UTILITY MAXIMIZATION WITH DISCRETIONARY STOPPING*

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Abstract

Utility maximization problems of mixed *optimal stopping /control* type are considered, which can be solved by reduction to a family of related *pure optimal stopping* problems. Sufficient conditions for the existence of optimal strategies are provided in the context of continuous-time, Itô process models for complete markets. The mathematical tools used are those of optimal stopping theory, continuous-time martingales, convex analysis and duality theory. Several examples are solved explicitly, including one which demonstrates that optimal strategies need not always exist.

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1 Introduction

Problems of expected utility maximization go back at least to the seminal articles of Samuelson & Merton (1969), Merton (1971), and have been studied extensively in recent years, for instance by Pliska (1986), Karatzas, Lehoczky & Shreve [KLS] (1987) and Cox & Huang (1989). Most of this literature shares the common setting of an agent who receives a deterministic initial capital, which he must then invest in a market (complete or incomplete) so as to maximize the expected utility of his wealth and/or consumption, up to a *prespecified* terminal time.

In this paper we consider a variant of these problems by allowing the agent freely to stop *before or at* a prespecified final time, in order to maximize the expected utility of his wealth and/or consumption up to the stopping time. The assets available to the agent can be traded continuously, without restrictions, frictions or transaction costs; they consist of a locally riskless money-market, and m risky stocks. (One can think, for example, of an investor or mutual fund manager, who tries to invest/consume as skilfully as possible, before "retiring" from the stock market and putting all his holdings in the money-market.) The stock-prices are driven by m independent Brownian motions; these represent the sources of uncertainty in the market model, which is assumed to be complete in the sense of Harrison & Pliska (1981). The market coëfficients, i.e., the money-market rate, the stock–appreciation rates, and the matrix of stock-volatilities, are bounded random processes adapted to the driving m-dimensional Brownian motion.

The utility maximization problem studied here involves aspects of both *optimal stopping* and *stochastic control*. Such problems also arise in situations like pricing American Contingent Claims under constraints, selecting trading strategies in the presence of transaction costs with an American option held in the portfolio, target-tracking followed by a decision (to engage the target, or not), etc.; see Karatzas & Kou (1998), Davis & Zariphopoulou (1995), Davis & Zervos (1994), as well as Karatzas & Sudderth (1999) for such problems in different contexts. The free–boundary problem approach, based on an associated Hamilton–Jacobi–Bellman (HJB) equation of dynamic programming, is inadequate for the analysis of the general version of our model, which is not Markovian. Instead, duality theory plays an important role, and leads to a family of pure optimal stopping problems which is even more amenable to analysis. Duality approaches have been used with success in treating portfolio optimization problems for financial markets which are incomplete or impose constraints on portfolio choice, as in Karatzas, Lehoczky, Shreve & Xu [KLSX] (1991), Shreve & Xu (1992) and Cvitanić & Karatzas (1992).

The model and the utility maximization problem are described in Sections 2–5. We present a solution in Section 6 using a *duality approach*. However, this solution is not quite satisfactory, in the sense that it leads to computationally tractable results only in very special cases and does not shed much light on the general question of existence of optimal strategies. We then introduce and analyze *a family of pure optimal stopping problems* in Sections 7–8. In terms of these, we are able to provide conditions which guarantee the existence of optimal strategies. In Section 9, several examples are presented, one of which demonstrates that optimal strategies need *not* always exist. For completeness, we treat in the Appendix (Section 10) an example which can be solved explicitly using a free–boundary problem for the associated HJB equation. In a second Appendix (Section 11) we formulate an open problem, suggested by the referee, where consumption continues past the time of retirement from the stock-market.

It is hoped that the analysis in this paper will serve as a step towards establishing a general theory for stochastic control problems with discretionary stopping in continuous time, possibly along the lines of the Dubins–Savage (1965) theory for discrete–time "leavable gambling problems" developed in Chapter 3 of Maitra & Sudderth (1996).

Remark 1.1: We denote by "standing assumption" those conditions that are always in force throughout the paper; they will not be cited in theorems. And "assumption" stands for those conditions which are in force only when theorems specifically cite them.

2 The market model

We adopt a model consisting of a money-market, with price $P_0(\cdot)$ given by

$$dP_0(t) = P_0(t)r(t)\,dt, \qquad P_0(0) = 1 \tag{2.1}$$

and of m stocks with prices-per-share $P_i(\cdot)$ satisfying the equations

$$dP_i(t) = P_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW_j(t)\right], \qquad i = 1, \cdots, m. \tag{2.2}$$

Here $W(\cdot) = (W_1(\cdot), \cdots, W_m(\cdot))^*$ is an m-dimensional Brownian motion on a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We shall denote by $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ the $\mathbb{P}-$augmentation of the filtration generated by $W(\cdot)$. The *coëfficients* of the model, that is, the scalar *interest rate* process $r(\cdot)$, the vector process $b(\cdot) = (b_1(\cdot), \cdots, b_m(\cdot))^*$ of *appreciation rates*, and the matrix-valued *volatility process* $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{1\le i,j\le m}$, are assumed to be bounded, and progressively measurable with respect to $\mathbb{F}$. All processes encountered throughout Sections 2–9 of the paper will be defined on the fixed, finite horizon $[0, T]$.

Standing Assumption 2.1: We assume that $\| b(t) \| \le L$, $|r(t)| \le L$, $\forall\ 0 \le t \le T$ *hold almost surely, for some given real constant* $L > 0$.

Standing Assumption 2.2: The process $\sigma(\cdot)$ *satisfies the strong non-degeneracy condition*

$$\xi^*\sigma(t)\sigma^*(t)\xi \ge \epsilon \parallel \xi \parallel^2, \qquad \forall\ (t, \xi) \in [0, T] \times \mathbb{R}^m$$

almost surely, for some given real constant $\epsilon > 0$.

From Standing Assumption 2.2 the matrices $\sigma(t)$, $\sigma^*(t)$ are invertible, and the norms of $(\sigma(t))^{-1}$ and $(\sigma^*(t))^{-1}$ are bounded from above and below by δ and δ^{-1}, respectively, for some $\delta \in (1, \infty)$; cf. Karatzas & Shreve (1991), page 372. We also define the "relative risk" process

$$\theta(t) \triangleq \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}_m] \tag{2.3}$$

where $\mathbf{1}_m = (1, \cdots, 1)^*$, the discount process

$$\gamma(t) \triangleq \frac{1}{P_0(t)} = \exp\left\{-\int_0^t r(s)\,ds\right\}, \tag{2.4}$$

the exponential martingale (or *likelihood ratio* process)

$$Z_0(t) \stackrel{\triangle}{=} \exp\left\{ -\int_0^t \theta^*(s)\,dW(s) - \frac{1}{2}\int_0^t \parallel \theta(s) \parallel^2 \, ds \right\}, \tag{2.5}$$

and the state-price-density process

$$H(t) \stackrel{\triangle}{=} \gamma(t) Z_0(t). \tag{2.6}$$

3 Portfolio and wealth processes

A *portfolio process* $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_m(\cdot))^*$ is $\mathbb{R}^m$–valued, and a *consumption process* $c(\cdot)$ takes values in $[0, \infty)$; these are both $\mathbb{F}$–progressively measurable, and satisfy

$$\int_0^T c(t)\,dt + \int_0^T \parallel \pi(t) \parallel^2 \, dt < \infty$$

almost surely. We regard $\pi_i(t)$ as the proportion of an agent's wealth invested in stock i at time t; the remaining proportion $1 - \pi^*(t)\mathbf{1}_m = 1 - \sum_{i=1}^m \pi_i(t)$ is invested in the money-market. These proportions are *not* constrained to take values in the interval $[0, 1]$; in other words, we allow both short-selling of stocks, and borrowing at the interest rate of the bond. For a given, nonrandom, initial capital $x > 0$, let $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$ denote the *wealth-process* corresponding to a portfolio/consumption process pair $\big(\pi(\cdot), c(\cdot)\big)$ as above. This wealth-process is defined by the initial condition $X^{x,\pi,c}(0) = x$ and the equation

$$\begin{aligned} dX(t) &= \sum_{i=1}^m \pi_i(t) X(t) \left\{ b_i(t)\,dt + \sum_{j=1}^m \sigma_{ij}(t) dW_j(t) \right\} + \left\{ 1 - \sum_{i=1}^m \pi_i(t) \right\} X(t) r(t)\,dt - c(t)\,dt \\ &= r(t)X(t)dt + X(t)\pi^*(t)\sigma(t)dW_0(t) - c(t)\,dt, \qquad X(0) = x > 0, \end{aligned} \tag{3.1}$$

where we have set

$$W_0(t) \stackrel{\triangle}{=} W(t) + \int_0^t \theta(s)\,ds, \qquad 0 \le t \le T. \tag{3.2}$$

In other words,

$$d\left(\gamma(t) X^{x,\pi,c}(t)\right) = \gamma(t) X^{x,\pi,c}(t) \pi^*(t) \sigma(t)\,dW_0(t) - \gamma(t)c(t)\,dt, \qquad 0 \le t \le T. \tag{3.3}$$

The process $W_0(\cdot)$ of (3.2) is Brownian motion under the *equivalent martingale measure*

$$\mathbb{P}_0(A) \stackrel{\triangle}{=} \mathbb{E}\big[Z_0(T)\mathbf{1}_A\big], \quad A \in \mathcal{F}_T\ , \tag{3.4}$$

by the Girsanov theorem (section 3.5 in Karatzas & Shreve (1991)). We shall say that a portfolio/consumption process pair (π, c) is *available at initial capital* $x > 0$, if the corresponding wealth-process $X^{x,\pi,c}(\cdot)$ of (3.3) is strictly positive on $[0, T]$, almost surely.

An application of Itô's rule to the product of the processes $Z_0(\cdot)$ and $\gamma(\cdot)X^{x,\pi,c}(\cdot)$ leads to

$$H(t)X^{x,\pi,c}(t) + \int_0^t H(s)c(s)\,ds = x + \int_0^t H(s)X^{x,\pi,c}(s)(\sigma^*(s)\pi(s) - \theta(s))^*\,dW(s). \tag{3.5}$$

This shows, in particular, that for any pair (π, c) available at initial capital $x > 0$, the process $H(\cdot)X^{x,\pi,c}(\cdot) + \int_0^{\cdot} H(s)c(s)\,ds$ is a continuous, positive local martingale, hence a supermartingale, under $\mathbb{P}$. Consequently, the optional sampling theorem gives

$$\mathbb{E}\left[H(\tau)X^{x,\pi,c}(\tau) + \int_0^{\tau} H(s)c(s)\,ds\right] \le x\ ; \qquad \forall\ \tau \in \mathcal{S}. \tag{3.6}$$

Here and in the sequel, we denote by $\mathcal{S}_{s,t}$ the class of $\mathbb{F}-$stopping times $\tau : \Omega \longrightarrow [s,t]$ for $0 \le s \le t \le T$, and let $\mathcal{S} \equiv \mathcal{S}_{0,T}$.

4 Utility function

A function $U : (0,\infty) \longrightarrow \mathbb{R}$ will be called *utility function*, if it is strictly increasing, strictly concave, continuously differentiable, and satisfies

$$U'(0+) \ \stackrel{\triangle}{=}\ \lim_{x\downarrow 0} U'(x) = \infty, \qquad U'(\infty)\ \stackrel{\triangle}{=}\ \lim_{x\uparrow\infty} U'(x) = 0. \tag{4.1}$$

We shall denote by $I(\cdot)$ the (continuous, strictly decreasing) inverse of the marginal-utility function $U'(\cdot)$; this function maps $(0,\infty)$ onto itself, and satisfies $I(0+) = \infty,\ I(\infty) = 0$. We also introduce the Legendre-Fenchel transform

$$\tilde{U}(y)\ \stackrel{\triangle}{=}\ \max_{x>0}\,[U(x) - xy]\ =\ U(I(y)) - yI(y), \qquad 0 < y < \infty \tag{4.2}$$

of $-U(-x)$; this function $\tilde{U}(\cdot)$ is strictly decreasing and strictly convex, and satisfies

$$\tilde{U}'(y)\ =\ -I(y), \qquad 0 < y < \infty, \tag{4.3}$$

$$U(x)\ =\ \min_{y>0}\,[\tilde{U}(y) + xy]\ =\ \tilde{U}(U'(x)) + xU'(x), \qquad 0 < x < \infty. \tag{4.4}$$

The inequality

$$U(I(y)) \ge U(x) + y[I(y) - x], \qquad \forall\ x > 0,\ y > 0 \tag{4.5}$$

is a direct consequence of (4.2).

5 The optimization problem

The agent in our model has time-dependent utility of the form $\int_0^t e^{-\beta s} U_1\big(c(s)\big)\,ds + e^{-\beta t} U_2(x)$, with $\beta \ge 0$ a real constant. The utility functions $U_1(\cdot)$, $U_2(\cdot)$ measure his utility from consumption and wealth, respectively, whereas β stands for a discount factor. If the agent uses the portfolio/consumption strategy (π, c) available at initial capital $x > 0$, and the stopping rule $\tau \in \mathcal{S}$, his *expected discounted utility* is

$$J(x;\pi,c,\tau)\ \stackrel{\triangle}{=}\ \mathbb{E}\left[\int_0^{\tau} e^{-\beta t} U_1\big(c(t)\big)\,dt + e^{-\beta\tau} U_2(X^{x,\pi,c}(\tau))\right]. \tag{5.1}$$

The optimization problem considered in this paper is the following: *to maximize the expected discounted utility in* (11.5), over the class $\mathcal{A}(x)$ of triples (π, c, τ) as above, for which the expectation in (5.1) is well-defined, i.e.,

$$\mathbb{E}\left[\int_0^\tau e^{-\beta t} U_1^-\big(c(t)\big)\, dt + e^{-\beta\tau} U_2^-(X^{x,\pi,c}(\tau))\right] < \infty. \tag{5.2}$$

[Here and in the sequel, x^- denotes the negative part of the real number x, namely $x^- = \max(-x, 0)$.] The value-function of this problem will be denoted by

$$V(x) \stackrel{\triangle}{=} \sup_{(\pi,c,\tau)\in\mathcal{A}(x)} J(x;\pi,c,\tau), \qquad x \in (0,\infty). \tag{5.3}$$

We say that the value $V(x)$ is "attainable", if we can find a triple $(\hat{\pi}, \hat{c}, \hat{\tau}) \in \mathcal{A}(x)$ with $V(x) = J(x, \hat{\pi}, \hat{c}, \hat{\tau})$; such a triple is then called "optimal" for the problem of (11.7). To ensure that this problem is meaningful, we impose the following assumption throughout.

Standing Assumption 5.1: $V(x) < \infty$*, for all* $x \in (0,\infty)$.

It is fairly straightforward that the function $V(\cdot)$ is increasing on $(0,\infty)$. However, it is not clear at this stage whether $V(\cdot)$ is concave or not. We shall discuss this issue in Section 8.
Remark 5.1: A sufficient condition for Standing Assumption 5.1 is that

$$\max\{U_1(x),\ U_2(x)\} \le k_1 + k_2 x^\delta, \qquad \forall\, x \in (0,\infty) \tag{5.4}$$

holds, for some $k_1 > 0$, $k_2 > 0$, $\delta \in (0,1)$; cf. Remark 3.6.8 in Karatzas & Shreve (1998).

6 Duality Approach

For any *fixed* stopping time $\tau \in \mathcal{S}$, we denote by $\Pi_\tau(x)$ the set of portfolio/consumption process pairs (π, c) for which $(\pi, c, \tau) \in \mathcal{A}(x)$. The solution of the utility maximization problem

$$V_\tau(x) \stackrel{\triangle}{=} \sup_{(\pi,c)\in\Pi_\tau(x)} J(x;\pi,c,\tau) \tag{6.1}$$

can be derived as in KLS (1987). We review briefly the results in this Section. For any triple $(\pi, c, \tau) \in \mathcal{A}(x)$ and any real number $\lambda > 0$, it follows from (4.2), (3.6) that

$$\begin{aligned} J(x;\pi,c,\tau) &= \mathbb{E}\left[\int_0^\tau e^{-\beta t} U_1\big(c(t)\big)\, dt + e^{-\beta\tau} U_2(X^{x,\pi,c}(\tau))\right] \\ &\le \mathbb{E}\left[\int_0^\tau e^{-\beta t} \tilde{U}_1(\lambda e^{\beta t} H(t))\, dt + e^{-\beta\tau} \tilde{U}_2(\lambda e^{\beta\tau} H(\tau))\right] + \lambda \cdot \mathbb{E}\left[H(\tau) X^{x,\pi,c}(\tau) + \int_0^\tau H(t) c(t)\, dt\right] \\ &\le \mathbb{E}\left[\int_0^\tau e^{-\beta t} \tilde{U}_1(\lambda e^{\beta t} H(t))\, dt + e^{-\beta\tau} \tilde{U}_2(\lambda e^{\beta\tau} H(\tau))\right] + \lambda x, \end{aligned}$$

with equality if and only if

$$X^{x,\pi,c}(\tau) = I_2(\lambda e^{\beta\tau} H(\tau)), \quad \text{and} \quad c(t) = I_1\big(\lambda e^{\beta t} H(t)\big),\ \forall\, 0 \le t \le \tau \qquad \text{a.s.} \tag{6.2}$$

$$(6.3) \qquad \mathbb{E}\left[H(\tau)X^{x,\pi,c}(\tau) + \int_0^\tau H(t)c(t)\,dt\right] = x$$

hold. It develops that we have $V_\tau(x) \le \inf_{\lambda>0}\left[\tilde{J}(\lambda;\tau) + \lambda x\right]$ for all $\tau \in \mathcal{S}$, as well as

$$(6.4) \qquad V(x) \; = \; \sup_{\tau\in\mathcal{S}} V_\tau(x) \; \le \; \sup_{\tau\in\mathcal{S}} \inf_{\lambda>0} \left[\tilde{J}(\lambda;\tau) + \lambda x\right]$$

with the notation

$$(6.5) \qquad \tilde{J}(\lambda;\tau) \triangleq \mathbb{E}\left[\int_0^\tau e^{-\beta t}\tilde{U}_1\big(\lambda e^{\beta t}H(t)\big)\,dt + e^{-\beta\tau}\tilde{U}_2(\lambda e^{\beta\tau}H(\tau))\right].$$

In order to proceed, we shall need the following assumption (see Remark 6.7 for discussion).

Assumption 6.1: $\mathbb{E}\left[\sup_{0\le t\le T}\big(H(t)\cdot I_2(\lambda e^{\beta t}H(t)\big) + \int_0^T H(t)I_1\big(\lambda e^{\beta t}H(t)\big)\,dt\right] < \infty$, for all $\lambda \in (0,\infty)$.

Under this assumption, for any given $\tau \in \mathcal{S}$, the function $\mathcal{X}_\tau : (0,\infty) \to (0,\infty)$ defined by

$$(6.6) \qquad \mathcal{X}_\tau(\lambda) \triangleq \mathbb{E}\left[\int_0^\tau H(t)I_1\big(\lambda e^{\beta t}H(t)\big)\,dt + H(\tau)\cdot I_2(\lambda e^{\beta\tau}H(\tau))\right], \qquad 0<\lambda<\infty$$

is a continuous, strictly decreasing mapping of $(0,\infty)$ onto itself with $\mathcal{X}_\tau(0+) = \infty$, $\mathcal{X}_\tau(\infty) = 0$; thus $\mathcal{X}_\tau(\cdot)$ has a continuous, strictly decreasing inverse $\mathcal{Y}_\tau(\cdot)$ from $(0,\infty)$ onto itself. We define

$$(6.7) \qquad \xi^x(\tau) \triangleq I_2\big(\mathcal{Y}_\tau(x)e^{\beta\tau}H(\tau)\big) \qquad \text{and} \qquad \eta^x(t) \triangleq I_1\big(\mathcal{Y}_\tau(x)e^{\beta t}H(t)\big), \qquad 0 \le t \le T$$

so that, in particular,

$$(6.8) \qquad \mathbb{E}\left[H(\tau)\xi^x(\tau) + \int_0^\tau H(t)\eta^x(t)\,dt\right] = x.$$

LEMMA 6.2: *For any $\tau \in \mathcal{S}$, the random variables of (6.7) satisfy*

$$(6.9) \qquad \mathbb{E}\left[e^{-\beta\tau}U_2^-\big(\xi^x(\tau)\big) + \int_0^\tau e^{-\beta t}U_1^-\big(\eta^x(t)\big)\,dt\right] < \infty,$$

and for every portfolio/consumption pair $(\pi, c) \in \Pi_\tau(x)$ we have

$$(6.10) \qquad \mathbb{E}\left[\int_0^\tau U_1\big(c(t)\big)\,dt + e^{-\beta\tau}U_2(X^{x,\pi,c}(\tau))\right] \le \mathbb{E}\left[\int_0^\tau U_1\big(\eta^x(t)\big)\,dt + e^{-\beta\tau}U_2(\xi^x(\tau)\right].$$

Lemma 6.2 can be proved by arguments similar to those used in the proof of Theorem 3.6.3 in Karatzas & Shreve (1998). We conclude from it that, if there exists a portfolio $\hat{\pi}_\tau(\cdot)$ such that $(\hat{\pi}_\tau, \hat{c}_\tau)$ is available at initial capital $x > 0$, where $\hat{c}_\tau(\cdot) \triangleq \eta^x(\cdot)\mathbf{1}_{[0,\tau[}(\cdot)$, and if

$$(6.11) \qquad X^{x,\hat{\pi}_\tau,\hat{c}_\tau}(\tau) = \xi^x(\tau),$$

holds almost surely, then the pair $(\hat{\pi}_\tau, \hat{c}_\tau)$ belongs to $\Pi_\tau(x)$ and is optimal for the utility maximization problem (6.1). The existence of such a portfolio will need the assumption of market completeness, as we shall see in the next lemma.

LEMMA 6.3: *For any $\tau \in \mathcal{S}$, any $\mathcal{F}_\tau$–measurable random variable B with $\mathbb{P}[B > 0] = 1$, and any progressively measurable process $c(\cdot) \geq 0$ that satisfies $c(\cdot) \equiv 0$ a.e. on $[\![\tau, T]\!]$ as well as $\mathbb{E}\left[H(\tau)B + \int_0^T H(t)c(t)\,dt\right] = x$, there exists a portfolio process $\pi(\cdot)$ such that, almost surely:*

$$X^{x,\pi,c}(t) > 0, \quad 0 \leq t \leq T \qquad \text{and} \qquad X^{x,\pi,c}(\tau) = B.$$

Proof: We begin with the strictly positive, continuous process $X(\cdot)$ defined by

$$X(t) \stackrel{\triangle}{=} \frac{1}{\gamma(t)} \cdot \mathbb{E}_0\left[\gamma(\tau)B + \int_{t\wedge\tau}^{\tau} \gamma(s)c(s)\,ds \middle| \mathcal{F}_t\right]; \qquad 0 \leq t \leq T.$$

This process satisfies

$$X(0) = \mathbb{E}_0\left[\gamma(\tau)B + \int_0^\tau \gamma(s)c(s)\,ds\right] = \mathbb{E}\left[H(\tau)B + \int_0^\tau H(s)c(s)\,ds\right] = x, \quad \text{and} \quad X(\tau) = B \quad \text{a.s.}$$

On the other hand, the $\mathbb{P}_0$–martingale $M(\cdot) \stackrel{\triangle}{=} \gamma(\cdot)X(\cdot) + \int_0^\cdot \gamma(s)c(s)\,ds = \mathbb{E}_0\left[\gamma(\tau)B + \int_0^\tau \gamma(s)c(s)\,ds | \mathcal{F}_\cdot\right]$ admits the stochastic integral representation

$$M(t) = x + \int_0^t \psi^*(s)\,dW_0(s), \qquad 0 \leq t \leq T$$

for some $\mathbb{F}$–adapted process $\psi(\cdot)$ that satisfies $\int_0^T \| \psi(s) \|^2 \, ds < \infty$ almost surely (e.g. Karatzas & Shreve (1998), Lemma 1.6.7). Define $\pi(t) \stackrel{\triangle}{=} (\sigma^*(t))^{-1}\psi(t)/M(t), \quad 0 \leq t \leq T$ and check from (3.3) that $X(\cdot) = X^{x,\pi,c}(\cdot)$, almost everywhere on $[0,T] \times \Omega$. $\diamondsuit$

Remark 6.4: Note that the martingale $M(\cdot)$ is constant, and thus we have $\psi(\cdot) \equiv 0, \ \pi(\cdot) \equiv 0$, a.e. on the stochastic interval $[\![\tau, T]\!]$; in particular, $X^{x,\pi,c}(t,\omega) = B(\omega)e^{\int_{\tau(\omega)}^t r(u,\omega)\,du}$, a.e. on $[\![\tau, T]\!]$. In other words, at the stopping time τ all investment in the stock-market ceases, and all proceeds are invested in the money-market from then on.

We have proved the following result.

PROPOSITION 6.5: *Under Assumption 6.1, for any $\tau \in \mathcal{S}$ we have*

$$V_\tau(x) = \inf_{\lambda>0}\left[\tilde{J}(\lambda;\tau) + \lambda x\right] = \tilde{J}(\mathcal{Y}_\tau(x);\tau) + x\mathcal{Y}_\tau(x), \tag{6.12}$$

and the supremum in (6.1) is attained by the consumption strategy $\hat{c}_\tau(t) = I_1\big(\mathcal{Y}_\tau(x)e^{\beta t}H(t)\big)\mathbf{1}_{[0,\tau)}(t)$ and some portfolio $\hat{\pi}_\tau(\cdot)$ that satisfies (6.11). Moreover,

$$V(x) = \sup_{\tau\in\mathcal{S}} V_\tau(x) = \sup_{\tau\in\mathcal{S}}\inf_{\lambda>0}\left[\tilde{J}(\lambda;\tau) + \lambda x\right] = \sup_{\tau\in\mathcal{S}}\left[\tilde{J}(\mathcal{Y}_\tau(x);\tau) + x\mathcal{Y}_\tau(x)\right]. \tag{6.13}$$

Example 6.6 (*Logarithmic utility functions*): $U_1(x) = \delta \log x,\ U_2(x) = \log x$ for $x > 0$ and some $\delta \in [0,1]$. In this case the Assumption 6.1 is satisfied, and we have $I_1(y) = \delta/y,\ \tilde{U}_1(y) = \delta \log \delta - \delta[1 + \log y]$ and $I_2(y) = 1/y,\ \tilde{U}_2(y) = -1 - \log y$. Hence, with

$$Q(t) \ \stackrel{\triangle}{=} \ \int_0^t \theta^*(s)\, dW(s) + \int_0^t \left(r(s) + \frac{\|\theta(s)\|^2}{2} - \beta \right) ds$$

and with the convention $\delta \log \delta \equiv 0$ for $\delta = 0$, we have

$$\tilde{J}(\lambda;\tau) = \mathbb{E}\left[e^{-\beta\tau} \left(Q(\tau) - (1 + \log \lambda) \right) \right] + \delta \cdot \mathbb{E} \int_0^\tau e^{-\beta t} \big(Q(t) - (1 + \log \lambda) \big)\, dt + \delta \log \delta \cdot \mathbb{E} \int_0^\tau e^{-\beta t}\, dt$$

for any stopping time τ. It develops that $\mathcal{X}_\tau(\lambda) = K_\tau/\lambda$ and thus $\mathcal{Y}_\tau(x) = K_\tau/x$, where

$$K_\tau \stackrel{\triangle}{=} \mathbb{E}\left[e^{-\beta\tau} + \delta \int_0^\tau e^{-\beta t}\, dt \right].$$

From Proposition 6.4, the value function of the problem (11.7) is given by

$$V(x) = \sup_{\tau \in \mathcal{S}} \ \mathbb{E}\left[e^{-\beta\tau} \big\{ \log\big(x/K_\tau\big) + Q(\tau) \big\} + \delta \cdot \int_0^\tau e^{-\beta t} \big\{ \log\big(x/K_\tau\big) + Q(t) \big\}\, dt \right],$$

a quantity that is, in general, very difficult to compute. It is not even clear whether the supremum in this expression is attained (see Example 9.3 in this regard). However, *in the special case* $\beta = 0$ *and* $\delta = 0$, the above expression can be reduced significantly to

$$V(x) = \log x + \sup_{\tau \in \mathcal{S}} \mathbb{E} \int_0^\tau \left[r(u) + \frac{1}{2} \parallel \theta(u) \parallel^2 \right] du,$$

and amounts to solving a standard optimal stopping problem. This latter has the trivial solution $\tau^* \equiv T$ for $r(\cdot) \geq 0$.

Remark 6.7: A sufficient condition for Assumption 6.1 is that

$$I_1(y) + I_2(y) \leq k_1 + k_2 y^{-\alpha}, \qquad \forall\ y \in (0, \infty) \tag{6.14}$$

holds for some constants $k_1 > 0$, $k_2 > 0$ and $\alpha > 0$. Indeed, under (6.14) we have

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} \left(H(s) \cdot I_j(\lambda e^{\beta s} H(s)) \right) \right] \leq k_1 \mathbb{E}\left[\sup_{0 \leq s \leq T} \big(H(s) \big) \right] + k_2 \lambda^{-\alpha} \mathbb{E}\left[\sup_{0 \leq s \leq T} \big(H(s) \big)^{1-\alpha} \right] < \infty$$

for $j = 1, 2$, as is easy to check using Hölder's inequality, Doob's maximal inequality, and the boundedness of market coefficients. This is because, for any $\rho \in \mathbf{R}$, there exist positive constants $C_1,\ C_2$ such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0 \leq t \leq T} \big(H(t) \big)^\rho \right] \ &= \ \mathbf{E}\left[\sup_{0 \leq t \leq T} \big(\gamma(t) Z_0(t) \big)^\rho \right] \ \leq \ C_1 \cdot \mathbb{E}\left[\sup_{0 \leq t \leq T} \big(Z_0(t) \big)^\rho \right] \\
&\leq \ C_1 \cdot \mathbb{E}\left[\sup_{0 \leq t \leq T} \left(e^{-\rho \int_0^t \theta^*(s)\, dW(s) - \frac{\rho^2}{2} \int_0^t \|\theta(s)\|^2\, ds} \right) \cdot \sup_{0 \leq t \leq T} \left(e^{\frac{\rho(\rho-1)}{2} \int_0^t \|\theta(s)\|^2\, ds} \right) \right] \\
&\leq \ C_2 \cdot \mathbb{E}\left[\sup_{0 \leq t \leq T} \left(e^{-\rho \int_0^t \theta^*(s)\, dW(s) - \frac{\rho^2}{2} \int_0^t \|\theta(s)\|^2\, ds} \right) \right] < \infty.
\end{aligned}$$

7 Pure Optimal Stopping Problems

The representation (6.13) for the solution of the utility maximization problem in Section 5 is not entirely satisfactory. It is not clear how the quantities $\mathcal{Y}_\tau(x)$ are related to each other for different stopping times $\tau \in \mathcal{S}$, except in some very special cases. Furthermore, it is not easy to compute the last supremum in (6.13), or even to decide whether it is attained or not. All these points are illustrated in the Example 6.6 of a logarithmic utility function. In this section, we shall try to convert the original problem into a family of *pure optimal stopping* problems, for which we can obtain a better understanding. To this end, we define, for every $\lambda \in (0,\infty)$, the following *dual optimization problem*

$$\tilde{V}(\lambda) \triangleq \sup_{\tau\in\mathcal{S}} \tilde{J}(\lambda;\tau) = \sup_{\tau\in\mathcal{S}} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\tilde{U}_1\big(\lambda e^{\beta t}H(t)\big)\,dt + e^{-\beta\tau}\tilde{U}_2\big(\lambda e^{\beta\tau}H(\tau)\big)\right] \tag{7.1}$$

of pure optimal stopping type, in the notation of (6.5), (4.2), (2.6). To ensure that the problem of (7.1) is meaningful, we impose the following assumption throughout.

Standing Assumption 7.1: For any $\lambda \in (0,\infty)$ we have $\tilde{V}(\lambda) < \infty$, and there exists some stopping time $\hat{\tau}_\lambda$ which is optimal in (7.1), i.e., such that $\tilde{V}(\lambda) = \tilde{J}(\lambda;\hat{\tau}_\lambda)$.

Here and in the sequel, we denote by $\hat{\mathcal{S}}_\lambda$ the set of stopping times that attain the supremum in (6.5), for every given $\lambda > 0$. It follows from (6.4) that we have, in the notation of (7.1):

$$V(x) \le \sup_{\tau\in\mathcal{S}} \inf_{\lambda>0} \left[\tilde{J}(\lambda;\tau) + \lambda x\right] \le \inf_{\lambda>0}\left[\sup_{\tau\in\mathcal{S}} \tilde{J}(\lambda;\tau) + \lambda x\right] = \inf_{\lambda>0}\left[\tilde{V}(\lambda) + \lambda x\right]. \tag{7.2}$$

We wish that the inequalities in (7.2) would always hold as equalities. Unfortunately, it turns out that the second inequality in (7.2) might be strict, depending on the coëfficients of the model and on the initial capital x. We shall see this more clearly in the following sections.

Remark 7.2: Standing Assumption 7.1 holds if condition (5.4) *is satisfied.* This is because the continuous process $Y^\lambda(t) \triangleq \int_0^t e^{-\beta s}\tilde{U}_1\big(\lambda e^{\beta s}H(s)\big) + e^{-\beta t}\tilde{U}_2(\lambda e^{\beta t}H(t)),\quad 0 \le t \le T$ satisfies in this case $\mathbb{E}[\sup_{0\le t\le T}|Y^\lambda(t)|] < \infty$. Indeed, it is easy to check that (5.4) implies

$$\max\{\tilde{U}_1(y), \tilde{U}_2(y)\} \le k_1 + k_3 y^{-\alpha}, \qquad \forall\ 0 < \lambda < \infty \tag{7.3}$$

with $\alpha = \delta/(1-\delta)$, $k_3 = (1-\delta)(k_2\delta^\delta)^{1/(1-\delta)}$ (cf. KLSX (1991)), and it follows from Remark 6.7 that $\tilde{V}(\lambda) \le \mathbb{E}\left[\sup_{0\le t\le T}|Y^\lambda(t)|\right] \le k_4 + k_5\lambda^{-\alpha}\cdot\mathbb{E}\left[\sup_{0\le t\le T}\big(H(t)\big)^{-\alpha}\right] < \infty$. Standard results in the theory of optimal stopping (e.g. Theorem D.12 in Karatzas & Shreve (1998)) guarantee then the existence of an optimal stopping time.

8 Analysis of the Optimal Stopping Problem

In this section we shall derive our main results for the optimization problem of (11.7), by first establishing several properties of the "dual" value function $\tilde{V}(\cdot)$ defined in (7.1). It is not a trivial matter, to decide whether the value function $V(\cdot)$ of our "primal" problem (11.7) inherits the concavity of $U(\cdot)$. Indeed, even the continuity of $V(\cdot)$ is not quite clear *a priori.* However, properties of convexity and monotonicity are relatively straightforward for the dual value function $\tilde{V}(\cdot)$ of (7.1).

LEMMA 8.1: *The function $\tilde{V}(\cdot)$ of (7.1) is strictly convex, strictly decreasing. In particular, it is continuous and almost everywhere differentiable.*

Proof: For any $0 < \lambda_1 < \lambda_2 < \infty,\ 0 < s < 1$, and $\lambda_0 \stackrel{\triangle}{=} s\lambda_1 + (1-s)\lambda_2$, we have $\tilde{V}(\lambda_2) = \tilde{J}(\lambda_2; \hat{\tau}_2) < \tilde{J}(\lambda_1; \hat{\tau}_2) \leq \tilde{V}(\lambda_1)$ from the Standing Assumption 7.1, where $\hat{\tau}_i \in \hat{\mathcal{S}}_{\lambda_i},\ i = 0,1,2$ are optimal stopping times, and $\tilde{V}(\lambda_0) = \tilde{J}(\lambda_0; \hat{\tau}_0) < s\tilde{J}(\lambda_1; \hat{\tau}_0) + (1-s)\tilde{J}(\lambda_2; \hat{\tau}_0) \leq s\tilde{V}(\lambda_1) + (1-s)\tilde{V}(\lambda_2)$. $\diamondsuit$

It follows from Lemma 8.1 that the right– and left– derivatives

$$\triangle^{\pm}\tilde{V}(\lambda) \stackrel{\triangle}{=} \lim_{h\to 0\pm} \frac{1}{h}[\tilde{V}(\lambda+h) - \tilde{V}(\lambda)] \tag{8.1}$$

of the convex function $\tilde{V}(\cdot)$ exist, and are finite for every $\lambda \in (0,\infty)$. Furthermore, the strict convexity of $\tilde{V}(\cdot)$ implies

$$\triangle^{+}\tilde{V}(\lambda_1) < \triangle^{-}\tilde{V}(\lambda_2) \leq \triangle^{+}\tilde{V}(\lambda_2) \leq 0, \quad \forall\ 0 < \lambda_1 < \lambda_2 < \infty, \tag{8.2}$$

and $\triangle^{+}\tilde{V}(\cdot)$ (respectively, $\triangle^{-}\tilde{V}(\cdot)$) is right– (respectively, left–) continuous.

LEMMA 8.2: *For every $\lambda \in (0,\infty)$ and any optimal stopping time $\hat{\tau}_\lambda \in \hat{\mathcal{S}}_\lambda$, we have*

$$\triangle^{-}\tilde{V}(\lambda) \leq -\mathcal{X}_{\hat{\tau}_\lambda}(\lambda) \leq \triangle^{+}\tilde{V}(\lambda). \tag{8.3}$$

Proof: The convexity of $\tilde{U}_j(\cdot),\ j = 1,2$ gives

$$\tilde{U}_j'(y)(x-y) \leq \tilde{U}_j(x) - \tilde{U}_j(y) \leq \tilde{U}_j'(x)(x-y), \quad \forall\ 0 < x, y < \infty, \tag{8.4}$$

and for any real number h with $|h| < \lambda$ we obtain

$$\begin{aligned}
\tilde{V}(\lambda+h) - \tilde{V}(\lambda) &= \tilde{V}(\lambda+h) - \tilde{J}(\lambda; \hat{\tau}_\lambda) \geq \tilde{J}(\lambda+h; \hat{\tau}_\lambda) - \tilde{J}(\lambda; \hat{\tau}_\lambda) \\
&\geq h \cdot \mathbb{E}\left[\int_0^{\hat{\tau}_\lambda} H(t)\tilde{U}_1'\big(\lambda e^{\beta t} H(t)\big)\, dt + H(\hat{\tau}_\lambda)\tilde{U}_2'(\lambda e^{\beta\hat{\tau}_\lambda} H(\hat{\tau}_\lambda))\right] = -h\mathcal{X}_{\hat{\tau}_\lambda}(\lambda).
\end{aligned}$$

The last equality follows from (4.3) and the definition (6.6) of $\mathcal{X}_{\hat{\tau}}(\cdot)$. Letting $h \to 0$, we deduce for arbitrary $\lambda \in (0,\infty)$:

$$\triangle^{+}\tilde{V}(\lambda) = \lim_{h\to 0+} \frac{1}{h}[\tilde{V}(\lambda+h) - \tilde{V}(\lambda)] \geq -\mathcal{X}_{\hat{\tau}_\lambda}(\lambda) \geq \lim_{h\to 0-} \frac{1}{h}[\tilde{V}(\lambda+h) - \tilde{V}(\lambda)] = \triangle^{-}\tilde{V}(\lambda).$$

COROLLARY 8.3: *If $\tilde{V}(\cdot)$ is differentiable at $\lambda > 0$, then $\tilde{V}'(\lambda) = -\mathcal{X}_{\hat{\tau}_\lambda}(\lambda)$.*

LEMMA 8.4: *We have $\lim_{\lambda\downarrow 0} \triangle^{\pm}\tilde{V}(\lambda) = -\infty$. Moreover, we also have $\lim_{\lambda\uparrow\infty} \triangle^{\pm}\tilde{V}(\lambda) = 0$ if Assumption 6.1 holds.*

Proof: From the decrease of the function $I(\cdot)$, the Monotone Convergence Theorem, and $I(0+) = \infty$, it follows that $\lim_{\lambda\downarrow 0} \mathcal{X}_{\hat{\tau}_\lambda}(\lambda) \geq \lim_{\lambda\downarrow 0} \mathbb{E}\left[\inf_{0\leq s\leq T}\left(H(s)\cdot I_2\left(\lambda e^{\beta T}\sup_{0\leq s\leq T} H(s)\right)\right)\right] = \infty$, so by Lemma 8.2 and the inequality (8.2) we obtain $\lim_{\lambda \downarrow 0} \triangle^{\pm} \tilde{V}(\lambda) = -\infty$. Now suppose that Assumption 6.1 holds; we have then

$$0 \leq \lim_{\lambda \uparrow \infty} \mathcal{X}_{\hat{\tau}_\lambda}(\lambda) \leq \lim_{\lambda \uparrow \infty} \mathbb{E}\left[\sup_{0 \leq s \leq T} \Big(H(s) \cdot I_2 \Big(\lambda e^{\beta s} H(s) \Big) \Big) + \int_0^T H(s) \cdot I_1\big(\lambda e^{\beta s} H(s) \big)\, ds \right] = 0$$

from the decrease of the functions $I_j(\cdot)$, the Dominated Convergence Theorem, and $I_j(\infty) = 0,\ j = 1, 2$. It follows again from Lemma 8.2 and (8.2) that $\lim_{\lambda \uparrow \infty} \triangle^{\pm} \tilde{V}(\lambda) = 0$. $\diamondsuit$

We shall define, for each given $\lambda > 0$, the following subset

$$\mathcal{G}_\lambda \triangleq \Big\{ \mathcal{X}_{\hat{\tau}_\lambda}(\lambda) \Big/ \ \hat{\tau}_\lambda \ \text{ is optimal in (7.1), i.e.,} \quad \hat{\tau}_\lambda \in \hat{\mathcal{S}}_\lambda \Big\} \tag{8.5}$$

of $\mathbb{R}^+$. It follows from (8.2) and (8.3) that the sets $\{\mathcal{G}_\lambda\}_{\lambda > 0}$ satisfy the following properties:

(i) $\mathcal{G}_\lambda$ is non-empty for every $\lambda > 0$,

(ii) $\mathcal{G}_\lambda \cap \mathcal{G}_\nu = \emptyset$, if $\lambda \neq \nu$,

(iii) for any $0 < \nu < \lambda < \infty$ and $x \in \mathcal{G}_\lambda,\ y \in \mathcal{G}_\nu$, we have $x < y$.

Let us also introduce the set

$$\mathcal{G} \triangleq \bigcup_{\lambda > 0} \mathcal{G}_\lambda. \tag{8.6}$$

We can state now the main result of the paper. This explains, in particular, when we can expect to find an optimal triple in (11.7), and to have equality in (7.2).

THEOREM 8.5: *For any $x \in \mathcal{G}$, the value $V(x)$ of* (11.7) *is attainable and we have*

$$V(x) = \inf_{\lambda > 0} \left[\tilde{V}(\lambda) + \lambda x \right]. \tag{8.7}$$

Conversely, for any $x \in (0, \infty)$ that satisfies (8.7) *and for which the value $V(x)$ of (5.3) is attainable, we have $x \in \mathcal{G}$, provided that Assumption 6.1 holds.*

Proof: Suppose $x \in \mathcal{G}_\nu$ for some $\nu > 0$, and $x = \mathcal{X}_{\hat{\tau}_\nu}(\nu)$ for some stopping time $\hat{\tau}_\nu \in \hat{\mathcal{S}}_\nu$ which is optimal in (7.1) with $\lambda = \nu$, i.e. with

$$\tilde{V}(\nu) = \tilde{J}(\nu; \hat{\tau}_\nu) = \mathbb{E}\Big[\int_0^{\hat{\tau}_\nu} e^{-\beta t} \tilde{U}_1\big(\nu e^{\beta t} H(t) \big)\, dt + e^{-\beta \hat{\tau}_\nu} \tilde{U}_2\big(\nu e^{\beta \hat{\tau}_\nu} H(\hat{\tau}_\nu) \big) \Big]. \tag{8.8}$$

Then we claim

$$V(x) = \tilde{V}(\nu) + \nu x = \inf_{\lambda > 0} [\tilde{V}(\lambda) + \lambda x]. \tag{8.9}$$

Indeed, by Lemma 8.2, we have $-x \in [\triangle^- \tilde{V}(\nu), \triangle^+ \tilde{V}(\nu)]$, so that $\tilde{V}(\lambda) - \tilde{V}(\nu) \geq (-x) \cdot (\lambda - \nu)$ or equivalently $\tilde{V}(\lambda) + \lambda x \geq \tilde{V}(\nu) + \nu x, \quad \forall \lambda > 0$.

Since $x = \mathcal{X}_{\hat{\tau}_\nu}(\nu) = \mathbb{E}\left[H(\hat{\tau}_\nu) I_2(\nu e^{\beta \hat{\tau}_\nu} H(\hat{\tau}_\nu)) + \int_0^{\hat{\tau}_\nu} H(t) I_1\big(\nu e^{\beta t} H(t) \big)\, dt \right]$, it follows from Lemma 6.3 and Lemma 6.2 that there exists a portfolio process $\hat{\pi}(\cdot)$ with $X^{x, \hat{\pi}, \hat{c}}(\hat{\tau}_\nu) = I_2(\nu e^{\beta \hat{\tau}_\nu} H(\hat{\tau}_\nu))$, where $\hat{c}(t) \stackrel{\triangle}{=} I_1\big(\nu e^{\beta t} H(t)\big)\mathbf{1}_{[0,\tau)}(t)$. The expected utility $J(x;\hat{\pi},\hat{c},\hat{\tau}_\nu)$, under the portfolio/consumption strategy $(\hat{\pi},\hat{c})$ and the stopping time $\hat{\tau}_\nu$, is thus

$$\begin{aligned} V(x) &\geq J(x;\hat{\pi},\hat{c},\hat{\tau}_\nu) = \mathbb{E}\left[\int_0^{\hat{\tau}_\nu} e^{-\beta t} U_1\big(I_1(\nu e^{\beta t} H(t))\big)\, dt + e^{-\beta \hat{\tau}_\nu} U_2(I_2(\nu e^{\beta \hat{\tau}_\nu} H(\hat{\tau}_\nu)))\right] \\ &= \mathbb{E}\left[\int_0^{\hat{\tau}_\nu} e^{-\beta t} \tilde{U}_1\big(\nu e^{\beta t} H(t)\big) + e^{-\beta \hat{\tau}_\nu} \tilde{U}_2(\nu e^{\beta \hat{\tau}_\nu} H(\hat{\tau}_\nu))\right] + \nu \cdot \mathbb{E}\left[H(\hat{\tau}_\nu) X^{x,\hat{\pi},\hat{c}}(\hat{\tau}_\nu) + \int_0^{\hat{\tau}_\nu} H(t)\hat{c}(t)\, dt\right] \\ &= \tilde{V}(\nu) + \nu x = \inf_{\lambda>0} [\tilde{V}(\lambda) + \lambda x], \end{aligned}$$

and (8.9) follows then from (7.2). In particular, the triple $(\hat{\pi},\hat{c},\hat{\tau}_\nu)$ in $\mathcal{A}(x)$ is optimal for the original optimization problem of (5.3).

Conversely, suppose that (8.7) holds for some positive real number x, for which the value $V(x)$ of (5.3) is attained by some optimal triple $(\pi^*, c^*, \tau^*) \in \mathcal{A}(x)$. In other words,

$$V(x) = \inf_{\lambda>0} [\, \tilde{V}(\lambda) + \lambda x \,] = J(x;\pi^*,c^*,\tau^*) \leq V_{\tau^*}(x) \tag{8.10}$$

in the notation of (6.1). Suppose also that Assumption 6.1 holds. By Lemma 8.1 the function $\lambda \longmapsto \tilde{V}(\lambda) + \lambda x =: G(\lambda)$ is strictly convex, with $G(0+) = \tilde{V}(0+)$ and $G(\infty) = \infty$. Thus, either there exists a unique $\nu > 0$ such that

$$\tilde{V}(\nu) + \nu x = \inf_{\lambda>0} [\, \tilde{V}(\lambda) + \lambda x \,], \tag{8.11}$$

or else we have $\tilde{V}(0+) \leq \tilde{V}(\lambda) + \lambda x,\ \forall \lambda > 0$. This latter possibility can be ruled out easily; it cannot hold if $\tilde{V}(0+) = \infty$, whereas with $\tilde{V}(0+) < \infty$ it leads to $\lim_{\lambda\downarrow 0}\big(-\triangle^+ \tilde{V}(\lambda)\big) \leq x$ which is impossible, by Lemma 8.4. Therefore, (8.11) holds for a unique $\nu > 0$ and leads, with (8.10) and Proposition 6.4, to

$$V(x) = \tilde{V}(\nu) + \nu x \geq \tilde{J}(\nu;\tau^*) + \nu x \geq \inf_{\lambda>0} [\tilde{J}(\lambda;\tau^*) + \lambda x] = V_{\tau^*}(x) \geq V(x). \tag{8.12}$$

We obtain $\tilde{V}(\nu) = \tilde{J}(\nu;\tau^*)$, as well as $\tilde{J}(\nu;\tau^*) + \nu x = \inf_{\lambda>0} [\tilde{J}(\lambda;\tau^*) + \lambda x]$ from (8.10), (8.12), or equivalently, $\tau^* \in \hat{\mathcal{S}}_\nu$ and $\nu = \mathcal{Y}_{\tau^*}(x)$. Thus $x = \mathcal{X}_{\tau^*}(\nu) \in \mathcal{G}_\nu$, which concludes the proof. ◊

Corollary 8.6: *Under Assumption 6.1, for any $x \notin \mathcal{G} \equiv \bigcup_{\lambda>0} \mathcal{G}_\lambda$, we have the strict inequality ("duality gap"): $V(x) < \inf_{\lambda>0} [\tilde{V}(\lambda) + \lambda x]$.*

Corollary 8.7: *Under Assumption 6.1, and if $\tilde{V}(\cdot)$ is differentiable everywhere, the value $V(x)$ of (5.3) is attainable and (8.7) holds for every $x \in (0,\infty)$.*

Proof: Since every differentiable convex function is continuously differentiable (cf. Rockafellar (1970), Corollary 25.5.1), $\tilde{V}'(\cdot)$ is continuous. By Lemma 8.4, the range of $\tilde{V}'(\cdot)$ is $(-\infty, 0)$. It follows from Corollary 8.3 that $\mathcal{G} = (0,\infty)$, and Theorem 8.5 applies. ◊

Corollary 8.8: *Under Assumption 6.1, suppose that for any $\lambda \in (0,\infty)$ there exist two sequences $\{\lambda_n^{(\pm)}\}$ with $\lambda_n^{(+)} \downarrow \lambda,\ \lambda_n^{(-)} \uparrow \lambda$, as well as stopping times $\hat{\tau} \in \hat{\mathcal{S}}_\lambda,\ \hat{\tau}_n^{(\pm)} \in \hat{\mathcal{S}}_{\lambda_n^{(\pm)}}$ such that*

$\hat\tau_n^{(\pm)} \to \hat\tau$ *a.s.; then the value* $V(x)$ *of* (11.7) *is attainable and* (8.7) *holds for every* $x > 0$.

Proof: By Corollary 8.7, we need only show that $\tilde V(\cdot)$ is differentiable everywhere. From (8.4) and (4.3) we have

$$
\begin{aligned}
\tilde V(\lambda_n^{(\pm)}) - \tilde V(\lambda) &\le \tilde J(\lambda_n^{(\pm)}; \hat\tau_n^{(\pm)}) - \tilde J(\lambda; \hat\tau_n^{(\pm)}) \\
&\le -(\lambda_n^{(\pm)} - \lambda)\cdot \mathbb{E}\left[\int_0^{\hat\tau_n^{\pm}} H(t) I_1\big(\lambda_n^{(\pm)} e^{\beta t} H(t)\big)\, dt + H(\hat\tau_n^{(\pm)}) I_2(\lambda_n^{(\pm)} e^{\beta \hat\tau_n^{(\pm)}} H(\hat\tau_n^{(\pm)}))\right] \\
&= -(\lambda_n^{(\pm)} - \lambda)\cdot \mathcal{X}_{\hat\tau_n^{(\pm)}}(\lambda_n^{(\pm)}),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\triangle^+ \tilde V(\lambda) &= \lim_{\lambda_n^{(+)}\downarrow\lambda} \frac{\tilde V(\lambda_n^{(+)}) - \tilde V(\lambda)}{\lambda_n^{(+)} - \lambda} \le \limsup_{\lambda_n^{(+)}\downarrow\lambda} \Big(-\mathcal{X}_{\hat\tau_n^{(+)}}(\lambda_n^{(+)})\Big) = -\mathcal{X}_{\hat\tau}(\lambda) \\
\triangle^- \tilde V(\lambda) &= \lim_{\lambda_n^{(-)}\uparrow\lambda} \frac{\tilde V(\lambda_n^{(-)}) - \tilde V(\lambda)}{\lambda_n^{(-)} - \lambda} \ge \liminf_{\lambda_n^{(-)}\downarrow\lambda} \Big(-\mathcal{X}_{\hat\tau_n^{(-)}}(\lambda_n^{(-)})\Big) = -\mathcal{X}_{\hat\tau}(\lambda)
\end{aligned}
$$

by the Dominated Convergence Theorem. From (8.2), $\tilde V'(\lambda) = \triangle^+\tilde V(\lambda) = \triangle^-\tilde V(\lambda) = -\mathcal{X}_{\hat\tau}(\lambda)$. $\diamondsuit$

Corollaries 8.7 and 8.8 provide simple sufficient (but *not* necessary) conditions, under which there is no "duality gap" in (7.2) – i.e., its leftmost and rightmost members are equal. The following proposition will characterize this kind of interchangeability of "inf" and "sup" operations from another point of view, namely, the concavity of the "primal" value function $V(\cdot)$.

PROPOSITION 8.9: *Under Assumption 6.1, the following two statements are equivalent:*
(*A*) $V(\cdot)$ *is concave on* $(0, +\infty)$,
(*B*) $V(x) = \inf_{\lambda>0}[\tilde V(\lambda) + \lambda x]$ *holds for every* $x \in (0,\infty)$.

Proof of (*B*) $\Longrightarrow$ (*A*) : Under condition (B), the number $-V(x)$ is the pointwise supremum of the affine functions $g(\lambda) = -\lambda x - \mu$ such that (x,μ) belongs to the epigraph of $\tilde V(\cdot)$. Hence $-V(\cdot)$ is a convex function (Rockafellar (1970), Theorem 12.1), or equivalently $V(\cdot)$ is concave.

Proof of (*A*) $\Longrightarrow$ (*B*) : By Lemma 8.4 and (8.2), it is sufficient to show that for any $(\nu, x) \in (0,\infty)\times(0,\infty)$ such that $-\triangle^+\tilde V(\nu) \le x \le -\triangle^-\tilde V(\nu)$, we have $V(x) = \tilde V(\nu) + \nu x$.

Let $x_0 \triangleq -\triangle^+\tilde V(\nu)$, $x_1 \triangleq -\triangle^-\tilde V(\nu)$. Since $\tilde V(\cdot)$ is strictly convex and differentiable except on a countable set, we can find a sequence of positive real numbers $\{\lambda_n\}$, such that $\lambda_n \downarrow \nu$ as $n\to\infty$, and $\tilde V(\cdot)$ is differentiable at each λ_n. Define $y_n \triangleq -\tilde V'(\lambda_n)$. It follows from the right–continuity of $\triangle^+\tilde V(\cdot)$ that $-y_n = \triangle^+\tilde V(\lambda_n) \downarrow \triangle^+\tilde V(\nu) = -x_0$. However, Theorem 8.5 and Corollary 8.3 assert that

$$
V(y_n) = \inf_{\lambda>0}[\tilde V(\lambda) + \lambda y_n] = \tilde V(\lambda_n) + \lambda_n y_n. \tag{8.13}
$$

Letting $n\to\infty$ we obtain

$$
V(x_0) = \tilde V(\nu) + \nu x_0, \tag{8.14}
$$

thanks to the continuity of $V(\cdot)$ (which is concave, by assumption (A)) and of $\tilde{V}(\cdot)$ (which is convex, by Lemma 8.1). Furthermore, we claim that $\triangle^- V(x_0) \le \nu$. Indeed, it follows from (8.13) and (8.14) that

$$V(y_n) - V(x_0) = \tilde{V}(\lambda_n) + \lambda_n y_n - \tilde{V}(\nu) - \nu x_0 \ge \triangle^+ \tilde{V}(\nu)(\lambda_n - \nu) + \lambda_n y_n - \nu x_0 = \lambda_n (y_n - x_0),$$

hence

$$\triangle^- V(x_0) = \lim_{n\to\infty} \frac{V(y_n) - V(x_0)}{y_n - x_0} \le \lim_{n\to\infty} \lambda_n = \nu. \tag{8.15}$$

Similarly, we obtain

$$V(x_1) = \tilde{V}(\nu) + \nu x_1 \qquad \text{and} \qquad \triangle^+ V(x_1) \ge \nu. \tag{8.16}$$

However, $\triangle^- V(x_0) \ge \triangle^+ V(x_1)$ holds from the concavity of $V(\cdot)$. It follows from (8.15) and (8.16) that $\triangle^- V(x_0) = \nu = \triangle^+ V(x_1)$, or equivalently, $\triangle^- V(x) = \triangle^+ V(x) = V'(x) = \nu$, $\forall\ x_0 \le x \le x_1$. It is clear now that $V(x) = \tilde{V}(\nu) + \nu x = \inf_{\lambda>0} [\tilde{V}(\lambda) + \lambda x]$ holds for any $x_0 \le x \le x_1$.

9 Examples

Using the technique developed in the preceding section, we study here several examples, including one which shows that optimal strategies need not always exist (see Example 9.3). The first of these examples can also be treated using the methods of Section 6, but for the second and third examples the methodology of Section 8 is indispensable. The reader of this section should not fail to notice the rarity of a setting, where utility functions of power-type are much easier to handle than logarithmic ones.

Example 9.1 (*Utility functions of power-type*): $U_j(x) = x^\alpha/\alpha$ where $0 < \alpha < 1$, $j = 1, 2$. In this case the condition (5.4) is satisfied and we have $I_j(y) = y^{-1/(1-\alpha)}$ and $\tilde{U}_j(y) = y^{-\gamma}/\gamma$ with $\gamma = \alpha/(1-\alpha)$, $j = 1, 2$, so that Assumption 6.1 is also satisfied (see Remark 6.7) and implies $K < \infty$ in (9.2) below. We obtain easily

$$\tilde{V}(\lambda) = \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_0^\tau e^{-\beta t} \tilde{U}_1\big(\lambda e^{\beta t} H(t)\big)\, dt + e^{-\beta\tau} \tilde{U}_2(\lambda e^{\beta\tau} H(\tau))\right] = \frac{K}{\gamma} \lambda^{-\gamma}, \tag{9.1}$$

with

$$K \triangleq \sup_{\tau\in\mathcal{S}} K_\tau, \qquad \text{where} \qquad K_\tau \triangleq \mathbb{E}\left[\int_0^\tau e^{-(1+\gamma)\beta t} \big(H(t)\big)^{-\gamma}\, dt + e^{-(1+\gamma)\beta\tau} \big(H(\tau)\big)^{-\gamma}\right]. \tag{9.2}$$

Clearly $\tilde{V}(\cdot)$ is differentiable everywhere, and it follows from Corollary 8.7 that $V(x) = \inf_{\lambda>0} [\tilde{V}(\lambda) + \lambda x] = K^{1-\alpha}\ x^\alpha/\alpha$. In other words, *with utility functions of power-type, the original optimization problem is reduced to the pure optimal stopping problem* (9.2). We can arrive at this conclusion also using Proposition 6.4, since we have $\mathcal{X}_\tau(\lambda) = K_\tau \lambda^{-1/(1-\alpha)}$, $\mathcal{Y}_\tau(x) = (K_\tau/x)^{1-\alpha}$, $\tilde{J}(\lambda;\tau) = \frac{K_\tau}{\gamma}\lambda^{-\gamma}$, and thus $V(x) = \frac{x^\alpha}{\alpha} K^{1-\alpha}$ from (6.12), (6.13).

The optimal stopping time $\hat{\tau}$ for the original problem is also optimal for the problem of (9.2), the corresponding optimal consumption $\hat{c}(\cdot)$ and wealth–level $X^{x,\hat{\pi},\hat{c}}(\hat{\tau}) \equiv \xi^x(\hat{\tau})$ are given as

$$\hat{c}(t) = \frac{x}{K} e^{-\frac{\beta t}{1-\alpha}} \big(H(t)\big)^{-\frac{1}{1-\alpha}}, \quad 0 \le t \le \hat{\tau}, \qquad \xi^x(\hat{\tau}) = \frac{x}{K} e^{-\frac{\beta \hat{\tau}}{1-\alpha}} \left(H(\hat{\tau})\right)^{-\frac{1}{1-\alpha}}$$

by (6.11), and the optimal portfolio process $\hat{\pi}(\cdot)$ can then be obtained from Lemma 6.3.

It is straightfoward to check that $\hat{\tau} \equiv 0,\ K = 1$ if

$$\beta \ge \gamma \left[\frac{r(t)}{1+\gamma} + \frac{1}{2} \parallel \theta(t) \parallel^2 \right], \qquad \forall\ 0 \le t \le T$$

holds almost surely, and that $\hat{\tau} \equiv T,\ K = K_T$ if

$$\beta \le \gamma \left[\frac{r(t)}{1+\gamma} + \frac{1}{2} \parallel \theta(t) \parallel^2 \right], \qquad \forall\ 0 \le t \le T$$

holds almost surely. This observation provides a complete solution to the optimal stopping problem of (9.2) in the case of constant interest-rate $r(t) \equiv r \in \mathbb{R}$ and relative risk $\theta(t) \equiv \theta \in \mathbb{R}^m$; in particular, if $\beta = \gamma \left(\frac{r}{1+\gamma} + \frac{\|\theta\|^2}{2} \right)$, every stopping time $\tau \in \mathcal{S}_{0,T}$ is optimal in (9.2) and $K = K_\tau = 1$.

EXAMPLE 9.2 (*Logarithmic utility function from terminal wealth only, with* $\beta > 0$): $U_2(x) = \log x$ for $x > 0$ and $U_1(\cdot) \equiv 0$. This is the setting of Example 6.6 with $\delta = 0$; Assumption 6.1 is now satisfied trivially.

(i) $b(\cdot) \equiv r(\cdot)\mathbf{1}_m$. Since we have $\theta(\cdot) \equiv 0$ in this case, it follows that $\tilde{J}(\lambda;\tau) = -\mathbb{E}[e^{-\beta\tau}(1 + \log\lambda + A(\tau))]$, where

$$A(t,\omega) \stackrel{\triangle}{=} \beta t - \int_0^t r(s,\omega)\, d\, s, \qquad \forall\ 0 \le t \le T.$$

We claim that

if $\frac{dA(t,\omega)}{dt} - \beta A(t,\omega)$ *is strictly increasing for almost every* $\omega \in \Omega$
(e.g., if $r(t) \equiv r > \beta$*), then (8.7) holds.*

In order to check this, let $\hat{\tau}_\lambda \stackrel{\triangle}{=} \inf \left\{ t \ge 0 \ \Big/ \ \frac{dA(t)}{dt} - \beta A(t) \ge \beta(1+\log\lambda) \right\} \wedge T$. It is not difficult to see that $\hat{\tau}_\lambda \in \hat{\mathcal{S}}_\lambda$, since $-e^{-\beta\hat{\tau}_\lambda(\omega)}(1 + \log\lambda + A(\hat{\tau}_\lambda(\omega),\omega))$ is then the minimum of the path $e^{-\beta t}(1 + \log\lambda + A(t,\omega)),\ 0 \le t \le T$. Moreover, the condition of Corollary 8.8 is satisfied, and $\hat{\tau}_{\lambda_n} \to \hat{\tau}_\lambda$ if $\lambda_n \to \lambda$. It follows that

$$V(x) = \inf_{\lambda > 0} [\tilde{J}(\lambda;\hat{\tau}_\lambda) + \lambda x].$$

The optimal stopping time for the original optimization problem is $\hat{\tau} \equiv \hat{\tau}_{\hat{\lambda}}$, where $\hat{\lambda} > 0$ attains the infimum in the above expression. The corresponding optimal level of wealth $X^{x,\hat{\pi},0}(\hat{\tau}) \equiv \xi^x(\hat{\tau})$ is given by (6.11) as

$$\xi^x(\hat{\tau}) = \frac{x}{\mathbb{E}\left(e^{-\beta\hat{\tau}}\right)}\ e^{\int_0^{\hat{\tau}} r(s)\, ds - \beta\hat{\tau}},$$

and the optimal portfolio process $\hat{\pi}(\cdot)$ can be derived from Lemma 6.3.

(ii) A general result for the logarithmic utility function seems difficult to obtain, as we saw already in Example 6.6. Nevertheless, using the theory of Section 8 we shall establish the following property:

$$\text{(9.3)} \qquad \left\{ \begin{array}{c} V(x) \textit{ is attainable and (8.7) holds for every } x>0, \textit{ if there exists a } \text{unique} \\ \textit{optimal stopping time solving the optimization problem (7.2) for every } \lambda>0. \end{array} \right\}$$

The rest of this paragraph is dedicated to the proof of the statement (9.3). Consider the continuous process

$$Y^\lambda(t) \triangleq e^{-\beta t}\tilde{U}(\lambda e^{\beta t}H(t)) = -e^{-\beta t}(1+\log\lambda+\beta t+\log H(t))$$

and its Snell envelope, given as an RCLL modification of the supermartingale

$$Z^\lambda(t) \triangleq \ \text{esssup}_{\tau\in\mathcal{S}_{t,T}}\,\mathbb{E}[Y^\lambda(\tau)|\mathcal{F}_t], \qquad 0\le t\le T$$

with $Z^\lambda(0) = \sup_{\tau\in\mathcal{S}_{0,T}}\mathbb{E}Y^\lambda(\tau) = \tilde{V}(\lambda)$. We claim that $Z^\lambda(\cdot)$ is actually continuous. Indeed, since the random variable $\sup_{0\le t\le T}Y^\lambda(t)$ is integrable by Remark 7.3, the Snell envelope $Z^\lambda(\cdot)$ admits the Doob–Meyer decomposition $Z^\lambda(\cdot) = Z^\lambda(0)+M^\lambda(\cdot)-A^\lambda(\cdot)$ (Karatzas & Shreve (1998), Theorem D.13) where $M^\lambda(\cdot)$ is an RCLL martingale and $A^\lambda(\cdot)$ is continuous and nondecreasing. But any RCLL martingale of the Brownian filtration is continuous (Karatzas & Shreve (1991), Problem 3.4.16), hence $M^\lambda(\cdot)$ is continuous, and thus so is $Z^\lambda(\cdot)$. The stopping time $\tau^*_\lambda \triangleq \inf\left\{t\in[0,T)\ /\ Z^\lambda(t)=Y^\lambda(t)\right\}\wedge T$ is the *smallest* optimal stopping time in $\hat{\mathcal{S}}_\lambda$, whereas the stopping time $\rho^*_\lambda \triangleq \inf\left\{t\in[0,T)\ /\ A^\lambda(t)>0\right\}\wedge T$ is the *largest* optimal stopping time in $\hat{\mathcal{S}}_\lambda$ (Karatzas & Shreve (1998), Theorems D.12 and D.9; El Karoui (1981)). In particular, the uniqueness property (9.3) amounts to the statement: $\mathbb{P}[\tau^*_\lambda=\rho^*_\lambda]=1$, for all $0<\lambda<\infty$.

Moreover, $\lambda\mapsto\tau^*_\lambda$ is *increasing*, that is, for any $\lambda\ge\nu$ we have $\tau^*_\lambda\ge\tau^*_\nu$ almost surely. To see this, observe that $Y^\lambda(t)-Y^\nu(t) = -e^{-\beta t}\log(\lambda/\nu)$ and obtain

$$\begin{aligned} Z^\lambda(t)-Z^\nu(t) &= \text{esssup}_{\tau\in\mathcal{S}_{t,T}}\mathbb{E}[Y^\lambda(\tau)|\mathcal{F}_t] - \text{esssup}_{\tau\in\mathcal{S}_{t,T}}\mathbb{E}[Y^\lambda(\tau)+e^{-\beta\tau}\log(\lambda/\nu)|\mathcal{F}_t] \\ &\ge \text{esssup}_{\tau\in\mathcal{S}_{t,T}}\mathbb{E}[Y^\lambda(\tau)|\mathcal{F}_t] - \left\{\text{esssup}_{\tau\in\mathcal{S}_{t,T}}\mathbb{E}[Y^\lambda(\tau)|\mathcal{F}_t]+e^{-\beta t}\log(\lambda/\nu)\right\} \\ &= Y^\lambda(t)-Y^\nu(t) \end{aligned}$$

almost surely, for any given $0\le t\le T$. By the continuity of $Z(\cdot)$ and $Y(\cdot)$, it follows that

$$\mathbb{P}\left[Z^\lambda(t)-Y^\lambda(t)\ge Z^\nu(t)-Y^\nu(t),\ \text{for all } 0\le t\le T\right]=1,$$

which implies that $\tau^*_\lambda\ge\tau^*_\nu$ almostly surely, since $Z(\cdot)$ always dominates $Y(\cdot)$. It is not difficult to see that $\tau^\pm_\lambda \triangleq \lim_{n\to\infty}\tau^*_{\lambda\pm\frac{1}{n}}$ are stopping times, thanks to the continuity of the filtration $\mathbb{F}$. Moreover, they both belong to $\hat{\mathcal{S}}_\lambda$, which is an easy exercise on the Dominated Convergence Theorem (we omit the details).

Now we can prove our assertion (9.3). Clearly it must hold that $\tau^*_\lambda=\tau^+_\lambda=\tau^-_\lambda$ by uniqueness of optimal stopping time. It follows from Corollary 8.8 that $V(x)$ is attainable and (8.7) holds for every $x>0$.

Example 9.3 (*A case where NO optimal strategy exists*): We present now an example which shows that optimal strategies *need not* always exist for every initial capital $x \in (0, \infty)$.

Consider the logarithmic utility functions as in Example 6.6 with $\delta = 0$, i.e. $U_1(\cdot) \equiv 0$ and $U_2(x) = \log x$, discount factor $\beta = 1$, and model parameters $m = 1$, $r(\cdot) \equiv 0$, $b(\cdot) \equiv 0$, $\sigma(\cdot) \equiv 1$ in (2.1), (2.2). In this case we may take $c(\cdot) \equiv 0$ since there is no utility from consumption, and for a given initial capital $x > 0$ the wealth process $X^{x,\pi}(\cdot) \equiv X^{x,\pi,0}(\cdot)$ corresponding to a portfolio $\pi(\cdot)$ satisfies

$$dX^{x,\pi}(t) = X^{x,\pi}(t)\pi(t)\,dW(t), \qquad X^{x,\pi}(0) = x. \tag{9.4}$$

It is not difficult to check that

$$\tilde{V}(\lambda) = \sup_{\tau \in \mathcal{S}} \tilde{J}(\lambda; \tau) = \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[-e^{-\tau}(1 + \log \lambda + \tau)\right] = \max_{0 \le t \le T} F(\lambda; t), \tag{9.5}$$

where $F(\lambda; t) \triangleq -e^{-t}(1 + \log \lambda + t)$, $\lambda > 0$, $t > 0$. Note that the function $t \mapsto F(\lambda; t)$ attains its maximum on the interval $[0, T]$ at one of its endpoints, that is, $\max_{0 \le t \le T} F(\lambda; t) = \max\{F(\lambda; 0), F(\lambda; T)\}$, since $e^t \frac{dF}{dt}(\lambda; t) = \log \lambda + t$ is increasing. It follows then from (9.5) that

$$\tilde{V}(\lambda) = \left\{ \begin{array}{ccc} -(1 + \log \lambda) & ; & 0 < \lambda \le \lambda^*(T) \\ -e^{-T}(1 + \log \lambda + T) & ; & \lambda^*(T) \le \lambda < \infty \end{array} \right\}, \tag{9.6}$$

where $\lambda^*(s) \triangleq \exp\left\{\big(s/(e^s - 1)\big) - 1\right\} \in (0, 1)$ is determined by the equation

$$1 + \log \lambda^*(s) = e^{-s}(1 + \log \lambda^*(s) + s). \tag{9.7}$$

Clearly, $\tilde{V}(\cdot)$ is *not* differentiable at $\lambda = \lambda^*(T)$. Moreover, it is easy to verify that $\mathcal{G}_\lambda = \{1/\lambda\}$ for $0 < \lambda < \lambda^*(T)$ and that $\mathcal{G}_\lambda = \left\{e^{-T}/\lambda\right\}$ for $\lambda > \lambda^*(T)$, thus

$$\mathcal{G} = \bigcup_{\lambda > 0} \mathcal{G}_\lambda = \big(0, x_0(T)\big] \cup \big[x_1(T), \infty\big) \tag{9.8}$$

with $x_0(s) \triangleq \frac{e^{-s}}{\lambda^*(s)} \in (0, 1)$ and $x_1(s) \triangleq \frac{1}{\lambda^*(s)} \in (1, \infty)$; we omit the details of these computations. It should be noted that $x_1(\cdot)$ is increasing with $x_1(0+) = 1$, $x_1(\infty) = e$, whereas $x_0(\cdot)$ is decreasing with $x_0(0+) = 1$, $x_0(\infty) = 0$.

Now with $V_0(x) \triangleq e^{-T} \log x$ and $V_1(x) \triangleq \log x$, let us consider the concave function

$$\begin{aligned} G(x) &\triangleq \inf_{\lambda > 0} [\tilde{V}(\lambda) + \lambda x] \\ &= \left\{ \begin{array}{ccc} V_0(x) & ; & 0 < x \le x_0(T) \\ V_0(x_0(T)) \frac{x_1(T) - x}{x_1(T) - x_0(T)} + V_1(x_1(T)) \frac{x - x_0(T)}{x_1(T) - x_0(T)} & ; & x_0(T) < x < x_1(T) \\ V_1(x) & ; & x_1(T) \le x < \infty \end{array} \right\}; \end{aligned}$$

see Remark 9.4 for discussion. We have $V(x) = G(x)$ for $x \in \mathcal{G}$ from Theorem 8.5, or

$$V(x) = \left\{ \begin{array}{ccc} V_0(x) & ; & 0 < x \le x_0(T) \\ V_1(x) & ; & x_1(T) \le x < \infty \end{array} \right\}. \tag{9.9}$$

In particular, the optimal strategy is to keep all the wealth in the money-market (i.e., $\pi(\cdot) \equiv 0$) and to wait until the terminal time T, if the initial capital x is in $(0, x_0(T)]$, whereas the optimal strategy for $x \geq x_1(T)$ is to stop immediately.

But how about an initial capital $x \in (\, x_0(T), x_1(T)\,)$? From Theorem 8.5 and Proposition 8.9 we know that, either $V(x) < G(x)$ for some $x \in (\, x_0(T), x_1(T)\,)$ (which will give us a nonconcave value function $V(\cdot)$), or else $V(x) \equiv G(x)$ for all $x \in (\, x_0(T), x_1(T)\,)$ (in which case no optimal strategy exists).

We claim that the latter is the case. In other words, $V(x) \equiv G(x)$ for all $x \in \mathbf{R}^+$, but *no* optimal strategy exists for $x \in (\, x_0(T), x_1(T)\,)$. Actually, for every $x \in (\, x_0(T), x_1(T)\,)$, a maximizing sequence of strategy pairs $\{(\pi_n, \tau_n)\}_{n=1}^{\infty}$ can be constructed so that $J(x; \pi_n, \tau_n) \to G(x)$ as $n \to \infty$; this proves, in particular, that $V(\cdot) \equiv G(\cdot)$ on $\big(x_0(T), x_1(T)\big)$. Indeed, consider the wealth process $dX^{x,n}(t) = nX^{x,n}(t)\, dW(t)$, $X^{x,n}(0) = x$, and let

$$T_0^n \stackrel{\triangle}{=} \inf\big\{t \geq 0 \;/\; X^{x,n}(t) \leq x_0(T-t)\big\} \wedge T, \tag{9.10}$$

$$T_1^n \stackrel{\triangle}{=} \inf\big\{t \geq 0 \;/\; X^{x,n}(t) \geq x_1(T-t)\big\} \wedge T. \tag{9.11}$$

Recall $x_0(0+) = x_1(0+) = 1$, so that $T_0^n \wedge T_1^n < T$ holds almost surely. We define the portfolio / stopping time pair (π_n, τ_n) by

$$\pi_n(t) \stackrel{\triangle}{=} n \cdot 1_{\{t < T_1^n \wedge T_0^n\}}, \quad 0 \leq t \leq T \qquad \text{and} \qquad \tau_n \stackrel{\triangle}{=} T_1^n \cdot 1_{\{T_1^n < T_0^n\}} + T \cdot 1_{\{T_1^n \geq T_0^n\}}. \tag{9.12}$$

This means: if the wealth reaches the curve $x_1(T-\cdot)$ before reaching the curve $x_0(T-\cdot)$, stop immediately when this happens; if the wealth reaches the curve $x_0(T-\cdot)$ before reaching the curve $x_1(T-\cdot)$, then put all the money in the bank account and wait until the terminal time T; and up until the first time that one of these curves is reached, keep an amount of n dollars invested in stock. Clearly,

$$X^{x,\pi_n}(\tau_n) = x_0(T - T_0^n) \cdot 1_{\{T_0^n < T_1^n\}} + x_1(T - T_1^n) \cdot 1_{\{T_1^n < T_0^n\}}. \tag{9.13}$$

Moreover, since $\pi_n(\cdot)$ is bounded, the wealth process $X^{x,\pi_n}(\cdot)$ is a martingale, and the optional sampling theorem gives

$$x = \mathbb{E}\left[X^{x,\pi_n}(\tau_n)\right]. \tag{9.14}$$

Because $T_0^n = \inf\left\{t \geq 0 \;/\; W(t) \leq \frac{1}{2}nt + \frac{1}{n}\log\left(\frac{x_0(T-t)}{x}\right)\right\} \wedge T \longrightarrow 0$ almost surely as $n \to \infty$, it follows from (9.13) and (9.14) that $x_0(T)p_n + x_1(T)(1-p_n) \longrightarrow x$ as $n \to \infty$, where $p_n \stackrel{\triangle}{=} \mathbb{P}(T_0^n < T_1^n) = 1 - \mathbb{P}(T_1^n < T_0^n)$, or equivalently

$$p_n \to \frac{x_1(T) - x}{x_1(T) - x_0(T)} \qquad \text{as} \quad n \to \infty. \tag{9.15}$$

On the other hand, the expected discounted utility corresponding to (π_n, τ_n) of (9.12) is

$$J(x; \pi_n, \tau_n) = \mathbb{E}\left[e^{-T}\log\left(x_0(T - T_0^n)\right) \cdot 1_{\{T_0^n < T_1^n\}} + \log\left(e^{-T_1^n} x_1(T - T_1^n)\right) \cdot 1_{\{T_1^n < T_0^n\}}\right].$$

We conclude the proof by noting from (9.15) and the Dominated Convergence Theorem, that

$$\lim_{n\to\infty} J(x;\pi_n,\tau_n) = e^{-T}\log x_0(T)\cdot\frac{x_1(T)-x}{x_1(T)-x_0(T)} + \log x_1(T)\cdot\frac{x-x_0(T)}{x_1(T)-x_0(T)} = G(x).$$

Remark 9.4: The tangent to the graph of $V_0(\cdot)$ at $x = x_0 \stackrel{\triangle}{=} x_0(T)$, and the tangent to the graph of $V_1(\cdot)$ at $x = x_1 \stackrel{\triangle}{=} x_1(T)$, coincide. Indeed, $V_1'(x) = \frac{1}{x}$ so that the tangent $f_1(\cdot)$ to the graph of $V_1(\cdot)$, at the point $x = x_1$, is given by

$$f_1(x) = \frac{x-x_1}{x_1} + f_1(x_1) = \left(\frac{x}{x_1} - 1\right) + \log x_1 = \lambda^*(T)x - (1+\log\lambda^*(T)).$$

On the other hand, $V_0'(x) = \frac{1}{x}e^{-T}$ so that the tangent $f_0(\cdot)$ to the graph of $V_0(\cdot)$, at the point $x = x_0$, is given by

$$\begin{aligned} f_0(x) &= \frac{x-x_0}{x_0}e^{-T} + f_0(x_0) = e^{-T}\left(x\lambda^*(T)e^{T} - 1\right) + e^{-T}\log x_0 \\ &= \lambda^*(T)x - e^{-T}(1+\log\lambda^*(T)+T). \end{aligned}$$

Thanks to (9.7), these two expressions are the same.

10 Appendix A

In this section we provide an example which illustrates briefly, in a Markovian setting and with logarithmic utility from wealth (we set $c(\cdot)\equiv 0$ and write $X^{x,\pi}(\cdot)\equiv X^{x,\pi,0}(\cdot)$ throughout), how the optimization problem of (5.3) can be cast in the form of a free–boundary problem for a suitable *Hamilton–Jacobi–Bellman* (HJB) *equation*, which can then be solved explicitly.

In order to obtain such an explicit solution, we place ourselves on an infinite time–horizon so that all stopping times $\tau \in \mathcal{S}_{0,\infty}$ are admissible, and denote the corresponding value function by

$$V_\infty(x) = \sup_{(\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[e^{-\beta\tau}\log X^{x,\pi}(\tau)\cdot \mathbf{1}_{\{\tau<\infty\}}\right] \tag{10.0}$$

with $\beta > 0$, for a given initial capital $x > 0$ in the notation of (9.4). Furthermore, we assume that the coëfficients of the model $r(\cdot)\equiv r > 0,\ b(\cdot)\equiv b,\ \sigma(\cdot)\equiv\sigma > 0$ are all constant, and impose the assumption $b \neq r\mathbf{1}_m$, or equivalently $\theta(\cdot)\equiv\theta\neq 0$. For the measure–theoretic subtleties associated with working on an infinite time–horizon, we refer the reader to Section 1.7 in Karatzas & Shreve (1998).

Consider the following differential operator

$$\begin{aligned} \mathcal{L}u\,(x) &\stackrel{\triangle}{=} -\beta u(x) + rxu'(x) + \max_{\pi\in\mathbb{R}^m}\left(xu'(x)\pi^*\sigma\theta + \frac{1}{2}x^2u''(x)\parallel \pi^*\sigma \parallel^2\right) \\ &= -\beta u(x) + rxu'(x) - \frac{(u'(x))^2\Theta^2}{2u''(x)}, \end{aligned} \tag{10.1}$$

acting on functions $u:(0,\infty)\to\mathbb{R}$, which are twice continuously differentiable with $u''(\cdot) < 0$; here $\Theta \stackrel{\triangle}{=} \parallel (\sigma^*)^{-1}\theta \parallel = \parallel (\sigma\sigma^*)^{-1}(b - r\mathbf{1}_m) \parallel > 0$. By analogy with Section 2.7 in Karatzas & Shreve

(1998), we cast the original optimization problem of (10.0) as a *variational inequality*, relying on the familar "principle of smooth–fit".

VARIATIONAL INEQUALITY A.1: *Find a number* $b \in (1,\infty)$ *and an increasing function* $g(\cdot)$ *in the space* $\mathcal{C}([0,\infty)) \cap \mathcal{C}^1((0,\infty)) \cap \mathcal{C}^2((0,\infty)\setminus\{b\})$, *such that*:

$$\mathcal{L}g\,(x) = 0 \quad ; \quad 0 < x < b \tag{10.2}$$

$$\mathcal{L}g\,(x) < 0 \quad ; \quad x > b \tag{10.3}$$

$$g(x) > \log x \quad ; \quad 0 < x < b \tag{10.4}$$

$$g(x) = \log x \quad ; \quad x \ge b \tag{10.5}$$

$$g(x) > 0 \quad ; \quad x > 0 \tag{10.6}$$

$$g''(x) < 0 \quad ; \quad x \in (0,\infty)\setminus\{b\}. \tag{10.7}$$

THEOREM A.2: *Suppose that the pair* $(b, g(\cdot))$ *solves the Variational Inequality A.1, that the ratio* $|g'(x)/(xg''(x))|$ *is bounded away from both zero and infinity on* $(0,\infty)$*, and that the stochastic differential equation*

$$d\hat{X}(t) = \hat{X}(t)\left[r\,dt - \frac{g'(\hat{X}(t))}{\hat{X}(t)g''(\hat{X}(t))}\;\theta^*\,dW_0(t)\right], \qquad \hat{X}(0) = x > 0 \tag{10.8}$$

has a pathwise unique, strictly positive strong solution $\hat{X}(\cdot)$*. In terms of this process, define*

$$\hat{\pi}(\cdot) \stackrel{\triangle}{=} -\big(\sigma^*\big)^{-1}\theta\,\frac{g'(\xi)}{\xi g''(\xi)}\Bigg|_{\xi=\hat{X}(\cdot)} , \qquad \hat{\tau} \stackrel{\triangle}{=} \inf\Big\{t \ge 0 \big/ \hat{X}(t) \ge b\Big\}. \tag{10.9}$$

Then the function $g(\cdot)$ *coincides with the optimal expected utility* $V_\infty(\cdot)$ *of (10.0), the pair* $\big(\hat{\pi}(\cdot),\, \hat{\tau}\big)$ *attains the supremum in (10.0), and we have* $\hat{X}^{x,\hat{\pi}}(\cdot) \equiv \hat{X}(\cdot)$*.*

Proof: Fix $x \in (0,\infty)$. For any available portfolio process $\pi(\cdot)$, an application of Itô's rule to $\mathcal{G}^{x,\pi}(t) \stackrel{\triangle}{=} e^{-\beta t} g(X^{x,\pi}(t)),\ 0 \le t < \infty$ yields, in conjuction with (3.1), (10.2) and (10.3):

$$\begin{aligned} & e^{-\beta t} g(X^{x,\pi}(t)) - g(x) - \int_0^t e^{-\beta s}\pi^*\sigma \cdot \xi g'(\xi)\Big|_{\xi = X^{x,\pi}(s)}\,dW(s) \;= \\ &= \int_0^t e^{-\beta s}\left((\pi^*\sigma\theta + r)\cdot \xi g'(\xi) + \frac{1}{2}g''(\xi)\xi^2 \parallel \pi^*\sigma \parallel^2 - \beta g(\xi)\right)\Bigg|_{\xi = X^{x,\pi}(s)} ds \\ &\le \int_0^t e^{-\beta s}\mathcal{L}g(X^{x,\pi}(s))\,ds \;\le\; 0. \end{aligned} \tag{10.10}$$

It follows that the process $\mathcal{G}^{x,\pi}(t) = e^{-\beta t} g(X^{x,\pi}(t)),\ 0 \le t < \infty$ is a local supermartingale under $\mathbf{P}$, hence also a true supermartingale because it is positive. In particular, $\mathcal{G}^{x,\pi}(\infty) \stackrel{\triangle}{=} \limsup_{t\to\infty} \mathcal{G}^{x,\pi}(t) \ge 0$ exists almost surely, and $\{\mathcal{G}^{x,\pi}(t),\ 0 \le t \le \infty\}$ is a $\mathbf{P}-$supermartingale. Thus

$$\mathbb{E}[e^{-\beta\tau}\log X^{x,\pi}(\tau)\cdot \mathbf{1}_{\{\tau<\infty\}}] \le \mathbb{E}[e^{-\beta\tau} g\big(X^{x,\pi}(\tau)\big)\cdot \mathbf{1}_{\{\tau<\infty\}}] \le \mathbb{E}[\mathcal{G}^{x,\pi}(\tau))] \le g(x) \tag{10.11}$$

holds for any stopping time $\tau \in \mathcal{S}_{0,\infty}$, by the optional sampling theorem and (10.4)–(10.5); in other words, $V_\infty(x) \le g(x)$. We complete the proof upon noticing that, thanks to (10.2) and (10.5), all the inequalities in (10.10) and (10.11) hold as equalities for the choice

$$\text{(10.12)} \qquad \hat{\pi}(t) \triangleq -\frac{g'(\hat{X}(t))}{\hat{X}(t)g''(\hat{X}(t))}(\sigma^*)^{-1}\theta, \qquad \hat{\tau}_b \triangleq \inf\left\{t \ge 0 / \hat{X}(t) \ge b\right\},$$

since we have $0 < g(\hat{X}(\hat{\tau}_b)) \le \log b$ and $e^{-\beta\hat{\tau}_b}\, g(\hat{X}(\hat{\tau}_b)) = 0$ on the event $\{\hat{\tau}_b = \infty\}$. ◊

We have now to construct the solution of the Variational Inequality A.1, and to verify the properties for the equation (10.8) assumed in Theorem A.2.

PROPOSITION A.3: *Let α be the unique solution of the quadratic equation*

$$\text{(10.13)} \qquad \alpha^2 - \left(1 + \frac{\Theta^2}{2r} + \frac{\beta}{r}\right)\alpha + \frac{\beta}{r} = 0$$

in the interval $(0,1)$, set $b \triangleq e^{1/\alpha}$ and consider the function

$$\text{(10.14)} \qquad g(x) \triangleq \left\{ \begin{array}{lcl} x^\alpha/e\alpha & ; & 0 \le x < b \\ \log x & ; & b \le x < \infty \end{array} \right\}.$$

Then the pair $(b, g(\cdot))$ solves the Variational Inequality A.1; and the stochastic differential equation (10.8) has a pathwise unique, strictly positive strong solution $\hat{X}(\cdot)$.

Proof: Note that the function

$$\text{(10.15)} \qquad F(u) \triangleq u^2 - \left(1 + \frac{\frac{\Theta^2}{2} + \beta}{r}\right)u + \frac{\beta}{r}\,, \qquad 0 \le u < \infty$$

is convex with $F(0) = \beta/r > 0$, $F(1) = -\Theta^2/2r < 0$. Thus $F(\cdot)$ has exactly one root in the interval $(0,1)$. It is clear now that (10.5)–(10.7) are satisfied since $b > 1$. Furthermore, notice from (10.14) that

$$\text{(10.16)} \qquad g'(x) = \left\{ \begin{array}{lcl} x^{\alpha-1}/e & ; & 0 < x < b \\ 1/x & ; & b < x < \infty \end{array} \right\}$$

is continuous across $x = b$ (principle of smooth-fit), which implies that the function $g(\cdot)$ belongs to the space of functions $\mathcal{C}([0,\infty)) \cap \mathcal{C}^1((0,\infty)) \cap \mathcal{C}^2((0,\infty)\setminus\{b\})$. It is fairly straightforward to check that (10.2) holds for $0 < x < b$, and that $|g'(x)/(xg''(x))|$ is bounded away from both zero and infinity on $(0,\infty)$ (cf. (10.18) below). As for (10.3), we need to prove that $-\beta\log x + r + \Theta^2/2 < 0$, $\forall\, x > b$. Since $\log b = 1/\alpha$ and $\beta > 0$, it is sufficient to verify $\alpha < \alpha^* \triangleq \beta/(r + \frac{\Theta^2}{2})$. Indeed

$$F(\alpha^*) = \alpha^*\left(\alpha^* - \frac{\frac{\Theta^2}{2} + \beta}{r} - 1\right) + \frac{\beta}{r} < \alpha^*\left(\frac{\beta}{r} - \frac{\frac{\Theta^2}{2} + \beta}{r} - 1\right) + \frac{\beta}{r} = \alpha^*\left(-\frac{\frac{\Theta^2}{2} + r}{r}\right) + \frac{\beta}{r} \;=\; 0,$$

which yields $\alpha < \alpha^*$. Finally, (10.4) follows readily from

$$g'(x) - (\log x)' = \frac{1}{x}\left(\frac{1}{e}x^\alpha - 1\right) < \frac{1}{x}\left(\frac{1}{e}b^\alpha - 1\right) = 0, \quad 0 < x < b.$$

It is now clear that the pair $(b, g(\cdot))$ solves the Variational Inequality A.1.

For the function $g(\cdot)$ of (10.14), the optimal wealth-process $\hat{X}(\cdot)$ of Theorem A.2 satisfies the stochastic differential equation (10.8), namely

$$d\hat{X}(t) = \hat{X}(t)\big[r\,dt + \nu\big(\hat{X}(t)\big)\theta^*\,dW_0(t)\big], \qquad \hat{X}(0) = x > 0 \tag{10.17}$$

where

$$\nu(x) \triangleq -\frac{g'(x)}{xg''(x)} = \left\{ \begin{array}{ccc} 1\big/(1-\alpha) & ; & 0 < x < b \\ 1 & ; & b \le x < \infty \end{array} \right\}. \tag{10.18}$$

Equivalently, the process $\hat{Y}(\cdot) \triangleq \log \hat{X}(\cdot)$ solves the stochastic differential equation

$$d\hat{Y}(t) = \left[r - \frac{||\theta||^2}{2}\cdot \nu^2\big(e^{\hat{Y}(t)}\big)\right] dt + \nu\big(e^{\hat{Y}(t)}\big)\theta^*\,dW_0(t), \qquad \hat{Y}(0) = \log x, \tag{10.19}$$

which has a pathwise unique, strong solution (cf. Nakao (1972)). This, in turn, means that *the equation (10.14) for* $\hat{X}(\cdot) \equiv e^{\hat{Y}(\cdot)}$ *also has a strictly positive, pathwise unique strong solution*, as postulated in Theorem A.2. ♢

Remark A.4: For $x \ge b$, we have $\hat{\tau} \equiv 0$; on the other hand, for $0 < x < b$, we can write the stopping time $\hat{\tau} \triangleq \inf\left\{t \ge 0 \big/ \hat{X}(t) \ge x\right\} = \inf\left\{t \ge 0 \big/ \hat{Y}(t) \ge \log b\right\}$ in the form of the time

$$\hat{\tau} = \inf\left\{ t \ge 0 \Big/ \left(r + \frac{||\theta||^2}{2}\frac{1-2\alpha}{(1-\alpha)^2}\right)t + \frac{\theta^*}{1-\alpha}W(t) \;\ge\; \log\big(\frac{b}{x}\big) \right\}$$

of first–passage to a positive level by a Brownian motion with drift. Clearly, we have $\mathbf{P}[\hat{\tau} < \infty] = 1$ if and only if $(1-\alpha)^2 + ||\theta||^2(1-2\alpha)/2r \ge 0$, and in light of the equation (10.13) this last condition is equivalent to

$$\left(\beta - r - ||\theta||^2 + \frac{\Theta^2}{2}\right)\cdot\alpha \;\ge\; \left(\beta - r - \frac{||\theta||^2}{2}\right). \tag{10.20}$$

In particular, if $\sigma = I_m$, the condition (10.20) amounts to

$$\beta \le r + ||b - r\mathbf{1}_m||^2. \tag{10.21}$$

Remark A.5: From (10.12), the optimal portfolio process is actually given as

$$\hat{\pi}(t) \equiv \frac{(\sigma^*)^{-1}}{1-\alpha}\theta = \frac{(\sigma\sigma^*)^{-1}}{1-\alpha}[b - r\mathbf{1}_m], \qquad 0 \le t < \hat{\tau}\;; \tag{10.22}$$

this means that the optimal strategy is to invest a *fixed* proportion of total wealth in every stock, given by (10.2), up to the optimal stopping time $\hat{\tau}$.

Remark A.6: The assumption $\theta \neq 0$ is crucial for solving the Variational Inequality A.1. When $\theta = 0$, we can have situations, as in Example 9.3, for which *no* optimal strategy exists. Actually, for $\theta = 0$ and $\beta > r$, it is easy to show that the Variational Inequality A.1 has no solution (see Example 9.2 for discussion of the case $\theta = 0$, $\beta < r$).

11 Appendix B

As the referee points out, it would be very interesting to study optimization over a consumption stream that extends beyond the stopping time τ. Consider, for instance, the situation of an investor who remains in the stock-market up until a "retirement" time τ of his choice. At that point he consumes a lump-sum amount $\xi \geq 0$ of his choice (say, to buy a new house, or to finance some other, "retirement-related", activity); and from then on keeps his holdings in the money-market, making withdrawals for consumption at some rate, up until $t = T$.

We can capture such a situation by changing the wealth-equation of (3.1), to read

$$dX(t) = r(t)X(t)dt + X(t)\pi^*(t)\sigma(t)dW_0(t) - dC(t), \qquad X(0) = x > 0. \tag{11.1}$$

Here

$$C(t) = \int_0^t c(u)\, du + \xi \cdot 1_{[\tau,T]}(t), \qquad 0 \leq t \leq T \tag{11.2}$$

is the "cumulative consumption up to time t". This process consists of a stopping time $\tau \in \mathcal{S}$, a consumption-rate process $c(\cdot)$ as before, and an $\mathcal{F}_\tau-$measurable random variable $\xi : \Omega \to [0, \infty)$ representing lump-sum consumption at time τ. We say that a portfolio / cumulative-consumption process pair (π, C) is "available" to an investor with initial capital x, if the portfolio process $\pi(\cdot)$ and the wealth-process $X(\cdot) \equiv X^{x,\pi,C}(\cdot)$ of (11.1) satisfy

$$\pi(t) = 0, \qquad \tau \leq t \leq T \tag{11.3}$$

$$X^{x,\pi,C}(t) > 0, \quad \forall\ 0 \leq t < T \qquad \text{and} \qquad X^{x,\pi,C}(T) \geq 0\ , \tag{11.4}$$

almost surely. For any such pair (π, C), the investor's expected discounted utility is given as

$$J^*(x; \pi, C)\ \stackrel{\triangle}{=}\ \mathbb{E}\left[\alpha \int_0^\tau e^{-\beta t} U_1\big(c(t)\big)\, dt + e^{-\beta\tau} U_2(\xi) + \gamma \int_\tau^T e^{-\beta t} U_1\big(c(t)\big)\, dt\right] \tag{11.5}$$

for some given constants $\alpha \geq 0,\ \gamma \geq 0$ and utility functions $U_1(\cdot),\ U_2(\cdot)$. With $\alpha = 1,\ \gamma = 0$ we recover the problem of Section 5. With $\alpha = 0,\ \gamma = 1,$ the expression of (11.5) tries to capture the situation of an investor who consumes nothing up until retirement, consumes a lump-sum amount ξ at that time, and afterwards keeps all holdings in the money-market while consuming at some rate $c(\cdot)$. The objective now, is to maximize the expression of (11.5), over the class $\mathcal{A}^*(x)$ of pairs (π, C) that satisfy the analogue

$$\mathbb{E}\left[\alpha \int_0^\tau e^{-\beta t} U_1^-\big(c(t)\big)\, dt + e^{-\beta\tau} U_2^-(\xi) + \gamma \int_\tau^T e^{-\beta t} U_1^-\big(c(t)\big)\, dt\right] < \infty \tag{11.6}$$

of (5.2), and to see whether the value-function

$$V^*(x)\ \stackrel{\triangle}{=}\ \sup_{(\pi,C)\in\mathcal{A}^*(x)} J^*(x; \pi, C), \qquad x \in (0, \infty) \tag{11.7}$$

is attained by some optimal $(\hat{\pi}, \hat{C}) \in \mathcal{A}^*(x)$. We have not yet been able to obtain a satisfactory answer to these questions, and would like to suggest their resolution as an interesting open problem.

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