

Vacuum behaviors around rarefaction waves to 1D compressible Navier-Stokes equations with density-dependent viscosity

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Abstract: In this paper, we study the large time asymptotic behavior toward rarefaction waves for solutions to the 1-dimensional compressible Navier-Stokes equations with density-dependent viscosities for general initial data whose far fields are connected by a rarefaction wave to the corresponding Euler equations with one end state being vacuum. First, a global-in-time weak solution around the rarefaction wave is constructed by approximating the system and regularizing the initial data with general perturbations, and some a priori uniform-in-time estimates for the energy and entropy are obtained. Then it is shown that the density of any weak solution satisfying the natural energy and entropy estimates will converge to the rarefaction wave connected to vacuum with arbitrary strength in super-norm time-asymptotically. Our results imply, in particular, that the initial vacuum at far fields will remain for all the time which are in contrast to the case of non-vacuum rarefaction waves studied in [18] where all the possible vacuum states will vanish in finite time. Finally, it is proved that the weak solution becomes regular away from the vacuum region of the rarefaction wave.

Key Words: compressible Navier-Stokes equations, density-dependent viscosity, rarefaction wave, vacuum, weak solution, stability

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1 Introduction

In this paper, we consider the following compressible and isentropic Navier-Stokes equations with density-dependent viscosities

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = (\mu(\rho)u_x)_x, \end{cases} \quad x \in \mathbf{R}, \quad t > 0, \quad (1.1)$$

where $\rho(t, x) \geq 0$, $u(t, x)$ represent the density and the velocity of the gas, respectively. Let the pressure and viscosity function be given by

$$p(\rho) = A\rho^\gamma, \quad \mu(\rho) = B\rho^\alpha, \quad (1.2)$$

respectively, where $\gamma > 1$ denotes the adiabatic exponent, $\alpha > 0$ and $A, B > 0$ are the gas constants. For simplicity, it is assumed that $A = B = 1$.

Consider the Cauchy problem (1.1) with the initial values

$$(\rho, \rho u)(0, x) = (\rho_0, m_0)(x) \rightarrow (\rho_\pm, m_\pm), \quad \text{as } x \rightarrow \pm\infty, \quad (1.3)$$

where $\rho_\pm \geq 0$, m_\pm are prescribed constants.

The large time behavior of solutions to (1.1)-(1.3) is expected to be closely related to that of the Riemann problem of the corresponding Euler system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \end{cases} \quad (1.4)$$

with Riemann initial data

$$(\rho, \rho u)(0, x) := (\rho_0^r, m_0^r)(x) = \begin{cases} (\rho_-, m_-), & x < 0, \\ (\rho_+, m_+), & x > 0. \end{cases} \quad (1.5)$$

Different initial states (1.5) produce different type of waves, namely, shock waves and rarefaction waves for the one-dimensional compressible isentropic Euler equations (1.4). However, as pointed out by Liu-Smoller [25], among the two nonlinear waves, i.e., shocks and rarefaction waves, only rarefaction waves can be connected to vacuum. When vacuum appears, the stability of rarefaction waves to the 1D compressible Navier-Stokes equations is an important issue.

When the viscosity $\mu(\rho)$ is a constant, there have been extensive studies on the stability of the rarefaction waves to the 1D compressible Navier-Stokes equations under the assumptions that the rarefaction waves and the solutions are away from the vacuum (see [8], [17], [21], [24], [26], [29], [30] and the references therein). However, when vacuum appears, the well-known results by Hoff-Serre [13], Xin [37] and Rozanova [34] show that the solutions of the compressible Navier-Stokes equations with constant viscosity may behave singularly, in particular, in the case that the fluids jump to far field vacuum. Liu, Xin and Yang first proposed in [27] some models of the compressible Navier-Stokes equations with density-dependent viscosities to investigate the dynamics of the vacuum. On the other hand, when deriving by Chapman-Enskog expansions from the Boltzmann equation, the viscosity of the compressible Navier-Stokes equations depends on the temperature and thus on the density for isentropic flows. Also, the viscous Saint-Venant system for the shallow water, derived from the incompressible Navier-Stokes equations with

a moving free surface, is expressed exactly as in (1.1)-(1.2) with $\alpha = 1$ and $\gamma = 2$ (see [9]). However, there appear new mathematical challenges in dealing with such systems. In particular, these systems become highly degenerate. The velocity cannot even be defined in the presence of vacuum and hence it is difficult to get uniform estimates for the velocity near vacuum. The global existence of general weak solutions to the compressible Navier-Stokes equations with density-dependent viscosities or the viscous Saint-Venant system for the shallow water model in the multi-dimensional case remains open, and one can refer to [4], [5], [10], [31] for recent developments along this line.

There are a large number of literatures on mathematical studies of (1.1)-(1.2) with various initial and boundary conditions. If the initial density is assumed to be connected to vacuum with discontinuities, Liu, Xin and Yang first obtained in [27] the local well-posedness of weak solutions. The global well-posedness was obtained later by [16], [17], [33], [38] respectively. The case of initial densities connecting to vacuum continuously was studied by [7], [36], [38] and [39] respectively. However, most of these results concern with free boundary problems. Recently, initial-boundary-value problems for the one-dimensional equations (1.1)-(1.2) with $\mu(\rho) = \rho^\alpha$ ($\alpha > 1/2$) was studied by Li, Li and Xin in [22] and the phenomena of vacuum vanishing and blow-up of solutions were found there. The global existence of weak solutions for the initial-boundary-value problems for spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity was proved by Guo, Jiu and Xin in [10]. More recently, there are some results on global existence of weak solutions to the Cauchy problem (1.1)-(1.3). The existence and uniqueness of global strong solutions to the compressible Navier-Stokes equations (1.1)-(1.3) were obtained by Mellet and Vasseur [31] where no vacuum is permitted in the initial density and for $0 \leq \alpha < \frac{1}{2}$. However, the a priori estimates obtained in [31] depend on the time interval thus do not give the time-asymptotic behavior of the solutions. The first result about the time-asymptotic behavior of the solutions to the Cauchy problem (1.1)-(1.3) is obtained by Jiu-Xin [19], where the global existence and large time-asymptotic behavior of the weak solutions were considered in the case that $\rho_+ = \rho_- \geq 0$ and $u_+ = u_- = 0$. In the case that $\rho_+ = \rho_- > 0$ and $u_+ = u_- = 0$, the vanishing of the vacuum and the blow-up phenomena of the weak solutions were also obtained in [19]. One of the key elements in the analysis in [19] is an interesting entropy estimate which was observed first in [20] for the one-dimensional case and later established in [1, 2, 3] for more general and multi-dimensional cases due to the structure that the viscosity coefficients vanish at the vacuum. This entropy estimate provides higher regularity of the density and played a crucial role in [10, 19, 22] for global existence and large time asymptotic behaviors of weak solutions.

The stability of rarefaction waves of the 1D compressible Navier-Stokes equations with density-dependent viscosity was studied in [18] under general initial perturbations such that the initial data and the solutions may contain the vacuum. However, in [18], the rarefaction wave itself is away from the vacuum. In this paper, we are concerned with the case when the rarefaction wave is permitted to be connected to vacuum.

For definiteness, we consider the case of a 2-rarefaction wave such that $\rho_- = 0, \rho_+ > 0$ in (1.3). Similar to [18], we will first construct a class of approximate solutions satisfying some uniform estimates and furthermore prove the global existence of weak solutions for the Cauchy problem of (1.1)-(1.3). To get the uniform energy and entropy estimates in time to the approximate solutions, we combine the elementary energy estimates and the entropy estimates in an elaborate way. Note that the elementary energy estimates and the entropy estimates are coupled to each other due to the underlying rarefaction wave. This is quite different from the previous works on the global existence and the time-asymptotic behavior of the solutions

to Navier-Stokes equations (1.1) with density-dependent viscosity where the elementary energy estimates and the entropy estimates can be derived independently. Moreover, compared with the case of non-vacuum rarefaction waves in [18], some new difficulties occur due to the degeneracies at the vacuum states in the 2-rarefaction wave. To overcome these difficulties, we first cut off the 2-rarefaction wave with vacuum along the rarefaction wave curve and then derive some uniform estimates with respect to both the approximations and the cut-off process. More precisely, for any $\nu > 0$, a suitably small parameter, the cut-off 2-rarefaction wave will connect the state $(\rho, u) = (\nu, u_\nu)$ and (ρ_+, u_+) where u_ν can be obtained explicitly and uniquely by the definition of the 2-rarefaction wave curve. For any fixed $\nu > 0$, one can obtain a weak solution to the compressible Navier-Stokes equations (1.1)-(1.3) with (ρ_-, m_-) replaced by $(\nu, \nu u_\nu)$ along the same line as in our previous paper [18]. Thus, in order to get the solution to the original problem (1.1)-(1.3), we will derive some uniform estimates with respect to both the approximations and the cut-off process. To this end, the approximation parameters ε and the cut-off parameter ν should be chosen in an appropriate way. Thus, as a limit of this approximate solution, a global weak solution to (1.1)-(1.3) is shown to exist with the uniform-in-time estimates (2.18) and (2.19).

Next, we study the large-time asymptotic behavior of any weak solutions to (1.1)-(1.3) under the uniform-in-time bounds (2.18) and (2.19). It is shown that time-asymptotically, the density function tends to the rarefaction wave connected to the vacuum in L^∞ norm. This time-asymptotic behavior of the density function implies that the vacuum in the far field is essential and will maintain for all the time. This is quite different from the previous results in [18] and [22] where all the possible vacuum states will vanish in finite time. At last, we prove that such a weak solution becomes regular away from the vacuum region of the rarefaction wave by using the Di Giorgi-Moser iteration and higher order energy estimates.

Notations. Throughout this paper, positive generic constants are denoted by c and C , which are independent of ε , ν and T , without confusion, and $C(\cdot)$ stands for some generic constant(s) depending only on the quantity listed in the parenthesis. For function spaces, $L^p(\Omega)$, $1 \leq p \leq \infty$, denote the usual Lebesgue spaces on $\Omega \subset \mathbf{R} := (-\infty, \infty)$. $W^{k,p}(\Omega)$ denotes the k^{th} order Sobolev space, $H^k(\Omega) := W^{k,2}(\Omega)$.

2 Preliminaries and Main Results

In this section we first describe the rarefaction wave connected to the vacuum to the compressible Euler system (1.4). Then an approximate rarefaction wave will be constructed through the Burger's equation and the main results of the paper will be given at last.

2.1 Rarefaction waves

The Euler system (1.4) is a strictly hyperbolic one for $\rho > 0$ whose characteristic fields are both genuinely nonlinear, that is, in the equivalent system

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ p'(\rho)/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0,$$

the Jacobi matrix

$$\begin{pmatrix} u & \rho \\ p'(\rho)/\rho & u \end{pmatrix}$$

has two distinct eigenvalues

$$\lambda_1(\rho, u) = u - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u) = u + \sqrt{p'(\rho)}$$

with corresponding right eigenvectors

$$r_i(\rho, u) = (1, (-1)^i \frac{\sqrt{p'(\rho)}}{\rho})^t, \quad i = 1, 2,$$

such that

$$r_i(\rho, u) \cdot \nabla_{(\rho, u)} \lambda_i(\rho, u) = (-1)^i \frac{\rho p''(\rho) + 2p'(\rho)}{2\rho \sqrt{p'(\rho)}} \neq 0, \quad i = 1, 2, \quad \text{if } \rho > 0.$$

Define the i -Riemann invariant ($i = 1, 2$) by

$$\Sigma_i(\rho, u) = u + (-1)^{i+1} \int^\rho \frac{\sqrt{p'(s)}}{s} ds,$$

such that

$$\nabla_{(\rho, u)} \Sigma_i(\rho, u) \cdot r_i(\rho, u) \equiv 0, \quad \forall \rho > 0, u.$$

There are two families of rarefaction waves to the Euler system (1.4)-(1.5). Here we consider the case of 2-rarefaction wave connected with vacuum, that is $\rho_- = m_- = 0, \rho_+ > 0$. Thus we can define the velocity at the positive far field $u_+ = \frac{m_+}{\rho_+}$. First we give the description of the 2-rarefaction wave connected with vacuum, see also in details in [25]. From the fact that 2-Riemann invariant is constant:

$$\Sigma_2(\rho_- = 0, u_-) = \Sigma_2(\rho_+, u_+),$$

we can define the velocity u_- which is the speed of the gas coming into the vacuum region. Then the entropy condition $\lambda_2(\rho_- = 0, u_-) < \lambda_2(\rho_+, u_+)$ is always satisfied. This 2-rarefaction wave connecting the vacuum $\rho_- = 0$ to (ρ_+, u_+) is the self-similar solution $(\rho^r, u^r)(\xi)$, ($\xi = \frac{x}{t}$) of (1.4)-(1.5) defined by

$$\lambda_2(\rho^r(\xi), u^r(\xi)) = \begin{cases} \rho^r(\xi) = 0, & \text{if } \xi < \lambda_2(0, u_-) = u_-, \\ \xi, & \text{if } u_- \leq \xi \leq \lambda_2(\rho_+, u_+), \\ \lambda_2(\rho_+, u_+), & \text{if } \xi > \lambda_2(\rho_+, u_+), \end{cases} \quad (2.1)$$

and

$$\Sigma_2(\rho^r(\xi), u^r(\xi)) = \Sigma_2(\rho_+, u_+) = \Sigma_2(0, u_-). \quad (2.2)$$

Thus we can define the momentum of 2-rarefaction wave by

$$m^r(\xi) = \begin{cases} \rho^r(\xi) u^r(\xi), & \text{if } \rho^r(\xi) > 0, \\ 0, & \text{if } \rho^r(\xi) = 0. \end{cases} \quad (2.3)$$

In this paper, we consider the time-asymptotic behavior toward such rarefaction waves of solutions to the compressible Navier-Stokes equations (1.1) with density-dependent viscosities.

2.2 Approximate rarefaction waves

Consider the Riemann problem for the inviscid Burgers equation:

$$\begin{cases} w_t + ww_x = 0, & t > 0, x \in \mathbf{R}, \\ w(x, 0) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases} \quad (2.4)$$

If $w_- < w_+$, then the Riemann problem (2.4) admits a rarefaction wave solution $w^r(x, t) = w^r(\frac{x}{t})$ given by

$$w^r\left(\frac{x}{t}\right) = \begin{cases} w_-, & \frac{x}{t} \leq w_-, \\ \frac{x}{t}, & w_- \leq \frac{x}{t} \leq w_+, \\ w_+, & \frac{x}{t} \geq w_+. \end{cases} \quad (2.5)$$

Consider the solution to the following Cauchy problem for Burgers equation

$$\begin{cases} w_t + ww_x = 0, & t > 0, x \in \mathbf{R}, \\ w(0, x) := w_0(x) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} K_q \int_0^{\eta x} (1 + y^2)^{-q} dy. \end{cases} \quad (2.6)$$

Here $q \geq 2$ is some fixed constant, and K_q is a constant such that $K_q \int_0^\infty (1 + y^2)^{-q} dy = 1$, and η is a small positive constant to be determined later. It is easy to see that the solution to this problem is given by

$$w(t, x) = w_0(x_0(t, x)), \quad x = x_0(t, x) + w_0(x_0(t, x))t. \quad (2.7)$$

Then the following properties hold (see [30]).

Lemma 2.1 *Let $w_- < w_+$, the Cauchy problem (2.6) has a unique smooth solution $w(t, x)$ satisfying*

- i) $w_- < w(t, x) < w_+$, $w_x(t, x) > 0$;
- ii) For any p ($1 \leq p \leq \infty$), there exists a constant C_{pq} such that

$$\begin{aligned} \|w(t, \cdot) - w^r(\frac{\cdot}{t})\|_{L^p(\mathbf{R})} &\leq C_p \delta_r \eta^{-\frac{1}{p}}, \\ \|w_x(t)\|_{L^p(\mathbf{R})} &\leq C_{pq} \min \left\{ \delta_r \eta^{1-\frac{1}{p}}, \delta_r^{\frac{1}{p}} t^{-1+\frac{1}{p}} \right\}, \\ \|w_{xx}(t)\|_{L^p(\mathbf{R})} &\leq C_{pq} \min \left\{ \delta_r \eta^{2-\frac{1}{p}}, \eta^{(1-\frac{1}{2q})(1-\frac{1}{p})} \delta_r^{-\frac{(p-1)}{2pq}} t^{-1-\frac{(p-1)}{2pq}} \right\}, \end{aligned}$$

where $\delta_r = w_+ - w_-$, and C_p, C_{pq} are independent of t ;

- iii) $\sup_{x \in \mathbf{R}} |w(t, x) - w^r(\frac{x}{t})| \rightarrow 0$, as $t \rightarrow \infty$.

We now turn to rarefaction waves to the Euler system (1.4)-(1.5). Set $\lambda_2(\rho_\pm, u_\pm) = w_\pm$ with $\rho_- = 0$ in (2.4). Then the unique solution $(\rho^r, u^r)(\xi)$ in (2.1)-(2.3) to the Riemann problem (1.4)-(1.5) can also be expressed in terms of $w^r(\frac{x}{t})$ in (2.5), by

$$\begin{aligned} \lambda_2(\rho^r(\frac{x}{t}), u^r(\frac{x}{t})) &= w^r(\frac{x}{t}), \\ \Sigma_2(\rho^r(\frac{x}{t}), u^r(\frac{x}{t})) &= \Sigma_2(\rho_\pm, u_\pm). \end{aligned} \quad (2.8)$$

Correspondingly, an approximate 2–rarefaction wave $(\bar{\rho}, \bar{u})(t, x)$ can be defined by

$$\begin{aligned}\lambda_2(\bar{\rho}(t, x), \bar{u}(t, x)) &= w(1 + t, x), \\ \Sigma_2(\bar{\rho}(t, x), \bar{u}(t, x)) &= \Sigma_2(\rho_{\pm}, u_{\pm}).\end{aligned}\tag{2.9}$$

It can be checked that $(\bar{\rho}, \bar{u})(t, x)$ also satisfies the Euler system

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0 \\ (\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + p(\bar{\rho}))_x = 0, \end{cases}\tag{2.10}$$

and properties listed in the following Lemma.

Lemma 2.2 *The approximate 2–rarefaction wave $(\bar{\rho}, \bar{u})(t, x)$ defined in (2.9) satisfies*

$$(1) \quad \bar{\rho}_x > 0, \quad \bar{u}_x > 0, \quad \bar{u}_x = \sqrt{\gamma} \bar{\rho}^{\frac{\gamma-3}{2}} \bar{\rho}_x;$$

(2) *For any p ($1 \leq p \leq \infty$), there exists a constant C_{pq} such that*

$$\begin{aligned}\|(\bar{\rho}, \bar{u})(t, \cdot) - (\rho^r, u^r)\left(\frac{\cdot}{t}\right)\|_{L^p(\mathbf{R})} &\leq C_p(w_+ - w_-)\eta^{-\frac{1}{p}}, \\ \|\bar{u}_x(t, \cdot)\|_{L^p(\mathbf{R})} &\leq C_{pq} \min\{\delta\eta^{1-\frac{1}{p}}, \delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\}, \\ \|\bar{u}_{xx}(t, \cdot)\|_{L^p(\mathbf{R})} &\leq C_{pq} \min\{\delta\eta^{2-\frac{1}{p}}, \eta^{(1-\frac{1}{2q})(1-\frac{1}{p})}\delta^{-\frac{p-1}{2pq}}(1+t)^{-1-\frac{p-1}{2pq}} + \delta^{\frac{1}{p}}(1+t)^{-2+\frac{1}{p}}\},\end{aligned}$$

where $\delta = |\rho_+ - \rho_-| + |u_+ - u_-|$ is the strength of the rarefaction wave;

$$(3) \quad \lim_{t \rightarrow \infty} \sup_{x \in \mathbf{R}} |\bar{\rho}(t, x) - \rho^r\left(\frac{x}{t}\right)| = 0.$$

Remark 2.1 *For any $1 < p \leq +\infty$,*

$$\int_0^T \|\bar{u}_{xx}(t, \cdot)\|_{L^p(\mathbf{R})} dt \leq C,$$

where C is independent of T . Note that in the case $p = 1$, the constant C in the above estimates is not uniform in T . Moreover, the following estimate holds:

$$\int_0^T \|\bar{u}_{xx}(t, \cdot)\|_{L^\infty(\mathbf{R})} dt \leq C\eta^{\frac{2}{4q+1}} \int_0^T (1+t)^{-1-\frac{1}{4q+1}} dt \leq C\eta^{\frac{2}{4q+1}},$$

where C is independent of T .

2.3 Main Results

Set

$$\begin{aligned}\Psi(\rho, \bar{\rho}) &= \int_{\bar{\rho}}^{\rho} \frac{p(s) - p(\bar{\rho})}{s^2} ds \\ &= \frac{1}{(\gamma-1)\rho} \left[\rho^\gamma - \bar{\rho}^\gamma - \gamma\bar{\rho}^{\gamma-1}(\rho - \bar{\rho}) \right].\end{aligned}\tag{2.11}$$

The initial values are assumed to satisfy:

$$\left\{ \begin{array}{l} \rho_0 \geq 0; \quad m_0 = 0 \quad a.e. \text{ on } \{x \in \mathbf{R} | \rho_0(x) = 0\}; \\ (\rho_0^{\alpha-\frac{1}{2}})_x \in L^2(\mathbf{R}), \quad \rho_0 \Psi(\rho_0, \rho_{\pm}) \in L^1(\mathbf{R}^{\pm}) \quad \text{with } \rho_- = 0; \\ \rho_0 \left(\frac{m_0}{\rho_0} - \frac{m_{\pm}}{\rho_{\pm}} \right)^2 \in L^1(\mathbf{R}^{\pm}), \quad \rho_0 \left(\frac{m_0}{\rho_0} - \frac{m_{\pm}}{\rho_{\pm}} \right)^3 \in L^1(\mathbf{R}^{\pm}), \end{array} \right. \quad (2.12)$$

where $u_- := \frac{m_-}{\rho_-}$ since $\rho_- = m_- = 0$. Note that (2.12) implies that $\rho_0 \in C_B(\mathbf{R})$ which is the space of bounded and continuous functions.

Equivalently, the assumptions (2.12) can be rewritten as

$$\left\{ \begin{array}{l} \rho_0 \geq 0; \quad m_0 = 0 \quad a.e. \text{ on } \{x \in \mathbf{R} | \rho_0(x) = 0\}; \\ (\rho_0^{\alpha-\frac{1}{2}})_x \in L^2(\mathbf{R}), \quad \rho_0 \Psi(\rho_0, \bar{\rho}_0) \in L^1(\mathbf{R}); \\ \rho_0 \left(\frac{m_0}{\rho_0} - \bar{u}_0 \right)^2 \in L^1(\mathbf{R}), \quad \rho_0 \left(\frac{m_0}{\rho_0} - \bar{u}_0 \right)^3 \in L^1(\mathbf{R}), \end{array} \right. \quad (2.13)$$

where $(\bar{\rho}_0, \bar{u}_0) := (\bar{\rho}, \bar{u})(0, x)$ is the initial values of the approximate 2-rarefaction wave $(\bar{\rho}, \bar{u})(t, x)$ constructed in (2.9).

Before stating the main results, we give the definition of weak solutions to (1.1)-(1.3) with the far fields (ρ_{\pm}, m_{\pm}) . Let $T > 0$ be given. For any far fields (ρ_{\pm}, u_{\pm}) satisfying $\rho_{\pm} \geq 0$ and any smooth functions $(\tilde{\rho}, \tilde{u})(t, x)$ connecting with them, we define

Definition 2.1 *A pair (ρ, u) is said to be a weak solution to (1.1)-(1.3) with the far fields (ρ_{\pm}, u_{\pm}) provided that there exists a smooth functions $(\tilde{\rho}, \tilde{u})(t, x)$ with the same far fields (ρ_{\pm}, u_{\pm}) and $\tilde{\rho} \geq 0$, such that*

(1) $\rho \geq 0$ a.e., and

$$\begin{aligned} \rho &\in L^\infty(0, T; L^\infty(\mathbf{R})) \cap C([0, T] \times \mathbf{R}), \quad \rho \Psi(\rho, \tilde{\rho}) \in L^\infty(0, T; L^1(\mathbf{R})), \\ (\rho^{\alpha-\frac{1}{2}})_x &\in L^\infty(0, T; L^2(\mathbf{R})), \quad \sqrt{\rho}(u - \tilde{u}) \in L^\infty(0, T; L^2(\mathbf{R})); \end{aligned}$$

(2) For any $t_2 \geq t_1 \geq 0$ and any $\zeta \in C_0^1(\mathbf{R} \times [t_1, t_2])$, the mass equation (1.1) holds in the following sense:

$$\int_{\mathbf{R}} \rho \zeta dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbf{R}} (\rho \zeta_t + \sqrt{\rho} \sqrt{\rho u} \zeta_x) dx dt; \quad (2.14)$$

(3) For any $\psi \in C_0^\infty(\mathbf{R} \times (0, T))$, it holds that

$$\int_0^T \int_{\mathbf{R}} \left\{ \sqrt{\rho} \sqrt{\rho u} \psi_t + [(\sqrt{\rho u})^2 + \rho^\gamma] \psi_x \right\} dx dt + \langle \rho^\alpha u_x, \psi_x \rangle = 0, \quad (2.15)$$

where the diffusion term makes sense if written as

$$\langle \rho^\alpha u_x, \psi_x \rangle = - \int_0^T \int_{\mathbf{R}} \rho^{\alpha-\frac{1}{2}} \sqrt{\rho u} \psi_{xx} dx dt - \frac{2\alpha}{2\alpha-1} \int_0^T \int_{\mathbf{R}} (\rho^{\alpha-\frac{1}{2}})_x \sqrt{\rho u} \psi_x dx dt. \quad (2.16)$$

The first main result in this paper reads as

Theorem 2.1 *Let α and γ satisfy that*

$$1 < \gamma \leq 2, \quad \text{and} \quad 1 \leq \alpha \leq \frac{\gamma+1}{2}, \quad (2.17)$$

and suppose that (2.13) holds. Then the Cauchy problem (1.1)-(1.3) admits a global weak solution $(\rho(x, t), u(x, t))$ in the sense of Definition 2.1 with the smooth function $(\tilde{\rho}, \tilde{u})$ replaced by the approximate rarefaction wave $(\bar{\rho}, \bar{u})$. Furthermore, this weak solution $(\rho(x, t), u(x, t))$ satisfies

$$\rho \geq 0, \quad \max_{(x,t) \in \mathbf{R} \times [0,T]} \rho \leq C, \quad \rho \in C(\mathbf{R} \times [0, T]), \quad (2.18)$$

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbf{R}} [|\sqrt{\rho}(u - \bar{u})|^2 + (\rho^{\alpha - \frac{1}{2}})_x^2 + \rho \Psi(\rho, \bar{\rho})] dx \\ & + \int_0^T \int_{\mathbf{R}} \{[(\rho^{\frac{\gamma+\alpha-1}{2}} - \bar{\rho}^{\frac{\gamma+\alpha-1}{2}})_x]^2 + \bar{u}_x \rho \Psi(\rho, \bar{\rho}) + \rho \bar{u}_x (u - \bar{u})^2 + \Lambda(x, t)^2\} dx dt \leq C, \end{aligned} \quad (2.19)$$

where C is an absolute constant depending on the initial data but independent of T and $\Lambda(x, t) \in L^2(\mathbf{R} \times (0, T))$ satisfies

$$\begin{aligned} & \int_0^T \int_{\mathbf{R}} \rho^{\frac{\alpha}{2}} \Lambda \varphi dx dt = - \int_0^T \int_{\mathbf{R}} \rho^{\alpha - \frac{1}{2}} \sqrt{\rho} (u - \bar{u}) \varphi_x dx dt \\ & - \frac{2\alpha}{2\alpha - 1} \int_0^T \int_{\mathbf{R}} (\rho^{\alpha - \frac{1}{2}})_x \sqrt{\rho} (u - \bar{u}) \varphi dx dt, \quad \forall \varphi \in C_0^\infty(\mathbf{R} \times (0, T)). \end{aligned} \quad (2.20)$$

Remark 2.2 *It should be noted that there is no requirement on the sizes of the strength of the rarefaction wave and the perturbations. The class of initial perturbations given by (2.13) is quite large compared with those for the case of constant viscosities, [26], [29], [30].*

Remark 2.3 *The important case of the shallow water model, i.e., $\alpha = 1, \gamma = 2$, is included in our theorem.*

Remark 2.4 *For any far fields (ρ_\pm, u_\pm) satisfying $\rho_\pm \geq 0$ and any smooth functions $(\tilde{\rho}, \tilde{u})(t, x)$ connecting with them, one can also obtain the existence of weak solutions in the sense of Definition 2.1 in a similar way (see [23] for the weak solutions in the case of $\rho_\pm > 0$ and $u_\pm = 0$). However, in order to get the uniform in time estimates in (2.18)-(2.19), it seems that the far fields (ρ_\pm, u_\pm) should be specified and in Theorem 2.1 the smooth function $(\tilde{\rho}, \tilde{u})(t, x)$ is replaced by the approximate rarefaction wave $(\bar{\rho}, \bar{u})$.*

The next result concerns on the asymptotic behaviors of the weak solution, which can be stated as

Theorem 2.2 *Let α and γ satisfy (2.17) and suppose that (2.12) holds. Suppose that $(\rho, u)(x, t)$ is a global weak solution of the Cauchy problem (1.1)-(1.3) in the sense of Definition 2.1 satisfying (2.18) and (2.19). Then it holds that*

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbf{R}} |\rho(t, x) - \bar{\rho}(t, x)| = 0. \quad (2.21)$$

Consequently,

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbf{R}} |\rho(x, t) - \rho^r(\frac{x}{t})| = 0. \quad (2.22)$$

Remark 2.5 *A direct consequence of Lemma 2.2 (2), (2.19) and (2.22) is the following time-asymptotic behavior of the density function:*

$$\lim_{t \rightarrow \infty} \|\rho(\cdot, t) - \rho^r(\frac{\cdot}{t})\|_{L^p(\mathbf{R})} = 0, \quad \forall 2 < p \leq +\infty. \quad (2.23)$$

Remark 2.6 *Theorem 2.2 implies that for any weak solution (ρ, u) to the Cauchy problem (1.1)-(1.3) with the far fields given by the vacuum state and (ρ_+, u_+) , if (ρ, u) satisfies the bounds (2.18) and (2.19), then the density function converges to the 2-rarefaction wave to the corresponding Euler equations connecting the vacuum state and (ρ_+, u_+) in sup-norm as t tends to infinity. Consequently, the initial vacuum at far field will remain for all the time, which is contrast to the case of non-vacuum rarefaction waves studied in [18] where all the possible vacuum states will vanish.*

Finally, we can obtain the following higher regularity to the velocity function $u(t, x)$ to a global weak solution $(\rho, u)(t, x)$ of the Cauchy problem (1.1)-(1.3) in the sense of Definition 2.1 satisfying (2.18) and (2.19) in the region away from the vacuum region of 2-rarefaction wave $(\rho^r, u^r)(\xi)$.

Theorem 2.3 *(Regularity of the solution away from the vacuum) Let (ρ, u) be a weak solution to the Cauchy problem (1.1)-(1.3) satisfying (2.18) and (2.19). For any fixed $\sigma > 0$, there exist a straight line $x = \lambda_2^\sigma t$ with $\lambda_2^\sigma = \lambda_2(\rho, u)|_{(\rho, u)=(\sigma, u_\sigma)}$ defined in (5.1) and a large time T_σ , such that if $(t, x) \in \Omega_\sigma := \{(t, x) | t > T_\sigma, x > \lambda_2^\sigma t\}$, then the density has the lower bound*

$$\rho(t, x) \geq \frac{\sigma}{2}.$$

Furthermore, for any $(t_*, x_*) \in \Omega_\sigma$ and for any $r, s > 0$ such that $Q_{r,s}^* := B_r(x_*) \times (t_*, t_* + s] \subset \Omega_\sigma$ with $B_r(x_*)$ being the ball with the radius r and the center x_* , there exists a constant $\alpha_0 \in (0, 1)$, such that

$$\begin{aligned} u &\in C_{\text{loc}}^{\alpha_0, \frac{\alpha_0}{2}}(Q_{r,s}^*), \quad u \in L_{\text{loc}}^\infty(t_*, t_* + s, H_{\text{loc}}^1(B_r(x_*))), \\ u_t &\in L_{\text{loc}}^2(Q_{r,s}^*), \quad u \in L_{\text{loc}}^2(t_*, t_* + s, H_{\text{loc}}^2(B_r(x_*))). \end{aligned} \quad (2.24)$$

3 Existence of a weak solution

We first study the following approximate system:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = (\mu_\varepsilon(\rho) u_x)_x, \end{cases} \quad x \in \mathbf{R}, \quad t > 0, \quad (3.1)$$

where $\mu_\varepsilon(\rho) = \rho^\alpha + \varepsilon \rho^\theta$ with $\varepsilon > 0$ and $\theta = \frac{1}{2}$. This kind of the approximation $\varepsilon \rho^\theta$ was first used in [17] and $\theta = \frac{1}{2}$ is crucial in getting the lower bound of the approximation density function to ensure the existence of approximate solutions.

To overcome the difficulty caused by the vacuum in the rarefaction wave, we first cut off the rarefaction wave along the wave curve. More precisely, for any $\nu > 0$ suitably small and to be determined, let $(\nu, u(\nu))$ be the state such that

$$\Sigma_2(\nu, u(\nu)) = \Sigma_2(\rho_+, u_+),$$

where 2-Riemann invariant $\Sigma_2(\rho, u) = u - \frac{2\sqrt{\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}$. Then $(\nu, u(\nu))$ is connected to (ρ_+, u_+) by a non-vacuum 2-rarefaction wave given by $(\rho_\nu^r, u_\nu^r)(\frac{x}{t})$. Then it holds that

$$|(\rho^r, m^r)(\frac{x}{t}) - (\rho_\nu^r, m_\nu^r)(\frac{x}{t})| \leq C\nu.$$

One can compute that $u(\nu) = \frac{2\sqrt{\gamma}}{\gamma-1}\nu^{\frac{\gamma-1}{2}} + \Sigma_2(\rho_+, u_+)$. So the corresponding smooth approximate 2-rarefaction wave $(\bar{\rho}_\nu, \bar{u}_\nu)$ described in Section 2.2 can be constructed by setting

$$\lambda_2(\nu, u(\nu)) = w_-, \quad \lambda_2(\rho_+, u_+) = w_+.$$

Consequently, $(\bar{\rho}_\nu, \bar{u}_\nu)$ will converge to $(\bar{\rho}, \bar{u})$ point-wisely as ν tends to zero. In fact, $\nu = \nu(\varepsilon)$ will be chosen suitably such that $\nu(\varepsilon) \rightarrow 0$ as ε tends to zero.

The initial values (ρ_0, m_0) can be regularized in a similar way as in [18] such that

$$(\rho, \rho u)(0, x) = (\rho_{0\varepsilon, \nu}, m_{0\varepsilon, \nu}) \rightarrow \begin{cases} (\nu, \nu u(\nu)), & \text{as } x \rightarrow -\infty, \\ (\rho_+, m_+), & \text{as } x \rightarrow +\infty, \end{cases} \quad (3.2)$$

and

$$\rho_{0\varepsilon, \nu}(x) \geq \min\{\nu, \frac{1}{2}\varepsilon^{\frac{2}{2\alpha-1}}\}, \quad \forall x \in \mathbf{R}, \quad (3.3)$$

for suitably small $\varepsilon, \nu > 0$.

Furthermore, $\rho_{0\varepsilon, \nu}$ satisfies

$$\rho_{0\varepsilon, \nu}\Psi(\rho_{0\varepsilon, \nu}, \bar{\rho}_{0\nu}) \rightarrow \rho_0\Psi(\rho_0, \bar{\rho}_0) \text{ in } L^1(\mathbf{R}), \quad (\rho_{0\varepsilon, \nu}^{\alpha-1/2})_x \rightarrow (\rho_0^{\alpha-1/2})_x \text{ in } L^2(\mathbf{R}).$$

Since in the following paper α, γ satisfy (2.17) and we choose $\nu = \varepsilon^{\frac{2}{3}}$, it holds that

$$\int_{\mathbf{R}} \varepsilon^2 [(\ln \rho_{0\varepsilon, \nu})_x]^2 dx = (\frac{\varepsilon}{\alpha - \frac{1}{2}})^2 \int_{\mathbf{R}} \rho_{0\varepsilon, \nu}^{1-2\alpha} [(\rho_{0\varepsilon, \nu}^{\alpha-\frac{1}{2}})_x]^2 dx \leq C.$$

While $m_{0\varepsilon, \nu}$ satisfies

$$\rho_{0\varepsilon, \nu}(\frac{m_{0\varepsilon, \nu}}{\rho_{0\varepsilon, \nu}} - \bar{u}_{0\nu})^2 \rightarrow \rho_0(\frac{m_0}{\rho_0} - \bar{u}_0)^2 \text{ in } L^1(\mathbf{R}),$$

and

$$\rho_{0\varepsilon, \nu}(\frac{m_{0\varepsilon, \nu}}{\rho_{0\varepsilon, \nu}} - \bar{u}_{0\nu})^3 \rightarrow \rho_0(\frac{m_0}{\rho_0} - \bar{u}_0)^3 \text{ in } L^1(\mathbf{R}).$$

For any fixed $T > 0$ and $\varepsilon, \nu > 0$, we will first construct smooth approximate solutions $(\rho_{\varepsilon, \nu}, u_{\varepsilon, \nu})(x, t)$ to (3.1) with initial values $(\rho, \rho u)(0, x) = (\rho_{0\varepsilon, \nu}, m_{0\varepsilon, \nu})(x)$ defined in $[0, T]$. To do this, a key step is to get the lower bound of the density. Then the global existence of weak solutions to (1.1)-(1.3) can be proved by compactness arguments. We intend to deduce the uniform energy and entropy estimates with respect to ε, ν such that one can pass to the limit $\varepsilon, \nu \rightarrow 0$. Due to the closeness to the vacuum of the rarefaction wave, we will have to combine suitably the elementary energy estimates with the entropy estimates to get the following estimates which are crucial to prove our main results.

Lemma 3.1 *Suppose that the conditions in Theorem 2.1 are satisfied and that $(\rho_{\varepsilon,\nu}, u_{\varepsilon,\nu})$ is a smooth solution to (3.1) satisfying $\rho_{\varepsilon,\nu} > 0$. Then for any $T > 0$ and $\varepsilon, \nu > 0$ with $\varepsilon \ln(1+T) \leq C$ and ε, ν sufficiently small, the following estimate holds*

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbf{R}} \left\{ \rho_{\varepsilon,\nu} (u_{\varepsilon,\nu} - \bar{u}_\nu)^2 + \left[(\rho_{\varepsilon,\nu}^{\frac{\alpha-1}{2}})_x \right]^2 + \varepsilon^2 [(\ln \rho_{\varepsilon,\nu})_x]^2 + \rho_{\varepsilon,\nu} \Psi(\rho_{\varepsilon,\nu}, \bar{\rho}_\nu) \right\} (x, t) dx \\ & + \int_0^T \int_{\mathbf{R}} \left\{ (\bar{u}_\nu)_x \left[p(\rho_{\varepsilon,\nu}) - p(\bar{\rho}_\nu) - p'(\bar{\rho}_\nu)(\rho_{\varepsilon,\nu} - \bar{\rho}_\nu) \right] + \rho_{\varepsilon,\nu} (\bar{u}_\nu)_x (u_{\varepsilon,\nu} - \bar{u}_\nu)^2 \right. \\ & \quad \left. + \rho_{\varepsilon,\nu}^\alpha \left[(u_{\varepsilon,\nu} - \bar{u}_\nu)_x \right]^2 + \left[(\rho_{\varepsilon,\nu}^{\frac{\gamma+\alpha-1}{2}} - \bar{\rho}_\nu^{\frac{\gamma+\alpha-1}{2}})_x \right]^2 \right\} (x, t) dx dt \leq C. \end{aligned} \quad (3.4)$$

where $C > 0$ is a universal constant independent of ε, ν and T .

In the following, the subscripts ε, ν in the approximate solution $(\rho_{\varepsilon,\nu}, u_{\varepsilon,\nu})(t, x)$ and the subscripts ν in the approximate solution $(\bar{\rho}_\nu, \bar{u}_\nu)(t, x)$ will be omitted for simplicity.

Proof: Step 1. Energy Equality

It follows from (3.1)₂ that

$$\rho u_t + \rho u u_x + p(\rho)_x = (\mu_\varepsilon(\rho) u_x)_x. \quad (3.5)$$

Subtracting (3.5) from the second equation of (2.10) gives

$$\rho(u - \bar{u})_t + \rho u(u - \bar{u})_x + (p(\rho) - p(\bar{\rho}))_x + (\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x = (\mu_\varepsilon(\rho)(u - \bar{u})_x)_x + (\mu_\varepsilon(\rho)\bar{u}_x)_x. \quad (3.6)$$

Multiplying (3.6) by $u - \bar{u}$ yields

$$\begin{aligned} & \left[\frac{\rho(u - \bar{u})^2}{2} \right]_t + \left[\frac{\rho u(u - \bar{u})^2}{2} \right]_x + (u - \bar{u})(p(\rho) - p(\bar{\rho}))_x - \left[\mu_\varepsilon(\rho)(u - \bar{u})(u - \bar{u})_x \right]_x \\ & + \mu_\varepsilon(\rho) \left[(u - \bar{u})_x \right]^2 = \left[\mu_\varepsilon(\rho) \right]_x \bar{u}_x (u - \bar{u}) + \mu_\varepsilon(\rho) \bar{u}_{xx} (u - \bar{u}) - \left[(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x \right] (u - \bar{u}). \end{aligned} \quad (3.7)$$

Note that $\Psi(\rho, \bar{\rho})$ defined in (2.11) satisfies

$$\begin{aligned} & \left[\rho \Psi(\rho, \bar{\rho}) \right]_t + \left[\rho u \Psi(\rho, \bar{\rho}) \right]_x + (u - \bar{u})_x (p(\rho) - p(\bar{\rho})) + \bar{u}_x \left[\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} (\rho - \bar{\rho}) \right] \\ & = -\frac{p(\bar{\rho})_x}{\bar{\rho}} (\rho - \bar{\rho})(u - \bar{u}). \end{aligned} \quad (3.8)$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} & \left[\frac{\rho(u - \bar{u})^2}{2} + \rho \Psi(\rho, \bar{\rho}) \right]_t + H_{1x}(t, x) + \mu_\varepsilon(\rho) \left[(u - \bar{u})_x \right]^2 + \bar{u}_x \left[\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} (\rho - \bar{\rho}) \right] \\ & = \left[\mu_\varepsilon(\rho) \right]_x \bar{u}_x (u - \bar{u}) + \mu_\varepsilon(\rho) \bar{u}_{xx} (u - \bar{u}) - \left[(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x + \frac{p(\bar{\rho})_x}{\bar{\rho}} (\rho - \bar{\rho}) \right] (u - \bar{u}), \end{aligned} \quad (3.9)$$

where

$$H_1(t, x) = \frac{\rho u(u - \bar{u})^2}{2} + \rho u \Psi(\rho, \bar{\rho}) + (u - \bar{u})(p(\rho) - p(\bar{\rho})) - \mu_\varepsilon(\rho)(u - \bar{u})(u - \bar{u})_x.$$

Since

$$(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x + \frac{p(\bar{\rho})_x}{\bar{\rho}} (\rho - \bar{\rho}) = \rho(u - \bar{u})\bar{u}_x,$$

we obtain

$$\begin{aligned} & \left[\frac{\rho(u - \bar{u})^2}{2} + \rho\Psi(\rho, \bar{\rho}) \right]_t + H_{1x}(t, x) + \mu_\varepsilon(\rho) \left[(u - \bar{u})_x \right]^2 + \bar{u}_x \left[\rho^\gamma - \bar{\rho}^\gamma - \gamma\bar{\rho}^{\gamma-1}(\rho - \bar{\rho}) \right] \\ & + \rho(u - \bar{u})^2 \bar{u}_x = \left[\mu_\varepsilon(\rho) \right]_x \bar{u}_x(u - \bar{u}) + \mu_\varepsilon(\rho) \bar{u}_{xx}(u - \bar{u}). \end{aligned} \quad (3.10)$$

Step 2. Entropy Equality

Rewrite (3.6) as

$$\rho(u - \bar{u})_t + \rho u(u - \bar{u})_x + (p(\rho) - p(\bar{\rho}))_x + (\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x = \left[(\rho^{\alpha-1} + \varepsilon\rho^{\theta-1})\rho u_x \right]_x. \quad (3.11)$$

Note that

$$\left[(\rho^{\alpha-1} + \varepsilon\rho^{\theta-1})\rho u_x \right]_x = -\rho(\varphi_\varepsilon^{\alpha,\theta}(\rho))_{xt} - \rho u(\varphi_\varepsilon^{\alpha,\theta}(\rho))_{xx}, \quad (3.12)$$

where $\varphi_\varepsilon^{\alpha,\theta}(\rho)$ with $\theta = \frac{1}{2}$ is defined by

$$\varphi_\varepsilon^{\alpha,\theta}(\rho) = \begin{cases} \frac{\rho^{\alpha-1}}{\alpha-1} + \varepsilon \frac{\rho^{\theta-1}}{\theta-1}, & \text{if } \alpha \neq 1, \alpha > 0, \\ \ln \rho + \varepsilon \frac{\rho^{\theta-1}}{\theta-1}, & \text{if } \alpha = 1. \end{cases}$$

Thus (3.11) becomes

$$\rho(u - \bar{u})_t + \rho u(u - \bar{u})_x + (p(\rho) - p(\bar{\rho}))_x + (\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x = -\rho(\varphi_\varepsilon^{\alpha,\theta}(\rho))_{xt} - \rho u(\varphi_\varepsilon^{\alpha,\theta}(\rho))_{xx}. \quad (3.13)$$

Multiplying (3.13) by $(\varphi_\varepsilon^{\alpha,\theta}(\rho))_x$ shows that

$$\begin{aligned} & \left[\frac{\rho(\varphi_\varepsilon^{\alpha,\theta}(\rho))_x^2}{2} \right]_t + \left[\frac{\rho u(\varphi_\varepsilon^{\alpha,\theta}(\rho))_x^2}{2} \right]_x + \left[\rho(u - \bar{u})(\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \right]_t + \left[\rho u(u - \bar{u})(\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \right]_x \\ & - (u - \bar{u}) \left[\rho(\varphi_\varepsilon^{\alpha,\theta}(\rho))_{xt} + \rho u(\varphi_\varepsilon^{\alpha,\theta}(\rho))_{xx} \right] + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x (p(\rho) - p(\bar{\rho}))_x \\ & + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \left[(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x \right] = 0. \end{aligned} \quad (3.14)$$

Combining (3.13) with (3.14) yields

$$\begin{aligned} & \left\{ \frac{1}{2}\rho \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \right]^2 \right\}_t + \left\{ \frac{1}{2}\rho u \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \right]^2 \right\}_x + (u - \bar{u})(p(\rho) - p(\bar{\rho}))_x \\ & + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x (p(\rho) - p(\bar{\rho}))_x + (u - \bar{u}) \left[(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x \right] \\ & + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \left[(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x \right] = 0. \end{aligned} \quad (3.15)$$

Step 3. A Priori Estimates

It follows from (3.8) and (3.15) that

$$\begin{aligned} & \left\{ \frac{1}{2}\rho \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \right]^2 + \rho\Psi(\rho, \bar{\rho}) \right\}_t + \left\{ \frac{1}{2}\rho u \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \right]^2 + \rho u\Psi(\rho, \bar{\rho}) \right. \\ & \left. + (u - \bar{u})(p(\rho) - p(\bar{\rho})) \right\}_x + \bar{u}_x \left[p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho}) \right] + \rho(u - \bar{u})^2 \bar{u}_x \\ & + (\varphi_\varepsilon^{\alpha,\theta}(\rho))_x \left[(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x + p(\rho)_x - p(\bar{\rho})_x \right] = 0. \end{aligned} \quad (3.16)$$

Now we deal with the last term on the left hand side of (3.16). Note that

$$(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x + p(\rho)_x - p(\bar{\rho})_x = \rho(u - \bar{u})\bar{u}_x + \left[p(\rho)_x - \frac{\rho p(\bar{\rho})_x}{\bar{\rho}} \right], \quad (3.17)$$

and

$$(\varphi_\varepsilon^{\alpha, \theta}(\rho))_x = \rho^{\alpha-2}\rho_x + \varepsilon\rho^{\theta-2}\rho_x. \quad (3.18)$$

Thus

$$\begin{aligned} & (\varphi_\varepsilon^{\alpha, \theta}(\rho))_x \left[(\rho - \bar{\rho})\bar{u}_t + (\rho u - \bar{\rho}\bar{u})\bar{u}_x + p(\rho)_x - p(\bar{\rho})_x \right] \\ &= \left(\frac{\rho^\alpha}{\alpha} + \varepsilon \frac{\rho^\theta}{\theta} \right)_x (u - \bar{u})\bar{u}_x + (\rho^{\alpha-2}\rho_x + \varepsilon\rho^{\theta-2}\rho_x) \left[p(\rho)_x - \frac{\rho p(\bar{\rho})_x}{\bar{\rho}} \right]. \end{aligned} \quad (3.19)$$

Direct computations show

$$\begin{aligned} & \rho^{\alpha-2}\rho_x \left[p(\rho)_x - \frac{\rho p(\bar{\rho})_x}{\bar{\rho}} \right] \\ &= \frac{4\gamma}{(\alpha + \gamma - 1)^2} \left[(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \right]^2 + \left[\frac{8\gamma}{(\alpha + \gamma - 1)^2} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x (\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \right. \\ & \quad \left. - \frac{2\gamma}{\alpha(\alpha + \gamma - 1)} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} (\rho^\alpha - \bar{\rho}^\alpha) \right]_x - \frac{8\gamma}{(\alpha + \gamma - 1)^2} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_{xx} (\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \\ & \quad + \frac{2\gamma}{\alpha(\alpha + \gamma - 1)} \left[(\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} \right]_x (\rho^\alpha - \bar{\rho}^\alpha). \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} & \rho^{\theta-2}\rho_x \left[p(\rho)_x - \frac{\rho p(\bar{\rho})_x}{\bar{\rho}} \right] \\ &= \frac{4\gamma}{(\theta + \gamma - 1)^2} \left[(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \right]^2 + \left[\frac{8\gamma}{(\theta + \gamma - 1)^2} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x (\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \right. \\ & \quad \left. - \frac{2\gamma}{\theta(\theta + \gamma - 1)} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\theta-1}{2}} (\rho^\theta - \bar{\rho}^\theta) \right]_x - \frac{8\gamma}{(\theta + \gamma - 1)^2} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_{xx} (\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \\ & \quad + \frac{2\gamma}{\theta(\theta + \gamma - 1)} \left[(\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\theta-1}{2}} \right]_x (\rho^\theta - \bar{\rho}^\theta). \end{aligned} \quad (3.21)$$

Substituting (3.19)-(3.21) into (3.16) gives

$$\begin{aligned} & \left\{ \frac{1}{2}\rho \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha, \theta}(\rho))_x \right]^2 + \rho\Psi(\rho, \bar{\rho}) \right\}_t + H_{2x}(t, x) + \bar{u}_x \left[p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho}) \right] \\ & + \rho(u - \bar{u})^2\bar{u}_x + \left(\frac{\rho^\alpha}{\alpha} + \varepsilon \frac{\rho^\theta}{\theta} \right)_x (u - \bar{u})\bar{u}_x + \frac{4\gamma}{(\alpha + \gamma - 1)^2} \left[(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \right]^2 \\ & + \varepsilon \frac{4\gamma}{(\theta + \gamma - 1)^2} \left[(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \right]^2 = \frac{8\gamma}{(\alpha + \gamma - 1)^2} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_{xx} (\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \\ & + \varepsilon \frac{8\gamma}{(\theta + \gamma - 1)^2} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_{xx} (\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) - \frac{2\gamma}{\alpha(\alpha + \gamma - 1)} \left[(\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} \right]_x (\rho^\alpha - \bar{\rho}^\alpha) \\ & - \varepsilon \frac{2\gamma}{\theta(\theta + \gamma - 1)} \left[(\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\theta-1}{2}} \right]_x (\rho^\theta - \bar{\rho}^\theta), \end{aligned} \quad (3.22)$$

where

$$\begin{aligned}
H_2(t, x) &= \frac{1}{2}\rho u \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha, \theta}(\rho))_x \right]^2 + \rho u \Psi(\rho, \bar{\rho}) + (u - \bar{u})(p(\rho) - p(\bar{\rho})) \\
&\quad + \frac{8\gamma}{(\alpha + \gamma - 1)^2} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x (\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \\
&\quad - \frac{2\gamma}{\alpha(\alpha + \gamma - 1)} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} (\rho^\alpha - \bar{\rho}^\alpha) \\
&\quad + \varepsilon \frac{8\gamma}{(\theta + \gamma - 1)^2} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x (\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \\
&\quad - \varepsilon \frac{2\gamma}{\theta(\theta + \gamma - 1)} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\theta-1}{2}} (\rho^\theta - \bar{\rho}^\theta).
\end{aligned} \tag{3.23}$$

Multiplying (3.22) by α and then adding up to (3.10) and noticing that $\left[\mu_\varepsilon(\rho) \right]_x = (\rho^\alpha)_x + \varepsilon(\rho^\theta)_x$ in the right hand side of (3.10), one can get

$$\begin{aligned}
&\left\{ \frac{\alpha}{2}\rho \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha, \theta}(\rho))_x \right]^2 + \frac{\rho(u - \bar{u})^2}{2} + (\alpha + 1)\rho\Psi(\rho, \bar{\rho}) \right\}_t + \left[\alpha H_2(t, x) + H_1(t, x) \right]_x \\
&+ (\alpha + 1)\bar{u}_x \left[p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho}) \right] + (\alpha + 1)\rho(u - \bar{u})^2 \bar{u}_x + (\rho^\alpha + \varepsilon\rho^\theta) \left[(u - \bar{u})_x \right]^2_x \\
&+ \frac{4\alpha\gamma}{(\alpha + \gamma - 1)^2} \left[(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \right]^2 + \varepsilon \frac{4\alpha\gamma}{(\theta + \gamma - 1)^2} \left[(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \right]^2 \\
&= \rho^\alpha \bar{u}_{xx}(u - \bar{u}) + \varepsilon \left[\rho^\theta \bar{u}_{xx}(u - \bar{u}) + \left(1 - \frac{\alpha}{\theta}\right)(\rho^\theta)_x(u - \bar{u})\bar{u}_x \right] \\
&+ \frac{8\alpha\gamma}{(\alpha + \gamma - 1)^2} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_{xx} (\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) + \varepsilon \frac{8\alpha\gamma}{(\theta + \gamma - 1)^2} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_{xx} (\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \\
&- \frac{2\gamma}{(\alpha + \gamma - 1)} \left[(\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} \right]_x (\rho^\alpha - \bar{\rho}^\alpha) - \varepsilon \frac{2\alpha\gamma}{\theta(\theta + \gamma - 1)} \left[(\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\theta-1}{2}} \right]_x (\rho^\theta - \bar{\rho}^\theta).
\end{aligned} \tag{3.24}$$

Integrating (3.24) over $[0, t] \times \mathbf{R}$ with respect to t, x gives

$$\begin{aligned}
&\int_{\mathbf{R}} \left\{ \frac{\alpha}{2}\rho \left[(u - \bar{u}) + (\varphi_\varepsilon^{\alpha, \theta}(\rho))_x \right]^2 + \frac{\rho(u - \bar{u})^2}{2} + (\alpha + 1)\rho\Psi(\rho, \bar{\rho}) \right\} (t, x) dx \\
&+ \int_0^t \int_{\mathbf{R}} \left\{ (\alpha + 1)\bar{u}_x \left[p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho}) \right] + (\alpha + 1)\rho(u - \bar{u})^2 \bar{u}_x \right. \\
&+ (\rho^\alpha + \varepsilon\rho^\theta) \left[(u - \bar{u})_x \right]^2 + \frac{4\alpha\gamma}{(\alpha + \gamma - 1)^2} \left[(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \right]^2 \\
&\left. + \varepsilon \frac{4\alpha\gamma}{(\theta + \gamma - 1)^2} \left[(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \right]^2 \right\} dx d\tau \\
&= \int_{\mathbf{R}} \left\{ \frac{\alpha}{2}\rho_0 \left[(u_0 - \bar{u}_0) + (\varphi_\varepsilon^{\alpha, \theta}(\rho_0))_x \right]^2 + \frac{\rho_0(u_0 - \bar{u}_0)^2}{2} + (\alpha + 1)\rho_0\Psi(\rho_0, \bar{\rho}_0) \right\} dx + I,
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
I &= \int_0^t \int_{\mathbf{R}} \left\{ \rho^\alpha \bar{u}_{xx}(u - \bar{u}) + \varepsilon \left[\rho^\theta \bar{u}_{xx}(u - \bar{u}) + \left(1 - \frac{\alpha}{\theta}\right)(\rho^\theta)_x(u - \bar{u})\bar{u}_x \right] \right. \\
&+ \frac{8\alpha\gamma}{(\alpha + \gamma - 1)^2} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_{xx} (\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) + \varepsilon \frac{8\alpha\gamma}{(\theta + \gamma - 1)^2} (\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_{xx} (\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \\
&\left. - \frac{2\gamma}{(\alpha + \gamma - 1)} \left[(\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} \right]_x (\rho^\alpha - \bar{\rho}^\alpha) - \varepsilon \frac{2\alpha\gamma}{\theta(\theta + \gamma - 1)} \left[(\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\theta-1}{2}} \right]_x (\rho^\theta - \bar{\rho}^\theta) \right\} dx d\tau \\
&:= \sum_{i=1}^6 I_i.
\end{aligned} \tag{3.26}$$

We now estimate the right hand side of (3.26) terms by terms. First,

$$\begin{aligned}
I_1 &= \int_0^t \int_{\mathbf{R}} \rho^\alpha \bar{u}_{xx} (u - \bar{u}) dx d\tau \\
&= \int_0^t \int_{\mathbf{R}} \sqrt{\rho} (u - \bar{u}) \rho^{\alpha - \frac{1}{2}} \bar{u}_{xx} dx d\tau \\
&= \int_0^t \int_{\mathbf{R}} \sqrt{\rho} (u - \bar{u}) \rho^{\alpha - \frac{1}{2}} \bar{u}_{xx} [\mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}} + \mathbf{1}_{\{\rho \geq 2\rho_+\}}] dx d\tau \\
&:= I_{11} + I_{12},
\end{aligned} \tag{3.27}$$

where and in the sequel $\mathbf{1}_\Omega$ denotes the characteristic function of a set $\Omega \subset (0, t) \times \mathbf{R}$.

Rewrite I_{12} as

$$\begin{aligned}
I_{12} &= \int_0^t \int_{\mathbf{R}} \sqrt{\rho} (u - \bar{u}) \bar{u}_{xx} [(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) + \bar{\rho}^{\alpha - \frac{1}{2}}] \mathbf{1}_{\{\rho \geq 2\rho_+\}} dx d\tau \\
&:= I_{12}^1 + I_{12}^2.
\end{aligned} \tag{3.28}$$

Using Lemma 2.2 (and its Remark 2.1) and noting that $\alpha \geq 1$, one has

$$\begin{aligned}
I_{11} + I_{12}^2 &\leq C \int_0^t \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \|\bar{u}_{xx}\|_{L^2(\mathbf{R})} d\tau \\
&\leq C \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \int_0^t \|\bar{u}_{xx}\|_{L^2(\mathbf{R})} d\tau \\
&\leq C \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})},
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
I_{12}^1 &\leq C \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \sup_{t \in [0, T]} \|(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^2(\mathbf{R})} \cdot \int_0^t \|\bar{u}_{xx}\|_{L^\infty(\mathbf{R})} d\tau \\
&\leq C \eta^{\frac{2}{4q+1}} \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \sup_{t \in [0, T]} \|(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^2(\mathbf{R})} \\
&\leq C \eta^{\frac{2}{4q+1}} \left[\sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}^2 + \sup_{t \in [0, T]} \|(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^2(\mathbf{R})}^2 \right].
\end{aligned} \tag{3.30}$$

Note that if α and γ satisfy

$$1 \leq \alpha \leq \frac{\gamma + 1}{2}, \tag{3.31}$$

then $2(\alpha - \frac{1}{2}) \leq \gamma$, and then

$$\lim_{\rho \rightarrow +\infty} \frac{(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})^2}{\rho \Psi(\rho, \bar{\rho})} = \lim_{\rho \rightarrow +\infty} \frac{(\gamma - 1)(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})^2}{\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1}(\rho - \bar{\rho})} \leq C. \tag{3.32}$$

Thus if $1 \leq \alpha \leq \frac{\gamma+1}{2}$, then

$$\sup_{t \in [0, T]} \|(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^2(\mathbf{R})}^2 \leq C \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}, \tag{3.33}$$

for some uniform constant $C > 0$.

Substituting (3.28), (3.29), (3.30) and (3.33) into (3.27) yields

$$I_1 \leq C\eta^{\frac{2}{4q+1}} \left[\sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}^2 + \sup_{t \in [0, T]} \|\rho\Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})} \right] + C\eta. \quad (3.34)$$

Next, I_2 can be rewritten as

$$\begin{aligned} I_2 &= \varepsilon \int_0^t \int_{\mathbf{R}} \left[\rho^\theta \bar{u}_{xx}(u - \bar{u}) + \left(1 - \frac{\alpha}{\theta}\right) (\rho^\theta)_x (u - \bar{u}) \bar{u}_x \right] dx d\tau \\ &= \varepsilon \int_0^t \int_{\mathbf{R}} \left[\frac{\alpha}{\theta} \rho^\theta \bar{u}_{xx}(u - \bar{u}) - \left(1 - \frac{\alpha}{\theta}\right) \rho^\theta (u - \bar{u})_x \bar{u}_x \right] dx d\tau \\ &:= I_{21} + I_{22}, \end{aligned} \quad (3.35)$$

First, since $\theta = \frac{1}{2}$, it follows that

$$\begin{aligned} I_{21} &= \varepsilon \int_0^t \int_{\mathbf{R}} 2\alpha \bar{u}_{xx} \sqrt{\rho}(u - \bar{u}) dx d\tau \\ &\leq C\varepsilon \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \int_0^t \|\bar{u}_{xx}\|_{L^2(\mathbf{R})} d\tau \\ &\leq C\varepsilon \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}. \end{aligned} \quad (3.36)$$

On the other hand,

$$I_{22} \leq \frac{\varepsilon}{4} \int_0^t \int_{\mathbf{R}} \rho^\theta [(u - \bar{u})_x]^2 dx d\tau + C\varepsilon \int_0^t \int_{\mathbf{R}} \rho^\theta \bar{u}_x^2 dx d\tau, \quad (3.37)$$

while

$$\begin{aligned} \varepsilon \int_0^t \int_{\mathbf{R}} \rho^\theta \bar{u}_x^2 dx d\tau &= \varepsilon \int_0^t \int_{\mathbf{R}} \rho^\theta [\mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}} + \mathbf{1}_{\{\rho \geq 2\rho_+\}}] \bar{u}_x^2 dx d\tau \\ &= \varepsilon \int_0^t \int_{\mathbf{R}} \left\{ \rho^\theta \mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}} + [(\rho^\theta - \bar{\rho}^\theta) + \bar{\rho}^\theta] \mathbf{1}_{\{\rho \geq 2\rho_+\}} \right\} \bar{u}_x^2 dx d\tau \\ &\leq C\varepsilon \ln(1 + T) + C\varepsilon \sup_{t \in [0, T]} \|(\rho^\theta - \bar{\rho}^\theta) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^1(\mathbf{R})} \int_0^t \|\bar{u}_x\|_{L^\infty(\mathbf{R})}^2 d\tau \\ &\leq C\varepsilon \ln(1 + T) + C\varepsilon \sup_{t \in [0, T]} \|\rho\Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}, \end{aligned} \quad (3.38)$$

where in the last inequality we have used the fact that

$$\lim_{\rho \rightarrow +\infty} \frac{(\rho^\theta - \bar{\rho}^\theta) \mathbf{1}_{\{\rho \geq 2\rho_+\}}}{\rho\Psi(\rho, \bar{\rho})} = 0,$$

since $\theta = \frac{1}{2} < 1 < \gamma$.

Substituting the estimations (3.36)-(3.38) into (3.35), one can get

$$\begin{aligned} I_2 &\leq C\varepsilon \ln(1 + T) + \frac{\varepsilon}{4} \int_0^t \int_{\mathbf{R}} \rho^\theta [(u - \bar{u})_x]^2 dx d\tau + C\varepsilon \sup_{t \in [0, T]} \|\rho\Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})} \\ &\quad + C\varepsilon \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}. \end{aligned} \quad (3.39)$$

It follows from the fact that

$$\lim_{\rho \rightarrow 0} \frac{|\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}|^{\frac{2\gamma}{\alpha+\gamma-1}}}{\rho\Psi(\rho, \bar{\rho})} = 1, \quad (3.40)$$

that for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$, such that if $0 \leq \rho \leq \delta_\epsilon$, then

$$\left| \frac{|\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}|^{\frac{2\gamma}{\alpha+\gamma-1}}}{\rho\Psi(\rho, \bar{\rho})} - 1 \right| < \epsilon. \quad (3.41)$$

Fix $\epsilon = \frac{1}{2}$, then there exists $\delta_{\frac{1}{2}} > 0$, such that if $0 \leq \rho \leq \delta_{\frac{1}{2}}$, then

$$\left| \frac{|\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}|^{\frac{2\gamma}{\alpha+\gamma-1}}}{\rho\Psi(\rho, \bar{\rho})} - 1 \right| < \frac{1}{2}, \quad (3.42)$$

thus for any $\bar{\rho} \geq 0$,

$$\frac{1}{2}\rho\Psi(\rho, \bar{\rho}) \leq |\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}|^{\frac{2\gamma}{\alpha+\gamma-1}} \leq \frac{3}{2}\rho\Psi(\rho, \bar{\rho}), \quad \text{if } 0 \leq \rho \leq \delta_{\frac{1}{2}}. \quad (3.43)$$

Similarly, it follows from the fact that

$$\lim_{\bar{\rho} \rightarrow 0} \frac{|\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}|^{\frac{2\gamma}{\alpha+\gamma-1}}}{\rho\Psi(\rho, \bar{\rho})} = 1, \quad (3.44)$$

that there exists $\bar{\delta}_{\frac{1}{2}} > 0$, such that if $0 \leq \bar{\rho} \leq \bar{\delta}_{\frac{1}{2}}$, then

$$\left| \frac{|\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}|^{\frac{2\gamma}{\alpha+\gamma-1}}}{\rho\Psi(\rho, \bar{\rho})} - 1 \right| < \frac{1}{2}, \quad (3.45)$$

thus one can choose $\nu < \bar{\delta}_{\frac{1}{2}}$ such that for any $\rho \geq 0$,

$$\frac{1}{2}\rho\Psi(\rho, \bar{\rho}) \leq |\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}|^{\frac{2\gamma}{\alpha+\gamma-1}} \leq \frac{3}{2}\rho\Psi(\rho, \bar{\rho}), \quad \text{if } \nu \leq \bar{\rho} \leq \bar{\delta}_{\frac{1}{2}}. \quad (3.46)$$

The term I_3 can be estimated as follows. Since

$$\begin{aligned} (\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_{xx} &= \left(\frac{\alpha+\gamma-1}{2} \bar{\rho}^{\frac{\alpha+\gamma-3}{2}} \bar{\rho}_x \right)_x = \frac{\alpha+\gamma-1}{2\sqrt{\gamma}} (\bar{\rho}^{\frac{\alpha}{2}} \bar{u}_x)_x \\ &= \frac{\alpha+\gamma-1}{2\sqrt{\gamma}} \bar{\rho}^{\frac{\alpha}{2}} \bar{u}_{xx} + \frac{\alpha(\alpha+\gamma-1)}{4\gamma} \bar{\rho}^{\frac{\alpha+1-\gamma}{2}} \bar{u}_x^2, \end{aligned}$$

and

$$\alpha+1-\gamma \geq 2-\gamma \geq 0,$$

one can rewrite I_3 as

$$\begin{aligned} I_3 &= \int_0^t \int_{\mathbf{R}} \left(\frac{4\alpha\sqrt{\gamma}}{\alpha+\gamma-1} \bar{\rho}^{\frac{\alpha}{2}} \bar{u}_{xx} + \frac{8\alpha^2}{\alpha+\gamma-1} \bar{u}_x^2 \bar{\rho}^{\frac{\alpha+1-\gamma}{2}} \right) (\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) dx d\tau \\ &\leq C \int_0^t \int_{\mathbf{R}} |(\bar{u}_{xx}, \bar{u}_x^2)| |(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}})| dx d\tau \\ &= C \int_0^t \int_{\mathbf{R}} |(\bar{u}_{xx}, \bar{u}_x^2)| |(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}})| (\mathbf{1}_{\{0 \leq \rho \leq \delta_{\frac{1}{2}}\}} + \mathbf{1}_{\{\delta_{\frac{1}{2}} \leq \rho \leq 2\rho_+, \nu \leq \bar{\rho} \leq \bar{\delta}_{\frac{1}{2}}\}} \\ &\quad + \mathbf{1}_{\{\delta_{\frac{1}{2}} \leq \rho \leq 2\rho_+, \bar{\delta}_{\frac{1}{2}} \leq \bar{\rho} \leq \rho_+\}} + \mathbf{1}_{\{\rho \geq 2\rho_+\}}) dx d\tau \\ &:= I_{31} + I_{32} + I_{33} + I_{34}. \end{aligned} \quad (3.47)$$

Direct computations lead to

$$\begin{aligned}
I_{31} &\leq C \int_0^t \int_{\mathbf{R}} \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^{\frac{2\gamma}{\gamma+1-\alpha}}(\mathbf{R})} \|(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \mathbf{1}_{\{0 \leq \rho \leq \delta_{\frac{1}{2}}\}}\|_{L^{\frac{2\gamma}{\alpha+\gamma-1}}(\mathbf{R})} dx d\tau \\
&\leq C \sup_{t \in [0, T]} \|(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \mathbf{1}_{\{0 \leq \rho \leq \delta_{\frac{1}{2}}\}}\|_{L^{\frac{2\gamma}{\alpha+\gamma-1}}(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^{\frac{2\gamma}{\gamma+1-\alpha}}(\mathbf{R})} d\tau \\
&\stackrel{(3.43)}{\leq} C \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}^{\frac{\alpha+\gamma-1}{2\gamma}} \\
&\leq \frac{\alpha+1}{8} \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})} + C_\alpha.
\end{aligned} \tag{3.48}$$

Similarly, due to (3.46), one has

$$I_{32} \leq \frac{\alpha+1}{8} \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})} + C_\alpha. \tag{3.49}$$

On the other hand,

$$\begin{aligned}
I_{33} &\leq C \int_0^t \int_{\mathbf{R}} \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} \|(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \mathbf{1}_{\{\delta_{\frac{1}{2}} \leq \rho \leq 2\rho_+, \delta_{\frac{1}{2}} \leq \bar{\rho} \leq \rho_+\}}\|_{L^2(\mathbf{R})} dx d\tau \\
&\leq C \sup_{t \in [0, T]} \|(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \mathbf{1}_{\{\delta_{\frac{1}{2}} \leq \rho \leq 2\rho_+, \delta_{\frac{1}{2}} \leq \bar{\rho} \leq \rho_+\}}\|_{L^2(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} d\tau \\
&\leq C \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}^{\frac{1}{2}},
\end{aligned} \tag{3.50}$$

where one has used the fact that

$$\frac{(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}})^2 \mathbf{1}_{\{\delta_{\frac{1}{2}} \leq \rho \leq 2\rho_+, \delta_{\frac{1}{2}} \leq \bar{\rho} \leq \rho_+\}}}{\rho \Psi(\rho, \bar{\rho})} \leq C.$$

Moreover, I_{34} can be estimated as

$$\begin{aligned}
I_{34} &\leq C \int_0^t \int_{\mathbf{R}} \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^\infty(\mathbf{R})} \|(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^1(\mathbf{R})} dx d\tau \\
&\leq C \sup_{t \in [0, T]} \|(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^1(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^\infty(\mathbf{R})} d\tau \\
&\leq C \eta^{\frac{2}{4q+1}} \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})},
\end{aligned} \tag{3.51}$$

due to the facts that

$$\lim_{\rho \rightarrow +\infty} \frac{|\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}}| \mathbf{1}_{\{\rho \geq 2\rho_+\}}}{\rho \Psi(\rho, \bar{\rho})} \leq C,$$

since $\alpha \leq \frac{\gamma+1}{2}$ implies

$$\frac{\alpha + \gamma - 1}{2} \leq \gamma, \quad \text{i. e.,} \quad \alpha \leq \gamma + 1.$$

Now we turn to the term I_5 . First,

$$\begin{aligned}
\left[(\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} \right]_x &= \frac{\alpha + \gamma - 1}{2\sqrt{\gamma}} (\bar{\rho}^{\frac{\gamma-1}{2}} \bar{u}_x)_x \\
&= \frac{\alpha + \gamma - 1}{2\sqrt{\gamma}} \bar{\rho}^{\frac{\gamma-1}{2}} \bar{u}_{xx} + \frac{(\alpha + \gamma - 1)(\gamma - 1)}{4\gamma} (\bar{u}_x)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
I_5 &= \int_0^t \int_{\mathbf{R}} \frac{2\gamma}{(\alpha + \gamma - 1)} \left[(\bar{\rho}^{\frac{\alpha+\gamma-1}{2}})_x \bar{\rho}^{\frac{\gamma-\alpha-1}{2}} \right]_x (\rho^\alpha - \bar{\rho}^\alpha) dx d\tau \\
&\leq C \int_0^t \int_{\mathbf{R}} |(\bar{u}_{xx}, \bar{u}_x^2)| |(\rho^\alpha - \bar{\rho}^\alpha)| (\mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}} + \mathbf{1}_{\{\rho \geq 2\rho_+\}}) dx d\tau \\
&:= I_{51} + I_{52}.
\end{aligned}$$

One has

$$\begin{aligned}
I_{51} &\leq C \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} \|(\rho^\alpha - \bar{\rho}^\alpha) \mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}}\|_{L^2(\mathbf{R})} dx d\tau \\
&\leq C \sup_{t \in [0, T]} \|(\rho^\alpha - \bar{\rho}^\alpha) \mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}}\|_{L^2(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} d\tau \\
&\leq C \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}^{\frac{1}{2}},
\end{aligned}$$

due to the fact that

$$\begin{aligned}
\frac{(\rho^\alpha - \bar{\rho}^\alpha)^2 \mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}}}{\rho \Psi(\rho, \bar{\rho})} &= \frac{(\gamma - 1)(\rho^\alpha - \bar{\rho}^\alpha)^2 \mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}}}{\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1}(\rho - \bar{\rho})} \\
&\leq C \frac{(\rho^{2\alpha-2} + \bar{\rho}^{2\alpha-2})(\rho - \bar{\rho})^2}{(\rho - \bar{\rho})^2} \mathbf{1}_{\{0 \leq \rho \leq 2\rho_+\}} \\
&\leq C,
\end{aligned}$$

where one has used $1 < \gamma \leq 2$ and $\alpha \geq 1$.

On the other hand,

$$\begin{aligned}
I_{52} &\leq C \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^\infty(\mathbf{R})} \|(\rho^\alpha - \bar{\rho}^\alpha) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^1(\mathbf{R})} dx d\tau \\
&\leq C \sup_{t \in [0, T]} \|(\rho^\alpha - \bar{\rho}^\alpha) \mathbf{1}_{\{\rho \geq 2\rho_+\}}\|_{L^1(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^\infty(\mathbf{R})} d\tau \\
&\leq C \eta^{\frac{2}{4q+1}} \sup_{t \in [0, T]} \|\rho \Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})},
\end{aligned}$$

due to the fact that

$$\begin{aligned}
\lim_{\rho \rightarrow +\infty} \frac{|\rho^\alpha - \bar{\rho}^\alpha| \mathbf{1}_{\{\rho \geq 2\rho_+\}}}{\rho \Psi(\rho, \bar{\rho})} &= \lim_{\rho \rightarrow +\infty} \frac{(\gamma - 1) |\rho^\alpha - \bar{\rho}^\alpha| \mathbf{1}_{\{\rho \geq 2\rho_+\}}}{\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1}(\rho - \bar{\rho})} \\
&\leq C, \quad \text{if } \alpha \leq \gamma.
\end{aligned}$$

Finally, we estimate the terms I_4 and I_6 . As for I_3 , one has

$$\begin{aligned}
(\bar{\rho}^{\frac{\theta+\gamma-1}{2}})_{xx} &= \left(\frac{\theta + \gamma - 1}{2} \bar{\rho}^{\frac{\theta+\gamma-3}{2}} \bar{\rho}_x \right)_x = \frac{\theta + \gamma - 1}{2\sqrt{\gamma}} (\bar{\rho}^{\frac{\theta}{2}} \bar{u}_x)_x \\
&= \frac{\theta + \gamma - 1}{2\sqrt{\gamma}} \bar{\rho}^{\frac{\theta}{2}} \bar{u}_{xx} + \frac{\theta(\theta + \gamma - 1)}{4\gamma} \bar{\rho}^{\frac{\theta+1-\gamma}{2}} \bar{u}_x^2,
\end{aligned}$$

therefore,

$$\begin{aligned}
I_4 &\leq C\varepsilon \int_0^t \int_{\mathbf{R}} |(\bar{u}_{xx}, \bar{\rho}^{\frac{\theta+1-\gamma}{2}} \bar{u}_x^2)| |(\rho^{\frac{\theta+1-\gamma}{2}} - \bar{\rho}^{\frac{\theta+1-\gamma}{2}})| dxdt \\
&\leq C\varepsilon \nu^{\frac{\theta-1}{2}} \int_0^t \int_{\mathbf{R}} |(\bar{u}_{xx}, \bar{u}_x^2)| |(\rho^{\frac{\theta+1-\gamma}{2}} - \bar{\rho}^{\frac{\theta+1-\gamma}{2}})| dxdt \\
&= C\varepsilon \nu^{\frac{\theta-1}{2}} \int_0^t \int_{\mathbf{R}} |(\bar{u}_{xx}, \bar{u}_x^2)| |(\rho^{\frac{\theta+1-\gamma}{2}} - \bar{\rho}^{\frac{\theta+1-\gamma}{2}})| [\mathbf{1}_{\{0 \leq \rho < \frac{\nu}{2}\}} + \mathbf{1}_{\{\frac{\nu}{2} \leq \rho \leq 2\rho_+\}} + \mathbf{1}_{\{\rho > 2\rho_+\}}] dxdt \\
&:= I_{41} + I_{42} + I_{43}.
\end{aligned} \tag{3.52}$$

Recall the following useful fact that for any given $C > 0$, there exists a constant $\beta \in [0, 1]$ such that

$$\begin{aligned}
\rho\Psi(\rho, \bar{\rho})\mathbf{1}_{\{0 \leq \rho \leq C\}} &= \frac{1}{\gamma-1} \left[\rho^\gamma - \bar{\rho}^\gamma - \gamma\bar{\rho}^{\gamma-1}(\rho - \bar{\rho}) \right] \mathbf{1}_{\{0 \leq \rho \leq C\}} \\
&= [\beta\rho + (1-\beta)\bar{\rho}]^{\gamma-2} (\rho - \bar{\rho})^2 \mathbf{1}_{\{0 \leq \rho \leq C\}} \\
&\geq \max\{\rho_+, C\}^{\gamma-2} (\rho - \bar{\rho})^2,
\end{aligned} \tag{3.53}$$

provided that $1 < \gamma \leq 2$.

One can compute that

$$\begin{aligned}
I_{41} &\leq C\varepsilon \nu^{\frac{\theta-1}{2}} \int_0^t \int_{\mathbf{R}} \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} \|(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})\mathbf{1}_{\{0 \leq \rho \leq \frac{\nu}{2}\}}\|_{L^2(\mathbf{R})} dx d\tau \\
&\leq C\varepsilon \nu^{\frac{\theta-1}{2}} \sup_{t \in [0, T]} \|(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})\mathbf{1}_{\{0 \leq \rho < \frac{\nu}{2}\}}\|_{L^2(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} d\tau \\
&\leq C\varepsilon \nu^{\frac{3}{2}(\theta-1)} \sup_{t \in [0, T]} \|\rho\Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}^{\frac{1}{2}},
\end{aligned} \tag{3.54}$$

where one has used the facts that

$$\frac{(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})^2 \mathbf{1}_{\{0 \leq \rho < \frac{\nu}{2}\}}}{\rho\Psi(\rho, \bar{\rho})} \leq \frac{(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})^2 \mathbf{1}_{\{0 \leq \rho < \frac{\nu}{2}\}}}{(\rho_+)^{\gamma-2} (\rho - \bar{\rho})^2}, \quad \text{if } \nu \ll 1,$$

and the function

$$\frac{(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})^2}{(\rho - \bar{\rho})^2}$$

is monotone decreasing in $\rho \in [0, \frac{\nu}{2}]$, that is,

$$\frac{(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})^2 \mathbf{1}_{\{0 \leq \rho < \frac{\nu}{2}\}}}{(\rho - \bar{\rho})^2} \leq \lim_{\rho \rightarrow 0} \frac{(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})^2}{(\rho - \bar{\rho})^2} = \bar{\rho}^{\theta+\gamma-3} \leq \nu^{\theta-1}.$$

Moreover, it holds that

$$\begin{aligned}
I_{42} &\leq C\varepsilon \nu^{\frac{\theta-1}{2}} \int_0^t \int_{\mathbf{R}} \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} \|(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})\mathbf{1}_{\{\frac{\nu}{2} \leq \rho \leq 2\rho_+\}}\|_{L^2(\mathbf{R})} dx d\tau \\
&\leq C\varepsilon \nu^{\frac{\theta-1}{2}} \sup_{t \in [0, T]} \|(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})\mathbf{1}_{\{\frac{\nu}{2} \leq \rho \leq 2\rho_+\}}\|_{L^2(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^2(\mathbf{R})} d\tau \\
&\leq C\varepsilon \nu^{\frac{3}{2}(\theta-1)} \sup_{t \in [0, T]} \|\rho - \bar{\rho}\|_{L^2(\mathbf{R})} \\
&\stackrel{(3.53)}{\leq} C\varepsilon \nu^{\frac{3}{2}(\theta-1)} \sup_{t \in [0, T]} \|\rho\Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}^{\frac{1}{2}},
\end{aligned} \tag{3.55}$$

where in the third inequality one has used the fact that there exists a constant $\beta \in [0, 1]$,

$$\begin{aligned} \frac{(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}})^2 \mathbf{1}_{\{\frac{\nu}{2} \leq \rho \leq 2\rho_+\}}}{(\rho - \bar{\rho})^2} &= (\theta + \gamma - 1)^2 [\beta\rho + (1 - \beta)\bar{\rho}]^{\theta+\gamma-3} \mathbf{1}_{\{\frac{\nu}{2} \leq \rho \leq 2\rho_+\}} \\ &\leq (\theta + \gamma - 1)^2 \left(\frac{\nu}{2}\right)^{\theta-1}. \end{aligned}$$

And then

$$\begin{aligned} I_{43} &\leq C\varepsilon\nu^{\frac{\theta-1}{2}} \int_0^t \int_{\mathbf{R}} \|(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \mathbf{1}_{\{\rho > 2\rho_+\}}\|_{L^1(\mathbf{R})} \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^\infty(\mathbf{R})} dx d\tau \\ &\leq C\varepsilon\nu^{\frac{\theta-1}{2}} \sup_{t \in [0, T]} \|(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \mathbf{1}_{\{\rho > 2\rho_+\}}\|_{L^1(\mathbf{R})} \int_0^t \|(\bar{u}_{xx}, \bar{u}_x^2)\|_{L^\infty(\mathbf{R})} d\tau \\ &\leq C\varepsilon\nu^{\frac{\theta-1}{2}} \sup_{t \in [0, T]} \|\rho\Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})}, \end{aligned} \quad (3.56)$$

since

$$\lim_{\rho \rightarrow +\infty} \frac{(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}}) \mathbf{1}_{\{\rho > 2\rho_+\}}}{\rho\Psi(\rho, \bar{\rho})} = 0.$$

In summary, by combining (3.54), (3.55) and (3.56), one can arrive at

$$I_4 \leq C\varepsilon\nu^{\frac{3}{2}(\theta-1)} \left[\sup_{t \in [0, T]} \|\rho\Psi(\rho, \bar{\rho})\|_{L^1(\mathbf{R})} + 1 \right]. \quad (3.57)$$

Finally, I_6 can be estimated similarly as for I_4 and the details will be omitted for brevity.

For definiteness, we take $\varepsilon\nu^{\frac{3}{2}(\theta-1)} = \varepsilon^{\frac{1}{2}}$, i.e., $\nu = \varepsilon^{\frac{2}{3}}$ since $\theta = \frac{1}{2}$. Consequently, choosing ε such that $\varepsilon \ln(1+T) \leq 1$ and ε suitably small and combining all the above estimates shows that for α and γ satisfying (2.17),

$$\begin{aligned} &\sup_{t \in [0, T]} \int_{\mathbf{R}} \left\{ \rho(u - \bar{u})^2 + \left[\left(\frac{\rho^{\alpha-\frac{1}{2}}}{\alpha - \frac{1}{2}} \right)_x \right]^2 + \varepsilon^2 [(\ln \rho)_x]^2 + \rho\Psi(\rho, \bar{\rho}) \right\} (x, t) dx \\ &+ \int_0^T \int_{\mathbf{R}} \left\{ \bar{u}_x \left[p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})(\rho - \bar{\rho}) \right] + \rho(u - \bar{u})^2 \bar{u}_x + (\rho^\alpha + \varepsilon\rho^\theta) \left[(u - \bar{u})_x \right]^2 \right. \\ &\quad \left. + \left[\left(\rho^{\frac{\alpha+\gamma-1}{2}} - \bar{\rho}^{\frac{\alpha+\gamma-1}{2}} \right)_x \right]^2 + \varepsilon \left[\left(\rho^{\frac{\theta+\gamma-1}{2}} - \bar{\rho}^{\frac{\theta+\gamma-1}{2}} \right)_x \right]^2 \right\} (x, t) dx dt \leq C. \end{aligned} \quad (3.58)$$

Thus Lemma 3.1 is proved. \square

The following lemma is the key point to get the existence of the approximate solution $(\rho_{\varepsilon, \nu}, u_{\varepsilon, \nu})(t, x)$ with $\nu = \varepsilon^{\frac{2}{3}}$.

Lemma 3.2 *There exist an absolutely constant C and a positive constant $C(\varepsilon, \nu, T)$ depending on ε, ν and T such that*

$$0 < C(\varepsilon, \nu, T) \leq \rho_{\varepsilon, \nu} \leq C. \quad (3.59)$$

Proof: From the Gagliardo-Nirenberg inequality:

$$\|f\|_{L^\infty(\mathbf{R})} \leq C \|f_x\|_{L^2(\mathbf{R})}^\beta \|f\|_{L^p(\mathbf{R})}^{1-\beta},$$

where $0 < \beta < 1$, $1 \leq p < \infty$ to be determined, and β, p satisfy

$$\frac{\beta}{2} = \frac{1 - \beta}{p},$$

we have

$$\begin{aligned} & \sup_{t \in [0, T]} \|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}\|_{L^\infty(\mathbf{R})} \\ & \leq C \sup_{t \in [0, T]} \|(\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})_x\|_{L^2(\mathbf{R})}^\beta \sup_{t \in [0, T]} \|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}\|_{L^p(\mathbf{R})}^{1 - \beta} \\ & \leq C \sup_{t \in [0, T]} \left[\|(\rho^{\alpha - \frac{1}{2}})_x\|_{L^2(\mathbf{R})}^\beta + \|(\bar{\rho}^{\alpha - \frac{1}{2}})_x\|_{L^2(\mathbf{R})}^\beta \right] \sup_{t \in [0, T]} \|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}\|_{L^p(\mathbf{R})}^{1 - \beta} \\ & \leq C \sup_{t \in [0, T]} \|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}\|_{L^p(\mathbf{R})}^{1 - \beta}, \end{aligned} \quad (3.60)$$

due to the fact that

$$\|(\bar{\rho}^{\alpha - \frac{1}{2}})_x\|_{L^2(\mathbf{R})} = \left\| \left(\alpha - \frac{1}{2} \right) \bar{\rho}^{\alpha - \frac{1}{2}} \bar{u}_x \right\|_{L^2(\mathbf{R})} \leq C \|\bar{u}_x\|_{L^2(\mathbf{R})} \leq C.$$

Since

$$\lim_{\rho \rightarrow 0} \frac{|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^{\frac{2\gamma}{2\alpha - 1}}}{\rho \Psi(\rho, \bar{\rho})} = 1, \quad (3.61)$$

there exists a positive constant $\delta_{\frac{1}{2}}^1$, such that if $0 \leq \rho \leq \delta_{\frac{1}{2}}^1$, then

$$\left| \frac{|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^{\frac{2\gamma}{2\alpha - 1}}}{\rho \Psi(\rho, \bar{\rho})} - 1 \right| < \frac{1}{2}, \quad (3.62)$$

thus, for any $\bar{\rho} \geq 0$,

$$\frac{1}{2} \rho \Psi(\rho, \bar{\rho}) \leq |\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^{\frac{2\gamma}{2\alpha - 1}} \leq \frac{3}{2} \rho \Psi(\rho, \bar{\rho}), \quad \text{if } 0 \leq \rho \leq \delta_{\frac{1}{2}}^1. \quad (3.63)$$

Similarly, there exists a positive constant $\bar{\delta}_{\frac{1}{2}}^1$, such that for any $\rho \geq 0$,

$$\frac{1}{2} \rho \Psi(\rho, \bar{\rho}) \leq |\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^{\frac{2\gamma}{2\alpha - 1}} \leq \frac{3}{2} \rho \Psi(\rho, \bar{\rho}), \quad \text{if } 0 \leq \bar{\rho} \leq \bar{\delta}_{\frac{1}{2}}^1. \quad (3.64)$$

Set

$$p = \frac{2\gamma}{2\alpha - 1} \in [2, 2\gamma].$$

Then for such $p = \frac{2\gamma}{2\alpha - 1}$ and $\gamma \in (1, 2]$, one has

$$\frac{|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^p}{\rho \Psi(\rho, \bar{\rho})} \mathbf{1}_{\{\delta_{\frac{1}{2}}^1 \leq \rho \leq 2\rho_+, \bar{\delta}_{\frac{1}{2}}^1 \leq \bar{\rho} \leq \rho_+\}} \leq C \frac{|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2}{(\rho - \bar{\rho})^2} \mathbf{1}_{\{\delta_{\frac{1}{2}}^1 \leq \rho \leq 2\rho_+, \bar{\delta}_{\frac{1}{2}}^1 \leq \bar{\rho} \leq \rho_+\}} \leq C, \quad (3.65)$$

and

$$\lim_{\rho \rightarrow +\infty} \frac{|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^p}{\rho \Psi(\rho, \bar{\rho})} = \lim_{\rho \rightarrow +\infty} \frac{(\gamma - 1) |\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^{\frac{2\gamma}{2\alpha - 1}}}{\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma - 1} (\rho - \bar{\rho})} = \gamma - 1. \quad (3.66)$$

Thus it follows from (3.66) that

$$|\rho^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^p \leq C \rho \Psi(\rho, \bar{\rho}), \quad \text{if } \rho \geq 2\rho_+. \quad (3.67)$$

Collecting (3.63), (3.64), (3.65) and (3.67) gives that

$$\begin{aligned}
& \|\rho^{\alpha-\frac{1}{2}} - \bar{\rho}^{\alpha-\frac{1}{2}}\|_{L^p(\mathbf{R})}^p = \int_{\mathbf{R}} |\rho^{\alpha-\frac{1}{2}} - \bar{\rho}^{\alpha-\frac{1}{2}}|^p dx \\
& = \int_{\mathbf{R}} |\rho^{\alpha-\frac{1}{2}} - \bar{\rho}^{\alpha-\frac{1}{2}}|^p (\mathbf{1}_{\{0 \leq \rho \leq \delta_{\frac{1}{2}}^1\}} + \mathbf{1}_{\{\delta_{\frac{1}{2}}^1 \leq \rho \leq 2\rho_+, 0 \leq \bar{\rho} \leq \delta_{\frac{1}{2}}^1\}} \\
& \quad + \mathbf{1}_{\{\delta_{\frac{1}{2}}^1 \leq \rho \leq 2\rho_+, \delta_{\frac{1}{2}}^1 \leq \bar{\rho} \leq \rho_+\}} + \mathbf{1}_{\{\rho \geq 2\rho_+\}}) dx \\
& \leq C \int_{\mathbf{R}} \rho \Psi(\rho, \bar{\rho}) dx \leq C.
\end{aligned} \tag{3.68}$$

Substituting (3.68) into (3.60) yields the uniform upper bound for $\rho_{\varepsilon, \nu}(t, x)$.

Next we derive a lower bound for $\rho_{\varepsilon, \nu}(t, x)$. Since $\lim_{\rho \rightarrow 0} \rho \Psi(\rho, \bar{\rho}) = \bar{\rho}^\gamma \geq \nu^\gamma$, then $\rho \Psi(\rho, \bar{\rho})$ is bounded away from 0 on $[0, \frac{1}{2}\bar{\rho}]$. Thus one can deduce from the bound on $\rho \Psi(\rho, \bar{\rho})$ in $L^\infty(0, T; L^1(\mathbf{R}))$ that there exists a constant $C_1 = C_1(\nu, T) > 0$, such that for all $t \in [0, T]$,

$$\text{meas}\{x \in \mathbf{R} | \rho(x, t) \leq \frac{1}{2}\bar{\rho}(x, t)\} \leq \frac{1}{\inf_{\rho \in [0, \frac{1}{2}\bar{\rho}]} \rho \Psi(\rho, \bar{\rho})} \int_{\{x \in \mathbf{R} | \rho(x, t) \leq \frac{1}{2}\bar{\rho}(x, t)\}} \rho \Psi(\rho, \bar{\rho})(x, t) dx \leq C_1.$$

Therefore, for every $x_0 \in \mathbf{R}$, there exists $M = M(\nu, T) > 0$ large enough, such that

$$\begin{aligned}
\int_{|x-x_0| \leq M} \rho_{\varepsilon, \nu}(x, t) dx & \geq \int_{\{|x-x_0| \leq M\} \cap \{x \in \mathbf{R} | \rho_{\varepsilon, \nu}(x, t) > \frac{1}{2}\bar{\rho}(x, t)\}} \rho_{\varepsilon, \nu}(x, t) dx \\
& \geq \frac{1}{2} \inf_{(x, t)} \bar{\rho}(x, t) \text{meas}\left\{\{|x-x_0| \leq M\} \cap \{x \in \mathbf{R} | \rho_{\varepsilon, \nu}(x, t) > \frac{1}{2}\bar{\rho}(x, t)\}\right\} \\
& = \frac{\nu}{2} \text{meas}\left\{\{|x-x_0| \leq M\} \cap \{x \in \mathbf{R} | \rho_{\varepsilon, \nu}(x, t) \leq \frac{1}{2}\bar{\rho}(x, t)\}^c\right\} \\
& \geq \frac{\nu}{2} (2M - C_1) > 0,
\end{aligned}$$

for all $t \in [0, T]$.

Due to the continuity of $\rho_{\varepsilon, \nu}$, there exists $x_1 \in [x_0 - M, x_0 + M]$ such that

$$\rho_{\varepsilon, \nu}(x_1, t) = \frac{1}{2M} \int_{|x-x_0| \leq M} \rho_{\varepsilon, \nu}(x, t) dx \geq \frac{\nu}{4M} (2M - C_1).$$

Thus,

$$\begin{aligned}
|\ln \rho_{\varepsilon, \nu}(x_0, t)| & = |\ln \rho_{\varepsilon, \nu}(x_1, t) + \int_{x_1}^{x_0} (\ln \rho_{\varepsilon, \nu})_x(x, t) dx| \\
& \leq |\ln \rho_{\varepsilon, \nu}(x_1, t)| + \|(\ln \rho_{\varepsilon, \nu})_x(\cdot, t)\|_{L^2(\mathbf{R})} |x_1 - x_0|^{\frac{1}{2}} \\
& \leq C(\varepsilon, \nu, M, T) + C_\varepsilon M^{\frac{1}{2}}.
\end{aligned}$$

Consequently, we can get that there exists a positive constant $C(\varepsilon, \nu, T)$ such that

$$\rho_{\varepsilon, \nu}(x_0, t) \geq C(\varepsilon, \nu, T),$$

for any $x_0 \in \mathbf{R}$ and $t \in [0, T]$. □

With the lower and upper bounds on $\rho_{\varepsilon, \nu}$, we can get the existence of the approximate solution $(\rho_{\varepsilon, \nu}, u_{\varepsilon, \nu})(t, x)$ by a similar argument as in [32]. In order to pass the limit $\varepsilon \rightarrow 0$ with $\nu = \varepsilon^{\frac{2}{3}}$, we need the following higher estimates on the momentum.

Lemma 3.3 *There exists a positive constant $C(T)$ independent of ε, ν , such that*

$$\sup_{t \in [0, T]} \int_{\mathbf{R}} \rho_{\varepsilon, \nu} |u_{\varepsilon, \nu} - \bar{u}_{\nu}|^3(t, x) dx + \int_0^T \int_{\mathbf{R}} \rho_{\varepsilon, \nu} [(u_{\varepsilon, \nu} - \bar{u}_{\nu})_x]^2 |u_{\varepsilon, \nu} - \bar{u}| dx dt \leq C(T).$$

The proof of Lemma 3.3 can be done along the same line as in our previous paper [18] and we omit the details for brevity.

Now with these uniform in ε, ν estimates at hand, we can pass the limit process $\varepsilon \rightarrow 0$ with $\nu = \varepsilon^{\frac{2}{3}}$, obtain the existence of the weak solution $(\rho, u)(t, x)$, and get the uniform in time estimates in Theorem 2.1.

4 Asymptotic behavior of weak solutions

In this section, we will study the asymptotic behavior of a given weak solution $(\rho, u)(t, x)$ to the Cauchy problem (1.1)-(1.3) in the sense of Definition 2.1 satisfying (2.18) and (2.19). We assume that the solution is smooth enough. The rigorous proof can be obtained by using the usual regularization procedure.

Proof of Theorem 2.2: For any $s \geq 1$, by the uniform upper bound of ρ , it holds that

$$|\rho^s - \bar{\rho}^s|^2 \leq C|\rho - \bar{\rho}|^2.$$

Hence it follows from (2.19) and (3.53) that

$$\int_{\mathbf{R}} |\rho^s - \bar{\rho}^s|^2 dx \leq C \int_{\mathbf{R}} (\rho - \bar{\rho})^2 dx \leq C. \quad (4.1)$$

Similarly,

$$\int_{\mathbf{R}} |\rho^s - \bar{\rho}^s|^{2\lambda} dx \leq C \int_{\mathbf{R}} |\rho - \bar{\rho}|^{2\lambda} dx \leq C \int_{\mathbf{R}} (\rho - \bar{\rho})^2 dx \leq C, \quad (4.2)$$

for any $\lambda \geq 1$.

Set $b = \frac{\alpha + \gamma - 1}{2}$. Then one gets from (2.19) and (3.53) that

$$\int_0^t \int_{\mathbf{R}} \{[(\rho^b - \bar{\rho}^b)_x]^2 + \bar{u}_x(\rho - \bar{\rho})^2\} dx d\tau \leq C.$$

For $s > b + 1$, it holds that

$$\begin{aligned} (\rho^s - \bar{\rho}^s)^2(t, x) &= \int_{-\infty}^x [(\rho^s - \bar{\rho}^s)_x]^2 dx \\ &= 2 \int_{-\infty}^x (\rho^s - \bar{\rho}^s)(\rho^s - \bar{\rho}^s)_x dx \\ &= 2s \int_{-\infty}^x (\rho^s - \bar{\rho}^s)[(\rho^b - \bar{\rho}^b)_x \rho^{s-1} + (\bar{\rho}^b)_x (\rho^{s-b} - \bar{\rho}^{s-b})] dx \\ &\leq C \|\rho^s - \bar{\rho}^s\|_{L^2(\mathbf{R})} \|(\rho^b - \bar{\rho}^b)_x\|_{L^2(\mathbf{R})} + C \int_{\mathbf{R}} \bar{\rho}^{\frac{\alpha}{2}} \bar{u}_x (\rho^s - \bar{\rho}^s)(\rho^{s-b} - \bar{\rho}^{s-b}) dx \\ &\leq \|\rho^s - \bar{\rho}^s\|_{L^2(\mathbf{R})} \|(\rho^b - \bar{\rho}^b)_x\|_{L^2(\mathbf{R})} + C \int_{\mathbf{R}} \bar{u}_x (\rho - \bar{\rho})^2 dx. \end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_0^t \sup_{x \in \mathbf{R}} (\rho^s - \bar{\rho}^s)^4 dt \\
& \leq C \sup_{t \in [0, T]} \|\rho^s - \bar{\rho}^s\|_{L^2(\mathbf{R})}^2 \int_0^t \|(\rho^b - \bar{\rho}^b)_x\|_{L^2(\mathbf{R})}^2 dt \\
& \quad + C \sup_{t \in [0, T]} \|\rho - \bar{\rho}\|_{L^2(\mathbf{R})}^2 \int_0^t \int_{\mathbf{R}} \bar{u}_x (\rho - \bar{\rho})^2 dx d\tau \\
& \leq C.
\end{aligned} \tag{4.3}$$

Moreover, applying (4.2) leads to

$$\begin{aligned}
& \int_0^t \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^4 (\rho^s - \bar{\rho}^s)^{2l} dx dt \\
& \leq \int_0^t \left[\sup_{x \in \mathbf{R}} (\rho^s - \bar{\rho}^s)^4 \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2l} dx \right] dt \\
& \leq \sup_{t \in [0, T]} \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2l} dx \int_0^t \sup_{x \in \mathbf{R}} (\rho^s - \bar{\rho}^s)^4 dt \leq C, \quad \forall l \geq 1.
\end{aligned} \tag{4.4}$$

Set

$$f(t) = \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{4+2l} dx.$$

Then

$$f(t) \in L^1(0, \infty) \cap L^\infty(0, \infty)$$

due to (4.2) and (4.4).

Furthermore, direct calculations show that

$$\begin{aligned}
\frac{d}{dt} f(t) &= (4 + 2l)s \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{3+2l} (\rho^{s-1} \rho_t - \bar{\rho}^{s-1} \bar{\rho}_t) dx \\
&= -(4 + 2l)s \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{3+2l} [\rho^{s-1} (\rho u)_x - \bar{\rho}^{s-1} (\bar{\rho} \bar{u})_x] dx \\
&= (4 + 2l)(3 + 2l)s \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2+2l} (\rho^s - \bar{\rho}^s)_x (\rho^{s-1} \rho u - \bar{\rho}^{s-1} \bar{\rho} \bar{u}) dx \\
& \quad + (4 + 2l)s(s - 1) \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{3+2l} (\rho^{s-2} \rho_x \rho u - \bar{\rho}^{s-2} \bar{\rho}_x \bar{\rho} \bar{u}) dx \\
&= (4 + 2l)(3 + 2l)s \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2+2l} (\rho^s - \bar{\rho}^s)_x \rho^s (u - \bar{u}) dx \\
& \quad + (4 + 2l)(3 + 2l)s \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2+2l} (\rho^s - \bar{\rho}^s)_x \bar{u} (\rho^s - \bar{\rho}^s) dx \\
& \quad + (4 + 2l)s(s - 1) \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{3+2l} \rho^{s-1} \rho_x (u - \bar{u}) dx \\
& \quad + (4 + 2l)s(s - 1) \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{3+2l} \bar{u} (\rho^{s-1} \rho_x - \bar{\rho}^{s-1} \bar{\rho}_x) dx \\
&:= J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned} \tag{4.5}$$

Now we claim that $J_i(t) \in L^2(0, +\infty)$, ($i = 1, 2, 3, 4$). In fact,

$$\begin{aligned}
J_1(t) &= \frac{(4+2l)(3+2l)s^2}{b} \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2+2l} \rho^s (u - \bar{u}) \left[\rho^{s-b} (\rho^b - \bar{\rho}^b)_x + (\rho^{s-b} - \bar{\rho}^{s-b}) (\bar{\rho}^b)_x \right] dx \\
&= \frac{(4+2l)(3+2l)s^2}{b} \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2+2l} \sqrt{\rho} (u - \bar{u}) \rho^{2s-b-\frac{1}{2}} (\rho^b - \bar{\rho}^b)_x dx \\
&\quad + \frac{(4+2l)(3+2l)s^2}{b} \int_{\mathbf{R}} (\rho^s - \bar{\rho}^s)^{2+2l} \sqrt{\rho} (u - \bar{u}) \rho^{s-\frac{1}{2}} (\rho^{s-b} - \bar{\rho}^{s-b}) (\bar{\rho}^b)_x dx \\
&\leq C \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \|(\rho^b - \bar{\rho}^b)_x\|_{L^2(\mathbf{R})} \\
&\quad + C \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \|(\rho^s - \bar{\rho}^s)^{2+2l} (\rho^{s-b} - \bar{\rho}^{s-b}) (\bar{\rho}^b)_x\|_{L^2(\mathbf{R})} \\
&\leq C \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \|(\rho^b - \bar{\rho}^b)_x\|_{L^2(\mathbf{R})} + C \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})} \|\bar{u}_x(\rho - \bar{\rho})\|_{L^2(\mathbf{R})}.
\end{aligned} \tag{4.6}$$

Thus,

$$\begin{aligned}
\int_0^t |J_1(t)|^2 dt &\leq C \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}^2 \int_0^t \|(\rho^b - \bar{\rho}^b)_x\|_{L^2(\mathbf{R})}^2 dt \\
&\quad + C \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}^2 \int_0^t \|\bar{u}_x(\rho - \bar{\rho})\|_{L^2(\mathbf{R})}^2 dt \\
&\leq C \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}^2 \int_0^t \|(\rho^b - \bar{\rho}^b)_x\|_{L^2(\mathbf{R})}^2 dt \\
&\quad + C \sup_{t \in [0, T]} \|\sqrt{\rho}(u - \bar{u})\|_{L^2(\mathbf{R})}^2 \int_0^t \int_{\mathbf{R}} \bar{u}_x(\rho - \bar{\rho})^2 dx dt \\
&\leq C.
\end{aligned} \tag{4.7}$$

The fact that $J_i(t) \in L^2(0, +\infty)$, ($i = 2, 3, 4$) can be shown similarly.

Thus

$$\frac{d}{dt} f(t) \in L^2(0, +\infty).$$

Combining the fact that $f(t) \in L^1(0, +\infty) \cap L^\infty(0, +\infty)$, one has

$$f(t) \rightarrow 0, \quad t \rightarrow +\infty. \tag{4.8}$$

Let $m \geq 1$ be any real number to be determined later. One has

$$\begin{aligned}
|(\rho^s - \bar{\rho}^s)^m| &= \left| \int_{-\infty}^x [(\rho^s - \bar{\rho}^s)^m]_x dx \right| \\
&= \left| m \int_{-\infty}^x (\rho^s - \bar{\rho}^s)^{m-1} (\rho^s - \bar{\rho}^s)_x dx \right| \\
&= \left| ms \int_{-\infty}^x (\rho^s - \bar{\rho}^s)^{m-1} \left[\frac{1}{\alpha - \frac{1}{2}} (\rho^{\alpha-\frac{1}{2}})_x \rho^{s-\alpha+\frac{1}{2}} - \bar{\rho}^{s-1} \bar{\rho}_x \right] dx \right| \\
&\leq C \|(\rho^s - \bar{\rho}^s)^{m-1}\|_{L^2(\mathbf{R})} \left[\|(\rho^{\alpha-\frac{1}{2}})_x\|_{L^2(\mathbf{R})} + \|\bar{\rho}^{\frac{3-\gamma}{2}} \bar{u}_x\|_{L^2(\mathbf{R})} \right] \\
&\leq C \|(\rho^s - \bar{\rho}^s)^{m-1}\|_{L^2(\mathbf{R})}.
\end{aligned} \tag{4.9}$$

Choosing $2(m-1) = 4 + 2l$ yields

$$\sup_{x \in \mathbf{R}} |(\rho^s - \bar{\rho}^s)^m| \leq C f^{\frac{1}{2}}(t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty. \tag{4.10}$$

Therefore,

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbf{R}} |\rho^s - \bar{\rho}^s| = 0.$$

Now we prove that $\lim_{t \rightarrow +\infty} \sup_{x \in \mathbf{R}} |\rho - \bar{\rho}| = 0$.

Since

$$\lim_{\bar{\rho} \rightarrow 0^+} \frac{|\rho - \bar{\rho}|^s}{|\rho^s - \bar{\rho}^s|} = 1, \quad \text{uniformly in } \rho,$$

then for any $\sigma > 0$, there exists $\delta_\sigma > 0$, such that if $0 \leq \bar{\rho} < \delta_\sigma$, then

$$\left| \frac{|\rho - \bar{\rho}|^s}{|\rho^s - \bar{\rho}^s|} - 1 \right| \leq \sigma.$$

Thus, fix $\sigma = \frac{1}{2}$, then if $0 \leq \bar{\rho} < \delta := \delta_{\frac{1}{2}}$, one has for any $\rho \geq 0$,

$$|\rho - \bar{\rho}|^s \leq \frac{3}{2} |\rho^s - \bar{\rho}^s|. \quad (4.11)$$

Now

$$\begin{aligned} |\rho - \bar{\rho}|^s &= |\rho - \bar{\rho}|^s (\mathbf{1}_{\{0 \leq \bar{\rho} < \delta\}} + \mathbf{1}_{\{\bar{\rho} \geq \delta, 0 \leq \rho < \frac{\bar{\rho}}{2}\}} + \mathbf{1}_{\{\bar{\rho} \geq \delta, \rho > \frac{\bar{\rho}}{2}\}}) \\ &\leq \frac{3}{2} |\rho^s - \bar{\rho}^s| \mathbf{1}_{\{0 \leq \bar{\rho} < \delta\}} + C_\delta |\rho^s - \bar{\rho}^s| \mathbf{1}_{\{\bar{\rho} \geq \delta, 0 \leq \rho < \frac{\bar{\rho}}{2}\}} + C_\delta |\rho^s - \bar{\rho}^s| \mathbf{1}_{\{\bar{\rho} \geq \delta, \rho > \frac{\bar{\rho}}{2}\}}. \end{aligned}$$

Therefore,

$$\sup_{x \in \mathbf{R}} |\rho - \bar{\rho}|^s \leq C_\delta \sup_{x \in \mathbf{R}} |\rho^s - \bar{\rho}^s| + C_\delta \sup_{x \in \mathbf{R}} |\rho^s - \bar{\rho}^s|^s \rightarrow 0,$$

as $t \rightarrow +\infty$, which implies that

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbf{R}} |\rho - \bar{\rho}| = 0.$$

The proof of Theorem 2.2 is finished.

5 Regularity of the solution away from the vacuum

In this section, we will prove Theorem 2.3, that is, we will show that away from the vacuum region of the 2-rarefaction wave $(\rho^r, u^r)(\xi)$, any weak solution $(\rho, u)(t, x)$ to the Cauchy problem (1.1)-(1.3) satisfying (2.18) and (2.19) becomes regular as stated in Theorem 2.3.

Due to the definition of the 2-rarefaction wave $\rho^r(t, x)$ in (2.8), for any fixed $\sigma > 0$, there exist a unique u_σ such that $(\rho, u) = (\sigma, u_\sigma)$ lies on the 2-rarefaction wave curve. In fact,

$$u_\sigma = \frac{2\sqrt{\gamma}}{\gamma-1} \sigma^{\frac{\gamma-1}{2}} + \Sigma_2(\rho_+, u_+), \quad (5.1)$$

since 2-Riemann invariant $\Sigma_2(\rho, u) = u - \frac{2\sqrt{\gamma}}{\gamma-1} \rho^{\frac{\gamma-1}{2}}$ is constant along the 2-rarefaction wave curve. Due to the expanding property of the 2-rarefaction wave, we have the lower bound of the density function $\rho^r(t, x)$ of 2-rarefaction wave on the right of the straight line $x = \lambda_2^\sigma t$ with $\lambda_2^\sigma = \lambda_2(\sigma, u_\sigma) = \lambda_2(\rho, u)|_{(\rho, u) = (\sigma, u_\sigma)}$, that is,

$$\rho^r(t, x) \geq \sigma, \quad \text{if } x \geq \lambda_2^\sigma t. \quad (5.2)$$

Thus it follows from the asymptotic behavior (2.22) of $\rho(t, x)$ that for $\frac{\sigma}{2} > 0$, there exists a large time T_σ such that if $t > T_\sigma$, then

$$\|\rho(t, \cdot) - \rho^r(t, \cdot)\|_{L^\infty(\mathbf{R})} \leq \frac{\sigma}{2}. \quad (5.3)$$

Therefore, in the domain

$$\Omega_\sigma := \{(t, x) | t > T_\sigma, x > \lambda_2^\sigma t\}, \quad (5.4)$$

it holds that

$$\rho(t, x) \geq \frac{\sigma}{2}, \quad \text{if } (t, x) \in \Omega_\sigma.$$

So in the domain Ω_σ , any vacuum states vanish and thus the higher regularity of the weak solution $(\rho, u)(t, x)$ can be expected as stated in Theorem 2.3. In the following, we give the proof of Theorem 2.3. First we establish the local uniform boundedness of the velocity $u(x, t)$ by the De Giorgi-Moser iteration method. To this end, we rewrite the momentum equation as

$$ut + uu_x + \gamma\rho^{\gamma-2}\rho_x = \rho^{\alpha-1}u_{xx} + \alpha\rho^{\alpha-2}\rho_x u_x, \quad (5.5)$$

For any $(t_*, x_*) \in \Omega_\sigma$, and for any $r, s > 0$ such that $Q_{r,s}^* := B_r(x_*) \times (t_*, t_* + s] \subset \Omega_\sigma$, and for any test function $\zeta(x, t) \in \dot{W}_2^{1,1}(Q_{r,s}^*)$ satisfying $0 \leq \zeta \leq 1$, one can get from the uniform estimates in (2.18) and (2.19) that

$$\sup_{t_* \leq t \leq t_* + s} \int_{B_r(x_*)} \left[(u - \bar{u})^2 + \rho_x^2 + (\rho - \bar{\rho})^2 \right] (x, t) dx + \int_{t_*}^t \int_{B_r(x_*)} \left[(u - \bar{u})_x^2 + (\rho - \bar{\rho})_x^2 \right] dx dt \leq C. \quad (5.6)$$

It follows from the construction of the rarefaction wave $(\bar{\rho}, \bar{u})(x, t)$ that

$$\sup_{t_* \leq t \leq t_* + s} \int_{B_r(x_*)} (u^2 + \rho_x^2 + \rho^2)(x, t) dx + \int_{t_*}^t \int_{B_r(x_*)} (u_x^2 + \rho_x^2) dx dt \leq C. \quad (5.7)$$

Multiplying the equation (5.5) by $\zeta^2(u - k)_+$ for any $k \in \mathbf{R}$ and integrating the resulted equation over $B_r(x_*) \times (t_*, t]$ for $t \in (t_*, t_* + s]$, one arrives

$$\begin{aligned} & \frac{1}{2} \int_{B_r(x_*)} \zeta^2 (u - k)_+^2(x, t) dx + \int_{t_*}^t \int_{B_r(x_*)} \rho^{\alpha-1} [\zeta (u - k)_+]_x^2 dx dt \\ &= \frac{1}{2} \int_{B_r(x_*)} \zeta^2 (u - k)_+^2(x, t_*) dx + \int_{t_*}^t \int_{B_r(x_*)} \left\{ \zeta \zeta_t (u - k)_+^2 + \frac{2}{3} \zeta \zeta_x [(u - k)_+]^3 \right. \\ & \quad \left. - \gamma \rho^{\gamma-2} \rho_x \zeta^2 (u - k)_+ + \rho^{\alpha-1} \zeta_x^2 (u - k)_+^2 + \rho^{\alpha-2} \rho_x u_x \zeta^2 (u - k)_+ \right\} dx dt \\ &\leq \frac{1}{2} \int_{B_r(x_*)} \zeta^2 (u - k)_+^2(x, t_*) dx + \frac{1}{8} \int_{t_*}^t \int_{B_r(x_*)} \rho^{\alpha-1} [\zeta (u - k)_+]_x^2 dx dt \\ & \quad + C \int_{t_*}^t \int_{B_r(x_*)} \left\{ (|\zeta_t| + |\zeta_x|^2) (u - k)_+^2 + \zeta |\zeta_x| [(u - k)_+]^3 + |\rho_x| \zeta^2 (u - k)_+ \right. \\ & \quad \left. + |\rho_x|^2 \zeta^2 (u - k)_+^2 \right\} dx dt, \end{aligned} \quad (5.8)$$

where in the last inequality one has used the fact

$$\begin{aligned} |\rho^{\alpha-2} \rho_x u_x \zeta^2 (u - k)_+| &\leq C |\rho_x| |\zeta u_x| \zeta (u - k)_+ \\ &= C |\rho_x| |(\zeta u)_x - \zeta_x u| \zeta (u - k)_+ \\ &\leq C |\rho_x| |(\zeta u)_x| \zeta (u - k)_+ + C |\rho_x| \zeta |\zeta_x| (u - k)_+^2 \\ &= C |\rho_x| |[\zeta (u - k)_+]_x| \zeta (u - k)_+ + C |\rho_x| \zeta |\zeta_x| (u - k)_+^2 \\ &\leq \frac{1}{8} \rho^{\alpha-1} [\zeta (u - k)_+]_x^2 + C |\rho_x|^2 \zeta^2 (u - k)_+^2 + C |\zeta_x|^2 (u - k)_+^2. \end{aligned}$$

Thus from (5.8), it holds that

$$\begin{aligned}
& \int_{B_r(x_*)} \zeta^2(u-k)_+^2(x,t)dx + \int_{t_*}^t \int_{B_r(x_*)} [\zeta(u-k)_+]_x^2 dxdt \\
& \leq C \int_{B_r(x_*)} \zeta^2(u-k)_+^2(x,t_*)dx + C \int_{t_*}^t \int_{B_r(x_*)} \left\{ (|\zeta_t| + |\zeta_x|^2)(u-k)_+^2 \right. \\
& \quad \left. + \zeta|\zeta_x|[(u-k)_+]^3 + |\rho_x|\zeta^2(u-k)_+ + |\rho_x|^2\zeta^2(u-k)_+^2 \right\} dxdt.
\end{aligned} \tag{5.9}$$

Now the last three terms in the last integral of (5.9) can be estimated by

$$\begin{aligned}
& \int_{t_*}^t \int_{B_r(x_*)} \zeta|\zeta_x|[(u-k)_+]^3 dxdt \leq C \int_{t_*}^t \|\zeta_x(u-k)_+\|_{L^2(B_r(x_*))} \|\zeta(u-k)_+\|_{L^2(B_r(x_*))}^2 dt \\
& \leq C \int_{t_*}^t \int_{B_r(x_*)} |\zeta_x|^2(u-k)_+^2 dxdt + C \int_{t_*}^t \int_{B_r(x_*)} \zeta^2(u-k)_+^4 dxdt \\
& \leq C \int_{t_*}^t \int_{B_r(x_*)} |\zeta_x|^2(u-k)_+^2 dxdt + C \int_{t_*}^t \|\zeta(u-k)_+\|_{L^\infty(B_r(x_*))}^2 \|(u-k)_+\|_{L^2(B_r(x_*))}^2 dt \\
& \leq C \int_{t_*}^t \int_{B_r(x_*)} |\zeta_x|^2(u-k)_+^2 dxdt + C \int_{t_*}^t \|\zeta(u-k)_+\|_{L^\infty(B_r(x_*))}^2 dt \\
& \leq C \int_{t_*}^t \int_{B_r(x_*)} |\zeta_x|^2(u-k)_+^2 dxdt + C \int_{t_*}^t \|\zeta(u-k)_+\|_{L^2(B_r(x_*))} \|\zeta(u-k)_+\|_x \|L^2(B_r(x_*))\| dt \\
& \leq C \int_{t_*}^t \int_{B_r(x_*)} |\zeta_x|^2(u-k)_+^2 dxdt + \frac{1}{8} \int_{t_*}^t \|[\zeta(u-k)_+]_x\|_{L^2(B_r(x_*))}^2 dt \\
& \quad + C \int_{t_*}^t \|\zeta(u-k)_+\|_{L^2(B_r(x_*))}^2 dt,
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
& \int_{t_*}^t \int_{B_r(x_*)} |\rho_x|\zeta^2(u-k)_+ dxdt \\
& = \iint_{Q_{r,s}^* \cap [u>k]} |\rho_x|\zeta^2(u-k)_+ dxdt \\
& \leq C \|\zeta(u-k)_+\|_{L^6(Q_{r,s}^*)} \|\rho_x\|_{L^{\frac{6}{5}}(Q_{r,s}^* \cap [u>k])} \\
& \leq C \|\zeta(u-k)_+\|_{V_2(Q_{r,s}^*)} \|\rho_x\|_{L^2(Q_{r,s}^*)} |Q_{r,s}^* \cap [u>k]|^{\frac{1}{3}} \\
& \leq \frac{1}{8} \|\zeta(u-k)_+\|_{V_2(Q_{r,s}^*)}^2 + C |Q_{r,s}^* \cap [u>k]|^{\frac{2}{3}},
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
& \int_{t_*}^t \int_{B_r(x_*)} |\rho_x|^2 \zeta^2(u-k)_+^2 dxdt \\
& \leq C \sup_t \|\rho_x\|_{L^2(B_r(x_*))} \int_{t_*}^t \|\zeta(u-k)_+\|_{L^\infty(B_r(x_*))}^2 dt \\
& \leq C \int_{t_*}^t \|\zeta(u-k)_+\|_{L^2(B_r(x_*))} \|\zeta(u-k)_+\|_x \|L^2(B_r(x_*))\| dt \\
& \leq \frac{1}{8} \int_{t_*}^t \|[\zeta(u-k)_+]_x\|_{L^2(B_r(x_*))}^2 dt + C \int_{t_*}^t \|\zeta(u-k)_+\|_{L^2(B_r(x_*))}^2 dt.
\end{aligned} \tag{5.12}$$

Note that in (5.11), the space $V_2(Q_{r,s}^*)$ is defined by $V_2(Q_{r,s}^*) = \{f \in L^2(Q_{r,s}^*) \mid \|f\|_{V_2(Q_{r,s}^*)} < +\infty\}$, with the norm

$$\|f\|_{V_2(Q_{r,s}^*)} = \operatorname{ess\,sup}_{t \in [t_*, t_*+s]} \|f(\cdot, t)\|_{L^2(B_r(x_*))} + \|f_x\|_{L^2(Q_{r,s}^*)}.$$

Substituting (5.10)-(5.12) into (5.8) implies

$$\begin{aligned}
& \int_{B_r(x_*)} \zeta^2(u-k)_+^2(x,t)dx + \int_{t_*}^t \int_{B_r(x_*)} [\zeta(u-k)_+]_x^2 dxdt \\
& \leq \int_{B_r(x_*)} \zeta^2(u-k)_+^2(x,t_*)dx + C \int_{t_*}^t \int_{B_r(x_*)} \zeta^2(u-k)_+^2 dxdt \\
& + \frac{1}{8} \|\zeta(u-k)_+\|_{V_2(Q_{r,s}^*)}^2 + C \int_{t_*}^t \int_{B_r(x_*)} (|\zeta_t| + |\zeta_x|^2)(u-k)_+^2 dxdt + C|Q_{r,s}^* \cap [u > k]|^{\frac{2}{3}}.
\end{aligned} \tag{5.13}$$

Applying Gronwall's inequality to (5.13) gives

$$\begin{aligned}
& \sup_{t_* \leq t \leq t_*+s} \int_{B_r(x_*)} \zeta^2(u-k)_+^2(x,t)dx + \int_{t_*}^{t_*+s} \int_{B_r(x_*)} [\zeta(u-k)_+]_x^2 dxdt \\
& \leq \int_{B_r(x_*)} \zeta^2(u-k)_+^2(x,t_*)dx + C \int_{t_*}^{t_*+s} \int_{B_r(x_*)} (|\zeta_t| + |\zeta_x|^2)(u-k)_+^2 dxdt + C|Q_{r,s}^* \cap [u > k]|^{\frac{2}{3}}.
\end{aligned} \tag{5.14}$$

With the estimate (5.14) at hand, one can show that u is bounded above locally by the classical De Giorgi-Moser iteration method and choosing suitably k . Similarly, one can obtain the estimates to $(u-k)_-$ as in (5.14). Thus we can get the lower bound for u locally. Furthermore, one can get the local Hölder estimates of u by the classical parabolic theory, that is, there exists a positive constant $\alpha_0 \in (0, 1)$, such that

$$u \in C_{\text{loc}}^{\alpha_0, \frac{\alpha_0}{2}}(Q_{r,s}^*).$$

In the following, we will further show that the weak solution $u(x, t)$ is in fact a strong solution locally as stated in Theorem 2.3. Rewrite the momentum equation as

$$\rho u_t + \rho u u_x + \gamma \rho^{\gamma-1} \rho_x = \rho^\alpha u_{xx} + \alpha \rho^{\alpha-1} \rho_x u_x. \tag{5.15}$$

Multiplying (5.15) by $\zeta^2 u_t$ and integrating the result over $B_r(x_*) \times (t_*, t]$ with $t \in (t_*, t_* + s)$, one can get

$$\begin{aligned}
& \int_{t_*}^t \int_{B_r(x_*)} \rho \zeta^2 u_t^2 dxdt \\
& = \int_{t_*}^t \int_{B_r(x_*)} \left(-\rho u \zeta^2 u_t u_x - \gamma \rho^{\gamma-1} \rho_x \zeta^2 u_t + \rho^\alpha \zeta^2 u_t u_{xx} + \alpha \rho^{\alpha-1} \rho_x u_x \zeta^2 u_t \right) dxdt \\
& \leq \frac{1}{2} \int_{t_*}^t \int_{B_r(x_*)} \rho \zeta^2 u_t^2 dxdt + C \int_{t_*}^t \int_{B_r(x_*)} \left(\rho u^2 \zeta^2 u_x^2 + \rho^{2\gamma-3} \rho_x^2 \zeta^2 \right. \\
& \quad \left. + \rho^{2\alpha-1} \zeta^2 u_{xx}^2 + \rho^{2\alpha-3} \zeta^2 \rho_x^2 u_x^2 \right) dxdt.
\end{aligned} \tag{5.16}$$

By the uniform upper bound and lower bound of the density $\rho(x, t)$ in the domain Ω_σ and the local boundedness of the velocity $u(x, t)$, it follows from (5.16) that

$$\begin{aligned}
& \int_{t_*}^t \int_{B_r(x_*)} \zeta^2 u_t^2 dxdt \leq C \int_{t_*}^t \int_{B_r(x_*)} \left(\zeta^2 u_x^2 + \zeta^2 \rho_x^2 + \zeta^2 u_{xx}^2 + \zeta^2 \rho_x^2 u_x^2 \right) dxdt \\
& \leq C \int_{t_*}^t \int_{B_r(x_*)} \left[\zeta^2 (u - \bar{u})_x^2 + \zeta^2 (\rho - \bar{\rho})_x^2 + \zeta^2 (\bar{u}_x^2 + \bar{\rho}_x^2) \right] dxdt \\
& \quad + C \int_{t_*}^t \int_{B_r(x_*)} \left(\zeta^2 u_{xx}^2 + \zeta^2 \rho_x^2 u_x^2 \right) dxdt.
\end{aligned} \tag{5.17}$$

Note that

$$\begin{aligned}
\int_{t_*}^t \int_{B_r(x_*)} \zeta^2 \rho_x^2 u_x^2 dx dt &\leq \int_{t_*}^t \|\rho_x\|_{L^2(B_r(x_*))}^2 \|\zeta u_x\|_{L^\infty(B_r(x_*))}^2 dt \\
&\leq \sup_{t \in (t_*, t_* + s]} \|\rho_x\|_{L^2(B_r(x_*))}^2 \int_{t_*}^t \|\zeta u_x\|_{L^2(B_r(x_*))} \|\zeta u_x\|_{L^2(B_r(x_*))} dt \\
&\leq \beta \int_{t_*}^t \int_{B_r(x_*)} \zeta^2 u_{xx}^2 dx dt + C_\beta \int_{t_*}^t \int_{B_r(x_*)} (\zeta^2 u_x^2 + \zeta_x^2 u_x^2) dx dt \\
&\leq \beta \int_{t_*}^t \int_{B_r(x_*)} \zeta^2 u_{xx}^2 dx dt + C_\beta,
\end{aligned} \tag{5.18}$$

where β is a small positive constant to be determined and C_β is the positive constant depending on β .

Thus it follows from (5.7), (5.18) with $\beta = 1$, and (5.17) that

$$\int_{t_*}^t \int_{B_r(x_*)} \zeta^2 u_t^2 dx dt \leq C + C \int_{t_*}^t \int_{B_r(x_*)} \zeta^2 u_{xx}^2 dx dt. \tag{5.19}$$

Multiplying (5.5) by $\zeta^2 u_{xx}$ and integrating over $B_r(x_*) \times (t_*, t]$ with $t \in (t_*, t_* + s)$, one can get

$$\begin{aligned}
&\int_{t_*}^t \int_{B_r(x_*)} \rho^{\alpha-1} \zeta^2 u_{xx}^2 dx dt \\
&= \int_{t_*}^t \int_{B_r(x_*)} \left(\zeta^2 u_t u_{xx} + \zeta^2 u u_x u_{xx} + \gamma \rho^{\gamma-2} \rho_x \zeta^2 u_{xx} - \alpha \rho^{\alpha-2} \rho_x u_x \zeta^2 u_{xx} \right) dx dt.
\end{aligned} \tag{5.20}$$

Note that

$$\begin{aligned}
\zeta^2 u_t u_{xx} &= (\zeta^2 u_t u_x)_x - \zeta^2 u_x u_{xt} - 2\zeta \zeta_x u_t u_x \\
&= (\zeta^2 u_t u_x)_x - \left(\frac{\zeta^2 u_x^2}{2} \right)_t + \zeta \zeta_t u_x^2 - 2\zeta \zeta_x u_t u_x,
\end{aligned} \tag{5.21}$$

thus it holds that

$$\int_{t_*}^t \int_{B_r(x_*)} \zeta^2 u_t u_{xx} dx dt = - \int_{B_r(x_*)} \frac{\zeta^2 u_x^2}{2}(x, t) dx + \int_{t_*}^t \int_{B_r(x_*)} (\zeta \zeta_t u_x^2 - 2\zeta \zeta_x u_t u_x) dx dt. \tag{5.22}$$

Substituting (5.22) into (5.20) gives

$$\begin{aligned}
&\frac{1}{2} \int_{B_r(x_*)} \zeta^2 u_x^2(x, t) dx + \int_{t_*}^t \int_{B_r(x_*)} \zeta^2 u_{xx}^2 dx dt \\
&= \int_{t_*}^t \int_{B_r(x_*)} \left(\zeta \zeta_t u_x^2 - 2\zeta \zeta_x u_t u_x + \zeta^2 u u_x u_{xx} + 2\rho_x \zeta^2 u_{xx} - \frac{\rho_x}{\rho} u_x \zeta^2 u_{xx} \right) dx dt \\
&\leq \beta \int_{t_*}^t \int_{B_r(x_*)} (\zeta^2 u_t^2 + \zeta^2 u_{xx}^2) dx dt + C_\beta \int_{t_*}^t \int_{B_r(x_*)} \left[(\zeta_t^2 + \zeta_x^2) u_x^2 + \zeta^2 (\rho_x^2 + u_x^2) + \zeta^2 \rho_x^2 u_x^2 \right] dx dt.
\end{aligned} \tag{5.23}$$

Combining the estimates (5.18), (5.19) and (5.23) and choosing both β suitably small, we can get

$$\frac{1}{2} \int_{B_r(x_*)} \zeta^2 u_x^2(x, t) dx + \int_{t_*}^t \int_{B_r(x_*)} \zeta^2 (u_t^2 + u_{xx}^2) dx dt \leq C. \tag{5.24}$$

This completes the proof of Theorem 2.3.

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