Utility Maximization With Habit Formation: Dynamic Programming and Stochastic PDE's

NIKOLAOS ENGLEZOS

Department of Banking and Financial Management

University of Piraeus

Joint work with

Professor IOANNIS KARATZAS

Toronto 24 June 2010

Outline

The Model

Probability space: $(\Omega, \mathcal{F}, P)$.

Brownian Motion: $W(\cdot) = (W_1(\cdot), ..., W_d(\cdot))^*$

Information filtration: $\mathbb{F} = (\mathcal{F}_t^W)_{t \geq 0}$

Complete Financial Market:
- finite time horizon $[0, T]$
- a riskless asset $B(\cdot)$
- d stocks $S_i(\cdot)$, $i = 1, 2, \; ..., d$

such that

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1,$$

$$dS_i(t) = S_i(t) \left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t) \right], \quad i = 1, ..., d.$$

Interest rate: $r(\cdot)$ is bounded.

Vector of rates-of-return: $b(\cdot) = (b_1(\cdot), ..., b_d(\cdot))^*$
is integrable almost surely.

Volatility matrix: $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \leq i,j \leq d}$
is square-integrable almost surely and
$\sigma(t)$ has full rank for every t.

All processes are $\mathbb{F}-$progressively measurable.
No anticipation of the future.

Economic agent: At each time t he can decide

- proportion (portfolio) $\pi_i(t)$ of his wealth $X(t)$ to be invested in the ith stock
- remaining amount is invested in the riskless asset
- consumption rate $c(t) \geq 0$
- initial endowment $x > 0$.

For $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$ we have the equation

$$dX(t) = X(t)\left[\sum_{i=1}^{d} \pi_i(t) \cdot \frac{dS_i(t)}{S_i(t)} + \left(1 - \sum_{i=1}^{d} \pi_i(t)\right) r(t)dt\right]$$
$$- c(t)dt$$
$$= [r(t)X(t) - c(t)]dt + X(t)\pi^*(t)\sigma(t)\Big[dW(t) + \vartheta(t)dt\Big],$$

subject to the initial condition $X(0) = x > 0$.

Market price of risk: $\vartheta(t) \triangleq \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}]$

Equivalently, we have

$$\frac{X(t)}{B(t)} + \int_0^t \frac{c(s)}{B(s)}ds = x + \int_0^t \frac{X(s)}{B(s)}\pi^*(s)\sigma(s)\Big[dW(s) + \vartheta(s)ds\Big]$$

Utility function: $u : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$,
such that $u(t,\cdot)$ is strictly increasing, strictly concave, of class $C^1(\mathbb{R}^+)$, and $u'(t,0^+) = \infty$, $u'(t,\infty) = 0$;

Standard of living: average of past consumption.
For $\alpha(\cdot), \delta(\cdot) \geq 0$ $\mathbb{F}$-adapted processes,

$$z(t) \equiv z(t;c) = z\ e^{-\int_0^t \alpha(v)dv} + \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c(s)ds$$

The Habit-Forming Maximization Problem

Value function: For given $(x, z) \in \mathcal{D}$

$$V(x,z) \triangleq \sup_{(\pi,c)\in\mathcal{A}(x,z)} E\left[\int_0^T u(t, c(t) - z(t;c))dt\right]$$

Admissible controls: $(\pi, c) \in \mathcal{A}(x, z)$ such that

- $c(\cdot) \geq 0$
- $X^{x,\pi,c}(t) \geq 0$, for all $t \in [0, T]$
- $c(t) - z(t; c) > 0$, $\rightsquigarrow$ **"addiction"**

Optimal Policies by Detemple & Zapatero (1992):

- $\exists$ optimal pair (π_0, c_0)
- For $z_0(\cdot) \equiv z(\cdot\, , c_0)$ and $I \triangleq (u')^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$

$$c_0(t) - z_0(t) = I(t, y_0\Gamma(t)) > 0 :$$

optimal scalar: $y_0 \in \mathbb{R}^+$
"adjusted" state-price density: for $E_t[\,\cdot\,] \triangleq E[\,\cdot\,|\mathcal{F}(t)]$

$$\Gamma(t) \triangleq H(t) + \delta(t) \cdot E_t\left(\int_t^T e^{\int_t^s (\delta(v)-\alpha(v))dv} H(s)ds\right)$$

state-price density: $H(t) \triangleq e^{-\int_0^t r(s)ds} Z(t)$

density: $Z(t) \triangleq e^{-\int_0^t \vartheta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\vartheta(s)\|^2 ds}$

Theorem 1. $V(\cdot, z)$ satisfies all the conditions of a utility function as defined previously, for any given $z \geq 0$.

Optimal wealth: $X_0(\cdot) \equiv X^{x,\pi_0,c_0}(\cdot)$ is given by

$$X_0(t) = \frac{1}{H(t)} E_t \left[\int_t^T H(s) c_0(s) ds \right],$$

and substituting the optimal $c_0(\cdot)$ we can show that

$$X_0(t) - \mathcal{W}(t) z_0(t) = \frac{1}{H(t)} E_t \left[\int_t^T \Gamma(s) I(s, y_0 \Gamma(s)) ds \right],$$

where

$$\mathcal{W}(t) \triangleq E_t \left[\int_t^T e^{\int_t^s (\delta(v) - \alpha(v)) dv} \frac{H(s)}{H(t)} \, ds \right] \text{ and } \mathcal{W}(0) = w.$$

Theorem 2. The **effective state space** of $\left(X_0(t), z_0(t)\right)$ is identified as the **random wedge**

$$\mathcal{D}_t \triangleq \{(x', z') \in \mathbb{R}^+ \times [0, \infty);\ x' > \mathcal{W}(t) z'\},$$
$$\mathcal{D}_T \triangleq \{(0, z');\ z' \in [0, \infty)\}, \text{ and } \mathcal{D}_0 = \mathcal{D}.$$

- $\mathcal{W}(\cdot)$ is the **"marginal" cost of subsistence consumption per unit of standard of living at t.**

Next Goal: Dependence of the optimal pair (π_0, c_0) on the wealth $X_0(\cdot)$ and standard of living $z_0(\cdot)$.

Assumption: $\vartheta(\cdot)$ bounded away from 0 and ∞

Change of measure: $P^0(A) \triangleq E[Z(T) \mathbf{1}_A], \quad A \in \mathcal{F}(T)$

New d–dim BM: $W_0(t) \triangleq W(t) + \int_0^t \vartheta(s) ds$ under P^0

For $(t,y) \in [0,T] \times \mathbb{R}^+$ and $t \le s \le T$, consider

- $Z^t(s) \triangleq Z(s)/Z(t)$,
- $H^t(s) \triangleq \exp\left\{-\int_t^s r(v)dv\right\} Z^t(s)$,
- $\Gamma^t(s) \triangleq H^t(s) + \delta(s) \cdot E_s\left(\int_s^T e^{\int_s^\theta (\delta(v)-\alpha(v))dv} H^t(\theta)d\theta\right)$

Assumption: $r(\cdot) - \delta(\cdot) + \alpha(\cdot)$ is deterministic

- $\Gamma^t(s) = H^t(s)\mu(s)$, with

$$\mu(t) \triangleq 1 + \delta(t)w(t)$$
$$w(t) \triangleq \int_t^T e^{\int_t^s (-r(v)+\delta(v)-\alpha(v))dv} ds = \mathcal{W}(t)$$

Key Process: $Y^{(t,y)}(s) \triangleq y\Gamma^t(s), \quad t \le s \le T$

Rewrite the relationship derived for $X_0(\cdot)$ and $z_0(\cdot)$ as

$$X_0(t) - w(t)z_0(t) = \mathfrak{X}\left(t, \frac{Y^{(0,y_0)}(t)}{\mu(t)}\right),$$

in terms of

Random field: $\mathfrak{X} : [0,T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ given by

$$\mathfrak{X}(t,y) \triangleq E_t^0\left[\int_t^T e^{-\int_t^s r(v)dv} \mu(s) I(s, yY^{(t,1)}(s))ds\right]$$

In the following slide, we recall a generalized **Itô-Kunita-Wentzell formula** for *random fields*.

Generalized Itô-Kunita-Wentzell (GIKW): Let the random field $\mathbf{F}$ be of class $C^{0,2}([0,T]\times\mathbb{R}^n)$ and satisfy

$$\mathbf{F}(t,x) = \mathbf{F}(0,x) + \int_0^t \mathbf{f}(s,x)ds + \int_0^t \mathbf{g}^*(s,x)dW(s),$$

where $\mathbf{g} = (\mathbf{g}^{(1)}, \dots, \mathbf{g}^{(d)})$ are $C^{0,2}$, $\mathbb{F}$-adapted random fields, and $\mathbf{f}$ is a $C^{0,1}$ random field. Furthermore, let $\mathbf{X} = (\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)})$ be a vector of continuous semimartingales with decompositions

$$\mathbf{X}^{(i)}(t) = \mathbf{X}^{(i)}(0) + \int_0^t \mathbf{b}^{(i)}(s)ds + \int_0^t (\mathbf{h}^{(i)}(s))^* dW(s),$$

where $\mathbf{h}^{(i)} = (\mathbf{h}^{(i,1)}, \dots, \mathbf{h}^{(i,d)})$ is an $\mathbb{F}$-progressively measurable, almost surely square integrable vector process, and $\mathbf{b}^{(i)}(\cdot)$ is an almost surely integrable process. Then $\mathbf{F}(\cdot, \mathbf{X}(\cdot))$ is also a continuous semimartingale, with decomposition

$$\begin{aligned}
\mathbf{F}\Big(t, \mathbf{X}(t)\Big) = \mathbf{F}\Big(0, \mathbf{X}(0)\Big) &+ \sum_{i=1}^n \int_0^t \mathbf{F}_{\mathbf{x}_i}\Big(s, \mathbf{X}(s)\Big) d\mathbf{X}^{(i)}(t) \\
&+ \int_0^t \mathbf{f}\Big(s, \mathbf{X}(s)\Big)ds + \int_0^t \mathbf{g}^*\Big(s, \mathbf{X}(s)\Big)dW(s) \\
&+ \sum_{j=1}^d \sum_{i=1}^n \int_0^t \mathbf{g}^{(j)}_{\mathbf{x}_i}\Big(s, \mathbf{X}(s)\Big)\mathbf{h}^{(i,j)}(s)ds \\
&+ \frac{1}{2} \sum_{i,k=1}^n \int_0^t \mathbf{F}_{\mathbf{x}_i\mathbf{x}_k}\Big(s, \mathbf{X}(s)\Big) d\langle \mathbf{X}^{(i)}, \mathbf{X}^{(k)}\rangle(s).
\end{aligned}$$

Notation: Consider the class of *pair* random fields

$$\begin{aligned}
\mathbf{G}_{\mathbb{F}} \triangleq C_{\mathbb{F}}\Big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\Big) \\
\times \mathbb{L}^2_{\mathbb{F}}\Big(0,T; \mathbb{L}^2(\Omega; C^2(\mathbb{R}^+; \mathbb{R}^d))\Big)
\end{aligned}$$

The Role of Stochastic PDE's

Under reasonable assumptions on the utility preferences $(u,\ I, \ldots)$, and the study of parabolic BSPDE's by Ma and Yong (1997), we reach the following result.

Proposition 3.
There exists a random field $\Psi^{\mathfrak{X}}$ such that the pair $(\mathfrak{X}, \Psi^{\mathfrak{X}}) \in \mathbf{G}_{\mathbb{F}}$ is the **unique** $\mathbb{F}$**-adapted solution** of the **linear** parabolic BSPDE

$$\begin{aligned}
-d\mathfrak{X}(t,y) =& \Big[\frac{1}{2}\|\vartheta(t)\|^2 y^2 \mathfrak{X}_{yy}(t,y) \\
&+ \Big(\|\vartheta(t)\|^2 - r(t)\Big) y \mathfrak{X}_y(t,y) - r(t)\mathfrak{X}(t,y) \\
&- \vartheta^*(t) y \Psi^{\mathfrak{X}}_y(t,y) + \mu(t) I(t, y\mu(t))\Big] dt \\
&- \Big(\Psi^{\mathfrak{X}}(t,y)\Big)^* dW_0(t) \quad \text{on } [0,T) \times \mathbb{R}^+, \\
\mathfrak{X}(T,y) =& 0 \quad \text{on } \mathbb{R}^+.
\end{aligned}$$

Remark 4.
Integrating over $[t,T]$, the above BSPDE yields the semimartingale decomposition of the process $\mathfrak{X}(\cdot, y)$, for all $y \in \mathbb{R}^+$.

For $t \in [0,T)$
- $\mathfrak{X}(t,\cdot)$ is strictly decreasing,
- $\mathfrak{X}(t,0^+) = \infty$, $\mathfrak{X}(t,\infty) = 0$

Inverse random field: $\mathfrak{Y}(t,\cdot) \triangleq \mathfrak{X}^{-1}(t,\cdot)$
of class $C_{\mathbb{F}}\Big([0,T); C^3(\mathbb{R}^+)\Big)$.

Feedback Formulae

Inverting $\mathfrak{X}$ in the equation of $X_0(\cdot)$ and $z_0(\cdot)$, we have

$$Y^{(0,y_0)}(t) = \mu(t)\mathfrak{J}(t),$$

where

$$\mathfrak{J}(t) \triangleq \mathfrak{Y}\Big(t, X_0(t) - w(t)z_0(t)\Big).$$

Then the optimal consumption process $c_0(\cdot)$ is expressed by

$$c_0(t) = z_0(t) + I\Big(t, \mu(t)\mathfrak{J}(t)\Big)$$

Theorem 5a. The optimal consumption policy $c_0(\cdot)$ admits the **stochastic feedback form** of

$$c_0(t) = \mathfrak{C}(t, X_0(t), z_0(t)), \quad 0 \le t < T,$$

determined by the random field

$$\mathfrak{C}(t,x,z) \triangleq z + I\Big(t, \mu(t)\mathfrak{Y}(t, x - w(t)z)\Big), \quad (x,z) \in \mathcal{D}_t.$$

Employing the GIKW's rule to the equation of $X_0(\cdot)$ and $z_0(\cdot)$, and using the semimartingale decomposition of Proposition 3, we obtain the integral equation

$$\frac{X_0(t)}{B(t)} + \int_0^t \frac{c_0(s)}{B(s)} ds$$
$$= x - \int_0^t \frac{1}{B(s)} \left[\vartheta(s) \frac{\mathfrak{J}(s)}{\mathfrak{J}_x(s)} - \Psi^{\mathfrak{X}}(s, \mathfrak{J}(s)) \right]^* dW_0(s),$$

where $\mathfrak{J}_x(t) \triangleq \mathfrak{Y}_x\Big(t, X_0(t) - w(t) z_0(t)\Big)$.

A comparison of the later with the wealth equation implies that

$$X_0(t) \pi_0^*(t) \sigma(t) = -\left[\vartheta(t) \frac{\mathfrak{J}(t)}{\mathfrak{J}_x(t)} - \Psi^{\mathfrak{X}}(t, \mathfrak{J}(t)) \right]^*.$$

Theorem 5b. The optimal portfolio strategy $\pi_0(\cdot)$ admits the **stochastic feedback form** of

$$\pi_0(t) = \mathfrak{P}(t, X_0(t), z_0(t)), \quad 0 \le t < T$$

determined by

$$\mathfrak{P}(t, x, z) \triangleq -\frac{1}{x} (\sigma^*(t))^{-1} \left[\vartheta(t) \frac{\mathfrak{Y}\Big(t, x - w(t) z\Big)}{\mathfrak{Y}_x\Big(t, x - w(t) z\Big)} \right.$$
$$\left. - \Psi^{\mathfrak{X}}\Big(t, \mathfrak{Y}\Big(t, x - w(t) z\Big)\Big) \right], \quad (x, z) \in \mathcal{D}_t.$$

Dynamic Programming

Generalized time horizon: $[t,T]$.

Value random field:

$$\mathbf{V}(t,x,z) \triangleq \operatorname*{ess\,sup}_{(\pi,c)} E_t\left[\int_t^T u\left(s, c(s)-z(s)\right) ds\right],$$

Extension: $\mathbf{V}(0,\cdot,\cdot) = V(\cdot,\cdot)$

By analogy with the previous analysis, we get

$$\mathbf{V}(t,x,z) = \mathfrak{G}\Big(t, \mathfrak{Y}(t, x-w(t)z)\Big), \ (x,z)\in\mathcal{D}_t, \quad t\in[0,T),$$

where

$$\mathfrak{G}(t,y) \triangleq E_t\left[\int_t^T u\Big(s, I(s, yY^{(t,1)}(s))\Big)ds\right]$$

Boundary conditions:

$$\mathbf{V}(T,x,z) = 0,$$

$$\lim_{(x,z)\to(\chi,\zeta)} \mathbf{V}(t,x,z) = \int_t^T u(s,0^+)ds, \quad \forall\ (\chi,\zeta)\in\partial\mathcal{D}_t.$$

Likewise with $\mathfrak{X}$,

- $\exists$ random field $\Phi^{\mathfrak{G}}$ such that
- $(\mathfrak{G},\Phi^{\mathfrak{G}})\in \mathbf{G}_{\mathbb{F}}$ is the $\mathbb{F}$-**adapted unique solution** of another **linear** parabolic BSPDE

Stochastic Hamilton-Jacobi-Bellman(HJB) Equation

Theorem 6.

The pair of random fields $(\mathbf{V}, \Xi)$, where

$$\Xi(t,x,z) \triangleq \Phi^{\mathfrak{G}}\Big(t, \mathfrak{Y}(t, x - w(t)z)\Big) - \mathfrak{Y}(t, x - w(t)z)\Psi^{\mathfrak{X}}\Big(t, \mathfrak{Y}(t, x - w(t)z)\Big),$$

belongs to $\mathbf{G}_{\mathbb{F}}$ and is an $\mathbb{F}$-**adapted classical solution** of the **stochastic HJB dynamic programming** PDE (cf. Peng (1992))

$$\begin{aligned}
-d\mathbf{V}(t,x,z) = \operatorname*{ess\,sup}_{\substack{0\le c<\infty \\ \pi\in\mathbb{R}^d}} \Bigg\{ & \frac{1}{2}\|\sigma^*(t)\pi\|^2 x^2 \mathbf{V}_{xx}(t,x,z) \\
& + \Big[r(t)x - c + \pi^*\sigma(t)\vartheta(t)x\Big]\mathbf{V}_x(t,x,z) \\
& + \Big[\delta(t)c - \alpha(t)z\Big]\mathbf{V}_z(t,x,z) \\
& + \pi^*\sigma(t)x\Xi_x(t,x,z) + u(t, c - z)\Bigg\} dt \\
& - \Xi(t,x,z)dW(t), \quad t \in [0,T), \ (x,z) \in \mathcal{D}_t
\end{aligned}$$

with the boundary conditions of the previous slide.

Remark 7.

The **optimal values** for the above maximization are provided by $(\mathfrak{P}(t,x,z), \mathfrak{C}(t,x,z))$.

Carrying out the maximization, the stochastic HJB takes the

Conventional form:

$$-d\mathbf{V}_t(t,x,z) = \mathcal{H}\Big(\mathbf{V}_{xx}, \mathbf{V}_x, \mathbf{V}_z, \Xi_x, t, x, z\Big)dt \\ - \Xi(t,x,z)dW(t),$$

where

$$\begin{aligned}\mathcal{H}(\mathsf{A}, p, q, \mathsf{B}, t, x, z) \triangleq & -\frac{1}{2\mathsf{A}}\|\vartheta(t)p + \mathsf{B}\|^2 \\ & + \Big[r(t)x - z - I(t, p - \delta(t)q)\Big]p \\ & + \Big[(\delta(t) - \alpha(t))z + \delta(t)I(t, p - \delta(t)q)\Big]q \\ & + u\Big(t, I(t, p - \delta(t)q)\Big)\end{aligned}$$

for $\mathsf{A} < 0$, $p > 0$, $q < 0$ and $\mathsf{B} \in \mathbb{R}$.

Remark 8.

We have achieved a **concrete solution** of the **strongly nonlinear** stochastic HJB equation by solving instead the two **linear** equations of $\mathfrak{X}$ and $\mathfrak{G}$.

Uniqueness: Necessary and sufficient condition for

Convex dual of $\mathbf{V}$:

$$\widetilde{\mathbf{V}}(t,y) \triangleq \operatorname*{ess\,sup}_{(x,z)\in\mathcal{D}_t} \Big\{\mathbf{V}(t,x,z) - \Big(x - w(t)z\Big)y\Big\}, \quad y \in \mathbb{R}^+.$$

Identity: $\widetilde{\mathbf{V}}(t,y) = \mathfrak{G}(t,y) - y\mathfrak{X}(t,y),$

Theorem 9.

The pair of random fields $(\widetilde{\mathbf{V}}, \Lambda)$, where

$$\Lambda(t,y) \triangleq \Phi^{\mathfrak{G}}(t,y) - y\Psi^{\mathfrak{X}}(t,y),$$

belongs to $\mathbf{G}_{\mathbb{F}}$ and is the **unique $\mathbb{F}$-adapted solution** of the **linear** BSPDE

$$-d\widetilde{\mathbf{V}}(t,y) = \Big[\frac{1}{2}\|\vartheta(t)\|^2 y^2 \widetilde{\mathbf{V}}_{yy}(t,y) - r(t)y\widetilde{\mathbf{V}}_y(t,y) \\ - \vartheta^*(t)y\Lambda_y(t,y) + \tilde{u}(t, y\mu(t))\Big]dt \\ - \Lambda^*(t,y)dW(t) \quad \text{on } [0,T) \times \mathbb{R}^+, \\ \widetilde{\mathbf{V}}(T,y) = 0 \quad \text{on } \mathbb{R}^+.$$

Remark 10.

To compute $\mathbf{V}$:

- resolve the above equation for $(\widetilde{\mathbf{V}}, \Lambda)$
- invert the dual transformation to recover $\mathbf{V}$ as

$$\mathbf{V}(t,x,z) = \operatorname*{ess\,inf}_{y\in\mathbb{R}} \Big\{\widetilde{\mathbf{V}}(t,y) + \big(x - w(t)z\big)y\Big\}, \quad (x,z) \in \mathcal{D}_t.$$

Deterministic Coefficients

Case of deterministic model coefficients:

- asset prices become Markov processes
- the **feedback formulae** for (π_0, c_0) reduce to the **deterministic functions**

$$C(t,x,z) \triangleq z + I\Big(t, \mu(t)\mathcal{Y}(t, x - w(t)z)\Big),$$

$$\Pi(t,x,z) \triangleq -(\sigma^*(t))^{-1}\vartheta(t) \cdot \frac{\mathcal{Y}\Big(t, x - w(t)z\Big)}{x\mathcal{Y}_x\Big(t, x - w(t)z\Big)},$$

where

$\mathcal{Y}(t,\cdot)$ is the **inverse** of the function

$$\mathcal{X}(t,y) \triangleq E^0\left[\int_t^T e^{-\int_t^s r(v)dv}\mu(s)I(s, yY^{(t,1)}(s))ds\right].$$

Corollary 11.

An investor

- needs only his *current* level of $X_0(t)$ and $z_0(t)$,
- doesn't need entire history of the market up to t.

Therefore $\Big(X_0(t), z_0(t)\Big)$ is a **sufficient statistic** for the optimization problem.

An Example: Logarithmic Utility

Let $\quad u(t,x) = \log x.$

We have $\quad I(t,y) = 1/y \quad$ and $\quad \tilde{u}(t,y) = -\log y - 1\,.$

Case 1: <u>Deterministic Coefficients</u>.

Unique solution of the Cauchy problem for $\widetilde{V}$:

$$\tilde{v}(t,y) \triangleq -\nu(t)\log\Big(y\mu(t)\Big) - m(t), \qquad \text{where}$$

$$\nu(t) = T - t$$

$$m(t) = \int_t^T \left[1 - (T-s)\left(\frac{1}{2}\|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)}\right)\right] ds.$$

Therefore inverting the dual transformation:

$$\mathcal{X}(t,y) = \frac{\nu(t)}{y}\,, \qquad \mathcal{Y}(t,x) = \frac{\nu(t)}{x},$$

$$V(t,x) = \nu(t)\log\left(\frac{x - w(t)z}{\nu(t)\mu(t)}\right) + \nu(t) - m(t),$$

and the feedback formulae are

$$C(t,x) = z + \frac{x - w(t)z}{\nu(t)\mu(t)},$$

$$\Pi(t,x) = \left(\sigma^*(t)\right)^{-1}\vartheta(t)\frac{x - w(t)z}{x}.$$

Case 2: Stochastic Coefficients.

By analogy with the deterministic case, we introduce in this case the $\mathbb{F}$-adapted random field

$$\widetilde{\mathfrak{v}}(t,y) \triangleq -\nu(t) \log\Big(y\mu(t)\Big) - \mathfrak{m}(t)$$

with

$$\nu(t) = T - t, \qquad \mathfrak{m}(t) = E_t\left[\int_t^T m(s)ds\right].$$

Moreover, the *completeness* of the market stipulates the existence of an $\mathbb{R}^d$-valued, $\mathbb{F}$-progressively measurable, square-integrable process $\ell(\cdot)$, such that the Brownian martingale (cf. Karatzas & Shreve (1991))

$$\mathfrak{M}(t) = E_t\left[\int_0^T m(s)ds\right]$$

has the representation $\mathfrak{M}(t) = \mathfrak{M}(0) + \int_0^t \ell^*(s)\, dW(s)$.

It is verified directly that the pair $(\widetilde{\mathfrak{v}}, \ell)$ solves the BSPDE for $(\widetilde{\mathbf{V}}, \Lambda)$. Consequently,

$$\mathfrak{X}(t,y) = \frac{\nu(t)}{y}, \qquad \mathfrak{Y}(t,x) = \frac{\nu(t)}{x}$$

and

$$\mathsf{V}(t,x) \;=\; \nu(t) \log\left(\frac{x - w(t)z}{\nu(t)\mu(t)}\right) + \nu(t) - \mathfrak{m}(t).$$

For this particular choice of utility preference, $\mathfrak{X}$ (and so $\mathfrak{Y}$) is deterministic, and the feedback formulas are the same as those of the previous case.

Open Problems

(1) Incomplete Markets:

- Number of stocks smaller than the dimension of $W(\cdot)$
- Compute the **optimal policies**

(2) Exponential utility function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$

- $c(t) > z(t; c)$ is removed (non-addictive habits)
- Optimal policies exist by Detemple & Karatzas (2003)
- Develop the **Dynamic Programming Theory**

(3) Generalized utility function $u(t, c(t), z(t))$

- Compute the **optimal policies**
- Develop the **Dynamic Programming Theory**

Basic References

- **Detemple, J. B. and Zapatero, F.** (1992).
  Optimal consumption-portfolio policies with habit formation.
  Econometrica **59**, 251–274.

- **Detemple, J. B. and Karatzas, I.** (2003).
  Non addictive habits: optimal portfolio-consumption policies.
  J. of Economic Theory **113**, 265–285.

- **Karatzas, I. and Shreve, S. E.** (1991).
  Brownian Motion and Stochastic Calculus.
  Second Edition, Springer-Verlag, New York.

- **Ma, J. and Yong, J.** (1997).
  Adapted solution of a degenerate backward SPDE, with applications.
  Stochastic Prosses. Appl. **70**, 59–84.

- **Peng, S.** (1992).
  Stochastic Hamilton-Jacobi-Bellman equations.
  SIAM J. Control Optim. **30**, 284–304.