Utility Maximization in Incomplete Markets for Unbounded Processes

Sara Biagini*and Marco Frittelli†

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Abstract

When the price processes of the financial assets are described by possibly unbounded semimartingales, the classical concept of admissible trading strategies may lead to a trivial utility maximization problem because the set of bounded from below stochastic integrals may be reduced to the zero process. However, it could happen that the investor is willing to trade in such a risky market, where potential losses are unlimited, in order to increase his/her expected utility. We translate this attitude into mathematical terms by employing a class $\mathcal{H}^W$ of $W-$admissible trading strategies which depend on a *loss* random variable W. These strategies enjoy good mathematical properties and the losses they could generate in trading are *compatible* with the preferences of the agent.

We formulate and analyze by duality methods the utility maximization problem on the new domain $\mathcal{H}^W$. We show that, for all loss variables W contained in a properly identified set $\mathcal{W}$, the optimal value on the class $\mathcal{H}^W$ is constant and coincides with the optimal value of the maximization problem over a larger domain K_Φ. The class K_Φ doesn't depend on the single $W \in \mathcal{W}$, but it depends on the utility function u through its conjugate function Φ.

By duality methods we show that the optimal solution exists in K_Φ and it can be represented as a stochastic integral that is a uniformly integrable martingale under the minimax measure.

We provide the economic interpretation of the larger class K_Φ and we analyze some examples that show that this enlargement of the class of trading strategies is indeed necessary.

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*Università degli Studi di Perugia, Italy. Email: sbiagini@unipg.it
†Università degli Studi di Firenze, Italy. Email: Marco.Frittelli@dmd.unifi.it
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1 Introduction

We shall be interested in a classical problem in Economic Theory: the expected utility maximization from terminal wealth in a continuous time stochastic security market.

The history of the problem traces back to two seminal papers by Merton [23] and [24]. Its study received a renovated impulse in the middle of the Eighties, when the so-called duality approach to the problem was first developed. During the past twenty years the theory has constantly improved (see e.g. [2], [5], [6], [10], [13], [17], [18], [19], [20], [21], [29], [30]). However, a case has been left apart which is exactly the challenging situation examined in this paper where the semimartingale X representing the price process can be possibly *not locally bounded.*

As usual, we denote with $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ a filtered probability space and we assume that the filtration satisfies the usual assumptions of right continuity and completeness. T is a fixed time horizon, which can be as well $+\infty$: if this is the case, when considering processes Y it is understood that $Y_\infty = \lim_t Y_t$ exists. We assume that the general, càd-làg semimartingale X defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ is $\mathbb{R}^d-$valued.

Assumption 1: Throughout the paper we will suppose that the utility function $u : \mathbb{R} \to \mathbb{R}$ is strictly concave increasing differentiable and it satisfies the Inada conditions

$$u'(-\infty) \triangleq \lim_{x\to-\infty} u'(x) = +\infty,\ u'(+\infty) \triangleq \lim_{x\to+\infty} u'(x) = 0. \tag{1}$$

We then analyze the following problem:

$$\sup_{H\in\mathcal{H}} E[u(x + (H \cdot X)_T)], \tag{P}$$

where $x \in \mathbb{R}$ is the constant initial endowment and $\mathcal{H}$ is a class of trading strategies that is appropriate for general semimartingales.

The class $\mathcal{H}^W$ of "admissible" trading strategies (Definition 1) that we will adopt in this paper depends on a random variable W that controls the losses admitted in trading. We will require that W is *suitable* to the market model, so that $\mathcal{H}^W$ will be rich enough for trading purposes, and that W is *compatible* with the preferences, so that the expected utility of terminal wealth is never equal to $-\infty$ (Definitions 2 and 3).

We show, in Theorem 8, that, for all *loss* variables W contained in a properly identified set $\mathcal{W}$, the optimal value on the class $\mathcal{H}^W$ is constant, i.e. it does not depend on which $W \in \mathcal{W}$ is selected, and it coincides with the optimal value of the maximization problem over a larger domain K_Φ. We prove the existence in K_Φ of the optimal solution and show that it can be represented as a stochastic integral that is a uniformly integrable martingale under the minimax measure.

We therefore extend existing results in literature in two ways since we allow the semimartingale X to be not locally bounded and we adopt the new class $\mathcal{H}^W$ of admissible trading strategies.

One other minor contribution of this paper concerns a technical condition (see Section 2.2) that we will require on both the preferences and the market model. This assumption is weaker than the condition of Reasonable Asymptotic Elasticity introduced by Schachermayer [29] and frequently used in literature.

Our approach to solve problem (P) is based on the dual methodology, which has been proved to be such a powerful tool. The method naturally leads to the selection of the set $M_\sigma \cap P_\Phi$ - of sigma martingale measures having finite generalized entropy - as the domain in the dual optimization problem, and of the polar cone $K_\Phi = co(M_\sigma \cap P_\Phi)^0$ as the domain in the primal optimization problem. The economic relevance and interpretation of both sets is described in Sections 2.1 and 4. The mathematical motivation for the introduction of the larger domain K_Φ is shown in Remark 14.

The paper is organized as follows. In Section 2 we introduce and discuss all the concepts and definitions needed to formulate our main result, which is stated in Theorem 8. The examples in Section 3 illustrate the advantages of this new setting and show that in the classical Merton model the class $\mathcal{H}^W$ is also useful to achieve the optimal solution. The dual formulation of the primal problem and a key result on the existence of the minimax sigma-martingale measure are shown in Section 5. In Section 6 we characterize the optimal solution to the dual problem and in Section 7 we complete the proof of Theorem 8.

2 The main results

Given a semimartingale X and a process H, we will use the standard notation $H \in L(X)(P)$ as a shorthand for "*H is predictable and X-integrable under P*". An $\mathbb{R}^d-$valued process $H = (H_t)_{t\in[0,T]}$ is called a *trading strategy* if $H \in L(X)(P)$.

It is common knowledge that some other restrictions must be put on the class of trading strategies. In the literature (Harrison and Pliska [16], Delbaen and Schachermayer [7] and many others) the standard definition is that a trading strategy H is *admissible* (we will say $\mathbf{1}-$*admissible*) if there exists a constant $c \in \mathbb{R}$ such that, for all $t \in [0,T]$, the wealth process satisfies $(H \cdot X)_t \geq -c = -c \cdot 1 \;\; P-a.s.$ The financial interpretation of c is a finite credit line which the investor must respect in his/her trading. We will denote with $\mathcal{H}^1$ the class of these $\mathbf{1}-$admissible strategies.

As pointed out by Schachermayer, [29] Remark 2.6, in the non locally bounded case it can happen that $\mathcal{H}^1 = \{0\}$. This fact forces us to introduce the less restrictive notion of $\mathbf{W}-$admissibility, in order to provide a non trivial enlargement of the class $\mathcal{H}^1$.

Definition 1 *Let $W \in L^0(P)$ be a fixed random variable. The process $H \in L(X)(P)$ is* **W***-admissible, or it belongs to $\mathcal{H}^W$, if there exists a nonnegative constant c such that $(H \cdot X)_t \geq -cW \quad \forall t \leq T$.*

So $\mathcal{H}^1 \subseteq \mathcal{H}^W$, the containment generally being strict. However it is clear that if $W = 1$ then $\mathcal{H}^W = \mathcal{H}^1$ and we are back with the standard concept of admissible trading strategies.

This natural extension of the notion of admissibility was already used in Schachermayer ([31] Section 4.1) in the context of the fundamental theorem of asset pricing, as well as in Delbaen and Schachermayer [9]. In a subsequent article [8], these two authors introduced a different notion of $\mathbf{W}-$admissibility: their goal was analyzing the super replication price of unbounded claims (see Remark 21 for a comparison).

We will work with $\mathcal{H}^W$ as the domain in problem (P). To the best of our knowledge, $\mathcal{H}^W$ as well as any other extension of $\mathcal{H}^1$ was *never used before this paper in the framework of utility maximization for general semimartingale models.*

To ensure that utility maximization on $\mathcal{H}^W$ is really a non-trivial problem, we put two conditions on the random amount W that controls the losses admitted in the trading. The first of these two conditions guarantees that $\mathcal{H}^W$ is rich enough for trading purposes:

Definition 2 *A random variable $W \in L^0(P)$ is $X-$**suitable** (or simply suitable) if $W \geq 1$ $P-$ a.s. and for all $1 \leq i \leq d$ there exists a process $H^i \in L(X^i)(P)$ such that $P(\{\omega \,|\, \exists t \geq 0\, H^i_t(\omega) = 0\}) = 0$ and*

$$-W \leq (H^i \cdot X^i)_t \leq W, \text{ for all } t \in [0,T]. \tag{2}$$

This implies that $H^i \neq 0$ and both investments H^i and $-H^i$ in the single asset X^i are "**W**-admissible". For each suitable W, we always have $\mathcal{H}^1 \subseteq \mathcal{H}^W$, since $W \geq 1$.

The second condition imposes that the $\mathbf{W}-$admissible trading strategies are *compatible* with the preferences of the investor, i.e. it assures that the expected utility of terminal wealths $x + (H \cdot X)_T$ from all W-admissible trading strategies is never equal to $-\infty$.

Definition 3 *A random variable $W \in L^0(P)$ is $u-$**compatible** (or simply compatible) if $W \geq 1$ $P-$ a.s. and*

$$E[u(-cW)] > -\infty \ \ \forall c > 0.$$

*We denote with $\mathcal{W}$ the convex set of $X-$suitable and $u-$compatible random variables and we call its elements **loss** variables.*

This compatibility condition is also one of the main novelties of the paper: it is a precise mathematical formulation of the situation in which an agent is willing to accept higher risk, in order to increase his/her utility, but only within a certain degree. We note for future use that if W is a loss variable then $E[u(x+(H\cdot X)_R)] > -\infty$, for all stopping time R, $H \in \mathcal{H}^W$ and $x \in \mathbb{R}$.

We now show that the condition $\mathcal{W} \neq \emptyset$ is automatically satisfied in the locally bounded case.

Proposition 4 *If X is locally bounded, then the constant $1 \in \mathcal{W}$.*

Proof. Since compatibility is obvious, we prove that 1 is $X-$suitable. Let $(T_n)_n$ be a sequence of stopping times increasing to $+\infty$ such that X^{T_n} is bounded. We can assume that for all i, $|X^i_{t\wedge T_n}| \leq n$. Put $T_0 = 0$ and define the real valued process φ as follows:

$$\varphi = I_{\{0\}} + \sum_{n\geq 0} 2^{-n} I_{]T_n, T_{n+1}]}$$

so that $\varphi \cdot X^i$, as well as its maximal functional $(\varphi \cdot X^i)^*$, is bounded for all i. So, $\sum_{i=1}^d (\varphi \cdot X^i)^*_T \leq C$, with a suitably chosen constant C. Then relation (2) is verified with $W = 1$ and $H^i = \frac{\varphi}{C}$ for all i. ■

However, in the general case $\mathcal{W} \neq \emptyset$ may not hold, even in models free of arbitrage opportunities, so that we will require it.

Example 5 *(On $\mathcal{W} = \emptyset$) Let $T < +\infty$ and consider the trading interval $[0, T]$. Let $(\Omega, \mathcal{G}, P)$ be a probability space and Y be a random variable unbounded from both sides. Set X equal to 0 if $t < T$ and $X_T = Y$ when $t = T$ and take as the filtration the P-augmentation of the natural one. Clearly $\mathcal{H}^1 = \{0\}$, so that the model is arbitrage free and 1 is not suitable. Modulo a scaling factor, the minimal suitable W is $1 + |Y|$. Suppose now that $E[\exp(c|Y|)] = +\infty$ for some $c > 0$ and that our investor has exponential preferences. Then $\mathcal{W} = \emptyset$.*

Remark 6 *(On compatibility) Let $W \in L^0$, $W \geq 1$ and consider*

$$W \in L^\infty, \tag{3}$$

$$\forall c > 0 \qquad E[u(-cW)] > -\infty, \tag{4}$$

$$\exists c > 0 \qquad E[u(-cW)] > -\infty. \tag{5}$$

*Obviously (5) is weaker than (4), which is weaker than (3). The strongest condition (3) leads to the well established notion of **1-**admissibility, since in this case $\mathcal{H}^W = \mathcal{H}^1$. The weaker compatibility condition (4) is studied in this paper and will lead to the results stated in Theorem 8 and in the Remarks after it.*

The weakest condition (5) could also be useful for the application to utility maximization. Indeed, together with the property of being suitable, (5) ensures that there exists some non-zero $H \in \mathcal{H}^W$ such that $E[u(x+(H\cdot X)_T)] > -\infty$. However, this condition would not guarantee the same uniformity obtained under the present assumption (4). We leave this theme open for future investigation.

For the dual approach we are going to follow, we need the convex conjugate $\Phi : \mathbb{R}_+ \to \mathbb{R}$ of u:

$$\Phi(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\}. \tag{6}$$

Then Φ is a strictly convex differentiable function, $\Phi(+\infty) = +\infty$, $\Phi(0^+) = u(+\infty)$ and $\Phi'(0^+) = -\infty, \Phi'(+\infty) = +\infty$. From the definition of Φ, the Fenchel inequality immediately follows:

$$u(x) \leq yx + \Phi(y), \ \forall x \in \mathbb{R}, \ \forall y \geq 0, \tag{7}$$

and

$$u(-\Phi'(y)) = -y\Phi'(y) + \Phi(y), \quad \forall y \geq 0. \tag{8}$$

where the usual rule $0 \cdot \infty = 0$ is applied.

In order to formalize the optimization problems that we will discuss, we need some more notations. Let

$$P_\Phi = \left\{ Q \ll P \mid E_P \left[\Phi \left(\frac{dQ}{dP} \right) \right] < +\infty \right\}$$

be the set of $P-$a.c. probability measures with finite generalized entropy and

$$M_\sigma = \{Q \ll P : X \text{ is a } \sigma - \text{martingale under } Q\}$$

be the set of $P-$a.c. σ-martingale measures. In Section 4 we provide more information about these concepts. Here we only note that when X is bounded (resp. locally bounded) then

$$M_\sigma = \{Q \ll P : X \text{ is a martingale (resp. local martingale) w.r.to } Q\},$$

i.e. M_σ is the set of $P-$a.c. *martingale* (resp. *local martingale*) measures.

Assumption 2: Through out the paper we will suppose that for all $Q \in M_\sigma \cap P_\Phi$

$$E_P \left[\Phi \left(\lambda \frac{dQ}{dP} \right) \right] < +\infty \text{ for all } \lambda > 0. \tag{9}$$

For expository reasons, we postpone to Section 2.2 the detailed analysis of this condition and of its relationship with other similar ones.

Let $W \in \mathcal{W}$ and let the optimization domains be defined by

$$\begin{aligned} \mathbf{K^W} &= \left\{ (H \cdot X)_T \mid H \in \mathcal{H}^W \right\}; \qquad \mathbf{K^{\mathcal{W}}} = \bigcup_{W \in \mathcal{W}} K^W; \\ \mathbf{K_\Phi} &= \left\{ f \in \bigcap_{Q \in M_\sigma \cap P_\Phi} L^1(Q) \mid E_Q[f] \leq 0 \, \forall Q \in M_\sigma \cap P_\Phi \right\}. \end{aligned} \tag{10}$$

The analog of the domain K_Φ was first explicitly introduced in the locally bounded case by Frittelli [12] and [13], who showed its relevance for the existence of the optimal solutions to both the primal utility maximization problem and its dual problem. We defer to Section 2.1 the interpretation of K_Φ and its characterization. Here we only note that K_Φ doesn't depend on a single $W \in \mathcal{W}$, but only on the utility function u through its conjugate function Φ.

Hereafter we have the corresponding optimal values:

$$\mathbf{U^W}(x) \triangleq \sup_{k \in K^W} E[u(x+k)]$$
$$\mathbf{U^{\mathcal{W}}}(x) \triangleq \sup_{k \in K^{\mathcal{W}}} E[u(x+k)]$$
$$\mathbf{U_\Phi}(x) \triangleq \sup_{k \in K_\Phi} E[u(x+k)]$$

The values $\mathbf{U^W}(x)$ and $\mathbf{U^{\mathcal{W}}}(x)$ are well defined, since the condition $E[u^-(x+k)] < +\infty$ is automatically satisfied if $W \in \mathcal{W}$. From the Fenchel inequality (7) we get, for $x \in \mathbb{R}$, $k \in K_\Phi$, $Q \in M_\sigma \cap P_\Phi$ and $\lambda > 0$

$$E[u(x+k)] \leq E\left[(x+k)\,\lambda\frac{dQ}{dP}\right] + E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] \leq \lambda x + E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] < +\infty. \tag{11}$$

Therefore, $E[u^+(x+k)] < \infty$ and also $\mathbf{U_\Phi}(x)$ is well defined, as soon as $M_\sigma \cap P_\Phi \neq \emptyset$.

In section 4 we will prove the following inequalities among the optimal values.

Proposition 7 *(a) If W is $u-$compatible, then $W \in L^1(Q)$ for all $Q \in P_\Phi$.*

(b) If $W \in \mathcal{W}$ then

$$K^W \subseteq K^{\mathcal{W}} \subseteq K_\Phi \quad \text{and} \quad U^W(x) \leq U^{\mathcal{W}}(x) \leq U_\Phi(x)$$

In Theorem 8 we prove that the above three optimal values coincide and we state our main result on the existence of the optimal solution in K_Φ and its representation as a stochastic integral.

Theorem 8 *Suppose that*

$$\text{there exist } W_0 \in \mathcal{W} \text{ and } x_0 \in \mathbb{R} \text{ such that } U^{W_0}(x_0) < u(+\infty). \tag{12}$$

Then:
(a) $M_\sigma \cap P_\Phi \neq \varnothing$;
(b) For all $W \in \mathcal{W}$ and all $x \in \mathbb{R}$ the optimal value $U^W(x)$ is less than $u(+\infty)$; it does not depend on the particular $W \in \mathcal{W}$ and

$$U^W(x) = U^{\mathcal{W}}(x) = U_\Phi(x) = \min_{\lambda>0, Q \in M_\sigma \cap P_\Phi} \left\{\lambda x + E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]\right\};$$

(c) For all $x \in \mathbb{R}$ there exists the optimal solution $f_x \in K_\Phi$:

$$\max\{E[u(x+f)] \,|\, f \in K_\Phi\} = E[u(x+f_x)] = U_\Phi(x) < u(+\infty).$$

Moreover, if we indicate with λ_x, Q_x the optimal solution of the dual problem in item (b), then:

$$f_x = -x - \Phi'\left(\lambda_x \frac{dQ_x}{dP}\right);$$

(d) There exists $H^x \in L(X)(Q_x)$ such that the optimal solution f_x coincides Q_x-a.s. with the terminal value $(H^x \cdot X)_T$ and $H^x \cdot X$ is a Q_x-uniformly integrable martingale. In case $Q_x \sim P$, this integral representation of f_x holds under P, that is $H^x \in L(X)(P)$ and $f_x = (H^x \cdot X)_T$ P-a.s.

Hereafter we make some comments on the results just stated.

Remark 9 *(On $Q_x \sim P$) Set $M_\sigma^e \triangleq M_\sigma \cap \{Q : Q \sim P\}$. When $u(+\infty) = +\infty$, then $Q_x \sim P$ as noted in Bellini and Frittelli [2].*

The condition $M_\sigma^e \cap P_\Phi \neq \emptyset$ also ensures $Q_x \sim P$, as first proved in Csiszar [4] with exponential utility and by Kabanov and Stricker [18] with general u. Their argument relies on the Inada condition $u'(-\infty) = +\infty$ (i.e. $\Phi'(0) = -\infty$) and it applies also with σ-martingale measures.

Remark 10 *(On $M_\sigma \cap P_\Phi \neq \varnothing$) Condition (12) is* equivalent *to*

$$\mathcal{W} \neq \emptyset \text{ and item } (a).$$

Indeed, suppose that $Q \in M_\sigma \cap P_\Phi \neq \varnothing$. Then it is always possible to find some $x_0 \in \mathbb{R}$ for which $x_0 + E[\Phi(\frac{dQ}{dP})] < u(+\infty)$. Taking $\lambda = 1$ in (11), we see that (11) implies $U_\Phi(x_0) < u(+\infty)$ and, from Proposition 7, $U^W(x_0) \leq U_\Phi(x_0) < u(+\infty)$, for any $W \in \mathcal{W}$.

If condition (12) does not hold true then, even with an arbitrarily large debt ($x_0 \downarrow -\infty$), the investor might become arbitrarily close to his supremum utility $u(+\infty)$, by investing in claims $k \in \mathbf{K}^{W_0}$ having at most zero cost (i.e. $E_Q[k] \leq 0$) under each $Q \in M_\sigma \cap P_\Phi$. Thus (12) can be regarded as a hypothesis of **absence of utility based arbitrage**.

Remark 11 *(On FLVR) Note that we never require $M_\sigma^e \neq \varnothing$. This means, thanks to the Fundamental Theorem of Asset Pricing by Delbaen and Schachermayer [8], that Theorem 8 holds true even if there are Free Lunches with Vanishing Risk (FLVR). Remember that a claim $g \in L_+^\infty$ is a FLVR if it is positive on a set of positive P probability and there exists a sequence $h_n \in (K^1 - L_+^0(P)) \cap L^\infty(P)$ s.t. $h_n \to g$ in $L^\infty(P)$. Since $(K^1 - L_+^0(P)) \cap L^\infty(P) \subseteq K_\Phi$, this readily implies that whenever such a g exists, it belongs to K_Φ.*

However, the presence of a FLVR does not preclude that condition (12) holds true (see Lemma 12), nor the existence of the optimal solution $f_x \in K_\Phi$. This optimal solution is equal to $+\infty$ on the set $\{dQ_x/dP = 0\}$, which can have positive P measure when $Q_x \not\sim P$. Under the condition (12), a FLVR g will

not be considered by the investor as an interesting opportunity, since g will not increment the optimal utility. Indeed, $x + f_x = x + f_x + g$ $P-$a.s. since $P(\{f_x < +\infty\} \cap \{g > 0\}) = 0$ (otherwise, f_x could not be optimal).

We warmly thank W. Schachermayer for suggesting us the counterexample in Lemma 12 and for pointing out an erroneous statement on the implications in Lemma 12 in a previous version of this paper.

Lemma 12 *NFLVR $\not\Rightarrow$ (12) $\not\Rightarrow$ NFLVR .*

Proof. In the context of a complete continuous financial market with $W = 1$, Schachermayer already showed in Lemma 3.8 [29] that (12) may not hold even under NFLVR.
For the second statement, it is not difficult (adapting e.g. the proof of Proposition 3.3 in [29]) to construct a model with a continuous, bounded underlying X such that: (i) there is precisely one martingale measure Q and (ii) $\frac{dQ}{dP} = 2I_A$, $P(A) = \frac{1}{2}$. Since $Q \not\sim P$, then there are free lunches with vanishing risk. Take $u(x) = -e^{-x}$, $W = 1, x = 0$. Then $U^1(0) < u(+\infty)$ and the utility maximization program leads to $U^1(0) = -\frac{1}{2}$ and $f_0 = 0$ on A, $f_0 = +\infty$ on A^c. ■

Remark 13 *(On the uniformity over all $W \in \mathcal{W}$) Point b) in Theorem 8 provides a desirable uniformity over all the $W \in \mathcal{W}$. From this we see once more why in the locally bounded case $\mathcal{H}^1$ is the right set to work with. In fact, $\mathcal{H}^1$ is the smallest set of strategies, the wealth processes generated by $\mathcal{H}^1$ are controlled from below by the smallest bound in $\mathcal{W}$ and at the same time the related maximization leads to the optimal value $U^1(x) = U^{\mathcal{W}}(x)$.*

Finally we give the mathematical motivation of the "extra enlargement" from $K^{\mathcal{W}}$ to K_Φ needed to catch the optimal solution f_x.

Remark 14 *Even when $Q_x \sim P$, in general we can not hope that the optimal solution f_x belongs to $K^{\mathcal{W}}$, that is f_x can not always be represented as terminal value of $H^x \cdot X$, with $H^x \in \mathcal{H}^W$ for some $W \in \mathcal{W}$.*

*What may go wrong is well understood in a complete market case, where we denote with Q the unique equivalent measure in $M_\sigma \cap P_\Phi$. It may happen that $H^x \cdot X$ is a Q-uniformly integrable martingale, but $\sup_{t \le T}(H^x \cdot X)_t^-$ **is not** Q-integrable. By Proposition 7 (a), we deduce that H^x cannot belong to any $\mathcal{H}^W$.*

An explicit example can be found in the model described by Schachermayer in [30], where it was constructed for a different purpose. In discrete times, with $T = +\infty$, the author constructs a locally bounded bidimensional semimartingale (X^1, X^2), which admits a unique equivalent local martingale measure Q. Moreover, X^1_∞ is the optimal solution to the exponential utility maximization on the "classical" domain K^1 (recall that $1 \in \mathcal{W} \neq \emptyset$). Now, in our terminology, $(H^x \cdot X)_\infty = X^1_\infty \in K_\Phi$, but $X^1_\infty \notin K^{\mathcal{W}}$ since $\sup_{n \le \infty}(X^1)_n^-$ is not Q-integrable.

Using classical techniques (as in Cvitanic et al. [5], Kramkov and Schachermayer [20], Schachermayer [29]) the following statements can be also proved.

Proposition 15 *Let V_Φ be the dual value function, that is:*

$$V_\Phi(y) = \inf_{Q \in M_\sigma \cap P_\Phi} E[\Phi(y\frac{dQ}{dP})]$$

Then, under the same hypotheses of Theorem 8:
(α) $U_\Phi(x) = \inf_{y>0} yx + V_\Phi(y)$, that is the primal and dual value functions are conjugate;
(β) For all $y > 0$, there exists the unique minimal probability $\widehat{Q}_y$ such that $V_\Phi(y) = E[\Phi(y\frac{d\widehat{Q}_y}{dP})]$; V_Φ is then strictly convex;
(γ) U_Φ and V_Φ are differentiable and:

$$V'_\Phi(y) = E[\Phi'(y\frac{d\widehat{Q}_y}{dP})\frac{d\widehat{Q}_y}{dP}],$$

$$xU'_\Phi(x) = E[u'(x + f_x)(x + f_x)].$$

2.1 On the economic interpretation of the set K_Φ

From Theorem 25 we will deduce, in Section 6.1, the following Proposition. The representation in (13) will also be needed in the proof of Theorem 8, item (d).

Proposition 16 *Suppose that $W \in \mathcal{W}$ and $M_\sigma \cap P_\Phi \neq \varnothing$. Then*

$$\bigcap_{Q \in M_\sigma \cap P_\Phi} \overline{K^W - L^1_+(Q)}^Q = \bigcap_{Q \in M_\sigma \cap P_\Phi} \overline{K^{\mathcal{W}} - L^1_+(Q)}^Q = K_\Phi, \qquad (13)$$

where $\overline{A}^Q$ denotes the $L^1(Q)-$closure of a set A.

The weak super replication price $\widehat{f}_\Phi$ of each fixed $f \in \bigcap_{Q \in M_\sigma \cap P_\Phi} L^1(Q)$ admits the dual representation:

$$\widehat{f}_\Phi \triangleq \inf\{x \in \mathbb{R} \mid f - x \in K_\Phi\} = \sup\{E_Q[f] \mid Q \in M_\sigma \cap P_\Phi\} \qquad (14)$$

and when the above quantities are finite, the infimum is a minimum.

Equation (13) tells us that the class K_Φ, which by definition is the polar cone of $co(M_\sigma \cap P_\Phi)$ (see Section 6 for the precise polarity relation), admits a representation directly based on the set $K^{\mathcal{W}}$. Indeed, the set $\overline{K^{\mathcal{W}} - L^1_+(Q)}^Q$ is the cone of $Q-$integrable claims that can be approximated, in the $L^1(Q)$ norm topology, by elements in $K^{\mathcal{W}} - L^1_+(Q) = (K^{\mathcal{W}} - L^0_+(P)) \cap L^1(Q)$, i.e. by $Q-$integrable claims that can be dominated by claims attainable with zero initial wealth from $W-$admissible trading strategies (from all $W \in \mathcal{W}$).

We end up with $\bigcap_{Q \in M_\sigma \cap P_\Phi} \overline{K^{\mathcal{W}} - L^1_+(Q)}^Q$, since only those $Q \in M_\sigma \cap P_\Phi$

are "allowed by the utility function u". To explain this last assertion (see also Frittelli [13]) note that: $Q \in M_\sigma$ belongs to P_Φ if and only if

$$\forall x \in \mathbb{R} \ \sup\left\{E[u(x+f)] \mid f \in L^1(Q) : E_Q[f] \leq 0\right\} < u(+\infty) \tag{15}$$

(the implication $Q \in P_\Phi \Rightarrow$ (15) follows from Proposition 27 and the reverse implication from Corollary 2.1 Bellini and Frittelli [2]). Thus, to avoid "utility based arbitrage opportunities", similar to those described in Remark 10, the measures $Q \in M_\sigma$ that are not in P_Φ should not be selected as pricing measures.

Let $f \in \bigcap_{Q \in M_\sigma \cap P_\Phi} L^1(Q)$ be a "sufficiently integrable claim". Its weak super replication price, defined as $\widehat{f}_\Phi \triangleq \inf\{x \in \mathbb{R} \mid f - x \in K_\Phi\}$, was introduced in Biagini and Frittelli [3] in order to define a less expensive concept of "maximum selling price".

In the "classical" case, the super replication price $\inf\left\{x \in \mathbb{R} \mid f - x \in K^1\right\}$ of a bounded from below claim f admits the representation: $\sup\{E_Q[f] \mid Q \in M_\sigma\}$. Equation (14) shows that the dual representation of the weak super replication price is: $\widehat{f}_\Phi = \sup\{E_Q[f] \mid Q \in M_\sigma \cap P_\Phi\}$. As a consequence, $f \in K_\Phi$ if and only if $\widehat{f}_\Phi \leq 0$ and so K_Φ is the set of claims in $\bigcap_{Q \in M_\sigma \cap P_\Phi} L^1(Q)$ having "weak super replication price" less than or equal to zero.

2.2 On Assumption 2

Growth conditions on convex differentiable functions, finite valued on $\mathbb{R}_+$, were often considered in the literature about the existence of the minimal divergence projection (see Section 6), as well as in Orlicz spaces theory. In particular, see Liese and Vajda [22] Section 8.7, the following condition was extensively studied:

- For any $\lambda > 1$ there exist $y_0 > 0$ and $c_1, c_2, c_3 \in \mathbb{R}_+$ such that:

$$\Phi(\lambda y) \leq c_1 \Phi(y) + c_2 y + c_3, \ \forall y > y_0. \tag{16}$$

If $\lambda = 2$ and $c_2 = c_3 = 0$, (16) is the well known Δ_2 growth property in Orlicz spaces theory.
- Relation (16) and the assumption that $\Phi(0)$ is finite clearly imply that (9) holds true for all $Q \in P_\Phi$, so that Assumption 2 is satisfied.

However, when Φ is the conjugate of a utility function u, as in our setting, $\Phi(0)$ finite is equivalent to $u(+\infty)$ finite, which is not always the case.
W.l.o.g. we will assume in this section that $u(0) > 0$, so that $\Phi(y) > 0 \ \forall y \geq 0$.
A fundamental step in the solution of the utility maximization problem was established by Schachermayer, who introduced in [29] the following concept: The utility u has Reasonable Asymptotic Elasticity $RAE(u)$ if

$$(i) \ AE_{-\infty}(u) \triangleq \liminf_{x \to -\infty} \frac{xu'(x)}{u(x)} > 1 \text{ and } (ii) \ AE_{+\infty}(u) \triangleq \limsup_{x \to +\infty} \frac{xu'(x)}{u(x)} < 1$$

(see also Kramkov and Schachermayer [20], where the notion $AE_{+\infty}(u) < 1$ was developed).

In the theory of Orlicz spaces, the relation between the condition (i) on the function u and the condition (16) on its conjugate Φ is well known (see Rao and Ren [26] Corollary 4, where the equivalence between Δ_2 and $AE_{-\infty}(u) > 1$ is exactly stated). Schachermayer showed (Corollary 4.2 [29]) that when u has $RAE(u)$ then the conjugate Φ satisfies:

- For each compact interval $[\lambda_0, \lambda_1]$ contained in $(0, +\infty)$ there exists a constant $\alpha > 0$ such that:

$$\Phi(\lambda y) \leq \alpha \Phi(y), \text{ for } y > 0 \text{ and } \lambda \in [\lambda_0, \lambda_1]. \tag{17}$$

- This is clearly another sufficient condition for (9) to be true for all $Q \in P_\Phi$, even without the assumption that $\Phi(0)$ is finite.

Furthermore, consider the following condition

$$P_{\Phi_\lambda} = P_\Phi \ \forall \lambda > 0, \tag{18}$$

where $\Phi_\lambda : (0, +\infty) \to \mathbb{R}$ is defined by $\Phi_\lambda(y) \triangleq \Phi(\lambda y)$, and notice first that (17) implies (18). Also, Assumption 2 can be rewritten as

$$P_{\Phi_\lambda} \cap M_\sigma = P_\Phi \cap M_\sigma \ \forall \lambda > 0. \tag{19}$$

Since $RAE(u)$ implies (17), we get:

$$RAE(u) \Rightarrow (18) \Rightarrow (19). \tag{20}$$

While $RAE(u)$ is a condition only on u (as (17) is a condition only on Φ), (18) concerns both P and Φ; and (19) involves P, Φ and the market model via the set M_σ. *In general none of the implications in* (20) *may be reversed.* One trivial case is when P assigns non zero probability only to a finite number of atoms: Then (18) always holds true, regardless whether u satisfies $RAE(u)$ or Φ satisfies (17). We provide now two non trivial cases where the weaker Assumption 2 holds, but $RAE(u)$ is not verified, nor (17).

Example 17 *(a) (complete market) Suppose that X is modelled as a multi-dimensional diffusion, that the market is complete and the unique martingale measure has square integrable density (this is the case when e.g. the market price of risk is bounded). If $\Phi(y) \leq y^2$ for all $y \geq 0$ then clearly (19) holds true. However, Φ does not necessarily satisfy (17) - and hence u does not necessarily satisfy $RAE(u)$. Such a Φ can easily be derived as in the example in Rao and Ren [26], page 27. This shows that even if Φ has at most polynomial growth, Φ may not satisfy Δ_2 - nor (17) or (16).*

(b) (incomplete market) Suppose that all the elements $Q \in M_\sigma$ have bounded densities wrt P. Then, whenever $\Phi(0) < +\infty$, the relation (19) holds true even if u does not satisfy $RAE(u)$ nor Φ satisfies (17) - take for example $\Phi(y) =$

$(y \ln y + 1) I_{\{0<y\leq 1\}} + e^{y-1} I_{\{y>1\}}$.

To build a market model where $\frac{dQ}{dP} \in L^\infty$ *for all* $Q \in M_\sigma$ *consider a sequence of independent bernoullian variables* $(Y_n)_{n\geq 2}$ *such that the possible states of* Y_n *are* $\frac{1}{n^2}, -\frac{1}{n^2}$, *each with* P*-probability* $\frac{1}{2}$. *Let* Y_1 *be independent from* $(Y_n)_{n\geq 2}$, *with the three possible states* $-1, 0, 1$, *(say) each with* P*-probability* $\frac{1}{3}$ *and let* $Y_0 = 0$. *Consider the filtration generated by* Y. *Define* $X_0 = 0$, $X_1 = Y_1$, $X_n = X_{n-1} + Y_n + \frac{1}{e^n}$ *if* $n \geq 2$, $X_\infty = \lim_n X_n$ *and* $T = \infty$. *The market is incomplete and it is easy to find out the martingale measures for* $(X_n)_{n\leq\infty}$*: take* $W_n^\alpha = Z_1(\alpha) Z_2 \cdots Z_n$, *where* $Z_1(\alpha) = 3\alpha I_{\{Y_1=1\}} + 3(1-2\alpha) I_{\{Y_1=0\}} + 3\alpha I_{\{Y_1=-1\}}$ *with* $0 < \alpha < \frac{1}{2}$ *and* $Z_n = (1 - \frac{n^2}{e^n}) I_{\{Y_n=1/n^2\}} + (1 + \frac{n^2}{e^n}) I_{\{Y_n=-1/n^2\}}$ *for* $n \geq 2$. *An application of Kakutani's theorem gives that* $M_\sigma = M_\sigma^e = \{Q^\alpha | \frac{dQ^\alpha}{dP} = \lim_n W_n^\alpha$, *with* $0 < \alpha < \frac{1}{2}\}$, *where the limit is in* L^1 *and also a.s. To show that each* $Q^\alpha \in L^\infty$, *notice that* $0 < \frac{dQ^\alpha}{dP} \leq \|Z_1(\alpha)\|_\infty \lim_n \prod_{k=2}^{n} (1 + k^2 e^{-k}) < \infty$, *since the series* $\sum_{k=2}^{n} \ln(1 + k^2 e^{-k})$ *converges.*

When X is a locally bounded semimartingale and $\mathcal{H} = \mathcal{H}^\mathbf{1}$, Schachermayer [29] showed that $RAE(u)$ (together with Assumption 1 and $U^1(x) < u(\infty)$) implies the existence of the optimal solution of problem (P). He also showed that if the utility function u does not satisfy $RAE(u)$ then it is always possible to find a probabilistic model $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ in which problem (P) has no solution.

This is not in contradiction with our main result: When X is a semimartingale - not necessarily locally bounded - Theorem 8 shows that Assumption 2 (together with Assumption 1 and $U^W(x) < u(\infty)$) is a sufficient condition - on P, u *and* M_σ - for the existence of an optimal solution to (P). Thus, under Assumption 2 we always obtain an optimal solution even if u does not satisfy $RAE(u)$.

We remark that one other condition, necessary and sufficient for the existence of optimal investments, involving P, u and the set of local martingale measures has been already introduced by Kramkov and Schachermayer [21] in the different setting of utility functions finite valued only on $\mathbb{R}_+$.

Another advantage of using Assumption 2 is that all our results will be presented in a self contained manner. Indeed, in all our proofs we will either directly apply Assumption 2 or use the very simple facts stated in the next Lemma and deduced only from (9).

Lemma 18 *Let* $\Theta : \mathbb{R}_+ \to \mathbb{R}$ *be a convex differentiable function and* $Q \ll P$. *If* $E_P[\Theta(\lambda \frac{dQ}{dP})] < +\infty \ \forall \lambda > 0$, *then :*
(a) $\Theta'(\lambda \frac{dQ}{dP}) \in L^1(Q) \ \forall \lambda > 0$;
(b) If $(g + g^{-1}) \in L_+^\infty(P)$ *then* $E[\Theta(g \frac{dQ}{dP})] \in \mathbb{R}$;
(c) If $\Theta'(0^+) = -\infty$, $\Theta'(+\infty) = +\infty$ *and* Θ *is strictly convex then* $F(\lambda) \triangleq E[\frac{dQ}{dP} \Theta'(\lambda \frac{dQ}{dP})]$ *defines a bijection between* $(0, +\infty)$ *and* $(-\infty, +\infty)$.

Proof. Set $\eta = \frac{dQ}{dP}$.

(a) From $\Theta'(y) \leq \frac{\Theta(\alpha y)-\Theta(y)}{(\alpha-1)y}$, $\alpha > 1$, and $\Theta'(y) \geq \frac{\Theta(\alpha y)-\Theta(y)}{(\alpha-1)y}$, $\alpha < 1$ it follows that $\Theta'(\eta) \in L^1(Q)$. We may apply the same argument to the function $\Theta_\lambda(y) \triangleq \Theta(\lambda y)$, $\lambda > 0$, and deduce that $\Theta'_\lambda(\eta) = \lambda\Theta'(\lambda\eta) \in L^1(Q)$.

(b) There exist $M > \epsilon > 0$ such that $\epsilon \leq g \leq M$, $P-$a.s. Therefore, if $A = \{\Theta(\epsilon\eta) \geq \Theta(M\eta)\}$, $\Theta(\epsilon\eta)I_{A^c} + \Theta(M\eta)I_A \leq \Theta(g\eta) \leq \Theta(\epsilon\eta)I_A + \Theta(M\eta)I_{A^c}$, $P-$a.s. and $-\infty < E[\Theta(g\eta)] < +\infty$.

(c) From (a) we deduce that F is finite valued on $(0, +\infty)$. Note that Θ' is strictly increasing from $-\infty$ to $+\infty$ and it is continuous on $(0, +\infty)$. Then $\lambda \to \eta\Theta'(\lambda\eta)$ is $P-$a.s. monotone on $(0, +\infty)$ and F is strictly increasing on $(0, +\infty)$. Since $\eta\Theta'(\lambda\eta) \in L^1(P)$, for each $\lambda > 0$, by the monotone convergence Theorem we deduce that F is continuous on $(0, +\infty)$ and $\lim_{\lambda\downarrow 0} F(\lambda) = E[\lim_{\lambda\downarrow 0} \eta\Theta'(\lambda\eta)] = -\infty$, $\lim_{\lambda\uparrow\infty} F(\lambda) = E[\lim_{\lambda\uparrow\infty} \eta\Theta'(\lambda\eta)] = +\infty$, and so item (c) follows. ■

3 Examples

Hereafter we give two examples of the application of the new class of strategies. In particular, the second one really shows the performances of $\mathcal{H}^W$. In both cases we deal with the exponential utility $u(x) = -e^{-x}$ and therefore $\Phi(y) = y \ln y - y$.

3.1 Merton's model

With Merton's model we mean a Black Scholes market model where we have an exponential utility maximizer agent. Then, if with B we indicate the standard Brownian motion, the discounted asset price follows the dynamics:

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

up to the (finite) horizon T. Here X is continuous and hence locally bounded (and so one may also apply the results in Schachermayer [29]). The hypothesis of our Theorem 8 are satisfied with $W_0 = 1$, x arbitrary, so that for any $W \in \mathcal{W}$ we get $U^W(x) = U^1(x)$.

Let $Z_t = B_t + \frac{\mu}{\sigma}t$ be the Brownian motion under the unique martingale measure Q. It is widely known that $U^1(x) = E[-e^{-(x+\frac{\mu}{\sigma}Z_T)}]$ and that the supremum is reached on the claim $f_x = \frac{\mu}{\sigma}Z_T$, which is independent from x *and does not belong to* K^1, because it is unbounded. But if we take $\overline{W} = 1 - \inf_{t\leq T} Z_t$, then $\overline{W} \in \mathcal{W}$ and $f_x \in K^{\overline{W}}$. Indeed $f_x = \frac{\mu}{\sigma}\int_0^T \frac{1}{\sigma X_t} dX_t$, with $\overline{H} = \frac{\mu}{\sigma^2 X} \in \mathcal{H}^{\overline{W}}$.

This classic setup thus provides an example in which the set of strategies $\mathcal{H}^1$ is strictly contained in $\mathcal{H}^{\overline{W}}$, while $U^1(x) = U^{\overline{W}}(x)$. *The enlargement of strategies does not increase the maximum expected utility, but in this case it is necessary to catch the optimal solution to the primal problem.*

3.2 Exponential utility maximization with infinitely many unbounded jumps

Let X now be a scalar Compound Poisson process, that is $X_t = \sum_{T_j \leq t} Y_j$, in which: (i) $T_0 = 0$ and $(T_j)_{j\geq 1}$ is the sequence of the jump times of a Poisson process N; (ii) $Y_0 = 0$ and $(Y_j)_{j\geq 1}$ is a sequence of i.i.d. random variables independent from $(T_j)_{j\geq 1}$.

Fix a finite horizon T and choose the filtration $\mathcal{F}$ to be the P-augmentation of the natural one. Since X is a strong Markov process, $\mathcal{F}$ is already right continuous.

Whenever the Y_j are unbounded from both sides $\mathcal{H}^1$ is trivial. So, we assume that the Y_j are normally distributed, with mean $m \neq 0$ and variance σ^2. Classical utility maximization is then trivial and $U^1(x) = -e^{-x}$, but let us see what happens if we maximize over some $\mathcal{H}^W$, with $W \in \mathcal{W}$. Set $W = \sum_{n\in\mathbb{N}}(1+|Y_1|+\cdots+|Y_n|)I_{\{N_T=n\}}$. In practice, we disintegrate according to the jumps of N and we sum the absolute value of the occurred jump sizes. Then $W \in \mathcal{W}$, since $W \geq X_t \geq -W$ for all t and $E[-e^{\alpha W}] = -e^{\lambda T(E[e^{\alpha|Y_1|}]-1)} > -\infty$ for all $\alpha \in \mathbb{R}$. The condition $\sup_{k\in K^W} E[-e^{-k}] < 0$ can also be verified by disintegration with respect to N_T. Thus, we can apply the results of Theorem 8:

$$\begin{aligned}\sup_{k\in K^W} E[-e^{-x-k}] &= \max_{k\in K_\Phi} E[-e^{-x-k}] \\ &= \min_{\lambda>0, Q\in M_\sigma\cap P_\Phi} \left\{\lambda E_Q[\ln(\frac{dQ}{dP})] + \lambda(x-1+\ln\lambda)\right\}\end{aligned}$$

the optima being $f_x = \frac{m}{\sigma^2}X_T$ (which belongs to K^W and it does not depend on x) and $\frac{dQ_x}{dP} = \exp(-\frac{m}{\sigma^2}X_T - \lambda T(e^{-\frac{m^2}{2\sigma^2}} - 1))$. Hence, by accepting more risk, the expected utility is strictly increased:

$$U_\Phi(x) = -\exp(-x + \lambda T(e^{-\frac{m^2}{2\sigma^2}} - 1)) > -e^{-x} = U^1(x).$$

4 Sigma martingale measures and the supermartingale property of $\mathcal{H}^W$

In their seminal work [8], Delbaen and Schachermayer first showed the financial importance of the existence of (equivalent) probabilities that make X a σ-martingale. Emery [11] proved a nice characterization of σ-martingale processes, which shows "how weak" this property is with respect to the local martingale property. Hereafter we state the characterization that will be used in subsequent proofs.

Proposition 19 ([11], Proposition 2) *Let X be a d-dimensional semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, Q)$. The following conditions are equivalent: a) X is a σ-martingale; b) there exist (scalar) processes K^i with paths that $Q-a.s.$*

never touch zero, such that $K^i \in L(X^i)(Q)$ and $K^i \cdot X^i$ is a local martingale; c) there exist a d-dimensional $H^1(Q)$ martingale N and a positive (scalar) process $\psi \in \cap_{1\leq i\leq d} L(N^i)(Q)$ such that $X^i = \psi \cdot N^i$.

Let $W \in L^0(P)$ and define:

$$\begin{aligned} M_{\sigma,W} &\triangleq \{Q \in M_\sigma \mid E_Q[W] < +\infty\}; \\ M_{sup,W} &\triangleq \{Q \ll P \,|\, E_Q[W] < +\infty \text{ and } H \cdot X \text{ is a } Q\text{-supermart. } \forall H \in \mathcal{H}^W\}; \\ M_{T,W} &\triangleq \{Q \ll P \,|\, E_Q[W] < +\infty,\ (H \cdot X)_T \in L^1(Q) \text{ and} \\ &\quad E_Q[(H \cdot X)_T] \leq 0 \,\forall H \in \mathcal{H}^W\}. \end{aligned}$$

Applying a classical result by Ansel and Stricker [1] we deduce:

Proposition 20 *Let $W \in L^0(P)$ and suppose that $M_{\sigma,W}$ is not empty. Then:*
(a) For all $H \in \mathcal{H}^W$, $H \cdot X$ is a local martingale and a supermartingale under each $Q \in M_{\sigma,W}$;
(b) If W is $X-$suitable, then

$$M_{\sigma,W} = M_{sup,W} = M_{T,W};$$

(c) If W is $u-$compatible, then

$$M_{\sigma,W} \cap P_\Phi = M_\sigma \cap P_\Phi.$$

Proof. (a) By definition, there exists a $c \geq 0$ such that $(H \cdot X)_t \geq -cW$. Now, if $Q \in M_{\sigma,W}$, there exists a positive predictable scalar process ψ (of course, depending on Q) such that $\psi^{-1} \cdot X$ is a Q uniformly integrable martingale[1]. So, under Q, $H \cdot X = (\psi H) \cdot (\psi^{-1} \cdot X)$ is a stochastic integral wrt the martingale $(\psi^{-1} \cdot X)$ and its negative part is controlled by the Q-integrable variable cW. Thanks to Ansel and Stricker [1], $H \cdot X$ is then a Q-local martingale and a supermartingale.
(b) Let H^i be as in Definition 2: then $-W \leq (H^i \cdot X^i)_t \leq W$, for all $t \leq T$.

First we prove that $M_{\sigma,W} = M_{sup,W}$. By item (a), we only have to prove that $M_{sup,W} \subseteq M_{\sigma,W}$. Fix $s < t \leq T$. If $Q \in M_{sup,W}$ and $A \in \mathcal{F}_s$, then it is easily seen that both $I_A I_{(s,t]} H^i$, $-I_A I_{(s,t]} H^i$ are in $\mathcal{H}^W$:

$$[(\pm I_A I_{(s,t]} H^i) \cdot X^i]_u = \pm I_A [\,(H^i \cdot X)_{u\wedge t} - (H^i \cdot X^i)_s] I_{\{u>s\}} \geq -2W$$

therefore $E_Q[((I_A I_{(s,t]} H^i) \cdot X^i)_T] = 0$ and thus $M^i \triangleq H^i \cdot X^i$ is a Q-martingale for all $i = 1, \cdots, d$. Since it verifies condition b) in Proposition 19, X is a σ-martingale under Q.

Then, we show that $M_{sup,W} = M_{T,W}$: we prove only the nontrivial containment $M_{T,W} \subseteq M_{sup,W}$, by the following standard argument.

Define the stopping times (increasing to T) $T_n = \inf\{t \leq T \,|\, (H \cdot X)_t > n\}$

[1] Note that sometimes we make a slight abuse of notation: while $\psi^{-1} \cdot X = (\psi^{-1} \cdot X_1, \cdots, \psi^{-1} \cdot X_d)$ with componentwise integration, $H \cdot X$ stands for vector stochastic integration. We use the same symbol for both types of integration, since no confusion arises.

and fix $s < t \leq T$ and $A \in F_s$. If $(H \cdot X)_t \geq -cW$, then also $I_A I_{]s,t\wedge T_n]} H \in \mathcal{H}^W$, since

$$((I_A H I_{]s,t\wedge T_n]}) \cdot X)_u \geq -cW - n \geq -(c+n)W.$$

When $Q \in M_{T,W}$ we have $E_Q[((I_A H I_{]s,t\wedge T_n]}) \cdot X)_T] \leq 0$, so that:

$$E_Q[I_A (H \cdot X)_{t\wedge T_n} I_{\{T_n > s\}}] \leq E_Q[I_A (H \cdot X)_s I_{\{T_n > s\}}].$$

Observe now that $|I_A (H \cdot X)_s I_{\{T_n > s\}}| \leq |(H \cdot X)_s|$ and $(H \cdot X)_s \in L^1(Q)$, since $I_{[0,s]} H \in \mathcal{H}^W$. In addition, $I_A (H \cdot X)_{t\wedge T_n} I_{\{T_n > s\}} \geq -cW$: hence an application of Lebesgue dominated convergence theorem on the rhs and of Fatou lemma on the lhs leads us to the final inequality $E_Q[I_A (H \cdot X)_t] \leq E_Q[I_A (H \cdot X)_s]$.

(c) Since $W \geq 1$ then $E_Q[W] \geq 1$. Replacing in the Fenchel inequality (7) x by $(-W)$, y by $\frac{dQ}{dP}$ and computing the expectation, we see that $W \in L^1(Q)$ for all $Q \in P_\Phi$. Then (c) follows. ■

Proof of Proposition 7 (a) follows from the proof of item (c) in Proposition 20. (b) Applying Proposition 20, we deduce that if $W \in \mathcal{W}$ and if $Q \in M_\sigma \cap P_\Phi = M_{\sigma,W} \cap P_\Phi = M_{T,W} \cap P_\Phi$ then $E_Q[k] \leq 0 \, \forall k \in K^W$.

Remark 21 *The definition of W-admissibility is inspired by Section 5 in Delbaen and Schachermayer [8] and by the extended notion of admissibility there introduced. From Definitions 5.1 and 5.4 in [8], if $W \geq 1$ is a* ***feasible weight function*** *for X then $M^e_{\sigma,W} \triangleq M_{\sigma,W} \cap \{Q \sim P\} \neq \emptyset$. And a strategy H is W-admissible in the sense of [8] (briefly $H \in \widetilde{\mathcal{H}}^W$) if the losses are controlled in the following way: $(H \cdot X)_t \geq -cE_Q[W|\mathcal{F}_t]$, for every $Q \in M^e_{\sigma,W}$.*

One can also check that if W is a feasible weight function, then it is also X-suitable. If that is the case, then $\mathcal{H}^W \subseteq \widetilde{\mathcal{H}}^W$, thanks to the supermartingale property shown in the above Proposition.

In our setting, we do not need the assumption $M^e_{\sigma,W} \neq \emptyset$. Also, we note that Delbaen and Schachermayer's control of the losses requires the heavy computation of the set of processes $(E_Q[W|\mathcal{F}_t])_t$, generated by all the $Q \in M^e_{\sigma,W}$. On the contrary, our condition of $\mathbf{W}$-admissibility relies only on the single random variable W. In addition, if R is a stopping time, the admissibility of H, in the sense of [8], does not imply that of $H I_{[0,R]}$. Our class $\mathcal{H}^W$ satisfies this property, which seems a natural request, both from a mathematical and a financial point of view. Hence when W is a feasible weight function, $\mathcal{H}^W$ is smaller than $\widetilde{\mathcal{H}}^W$, and easier to handle in practice.

5 Dual formulation of the primal problem

In this section we consider a general setting, where $\mathcal{K}$ is a convex cone ($0 \in \mathcal{K}$) contained in the set $L^{bb}(P) \triangleq \{f \in L^0(P) \,|\, \exists c \in \mathbb{R} \text{ s.t. } f \geq c \; P\text{−a.s.}\}$ of bounded from below random variables.

We use the notation $\mathcal{K}W = \{f = kW \,|\, k \in \mathcal{K}\}$ and similarly for $\mathcal{K}/W$. Define

$$\begin{aligned} \mathcal{M}_1(\mathcal{K}) &\triangleq \left\{Q \ll P : k \in L^1(Q) \text{ and } E_Q[k] \leq 0 \text{ for all } k \in \mathcal{K}\right\}, \qquad (21) \\ \mathcal{M}_W(\mathcal{K}) &\triangleq \left\{Q \ll P : W \in L^1(Q) \text{ and } h \in L^1(Q),\, E_Q[h] \leq 0 \ \forall h \in \mathcal{K}W\right\}. \end{aligned}$$

The following Theorem proves the existence of the minimax measure in $\mathcal{M}_1(\mathcal{K})$ (see Bellini and Frittelli [2] for a detailed analysis of this topic) in a general framework, since W may be unbounded. From a simple check of its proof, we see that Assumption 2 is not necessary in Theorem 22.

Theorem 22 *Let $\mathcal{K} \subseteq L^{bb}(P)$ be a convex cone and let W be $u-$compatible. If $\sup_{f\in\mathcal{K}} E[u(x+fW)] < u(+\infty)$ then $\mathcal{M}_1(\mathcal{K})$ and $\mathcal{M}_W(\mathcal{K})$ are not empty and*

$$\begin{aligned} \sup_{k\in\mathcal{K}} E[u(x+kW)] &= \min_{\alpha>0, Q\in\mathcal{M}_1(\mathcal{K})} \left\{ E_Q\left[\frac{\alpha x}{W}\right] + E\left[\Phi\left(\frac{\alpha}{W}\frac{dQ}{dP}\right)\right]\right\} \\ &= \min_{\lambda>0, R\in\mathcal{M}_W(\mathcal{K})} \left\{ \lambda x + E\left[\Phi\left(\lambda\frac{dR}{dP}\right)\right]\right\}. \end{aligned}$$

Proof. We will follow the proof of Theorem 2.1 in [2] with a key device consisting of thinking $\tilde{u}(\omega,\xi) = u(x+\xi W(\omega))$, with $\xi \in \mathbb{R}$, as argument of the expectation. Let $I_{\tilde{u}}(h) \triangleq E[\tilde{u}(\omega, h(\omega))]$, $\mathcal{D} \triangleq \{h \in L^\infty(P) \mid I_{\tilde{u}}(h) > -\infty\}$ be its effective domain and set $\mathcal{C} \triangleq (\mathcal{K} - L^0_+(P)) \cap L^\infty(P)$. Since W is $u-$compatible, then for all $h \in L^\infty(P)$ we have $I_{\tilde{u}}(h) \geq E[u(x- \parallel h \parallel_\infty W)] > -\infty$. Therefore $\mathcal{D} = L^\infty(P)$ and we may apply Fenchel's duality theorem:

$$\sup_{k\in\mathcal{K}} E[u(x+kW)] = \sup_{h\in\mathcal{C}} E[u(x+hW)] = \min_{z\in ba(P)} \left\{\delta_{\mathcal{C}^0}(z) - (I_{\tilde{u}})^*(z)\right\}, \qquad (22)$$

where the first equality follows from the Fatou Lemma and the assumption $\mathcal{K} \subseteq L^{bb}(P)$. In (22) the polarity refers to the dual system $(L^\infty(P), ba(P))$, $\delta_{\mathcal{C}^0}$ is the indicator functional of the convex cone $\mathcal{C}^0$, the polar of $\mathcal{C}$ with respect to $(L^\infty(P), ba(P))$, and $(I_{\tilde{u}})^*$ is the concave conjugate of $I_{\tilde{u}}$.
Define the pointwise convex conjugate of $\tilde{u}$ as:

$$\tilde{\Phi}(\omega, y) = \sup_{\xi\in\mathbb{R}} \{\tilde{u}(\omega,\xi) - y\xi\} = \frac{xy}{W(\omega)} + \Phi\left(\frac{y}{W(\omega)}\right),$$

where Φ is the convex conjugate of u (see equation 6). Set $I_{-\tilde{\Phi}}(h) = E[-\tilde{\Phi}(h)]$ (and $I_{\tilde{\Phi}}(h) = E[\tilde{\Phi}(h)]$). Since $\tilde{u}$ is a normal concave integrand in the sense of Rockafellar [27], we can write:

$$(I_{\tilde{u}})^*(z) = I_{-\tilde{\Phi}}(z_a) - \delta^*_{\mathcal{D}}(-z_s),$$

in which z_a, z_s are the countably additive and purely singular parts of z and $\delta^*_{\mathcal{D}}(z) = \sup_{f\in\mathcal{D}} z(f)$ is the convex conjugate of $\delta_{\mathcal{D}}$. But $\mathcal{D} = L^\infty(P)$ so that

$\delta^*_{\mathcal{D}} = \delta_{\{0\}}$ and

$$\begin{aligned}\min_{z\in ba(P)}\{\delta_{\mathcal{C}^0}(z) - (I_{\tilde{u}})^*(z)\} &= \min_{z\in\mathcal{C}^0\cap L^1} I_{\tilde{\Phi}}(z)\\ &= \min_{z\in\mathcal{C}^0\cap L^1}\left\{E\left[\frac{zx}{W}\right] + E\left[\Phi\left(\frac{z}{W}\right)\right]\right\}\\ &= \min_{\alpha>0, Q\in\mathcal{M}_1(\mathcal{K})}\left\{E_Q\left[\frac{\alpha x}{W}\right] + E\left[\Phi\left(\frac{\alpha}{W}\frac{dQ}{dP}\right)\right]\right\},\end{aligned}$$

since $\mathcal{M}_1(\mathcal{K}) = \{Q \ll P : E_Q[g] \leq 0 \text{ for all } g \in \mathcal{C}\}$. Note that the minimum is not attained by $z = 0$, because $\sup_{f\in\mathcal{K}} E[u(x+fW)] < u(+\infty)$ and $\Phi(0) = u(+\infty)$. Since $Q \in \mathcal{M}_1(\mathcal{K})$ iff $R \in \mathcal{M}_W(\mathcal{K})$, where $\frac{dR}{dP} = \beta\frac{1}{W}\frac{dQ}{dP}$ and $\beta > 0$ is the normalizing constant, the proof is completed. ■

5.1 Application to $W-$admissible strategies

We come back to the setting of the Sections 1-4 and let $\mathcal{K} \triangleq \frac{K^{W_0}}{W_0}$ with $W_0 \in \mathcal{W}$. From the definition of $\mathcal{M}_{W_0}(\mathcal{K})$ and of M_{T,W_0} we immediately get:

$$\begin{aligned}\mathcal{M}_{W_0}(\mathcal{K}) &= \left\{Q \ll P \mid W_0 \in L^1(Q),\, g \in L^1(Q) \text{ and } E_Q[g] \leq 0\, \forall g \in K^{W_0}\right\}\\ &= M_{T,W_0} = M_{\sigma,W_0},\end{aligned}$$

where the last equality follows from Proposition 20, since W_0 is $X-$suitable.

Corollary 23 *If there exist $W_0 \in \mathcal{W}$, $x \in \mathbb{R}$ such that $U^{W_0}(x) = \sup_{k\in K^{W_0}} E[u(x+k)] < u(+\infty)$, then $M_\sigma \cap P_\Phi$ is not empty and for all $W \in \mathcal{W}$*

$$U^{W_0}(x) = U^W(x) = U^{\mathcal{W}}(x) = \min_{\lambda>0, Q\in M_\sigma\cap P_\Phi}\left\{\lambda x + E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]\right\}, \quad (23)$$

Proof. If $\mathcal{K} \triangleq \frac{K^{W_0}}{W_0}$, then $\mathcal{K}$ is contained in $L^{bb}(P)$, $\mathcal{M}_{W_0}(\mathcal{K}) = M_{\sigma,W_0}$ and we may apply Theorem 22:

$$\sup_{k\in K^{W_0}} E[u(x+k)] = \sup_{k\in\mathcal{K}} E[u(x+kW_0)] = \min_{\lambda>0, Q\in M_{\sigma,W_0}}\left\{\lambda x + E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]\right\}. \quad (24)$$

Using Assumption 2 and Proposition 20, we see that the minimax measure attaining the minimum in the above equation belongs to $M_{\sigma,W_0}\cap P_{\Phi_\lambda} = M_{\sigma,W_0}\cap P_\Phi = M_\sigma \cap P_\Phi$. For any other $W \in \mathcal{W}$ we deduce, as in Remark 10, that $U^W(x) < u(+\infty)$. Therefore, (24) holds true for an arbitrary $W \in \mathcal{W}$ and so (23) follows. ■

6 Solution to the dual problem

In this Section, the convex cone $\mathcal{K} \subseteq L^0(P)$ is not necessarily contained in $L^{bb}(P)$. Recall the definition of $\mathcal{M}_1(\mathcal{K})$ given in (21) and suppose that:

$$\mathcal{N}(\mathcal{K}) \triangleq \mathcal{M}_1(\mathcal{K}) \cap P_\Phi \neq \emptyset.$$

To simplify the notation, we drop the dependence of these sets of measures on $\mathcal{K}$. Let's define the linear spaces

$$L = \bigcap_{Q \in \mathcal{N}} L^1(Q) \text{ and } L' = Lin\{\mathcal{N}\} \subseteq L^1(P).$$

The map $< l, l' >: L \times L' \to \mathbb{R}$ given by $< l, l' > \triangleq E[ll']$ is a well-defined bilinear form on $L \times L'$. We put on both L, L' the weak topologies: $\sigma(L, L')$ and $\sigma(L', L)$ respectively. All the polars are defined with respect to this dual system which is not necessarily separated (see Grothendieck [15], as a standard reference for details on general dual systems). We remark that $\mathcal{K} \subseteq L$ by construction.

The following theorem provides a necessary and sufficient condition for the existence of the projection of P on $\mathcal{M}_1$ with respect to the $\Phi-$divergence distance. Theorem 24 is essentially Theorem 5 in Rüschendorf [28], when $\lambda = 1$. The proof we present here is the generalization of the proof of Theorem 2.3 in Frittelli [12]. Theorem 24 (in the formulation given in Corollary 26) is crucial for the proof of Theorem 8.

Theorem 24 *Suppose $\mathcal{N} = \mathcal{M}_1 \cap P_{\Phi_\lambda}$, $\forall \lambda > 0$. Then $Q_\lambda \in \mathcal{N}$ is optimal for*

$$\inf \left\{ E\left[\Phi\left(\lambda \frac{dQ}{dP} \right) \right] \mid Q \in \mathcal{N} \right\} \tag{25}$$

if and only if $f_\lambda \triangleq E[\eta_\lambda \Phi'(\lambda \eta_\lambda)] - \Phi'(\lambda \eta_\lambda) \in (co(\mathcal{N}))^0$, where $\eta_\lambda = dQ_\lambda / dP$.

Proof. First consider $\lambda = 1$. Suppose that Q_1 is optimal for (25) and let $Q_0 \in \mathcal{N}$. Set $\eta_0 = \frac{dQ_0}{dP}$, $\xi_x = x\eta_0 + (1-x)\eta_1$, $x \in [0,1]$. By optimality of Q_1, we necessarily have $\left(\frac{d}{dx} E[\Phi(\xi_x)]|_{x=0} \right) \geq 0$. From the convexity of Φ we derive

$$\eta_0 \Phi'(\eta_1) \leq \eta_1 \Phi'(\eta_1) + \Phi(\eta_0) - \Phi(\eta_1) \quad P - a.s. \tag{26}$$

From Lemma 18 (a) and from (26) we obtain

$$\eta_1 \Phi'(\eta_1) \in L^1(P) \text{ and } (\eta_0 \Phi'(\eta_1))^+ \in L^1(P). \tag{27}$$

Set $H(x) = \Phi(\xi_x)$, $x \in [0,1]$. By convexity of H, $\left(\frac{H(x) - H(0)}{x} \right)$ is non decreasing. Since $E[H(1) - H(0)] < +\infty$, we apply the monotone convergence Theorem to get:

$$\begin{aligned} 0 &\leq \frac{d}{dx} E[\Phi(\xi_x)]|_{x=0} = \lim_{x \downarrow 0} E\left[\frac{H(x) - H(0)}{x} \right] \\ &= E[H'(0)] = E[\Phi'(\eta_1)(\eta_0 - \eta_1)]. \end{aligned} \tag{28}$$

From (27) and (28) we deduce $\eta_0 \Phi'(\eta_1) \in L^1(P)$ and $f_1 \in L^1(Q_0)$. From equation (28) we get: $0 \leq E_{Q_0}[\Phi'(\eta_1)] - E_{Q_1}[\Phi'(\eta_1)] = -E_{Q_0}[f_1]$. This holds for all $Q_0 \in \mathcal{N}$ and so $f_1 \in (co(\mathcal{N}))^0$.

Conversely, suppose $f_1 \in (co(\mathcal{N}))^0$. Then, for any $Q_0 \in \mathcal{N}$

$$\begin{aligned} E[\Phi(\eta_0)] &\geq E[\Phi(\eta_1)] + E_{Q_0}[\Phi'(\eta_1)] - E_{Q_1}[\Phi'(\eta_1)] \\ &= E[\Phi(\eta_1)] - E_{Q_0}[f_1] \geq E[\Phi(\eta_1)]. \end{aligned}$$

Now fix $\lambda > 0$ and set $\eta_\lambda \triangleq \frac{dQ_\lambda}{dP}$. Note that $Q_\lambda \in \mathcal{N}$ is optimal for (25) iff it attains the infimum in: $\inf\left\{E\Phi_\lambda(\frac{dQ}{dP}) \mid Q \in \mathcal{N}\right\}$. From the case $\lambda = 1$, applied to the function Φ_λ, then the last statement is equivalent to $f_1 \triangleq E[\eta_\lambda \Phi'_\lambda(\eta_\lambda)] - \Phi'_\lambda(\eta_\lambda) \in (co(\mathcal{N}))^0$, and the thesis follows from the cone property of $(co(\mathcal{N}))^0$. ■

The following Theorem is the abstract version of Proposition 16, which is used in Section 2.1. It was first proved in Biagini and Frittelli [3], in the case $\mathcal{K} \subseteq L^{bb}$ and $\Phi(0) < +\infty$ and by Owen [25] in the case $\mathcal{K} \subseteq L^{bb}$, Φ generic. We prove the version with general $\mathcal{K}$ and Φ.

Theorem 25 *Suppose that $\mathcal{N} = \mathcal{M}_1 \cap P_{\Phi_\lambda}$ for all $\lambda > 0$. Then*

$$\bigcap_{Q \in \mathcal{N}} \overline{\mathcal{K} - L^1_+(Q)}^Q = \{f \in L \mid E_Q[f] \leq 0 \text{ for all } Q \in \mathcal{N}\} \triangleq (co(\mathcal{N}))^0; \quad (29)$$

$$\widehat{f}_\Phi \triangleq \inf\left\{x \in \mathbb{R} \mid f - x \in \bigcap_{Q \in \mathcal{N}} \overline{\mathcal{K} - L^1_+(Q)}^Q\right\} = \sup\{E_Q[f] \mid Q \in \mathcal{N}\}$$

$\forall f \in L$, *where $\overline{A}^Q$ denotes the $L^1(Q)-$closure of a set A.*

Proof. Set $C = \bigcap_{Q \in \mathcal{N}} \overline{\mathcal{K} - L^1_+(Q)}^Q$. We need to show that $(co(\mathcal{N}))^0 \subseteq C$, since the opposite inclusion is obvious. Let $k \in (co(\mathcal{N}))^0$ and suppose by contradiction that there exists $Q_0 \in \mathcal{N}$ such that $k \notin \overline{\mathcal{K} - L^1_+(Q_0)}^{Q_0}$.

By the Hahn-Banach Theorem, there exists $\widetilde{\xi} \in L^\infty(Q_0)$ such that:

$$\sup_{f \in \mathcal{K} - L^1_+(Q_0)} E_{Q_0}[\widetilde{\xi} f] \leq 0 < E_{Q_0}[\widetilde{\xi} k]. \quad (30)$$

Since $-I_{\left\{\widetilde{\xi} < 0\right\}} \in (\mathcal{K} - L^1_+(Q_0))$, we deduce that $\widetilde{\xi} \geq 0$ $Q_0 - a.s.$ Since $\widetilde{\xi}\frac{dQ_0}{dP} \geq 0$ $P - a.s.$, we normalize $\widetilde{\xi}$, call it ξ, and define a probability $Q_1 \ll P$ by setting: $\frac{dQ_1}{dP} = \xi \frac{dQ_0}{dP}$. From equation (30) we then derive $Q_1 \in \mathcal{M}_1$ and $E_{Q_1}[k] > 0$.

First consider the case $\Phi(0^+) < +\infty$. We claim that $Q_1 \in P_\Phi$, so that $Q_1 \in \mathcal{N}$, which is in contradiction with $k \in (co(\mathcal{N}))^0$. In fact, fix $\lambda_0 > 0$ and let c^* be the unique solution of $\Phi' = 0$. Since Φ increases after c^* we have:

$$E\left[\Phi(\frac{dQ_1}{dP})\right] \leq E\left[\Phi(\xi\frac{dQ_0}{dP})I_{\{\xi > \lambda_0\}}\right] + \Phi(0) + E\left[\Phi(\lambda_0\frac{dQ_0}{dP})I_{\{\xi\frac{dQ_0}{dP} > c^*\}}I_{\{\xi \leq \lambda_0\}}\right]$$

and the rhs is finite since $Q_0 \in \mathcal{N} = \mathcal{M}_1 \cap P_{\Phi_\lambda} \forall \lambda > 0$ and we may apply Lemma 18 (b) to $E\left[\Phi(\xi\frac{dQ_0}{dP})I_{\{\xi > \lambda_0\}}\right]$ because ξ is bounded $P-$a.s on $\left\{\frac{dQ_0}{dP} > 0\right\}$.

Now suppose that $\Phi(0^+) = +\infty$. Then the condition $Q_0 \in P_\Phi$ implies $Q_0 \sim P$ and therefore $\frac{dQ_1}{dQ_0} \in L^\infty(P)$. Consider the convex combination $Q^\lambda = \lambda Q^1 + (1-\lambda)Q^0 \in \mathcal{M}_1$. If $\lambda \in [0,1)$ then $0 < 1-\lambda \leq \frac{dQ^\lambda}{dQ^0} \leq \lambda \frac{dQ_1}{dQ_0} + 1 - \lambda$ holds $P-a.s$ and therefore Lemma 18 (b) and $Q^0 \in P_\Phi$ imply again $\Phi(\frac{dQ^\lambda}{dP}) \in L^1(P)$. So $\forall \lambda \in [0,1)$ $Q_\lambda \in \mathcal{N}$. However, since $E_{Q_\lambda}[k] = \lambda E_{Q_1}[k] + (1-\lambda)E_{Q_0}[k]$ and $E_{Q_1}[k] > 0$, there exists $\lambda^* \in [0,1)$ such that $E_{Q_{\lambda^*}}[k] > 0$, which contradicts $k \in (co(\mathcal{N}))^0$. The proof of (29) is now completed.

As an immediate consequence of (29) we have:

$$C \text{ is } \sigma(L, L') - \text{closed}, \; C^{00} = C, \; C^0 = \overline{co(\mathcal{N})}^{\sigma(L',L)}. \tag{31}$$

For any subset $G \subseteq L^1(P)$ we set: $G_1 \triangleq \{z \in G : E[z] = 1\}$. An easy application of the Hahn-Banach theorem permits also to conclude that

$$(\overline{co(\mathcal{N})}^{\sigma(L',L)})_1 = \overline{\mathcal{N}}^{\sigma(L',L)} \text{ and so } (C^0)_1 = \overline{\mathcal{N}}^{\sigma(L',L)}. \tag{32}$$

Theorem 10 [3] states that if $G \subseteq L$ is a convex cone satisfying $G^{00} = G$, $(G^0)_1 \neq \varnothing$ and $-I_\Omega \in G$ then for all $f \in L$ we have:

$$\inf\{x \in \mathbb{R} \mid f - x \in G\} = \sup\left\{E[zf] \mid z \in (G^0)_1\right\}. \tag{33}$$

Applying (33) to the set $G = C$ and using (32), we get for all $f \in L$:

$$\begin{aligned} \inf\{x \in \mathbb{R} \mid f - x \in C\} &= \sup\left\{E_Q[f] \mid Q \in \overline{\mathcal{N}}^{\sigma(L',L)}\right\} \\ &= \sup\{E_Q[f] \mid Q \in \mathcal{N}\}, \end{aligned}$$

since $E_\bullet[f] : L' \to \mathbb{R}$ is $\sigma(L', L)-$continuous. ■

6.1 Application to W-admissible strategies

We come back to the setting of the Sections 1-4 and we let $\mathcal{K} = K^W$ with $W \in \mathcal{W}$. From the definition of $\mathcal{M}_1(\mathcal{K})$ and of $M_{T,W}$ and by Propositions 20 and 7, we immediately get $\mathcal{M}_1(K^W) = \left\{Q \ll P \mid E_Q[k] \leq 0 \, \forall k \in K^W\right\}$ and $\mathcal{N}(K^W) = \mathcal{M}_1(K^W) \cap P_\Phi = M_{T,W} \cap P_\Phi = M_\sigma \cap P_\Phi$. From the definitions of L, L' and of K_Φ (see (10)) we then have: $K_\Phi = (co(M_\sigma \cap P_\Phi))^0$.

Recalling from Proposition 7 that $K^W \subseteq K^{\mathcal{W}} \subseteq K_\Phi$, one may now easily deduce Proposition 16 from Theorem 25. Moreover, from Theorem 24 we get:

Corollary 26 *Suppose that $M_\sigma \cap P_\Phi \neq \varnothing$. If $\lambda > 0$ then $Q_\lambda \in M_\sigma \cap P_\Phi$ is optimal for*

$$\inf\left\{E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right] \mid Q \in M_\sigma \cap P_\Phi\right\}$$

if and only if $f_\lambda \triangleq E[\eta_\lambda \Phi'(\lambda \eta_\lambda)] - \Phi'(\lambda \eta_\lambda) \in K_\Phi$, where $\eta_\lambda = dQ_\lambda / dP$.

7 Solution to the primal problem

Proposition 27 *If $Q \in P_\Phi$ satisfies (9), then for all $x \in \mathbb{R}$ the optimal $\lambda(x,Q)$ solution of*

$$\min_{\lambda>0}\left\{\lambda x + E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]\right\}$$

is the unique solution of the first order condition

$$x + E\left[\frac{dQ}{dP}\Phi'\left(\lambda\frac{dQ}{dP}\right)\right] = 0, \tag{34}$$

and $f_x \triangleq -x - \Phi'(\lambda(x,Q)\frac{dQ}{dP}) \in \left\{f \in L^1(Q) : E_Q[f] = 0\right\}$ satisfies

$$\sup\left\{E[u(x+f)] \mid f \in L^1(Q) \text{ and } E_Q[f] \leq 0\right\} = E[u(x+f_x)] < u(+\infty). \tag{35}$$

Proof. The existence and uniqueness of the solution to (34) follow from Lemma 18 (c). Set $\eta = \frac{dQ}{dP}$ and $\lambda^* \triangleq \lambda(x,Q)$ and recall from Lemma 18 (a) that, for all $\lambda > 0$, $\eta\Phi'(\lambda\eta) \in L^1(P)$, which implies, thanks to (8) and (9), that $u(-\Phi'(\lambda\eta)) \in L^1(P)$. From Fenchel inequality (7) and from (9), we deduce that for all $f \in L^1(Q)$ with $E_Q[f] \leq 0$ and for all $\lambda > 0$:

$$E[u(x+f)] \leq E[\lambda\eta(x+f)] + E[\Phi(\lambda\eta)] \leq \lambda x + E[\Phi(\lambda\eta)] < +\infty,$$

and therefore:

$$\begin{aligned} E[u(x+f)] \leq \inf_{\lambda>0} E[\lambda x + \Phi(\lambda\eta)] &= \lambda^* x + E[\Phi(\lambda^*\eta)] = \\ &\overset{(a)}{=} E[u(-\Phi'(\lambda^*\eta))] = E[u(x+f_x)], \end{aligned}$$

where in the equality (a) we used (34) and (8). However, $f_x \in L^1(Q)$, since $\Phi'(\lambda\eta) \in L^1(Q)$, and $E_Q[f_x] = 0$ (from equation (34)). Hence, f_x attains the supremum in (35) and, from the strict monotonicity of u, we deduce $E[u(x+f_x)] < u(+\infty)$. ■

Proof of Theorem 8

(a) and the main part of **(b)** follow directly from Corollary 23. Since we know that there exists $Q \in M_\sigma \cap P_\Phi$, from Proposition 27 we deduce that *for all* $x \in \mathbb{R}$ we have:

$$U_\Phi(x) \leq \sup\left\{Eu(x+f) \mid f \in L^1(Q) \text{ and } E_Q[f] \leq 0\right\} < u(+\infty)$$

and this completes the proof of point (b).

(c) We show first that $U_\Phi(x) = U^{\mathcal{W}}(x)$. As an immediate consequence of (11) and of Proposition 7 we have:

$$U^{\mathcal{W}}(x) \leq U_\Phi(x) \leq \min_{\lambda>0, Q\in M_\sigma\cap P_\Phi}\left\{\lambda x + E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]\right\}, \tag{36}$$

and by Corollary 23 we get that equalities must hold in the above formula.

Now, we show that there exists the optimal solution f_x in K_Φ. Consider the dual problem in (36) and think about it as:

$$\min_{Q\in M_\sigma\cap P_\Phi}\left\{\min_{\lambda>0}\left\{\lambda x+E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]\right\}\right\}.$$

Let us first minimize over λ with fixed $Q\in M_\sigma\cap P_\Phi$. The minimum in $\min_{\lambda>0}\left\{\lambda x+E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right]\right\}$ is reached when the optimal $\lambda(x,Q)$ is the unique solution of the first order condition in (34).

Then we minimize over Q and we face:

$$\min_{Q\in M_\sigma\cap P_\Phi}\left\{\lambda(x,Q)x+E\left[\Phi\left(\lambda(x,Q)\frac{dQ}{dP}\right)\right]\right\}.$$

Denote with Q_x the optimal probability measure, with η_x its density and set $\lambda_x=\lambda(x,Q_x)$. Now, consider the two obvious facts:

1. Q_x is also the optimal solution to $\min_{Q\in M_\sigma\cap P_\Phi}\{\lambda_x x+E[\Phi(\lambda_x dQ/dP)]\}$
2. λ_x is also the optimal solution to $\min_{\lambda>0}\{\lambda x+E[\Phi(\lambda\eta_x)]\}$

We can apply Corollary 26 to the problem in the first item to get that $f_x\triangleq E[\eta_x\Phi'(\lambda_x\eta_x)]-\Phi'(\lambda_x\eta_x)$ belongs to K_Φ. Looking at the second item, we realize, as in equation (34), that

$$x+E[\eta_x\Phi'(\lambda_x\eta_x)]=0, \tag{37}$$

so that $f_x=-x-\Phi'(\lambda_x\eta_x)$. From (8) we get:

$$u(x+f_x)=-\lambda_x\eta_x\Phi'(\lambda_x\eta_x)+\Phi(\lambda_x\eta_x).$$

Finally, by taking expectations on both sides, we obtain $E[u(x+f_x)]=\lambda_x x+E[\Phi(\lambda_x\eta_x)]$, since (37) holds.

(d) We now prove that f_x can be Q_x-represented as the terminal value of a stochastic integral $H^x\cdot X$, which is a Q_x-martingale. We have just shown that $f_x\in K_\Phi$ and that $E_{Q_x}[f_x]=0$ (in (37)). Fix $W\in\mathcal{W}$. Then, from the representation of K_Φ in (13), $f_x\in\overline{K^W}^{Q_x}$, so that we can select a sequence of integrals $Y^n=H^n\cdot X$ with $H^n\in\mathcal{H}^W$ such that $(Y^n)_T\to f_x$ in $L^1(Q_x)$ and Q_x-a.s. After this key observation, we can proceed in a standard way (see e.g. Schachermayer [29], Step 10 in the proof of Theorem 2.2).

The sequence $(Y^n)_n$ is made of Q_x supermartingales, but a priori we can not control the losses *uniformly* on n.

However, by passing to a subsequence if necessary, $\sum_n\parallel Y_T^n-Y_T^{n+1}\parallel_{L^1(Q_x)}<+\infty$. Take the supremum over the negative parts (here and in what follows the inequalities are intended to hold Q_x-a.s.):

$$Z=\sup_n(Y_T^n)^-\leq\mid Y_T^1\mid+\sum_n\mid Y_T^n-Y_T^{n+1}\mid$$

then $Y_T^n \geq -Z$ and Z is Q_x-integrable. Consider now the (càdlàg version of the) martingale generated by Z: $Z_t = E_{Q_x}[Z \mid \mathcal{F}_t]$. This process $(Z_t)_t$ controls the losses of the Y^n at every instant t. In fact, just use the supermartingale property of the Y^n:

$$Y_T^n \geq -Z \Rightarrow Y_t^n \geq E_{Q_x}[Y_T^n \mid \mathcal{F}_t] \geq -E_{Q_x}[Z \mid \mathcal{F}_t].$$

Hence, since $Q_x \in M_\sigma$, by Theorem D in Delbaen and Schachermayer, [9] (in the version stated in [8] section 5), we can find a $H^x \in L(X)(Q_x)$ and a supermartingale V such that: (i) $(H^x \cdot X)_t \geq V_t$ Q_x-a.s. for all t ; (ii) $V_T = f_x$ Q_x-a.s. But then, again by the result of Ansel and Stricker [1], $H^x \cdot X$ is a Q_x-local martingale and a supermartingale. To end up, $0 \geq E_{Q_x}[(H^x \cdot X)_T] \geq E_{Q_x}[f_x] = 0$ implies $f_x = (H^x \cdot X)_T$ Q_x-a.s. and that the process $H^x \cdot X$ is a true Q_x-martingale. □

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