientifold compactification. [11),12)]

large k limit captures string physics in the sense of 't Hooft. The model is inspired in the $n_f = 4$ case by the supermultiplets of the UV finite $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions with the gauge group $USp(2k)$. We are motivated by the exact result of Sen[1)] in F theory in eight spacetime dimensions on a certain elliptic fibered $K3$. The exact description of the axion/dilaton sector on this F theory compactification has been found to be mathematically identical to that of the quantum moduli space[13)] of the susy gauge theory mentioned above. (See Refs. 14) and 15).) We thus see that our model (the $n_f = 4$ case) after four-dimensional compactification under the procedure of Ref. 9) possesses the above quantum moduli space and naturally reproduces Sen's result in F theory in eight spacetime dimensions. We will discuss this again at the end of the paper.

In the next section, we define the $USp(2k)$ matrix model in zero dimensions. Eight dynamical and eight kinematical supercharges are shown to exist in our model in §3. This is necessary for this model to be interpreted as a critical string theory, i.e., a unified theory of gravity and other forces.

In §4, we determine the classical vacuum, which is found to be labelled by ten coordinates. The one-loop stability of this geometry is ensured by supersymmetry. We discuss the case in which the model in a particular large k limit produces an exact F theory compactification. This is the compatification whose perturbative limit is described by the eight-dimensional type-IIB string on a T^2/Z^2 orientifold. The issue of twelve versus ten spacetime dimensions naturally emerges. We clarify the sense of the spacetime dimensions designated by the model in the particular large k limit (compactification).

We adopt notation in which the inner product of two $2k$-dimensional vectors u_i and v_i invariant under $USp(2k)$ is

$$\langle u, v\rangle = u_i F^{ij} v_j \ , \quad \text{with} \quad F^{ij} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} . \tag{1}$$

Here I_k is the unit matrix. We can define $(u_i)^* \equiv (u^*)^i$. Raising and lowering of the indices are accomplished by $F = F^{ij}$ and $F^{-1} = F_{ij}$. Any element X of the $usp(2k)$ Lie algebra satisfying $X^t F + FX = 0$ and $X^\dagger = X$ can be represented as

$$X = \begin{pmatrix} M & N \\ N^* & -M^t \end{pmatrix}, \quad \text{with} \ M^\dagger = M, \ \ N^t = N. \tag{2}$$

Chiral superfields are expanded by the generators of $usp(2k)$ with complex coefficients.

§2. Definition of the zero dimensional matrix model

Our zero-dimensional model can be written by borrowing $N = 1$, $d = 4$ superfield notation in the Wess-Zumino gauge. One simply drops all spacetime dependence of the fields while keeping all Grassmann coordinates unchanged:

$$S \equiv S_{\rm vec} + S_{\rm asym} + S_{\rm fund},$$

$$S_{\rm vec} = \frac{1}{4g^2}\,{\rm Tr}\left(\int d^2\theta W^\alpha W_\alpha + {\rm h.c.} + 4\int d^2\theta d^2\bar\theta \Phi^\dagger e^{2V}\Phi e^{-2V}\right),$$

$$S_{\rm asym} = \frac{1}{g^2}\int d\theta^2 d\bar\theta^2\left(T^{*\,ij}\left(e^{2V({\rm asym})}\right)_{ij}^{\;\;k\ell} T_{k\ell} + \tilde T^{ij}\left(e^{-2V({\rm asym})}\right)_{ij}^{\;\;k\ell}\tilde T^*_{k\ell}\right)$$
$$+\frac{1}{g^2}\left\{\sqrt2\int d\theta^2\tilde T^{ij}\left(\Phi_{({\rm asym})}\right)_{ij}^{\;\;k\ell} T_{k\ell} + {\rm h.c.}\right\},$$

$$S_{\rm fund} = \frac{1}{g^2}\sum_{f=1}^{n_f}\left[\int d^2\theta d^2\bar\theta\left(Q^{*\,i}_{(f)}\left(e^{2V}\right)_i^{\;j} Q_{(f)\,j} + \tilde Q^i_{(f)}\left(e^{-2V}\right)_i^{\;j}\tilde Q^*_{(f)\,j}\right)\right.$$
$$\left. + \left\{\int d^2\theta\left(m_{(f)}\tilde Q_{(f)}^{\;i} Q_{(f)\,i} + \sqrt2\tilde Q_{(f)}^{\;i}(\Phi)_i^{\;j} Q_{(f)\,j}\right) + {\rm h.c.}\right\}\right]. \quad (3)$$

The chiral superfields introduced above are

$$W_\alpha = -\frac{1}{8}\bar D\bar D e^{-2V} D_\alpha e^{2V},\ \Phi = \Phi + \sqrt2\theta\psi_\Phi + \theta\theta F_\Phi, \quad (4)$$

$$Q_i = Q_i + \sqrt2\theta\psi_{Q\,i} + \theta\theta F_{Q\,i}\ ,\ T_{ij} = T_{ij} + \sqrt2\theta\psi_{T\,ij} + \theta\theta F_{T\,ij}, \quad (5)$$

while

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha},\ \ \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}, \quad (6)$$

$$V = -\theta\sigma^m\bar\theta v_m + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac12\theta\theta\bar\theta\bar\theta D\ . \quad (7)$$

We represent the antisymmetric tensor superfield T_{ij} as

$$Y \equiv (TF)_i^{\,j} = \begin{pmatrix} A & B \\ C & A^t\end{pmatrix} \quad (8)$$

with $B^t = -B$, $C^t = -C$. We define $\tilde Y$ similarly.

In terms of components, the action reads (with indices suppressed)

$$S_{\rm vec} = \frac{1}{g^2}{\rm Tr}\Big(-\frac14 v_{mn}v^{mn} - [\mathcal D_m,\Phi]^\dagger[\mathcal D^m,\Phi] - i\lambda\sigma^m[\mathcal D_m,\bar\lambda] - i\bar\psi\bar\sigma^m[\mathcal D_m,\psi]$$
$$-i\sqrt2[\lambda,\psi]\Phi^\dagger - i\sqrt2[\bar\lambda,\bar\psi]\Phi\Big)$$
$$+\frac{1}{g^2}{\rm Tr}\left(\frac12 DD - D[\Phi^\dagger,\Phi] + F_\Phi^\dagger F_\Phi\right), \quad (9)$$

$$S_{\rm asym} = \frac{1}{g^2}\Big\{-(\mathcal D_m T)^*(\mathcal D^m T) - i\bar\psi_T\bar\sigma^m\mathcal D_m\psi_T$$
$$-i\sqrt2 T^*\lambda^{({\rm asym})}\psi_T + i\sqrt2\bar\psi_T\bar\lambda^{({\rm asym})}T$$
$$-(\mathcal D_m\tilde T)(\mathcal D^m\tilde T)^* - i\bar\psi_{\tilde T}\bar\sigma^m\mathcal D_m\psi_{\tilde T} - i\sqrt2\tilde T^*\lambda^{({\rm asym})}\psi_{\tilde T} + i\sqrt2\bar\psi_{\tilde T}\bar\lambda^{({\rm asym})}\tilde T$$
$$-2(\Phi^*_{({\rm asym})}T^*)(\Phi_{({\rm asym})}T) - 2(\tilde T\Phi_{({\rm asym})})(\tilde T^*\Phi^*_{({\rm asym})})$$

$$-\sqrt{2}(\psi_{\tilde{T}}\psi^{(\mathrm{asym})}T + \tilde{T}\psi^{(\mathrm{asym})}\psi_T + \psi_{\tilde{T}}\Phi_{(\mathrm{asym})}\psi_T)$$
$$-\sqrt{2}(\overline{\psi}_T\overline{\psi}^{(\mathrm{asym})}\tilde{T}^* + T^*\overline{\psi}^{(\mathrm{asym})}\overline{\psi}_{\tilde{T}} + \psi_T\Phi^*_{(\mathrm{asym})}\overline{\psi}_{\tilde{T}})$$
$$+\sqrt{2}\tilde{T}F_\Phi^{(\mathrm{asym})}T + \sqrt{2}\tilde{T}^*F_\Phi^{*(\mathrm{asym})}T^* + \tilde{T}D^{(\mathrm{asym})}T + \tilde{T}^*D^{(\mathrm{asym})}T^*\Big\}, \quad (10)$$

$$S_{\mathrm{fund}} = +\frac{1}{g^2}\sum_{f=1}^{n_f}(-(\mathcal{D}_mQ_{(f)})^*(\mathcal{D}^mQ_{(f)}) - i\overline{\psi}_{Q(f)}\overline{\sigma}^m\mathcal{D}_m\psi_{Q(f)}$$
$$+i\sqrt{2}Q^*_{(f)}\lambda\psi_{Q(f)} - i\sqrt{2}\overline{\psi}_{Q(f)}\overline{\lambda}Q_{(f)})$$
$$+\frac{1}{g^2}\sum_{f=1}^{n_f}(-(\mathcal{D}_m\tilde{Q}_{(f)})(\mathcal{D}^m\tilde{Q}_{(f)})^* - i\overline{\psi}_{\tilde{Q}(f)}\overline{\sigma}^m\mathcal{D}_m\psi_{\tilde{Q}(f)}$$
$$-i\sqrt{2}\tilde{Q}_{(f)}\lambda\psi_{\tilde{Q}(f)} + i\sqrt{2}\overline{\psi}_{\tilde{Q}(f)}\overline{\lambda}\tilde{Q}^*_{(f)})$$
$$+\frac{1}{g^2}\sum_{f=1}^{n_f}(Q^*_{(f)}DQ_{(f)} + \tilde{Q}_{(f)}D\tilde{Q}^*_{(f)})$$
$$+\frac{1}{g^2}\sum_{f=1}^{n_f}\Big\{-(m_{(f)})^2(Q^*_{(f)}Q_{(f)} + \tilde{Q}_{(f)}\tilde{Q}^*_{(f)}) - m_{(f)}(\tilde{\psi}_{Q(f)}\psi_{Q(f)} + \bar{\tilde{\psi}}_{Q(f)}\bar{\psi}_{Q(f)})$$
$$-\sqrt{2}(Q^*_{(f)}\Phi^\dagger Q_{(f)} + \tilde{Q}_{(f)}\Phi^\dagger\tilde{Q}^* + Q^*_{(f)}\Phi Q_{(f)} + \tilde{Q}_{(f)}\Phi\tilde{Q}^*_{(f)})$$
$$-2Q^*_{(f)}\Phi^\dagger\Phi Q_{(f)} - 2\tilde{Q}_{(f)}\Phi^\dagger\Phi\tilde{Q}^*_{(f)}$$
$$-\sqrt{2}(\psi_{\tilde{Q}(f)}\psi Q_{(f)} + \tilde{Q}_{(f)}\psi\psi_{Q(f)} + \psi_{\tilde{Q}(f)}\Phi\psi_{Q(f)})$$
$$-\sqrt{2}(\overline{\psi}_{Q(f)}\overline{\psi}\tilde{Q}^*_{(f)} + Q^*_{(f)}\overline{\psi}\overline{\psi}_{\tilde{Q}(f)} + \psi_{Q(f)}\Phi^\dagger\overline{\psi}_{\tilde{Q}(f)})$$
$$+\sqrt{2}\tilde{Q}_{(f)}F_\Phi Q_{(f)} + \sqrt{2}\tilde{Q}^*_{(f)}F^\dagger_\Phi Q^*_{(f)}\Big\}, \quad (11)$$

where

$$D_i{}^j = [\Phi^\dagger,\Phi]_i{}^j + \sum_{f=1}^{n_f}(Q^{*j}_{(f)}Q_{(f)\,i} + \tilde{Q}^j_{(f)}\tilde{Q}^*_{(f)\,i}) + 2T^{*\,jk}T_{ki} + 2\tilde{T}^{jk}\tilde{T}^*_{ki}, \quad (12)$$

$$F_{\Phi\,i}{}^j = -\sum_{f=1}^{n_f}(\sqrt{2}Q^{*j}_{(f)}\tilde{Q}^*_{(f)\,i}) - \sqrt{2}T^{*\,jk}T^*_{ki}. \quad (13)$$

Here $\mathcal{D}_m = iv_m$ with v_m in appropriate representations, and $\Phi_{(\mathrm{asym})}$, $\psi^{(\mathrm{asym})}$ and $F_\Phi^{(\mathrm{asym})}$ are the fields in the anti-symmetric representation.

As is discussed in the Introduction, after the four-dimensional compactification the model possesses the exact quantum moduli space which describes the local deformation of four 7-branes away from an orientifold surface in F theory on $K3$. For this to hold, we have to set $n_f = 4$ and keep nonvanishing mass parameters. In a more generic situation, only the global cancellation of the charge associated with an eight-form potential is required, and in this case $n_f = 16$. It remains to be seen whether our model is able to provide a constructive framework for this general situation.

§3. Dynamical and kinematical supercharges

Let us determine the number of supercharges our model possesses. It is straightforward to confirm that the action of our $USp(2k)$ matrix model is invariant under the following dynamical supersymmetry transformations:

$$
\begin{aligned}
\delta^{(1)} v_m &= -i\bar{\xi}\bar{\sigma}_m\lambda + i\bar{\lambda}\bar{\sigma}_m\xi - i\bar{\eta}\bar{\sigma}_m\psi + i\bar{\psi}\bar{\sigma}_m\eta, \\
\delta^{(1)}\lambda &= \sigma^{mn}\xi v_{mn} + i\xi D - i\sqrt{2}\sigma^m\bar{\eta}\mathcal{D}_m\Phi - \sqrt{2}\eta F_\Phi, \\
\delta^{(1)}\Phi &= \sqrt{2}\xi\psi - \sqrt{2}\eta\lambda, \\
\delta^{(1)}\psi &= i\sqrt{2}\sigma^m\bar{\xi}\mathcal{D}_m\Phi + \sigma^{mn}\eta v_{mn} + i\eta D + \sqrt{2}\xi F_\Phi, \\
\delta^{(1)} T &= \sqrt{2}\xi\psi_T - \sqrt{2}\bar{\eta}\bar{\psi}_{\tilde{T}}, \\
\delta^{(1)}\tilde{T}^* &= \sqrt{2}\bar{\xi}\bar{\psi}_{\tilde{T}} + \sqrt{2}\eta\psi_T, \\
\delta^{(1)}\psi_T &= +i\sqrt{2}\sigma^m\bar{\xi}\mathcal{D}_m T + i\sqrt{2}\sigma^m\bar{\eta}\mathcal{D}_m\tilde{T}^* + \sqrt{2}\xi F_T + \sqrt{2}\eta F_{T(T\to\tilde{T}^*,\tilde{T}^*\to -T)}, \\
\delta^{(1)}\bar{\psi}_{\tilde{T}} &= -i\sqrt{2}\xi\sigma^m\mathcal{D}_m\tilde{T}^* + i\sqrt{2}\eta\sigma^m\mathcal{D}_m T + \sqrt{2}\bar{\xi}F^*_{\tilde{T}} + \sqrt{2}\bar{\eta}F^*_{\tilde{T}(T\to\tilde{T}^*,\tilde{T}^\dagger\to -T)}, \\
\delta^{(1)} Q &= \sqrt{2}\xi\psi_Q - \sqrt{2}\bar{\eta}\bar{\psi}_{\tilde{Q}}, \\
\delta^{(1)}\tilde{Q}^* &= \sqrt{2}\bar{\xi}\bar{\psi}_{\tilde{Q}} + \sqrt{2}\eta\psi_Q, \\
\delta^{(1)}\psi_Q &= +i\sqrt{2}\sigma^m\bar{\xi}\mathcal{D}_m Q + i\sqrt{2}\sigma^m\bar{\eta}\mathcal{D}_m\tilde{Q}^* + \sqrt{2}\xi F_Q + \sqrt{2}\eta F_{Q(Q\to\tilde{Q}^*,\tilde{Q}^*\to -Q)}, \\
\delta^{(1)}\bar{\psi}_{\tilde{Q}} &= -i\sqrt{2}\xi\sigma^m\mathcal{D}_m\tilde{Q}^* + i\sqrt{2}\eta\sigma^m\mathcal{D}_m Q + \sqrt{2}\bar{\xi}F^*_{\tilde{Q}} + \sqrt{2}\bar{\eta}F^*_{\tilde{Q}(Q\to\tilde{Q}^*,\tilde{Q}^*\to -Q)},
\end{aligned}
\tag{14}
$$

where D and F_Φ are given by (12) and (13), and

$$F_{T\,ij} = -\sqrt{2}\left(\Phi^*_{(\text{asym})}\right)_{ij}^{\ \ k\ell}\tilde{T}^*_{k\ell},\ \ F_{Q\,i} = -m\tilde{Q}^*_i - \sqrt{2}\tilde{Q}^*_k\Phi^{*k}{}_i, \tag{15}$$

$$F^*_{\tilde{T}\,ij} = -\sqrt{2}\left(\Phi_{(\text{asym})}\right)_{ij}^{\ \ k\ell}\tilde{T}_{k\ell},\ \ F^*_{\tilde{Q}\,i} = -m\tilde{Q}_i - \sqrt{2}\Phi_i{}^j\tilde{Q}_j. \tag{16}$$

The kinematical supersymmetry transformations are

$$
\begin{aligned}
&\delta^{(2)} v_m = 0, \quad \delta^{(2)}\lambda = 0, \quad \delta^{(2)}\Phi = 0, \quad \delta^{(2)}\psi = 0, \\
&\delta^{(2)} Q = 0, \quad \delta^{(2)}\tilde{Q}^* = 0, \quad \delta^{(2)}\psi_Q = 0, \quad \delta^{(2)}\bar{\psi}_{\tilde{Q}} = 0, \\
&\delta^{(2)} T = 0, \quad \delta^{(2)}\tilde{T}^* = 0, \quad \delta^{(2)}\psi_T = \zeta, \quad \delta^{(2)}\bar{\psi}_{\tilde{T}} = \bar{\tilde{\zeta}}.
\end{aligned}
\tag{17}
$$

Our model has eight dynamical supercharges and eight kinematical supercharges. This is the proper number of supercharges in order for this model to be interpretable as critical strings. Up to field dependent gauge transformations and equations of motion for the fermionic fields, we obtain the following commutation relations:

$$
\begin{aligned}
[\delta^{(1)}_{\xi,\eta}, \delta^{(1)}_{\xi',\eta'}] T &= 0, \\
[\delta^{(1)}_{\xi,\eta}, \delta^{(1)}_{\xi',\eta'}] \psi_T &= 0, \\
[\delta^{(1)}_{\xi,\eta}, \delta^{(1)}_{\xi',\eta'}] \tilde{T}^* &= 0, \\
[\delta^{(1)}_{\xi,\eta}, \delta^{(1)}_{\xi',\eta'}] \bar{\psi}_{\tilde{T}} &= 0.
\end{aligned}
\tag{18}
$$

We also have the following commutation relations:

$$\begin{aligned}
[\delta^{(1)}_{\xi,\eta}, \delta^{(2)}_{\zeta,\bar{\zeta}}]T &= \sqrt{2}(\xi\zeta - \bar{\eta}\bar{\bar{\zeta}}), \\
[\delta^{(1)}_{\xi,\eta}, \delta^{(2)}_{\zeta,\bar{\zeta}}]\psi_T &= 0, \\
[\delta^{(1)}_{\xi,\eta}, \delta^{(2)}_{\zeta,\bar{\zeta}}]\tilde{T}^* &= \sqrt{2}(\xi\zeta + \bar{\eta}\bar{\bar{\zeta}}), \\
[\delta^{(1)}_{\xi,\eta}, \delta^{(2)}_{\zeta,\bar{\zeta}}]\bar{\psi}_{\tilde{T}} &= 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
[\delta^{(2)}_{\zeta,\bar{\zeta}}, \delta^{(2)}_{\zeta',\bar{\zeta}'}]T &= 0, \\
[\delta^{(2)}_{\zeta,\bar{\zeta}}, \delta^{(2)}_{\zeta',\bar{\zeta}'}]\psi_T &= 0, \\
[\delta^{(2)}_{\zeta,\bar{\zeta}}, \delta^{(2)}_{\zeta',\bar{\zeta}'}]\tilde{T}^* &= 0, \\
[\delta^{(2)}_{\zeta,\bar{\zeta}}, \delta^{(2)}_{\zeta',\bar{\zeta}'}]\bar{\psi}_{\tilde{T}} &= 0.
\end{aligned} \tag{20}$$

The combination $\delta^{(1)} \pm \delta^{(2)}$, therefore, forms the supersymmetry algebra of sixteen supercharges which closes into translation of the four of the bosonic matrices in the antisymmetric representation.

§4. Vacuum configuration

Let us find a configuration with vanishing action, which is a particular classical solution of the model. This tells us how many spacetime coordinates are generated from our model. We set all fermions to zero in the action. We first demand

$$\begin{aligned}
v_{mn} &= 0, \\
[\mathcal{D}_m, \Phi] &= i\,[v_m, \Phi] = 0, \\
\mathcal{D}_m Q_f &= \mathcal{D}_m \tilde{Q}_f = 0.
\end{aligned} \tag{21}$$

We see that all of the v_m, Φ and $\Phi^\dagger$ lie on the Cartan subalgebra of $usp(2k)$. Namely,

$$N = 0 \text{ and } M = d = \text{diagonal} \tag{22}$$

in X of Eq. (2). In addition,

$$Q_f = \tilde{Q}_f = 0. \tag{23}$$

As for the antisymmetric tensor fields, we first examine

$$||\, \mathcal{D}^m\, (\text{asym})_{ij}^{k\ell}\, T_{k\ell}\, ||^2, \tag{24}$$

where $\mathcal{D}^m\,(\text{asym}) = iv^{m\,(r)}t^{(r)}(\text{asym})$. This can be written as

$$\text{tr}\,[\mathcal{D}_m, Y]\left[\mathcal{D}_m, Y^\dagger\right]. \tag{25}$$

In general, the commutator $[X,Y]$ with $X \in usp(2k)$ is written in terms of $k \times k$ blocks as

$$[X,Y] = \begin{pmatrix} [M,A] - (-NC + BN^*) & MB - (MB)^t - AN + (AN)^t \\ N^*A - (N^*A)^t - CM + (CM)^t & [M,A]^t - (-NC + BN^*)^t \end{pmatrix}. \tag{26}$$

When X is restricted to the Cartan subalgebras, the condition that the commutator vanish, $[X,Y]=0$, implies

$$[d,A]=0,\ \ (d_i+d_j)\,B_{ij}=0 \ \text{ and } \ (d_i+d_j)\,C_{ij}=0 \ \text{ not summed,} \tag{27}$$

and therefore

$$\begin{aligned} A = a &= \text{diagonal},\ \ B = C = 0, \\ \tilde{A} = \tilde{a} &= \text{diagonal},\ \ \tilde{B} = \tilde{C} = 0 \end{aligned} \tag{28}$$

in Eq. (8).

Under Eqs. (21) and (23), with all fermions set zero, the remaining part of the action $S_{\rm res}$ is

$$\begin{aligned} S_{\rm res} &= \frac{1}{g^2}{\rm Tr}\left\{\frac{1}{2}DD - D[\Phi^\dagger,\Phi] + F_\Phi^\dagger F_\Phi\right\} \\ &\quad -2(\Phi^*_{\rm (asym)}T^*)(\Phi_{\rm (asym)}T) - 2(\tilde{T}\Phi_{\rm (asym)})(\tilde{T}^*\Phi^*_{\rm (asym)}) \\ &\quad +\sqrt{2}\tilde{T}F_\Phi^{\rm (asym)}T + \sqrt{2}\tilde{T}^*F_\Phi^{*\rm (asym)}T^* + \tilde{T}D^{\rm (asym)}T + \tilde{T}^*D^{\rm (asym)}T^* \\ &= \frac{1}{g^2}{\rm Tr}\Big\{ -\frac{1}{2}([\Phi^\dagger,\Phi] + [Y^\dagger,Y] + [\tilde{Y}^\dagger,\tilde{Y}])^2 \\ &\quad -2([\tilde{Y}^\dagger,Y^\dagger][Y,\tilde{Y}] + [\tilde{Y},\Phi][\Phi^\dagger,\tilde{Y}^\dagger] + [Y^\dagger,\Phi^\dagger][\Phi,Y])\Big\}. \end{aligned} \tag{29}$$

Equation (29) vanishes for the configuration satisfying Eqs. (22) and (28).

We conclude that the vacuum configuration is represented by

$$v_M = \begin{pmatrix} p_M^1 & & & & & \\ & \ddots & & & & \\ & & p_M^k & & & \\ & & & {\rm sgn}(M)p_M^1 & & \\ & & & & \ddots & \\ & & & & & {\rm sgn}(M)p_M^k \end{pmatrix} \equiv {\rm diag}\ v_M^{\rm (class)} \tag{30}$$

and

$$Q_f = \tilde{Q}_f = Q_f^* = \tilde{Q}_f^* = 0, \tag{31}$$

where

$${\rm sgn}(M) = \begin{cases} -1 & M = 0,\ \cdots,\ 5 \\ +1 & M = 6,\ \cdots,\ 9 \end{cases}, \tag{32}$$

$$v_M \equiv \left(v_m, \frac{\Phi+\Phi^\dagger}{\sqrt{2}}, \frac{\Phi-\Phi^\dagger}{\sqrt{2}i}, \frac{Y+Y^\dagger}{\sqrt{2}}, \frac{Y-Y^\dagger}{\sqrt{2}i}, \frac{\tilde{Y}+\tilde{Y}^\dagger}{\sqrt{2}}, \frac{\tilde{Y}-\tilde{Y}^\dagger}{\sqrt{2}i}\right). \tag{33}$$

The spacetime coordinates generated extend not only to the six directions obtained from the gauge fields and the adjoint scalars lying on the Cartan subalgebra of the $usp(2k)$ but also to the four additional directions from the antisymmetric tensor fields.

It is relatively clear that the one-loop stability of this vacuum is ensured by supersymmetry. We consider the second order fluctuation from diag $v_M^{(\mathrm{class})}$ and compute determinants. Let the adjoint action $\hat{P}_M$ on the matrix X be $\hat{P}_M X = [\mathrm{diag}\, v_M^{(\mathrm{class})}, X]$. Following Ref. 5), the one-loop effective action obtained from the bosonic, fermionic and ghost degrees of freedom is $(\frac{1}{2}\cdot 10 - \frac{1}{4}\cdot 16 - 1)\mathrm{Tr}\log(\hat{P}^M\hat{P}_M)$ and vanishes.

We would now like to have a more definite physical interpretation of this model than those discussed briefly in §§1 and 2. At the same time we would like to clarify the issue of the number of spacetime dimensions. Let us recall that ten of the noncommuting coordinates v_M (Eqs. (33), (2) and (8)), which are dynamical variables, satisfy

$$\begin{aligned} v_i^t &= -F v_i F^{-1}, \quad i = 0 \sim 5, \\ v_I^t &= \ \ F v_I F^{-1}, \quad I = 6 \sim 9. \end{aligned} \tag{34}$$

The v_M represent the noncommuting analog of the ten string coordinates X_M in the standard first quantized approach. In addition, the operation F is the matrix analog of the twist operation Ω. The classical counterpart of Eq. (34) is therefore

$$\begin{aligned} X_i &= -\Omega X_i \Omega^{-1}, \quad i = 0 \sim 5, \\ X_I &= \ \ \Omega X_I \Omega^{-1}, \quad I = 6 \sim 9. \end{aligned} \tag{35}$$

The presence of a four-dimensional fixed surface (orientifold surface) becomes clear from Eq. (35).

Equation (34) also constitutes the relations of embedding the v_M into $U(2k)$ matrices. In fact, via this embedding, the part of the action which do not invlove fundamentals is obtained from zero-dimensional reduction of the ten-dimensional super Yang-Mills theory by the projection. This dimensionally reduced model has been interpreted as a matrix model of type-IIB superstrings.[5)] We conclude that our model not only provides an exact F theory compactification through the procedure of Ref. 9) applied to the v_m $(m = 0, 1, 2, 3)$ but also is a matrix model representing type-IIB superstrings on a large volume T^6/Z^2 orientifold.

Let us now impose periodicities on the infinite size matrices v_m $(m = 0, 1, 2, 3)$ in all four directions. For this, we decompose the v_m into blocks of $n \times n$ matrices. We specify each individual block by a row vector $\vec{a} = (a_1, \cdots a_4)$ and a column vector $\vec{b} = (b_1, \cdots b_4)$: $(v_m)_{\vec{a},\vec{b}} \equiv \sqrt{\alpha'}\langle \vec{a} \mid \hat{v}_m \mid \vec{b}\rangle$. Let the shift vector be

$$(U(i))_{\vec{a},\vec{b}} = \left(\prod_{j(\neq i)} \delta_{a_j,b_j}\right) \delta_{a_i,b_i+1}. \tag{36}$$

The condition to be imposed is

$$U(i) v_m U(i)^{-1} = v_m - \delta_{m,i} R/\sqrt{\alpha'}. \tag{37}$$

The solution in Fourier transformed space is

$$\langle \vec{x} \mid \hat{v}_m \mid \vec{x}' \rangle = -i \left(\frac{\partial}{\partial x^m} + i\tilde{v}_m(\vec{x}) \right) \delta^{(4)} (\vec{x} - \vec{x}') \,, \tag{38}$$

$$\begin{aligned} \tilde{v}_m(\vec{x}) &= \sum_{\vec{\ell} \in Z^4} \tilde{\tilde{v}}_m(\vec{\ell}) \exp \left(\frac{-i\vec{\ell} \cdot \vec{x}}{\tilde{R}} \right), \\ \tilde{R} &\equiv \alpha'/R. \end{aligned} \tag{39}$$

The Fourier transform acts as the T dual transformation: it interchanges the radius parameter R setting the period of the original matrix index with the dual radius $\tilde{R}$ which is the period of the space Fourier conjugate to the matrix index. The point is that we take the $R \to 0$ limit after the Fourier transform. The resulting description of the model in the dual coordinates v_m $(m = 0, 1, 2, 3)$ is the large k limit of the $\mathcal{N} = 2$ supersymmetric $USp(2k)$ gauge theory with an antisymmetric hypermultiplet and n_f fundamental hypermultiplets. The ten noncommuting coordinates have classical counterparts which are

$$\begin{aligned} X_i &= -\Omega X_i \Omega^{-1}, \;\; i = 4, 5, \\ \tilde{X}_\ell &= \;\; \Omega \tilde{X}_\ell \Omega^{-1}, \;\; \ell = 0, 1, 2, 3, \\ X_I &= \;\; \Omega X_I \Omega^{-1}, \;\; I = 6 \sim 9. \end{aligned} \tag{40}$$

Here $\tilde{X}_\ell \equiv \tilde{X}_{\ell\,R} - \tilde{X}_{\ell\,L}$ denotes the dual coordinates of $X_\ell \equiv \tilde{X}_{\ell\,R} + \tilde{X}_{\ell\,L}$. The fixed orientifold surface is now eight dimensional, as a minus sign appears twice ($i = 4, 5$) in Eq. (40). The model in this limit therefore represents the nonperturbative completion of type-IIB on a large volume T^2/Z^2 orientifold, namely on CP^1.

As it stands, the model produces only ten spacetime dimensions as eigenvalue distributions of the ten matrix coordinates. The original argument of Ref. 10) that F theory is a theory in twelve spacetime dimensions takes the following form in the present context of our matrix model representing F theory. The low energy effective action of the four-dimensional $USp(2k)$ $\mathcal{N} = 2$ susy gauge theory above, which is our matrix model in the compactification described, has been exactly determined.[13)] The effective running coupling τ depends on $\mathrm{tr}\Phi^2/2 \equiv u \,:\, \tau = \tau(u)$. The work of Ref. 13) exactly determines this function, and this is precisely the moduli of the torus which is the fiber of the elliptic fibered $K3$ surface with CP^1 the base labelled by u. This is the only rationale for regarding this torus as representing two additional dimensions. Coordinates which parametrize this torus do not, however, manifest themselves in the present framework: only its moduli appear. Even if one is willing to take this twelve-dimensional viewpoint, the two additional dimensions are treated very differently from the remaining ten dimensions, which are directly related to the noncommuting coordinates as dynamical variables.[*)]

[*)] We should, however, note that the possibility of twelve spacetime dimensions is suggested in Ref. 16), as well as in the recent work, based on current algebra[17)] and also in the work on the topological matrix model.[18)]

Acknowledgements

We thank S. Hirano and A. Tsuchiya for helpful discussions on this subject. This work is supported in part by a Grant-in-Aid for Scientific Research Fund (2126) from the Ministry of Education, Science and Culture, Japan.

References

1) A. Sen, Nucl. Phys. **B475** (1996), 562.
2) E. Witten, Nucl. Phys. **B460** (1996), 335.
3) T. Banks, W. Fischler, S. H. Shenker, L. Susskind, Phys. Rev. **D55** (1997), 5112.
4) E. Witten, Nucl. Phys. **B443** (1995), 85.
5) N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. **B498** (1997), 467.
M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, Preprint hep-th/9705128.
6) A. Fayyazuddin, Y. Makeenko, P. Olsen, D. J. Smith and K. Zarembo, Preprint hep-th/9703038.
I. Chepelev and A. A. Tseytlin, Preprint hep-th/9705120.
7) Partial list includes
L. Motl, Preprint hep-th/9612198;
N. Kim and S.J. Rey, Preprint hep-th/9701139;
R. Dijkgraaf, E. Verlinde and H. Verlinde, Preprint hep-th/9703030;
A. Fayyazuddin and D.J. Smith, Preprint hep-th/9703208;
T. Banks and L. Motl, Preprint hep-th/9703218;
P. Horava, Preprint hep-th/9705055;
N. Kim and S.J. Rey, Preprint hep-th/9705132;
S. Sethi and L. Susskind, Preprint hep-th/9702101;
T. Banks and N. Seiberg, Preprint hep-th/9702187.
8) For the heterotic case,
N. Kim and S. J. Rey, Preprint hep-th/9701139;
T. Banks and L. Motl, Preprint hep-th/9703218;
D. A. Lowe, Preprint hep-th/9704041;
S. J. Rey, Preprint hep-th/9704158;
P. Horava, Preprint hep-th/9705055.
9) W. Taylor IV, Phys. Lett. **B394** (1997), 283.
O. J. Ganor, S. Ramgoolam and W. Taylor IV, Nucl. Phys. **B492** (1997), 191.
10) C. Vafa, Nucl. Phys. **B469** (1996), 403.
11) J. Polchinski and E. Witten, Nucl. Phys. **B460** (1996), 525.
12) Y. Arakane, H. Itoyama, H. Kunitomo and A. Tokura, Nucl. Phys. **B486** (1997), 149.
13) N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994), 19; **B431** (1994), 484.
14) T. Banks, M. R. Douglas and N. Seiberg, Phys. Lett. **B387** (1996), 278.
15) M. R. Douglas, D. A. Lowe and J. H. Schwarz, Phys. Lett. **B394** (1997), 297.
O. Aharony, J. Sonnenschein, S. Yankielowicz and S. Theisen, Nucl. Phys. **B493** (1997), 177.
16) V. Periwal, Preprint hep-th/9611103.
A. A. Tseytlin, Preprint hep-th/9612164.
17) H. Sugawara, Preprint hep-th/9708029.
18) S. Hirano and M. Kato, Preprint hep-th/9708039.