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Vacuum Energies of String Compactified on Torus

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Computation of the one-loop vacuum energy is attempted for closed bosonic string compactified on various tori. Modular invariance of the one-loop vacuum energy is shown. The divergent tachyon contribution forces us to employ a subtraction prescription. For one-dimensional torus, the affine Kac-Moody algebra for $SU(2)\times SU(2)$ is realized at the absolute minimum of the vacuum energy. For general r-dimensional torus, the algebra for $[SU(2)\times SU(2)]^r$ is found to be an unstable saddle point. A detailed study of $r=2$ case shows that $SU(3)\times SU(3)$ has the lowest vacuum energy.

§ 1. Introduction

At present string theories are best candidates for a unified theory including gravity. Since string theories can exist only in specific dimensions of space-time,[1),2)] compactification of extra dimensions is of great interest. The string compactification is severely constrained by the requirement of no anomaly for the reparametrization invariance in the string world sheet. Generators of infinitesimal reparametrizations satisfy the Virasoro algebra.[3)] The central charge of the Virasoro algebra should be unchanged under compactification in order to avoid quantum anomaly for the infinitesimal reparametrizations. Moreover the compactified string must be invariant under large reparametrizations which cannot be reached continuously from the identity. This requirement manifests itself as the modular invariance in one-loop amplitudes.[4),5)]

Among many possible manifolds, compactifications on torus[6),7)] satisfy at least the consistency for the infinitesimal reparametrization invariance. This is because the commutation relations among string variables are a local property and unchanged by the compactification on torus (flat space). Orientable closed strings compactified on tori corresponding to special lattices realize an extremely interesting structure —— the affine Kac-Moody algebras: They give rise to solitons which become gauge bosons to form a semi-simple group together with the Kaluza-Klein $U(1)$ gauge bosons.[8),9)] In particular, tori associated with self-dual lattices are used to construct the the heterotic string.[10)]

The true vacuum state in quantum field theory is given by the configuration with the lowest vacuum energy. The purpose of this paper is to compute and to compare the one-loop vacuum energy of string compactified on various tori. Superstrings give vanishing vacuum energy for torus compactifications if supersymmetry is intact.[11),12)] Hence we take the bosonic orientable closed string in 26 dimensions and calculate the vacuum energy density in the $(26-r)$ dimensions when r dimensions are compactified on various tori. We shall see that there are two classes of r-dimensional lattices which give consistent torus compactifications. The first class is based on our intuitive concept of torus compactification, and allows tori with arbitrary sizes and angles. We shall demonstrate the modular invariance explicitly for this class of compactifications. The second class consists of only special lattices corresponding to the affine Kac-Moody algebras.[9),13)] One should note that the first class allows sizes and angles of the lattice to be changed continuously, whereas only isolated values are admitted for the second class.

In closed string one-loop amplitudes, integration regions of string parameters related by the modular transformation give equivalent contributions of a single string configuration which must not be double-counted.[4),5)] After being restricted to one integration region such as the fundamental region, one-loop amplitudes with external particles have no divergences except those interpretable as self-energy insertions in external lines.[4),14)] The vacuum energy, however, is divergent due to tachyons in the loop even after restricted to the fundamental region. Having no new ideas to solve the tachyon problem properly, we shall adopt a simple prescription of subtracting contributions of the tachyon in the fundamental region.

With this tentative prescription for the vacuum energy subtraction, we find that the affine Kac-Moody algegra for $SU(2)\times SU(2)$ is realized at the minimum of the vacuum energy when one dimension is compactified. If r dimensions are compactified on torus of the first class, we find an unstable saddle point of vacuum energy at the symmetric point where the affine Kac-Moody algebra for $[SU(2)\times SU(2)]^r$ is realized. A detailed analysis for the $r=2$ case shows that the lattice realizing the affine Kac-Moody algebra for $SU(3)\times SU(3)$ has the lowest vacuum energy.

In the next section string compactification on one-dimensional torus is summarized and its vacuum energy is derived. In § 3, compactification on various r-dimensional tori is treated and the two-dimensional case is analyzed in detail. Useful formulas for numerical analysis of the vacuum energy is summarized in the Appendix.

§ 2. Compactification on one-dimensional torus

Although the compactification on one-dimensional torus is well-known,[6),7),10)] we shall briefly summarize the formalism. When we identify points which differ by $2\pi R$ in a spatial direction, the corresponding momenta become discrete

$$p=m/R\,, \qquad m=\text{integer}\,, \tag{2·1}$$

because of $[x, p]=i$. The boundary condition for an orientable closed string*)

$$X(\tau, \sigma=\pi)=X(\tau, \sigma=0) \qquad \text{mod. } 2\pi R\,, \tag{2·2}$$

admits winding numbers L in units of R

$$L=lR\,, \qquad l=\text{integer}\,. \tag{2·3}$$

Hence an additional term appears in the normal mode expansion in the compactified direction

$$X(\tau, \sigma)=x+2\alpha' p\tau+2L\sigma$$
$$+\sqrt{2\alpha'}\frac{i}{2}\sum_{n\neq 0}\frac{1}{n}(\alpha_n e^{-2in(\tau-\sigma)}+\tilde{\alpha}_n e^{-2in(\tau+\sigma)})\,. \tag{2·4}$$

The orthonormal gauge conditions in the light-cone gauge $X^+(\tau, \sigma)=x^+ + 2\alpha' p^+\tau$

*) The winding numbers for the nonorientable closed string may be characterized by $l=0$ or 1 mod. 2. Open strings do not possess winding numbers and have tendency against compactification.[11)]

$$(\partial X^\mu/\partial\sigma)^2+(\partial X^\mu/\partial\tau)^2=0\,, \tag{2·5}$$

$$(\partial X^\mu/\partial\sigma)\cdot(\partial X_\mu/\partial\tau)=0\,, \tag{2·6}$$

can be integrated over to give the mass-shell condition and a constraint

$$\begin{aligned}\mathcal{M}=\alpha'(\mathrm{mass})^2/2&=N+\tilde{N}-2+2\alpha'\left(\frac{p}{2}\right)^2+\frac{1}{2\alpha'}L^2\\ &=N+\tilde{N}-2+\frac{\alpha'}{2R^2}m^2+\frac{R^2}{2\alpha'}l^2\,,\end{aligned} \tag{2·7}$$

$$N-\tilde{N}=p\cdot L=ml\,, \tag{2·8}$$

where the oscillator sum $N,\tilde{N}$ includes both compactified and uncompactified transverse dimensions

$$N=\sum_{\mu=1}^{24}\sum_{n=1}^{\infty}\alpha_{-n}^{\mu}\alpha_n^{\mu}\,,\qquad \tilde{N}=\sum_{\mu=1}^{24}\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_n^{\mu}\,. \tag{2·9}$$

The c-number ambiguity due to operator ordering in Eq. (2·7) is fixed by requiring the closure of the Lorentz algebra in uncompactified dimensions. Constraint (2·8) assures the independence on the choice of the origin in the coordinate.[10] There are two massless vector particles $\alpha_{-1}^{i}\tilde{\alpha}_{-1}|0\rangle$ and $\alpha_{-1}\tilde{\alpha}_{-1}^{i}|0\rangle$ corresponding to $U(1)$ gauge bosons due to the Kaluza-Klein mechanism. Tachyons occur only when $N=\tilde{N}=0$ (and consequently $m=0$ or $l=0$). Among tori of various sizes, only one special torus with $R=\sqrt{\alpha'}$ realizes the affine Kac-Moody algebra, and gives rise to massless solitons which form $SU(2)\times SU(2)$ gauge bosons together with the two $U(1)$ gauge bosons.

The vacuum energy density in the remaining flat 25-dimensional space is denoted as V_{25}. Although we shall later find necessary modifications for the correct string vacuum energy, let us start to write V_{25} as a sum of the vacuum energies of infinitely many modes of string (trace of logarithm of the inverse propagator)

$$\begin{aligned}V_{25}&=\frac{-i}{2}\mathrm{Tr}\ln\left[\frac{\alpha'}{2}p^2+\mathcal{M}\right]\\ &=\frac{i}{2}\mathrm{Tr}\left[\int_0^1 dx\; x^{(\alpha'/2)p^2+\mathcal{M}-1}\left(\ln\frac{1}{x}\right)^{-1}\right],\end{aligned} \tag{2·10}$$

where $\mathcal{M}$ is the mass operator (2·7) and the trace represents the integral over 25-dimensional momentum p and discrete sum over modes including discrete momenta and winding numbers.*) The constraint (2·8) can be implemented by**)

$$\delta_{N,\tilde{N}+ml}=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma'\,e^{i\sigma'(N-\tilde{N}-ml)}\,. \tag{2·11}$$

Summation over oscillator modes N and $\tilde{N}$ in Eq. (2·9) can be done by introducing[2]

$$f(z)=\prod_{n=1}^{\infty}(1-z^n)=[\prod_{n=1}^{\infty}\sum_{k=0}^{\infty}(z^n)^k]^{-1}\,. \tag{2·12}$$

*) We suppose that tree level vacuum energy dependent on R is absent. There have been attempts to compute the "shift" on the tachyon field.[15]

**) The variable σ' is related to the string coordinate σ in Eq. (2·4) as $\sigma'=2\sigma-\pi$.

Similarly to the other one-loop amplitudes with external particles,[6),7)] the vacuum energy can be expressed as an integral with the "correction factor" F_1 which arises from the replacement of momentum integration by a sum over discrete momenta and winding numbers

$$V_{25}=\frac{-1}{4\pi(2\pi\alpha')^{25/2}}\int_0^1 dx\int_{-\pi}^{\pi}d\sigma'\left(\ln\frac{1}{x}\right)^{-14}x^{-3}|f(xe^{i\sigma'})|^{-48}\cdot F_1\,, \tag{2·13}$$

$$F_1=\left(\ln\frac{1}{x}\right)^{1/2}\sum_{m,l=-\infty}^{\infty}e^{-i\sigma' ml}x^{(a^2/2)m^2+(1/2a^2)l^2}\,, \tag{2·14}$$

$$a=\sqrt{\alpha'}/R\,. \tag{2·15}$$

Changing x, σ' to a complex variable τ is more convenient to examine the modular invariance

$$z=xe^{i\sigma'}=e^{2\pi i\tau}\,,\qquad \tau=\tau_1+i\tau_2\,, \tag{2·16}$$

$$V_{25}=\frac{-\pi}{(2\pi\alpha')^{25/2}}\int d^2\tau(2\pi\tau_2)^{-14}e^{4\pi\tau_2}|f(e^{2\pi i\tau})|^{-48}\cdot F_1\,, \tag{2·17}$$

$$\begin{aligned}F_1&=\sqrt{2\pi\tau_2}\sum_{m,l}\exp\left\{\frac{i\pi\tau}{2}\left(ma-\frac{l}{a}\right)^2-\frac{i\pi\tau^*}{2}\left(ma+\frac{l}{a}\right)^2\right\}\\&=\sqrt{2\pi\tau_2}\sum_{m}e^{-\pi\tau_2a^2m^2}\Theta_3\left(m\tau_1\middle|\frac{i\tau_2}{a}\right)\\&=\sqrt{2\pi\tau_2}\sum_{l}e^{-\pi\tau_2l^2/a^2}\Theta_3(l\tau_1|i\tau_2a)\,,\end{aligned} \tag{2·18}$$

where Θ_3 is the Jacobi theta function.[16)]

Modular transformations $SL(2, Z)$ are given by

$$\tau\longrightarrow\frac{a\tau+b}{c\tau+d}\,,\qquad ad-bc=1 \tag{2·19}$$

with a, b, c, d integers and are generated by[16)]

$$\tau\longrightarrow\tau+1\,, \tag{2·20}$$

$$\tau\longrightarrow-1/\tau\,. \tag{2·21}$$

The vacuum energy density (2·17) is trivially invariant under (2·20), Invariance under the second transformation (2·21) can be shown by applying Jacobi's imaginary transformation[16)] once for the sum over l and once over m

$$\Theta_3(\nu/\tau|-1/\tau)=(-i\tau)^{1/2}\exp(i\pi\nu^2/\tau)\,\Theta_3(\nu|\tau)\,, \tag{2·22}$$

or by the method of Fourier transform[7)] whose generalization will be given in the next section for the general case of r-dimensional compactification. The original integration region

$$-\pi\leq\sigma'\leq\pi,\qquad 0\leq x\leq 1\,, \tag{2·23}$$

or equivalently

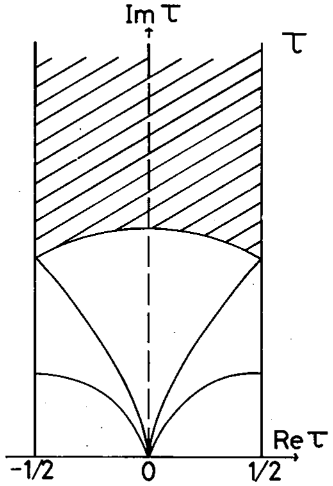

Fig. 1. The integration region in the τ-plane. The shaded area is the fundmental region of the modular group.

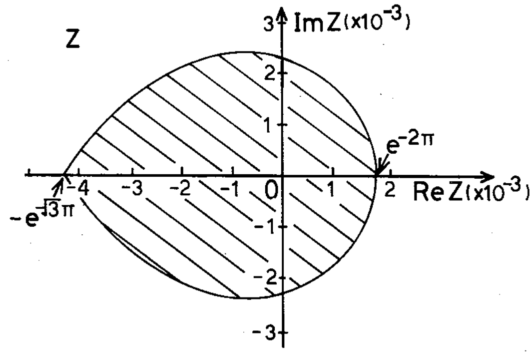

Fig. 2. The integration region in the z-plane. The shaded area is the fundamental region.

$$-\frac{1}{2}\leq\tau_1\leq\frac{1}{2},\qquad \tau_2>0\,, \tag{2·24}$$

is divided into infinitely many equivalent regions which can be mapped each other by the modular transformations (2·19) as illustrated in Figs. 1 and 2. In order to avoid multiple counting of the same string configurations we have to restrict the integral to one of these equivalent regions, usually taken as the fundamental region[4)]

$$-\frac{1}{2}\leq\tau_1\leq\frac{1}{2},\qquad \tau_2>\sqrt{1-\tau_1{}^2}\,, \tag{2·25}$$

or equivalently

$$-\pi\leq\sigma'\leq\pi\,,\qquad 0\leq x\leq\exp(-\sqrt{4\pi^2-\sigma'^2})\,. \tag{2·26}$$

Let us stress that dividing out the infinite volume $(Z\times Z)$ of the modular invariance group is an unavoidable characteristic feature of string theories. As a consequence the correct vacuum energy of string is different from a naive sum of the vacuum energies of each individual mode as we originally started in Eq. (2·10). In fact after being restricted to the fundamental region, the string vacuum energy is free from ultraviolet divergences (singularity near $x=1$), in contrast to the case of the naive sum of individual modes (2·10).

Even after being restricted to the fundamental region, the vacuum energy is divergent due to tachyons circulating in the loop. This is presumably due to either i) inappropriate choice of vacuum for the bosonic string, or ii) inconsistency of the bosonic string itself. At the present level of understanding of string theories, we shall tentatively employ a simple prescription:

$$|f(xe^{i\sigma'})|^2\longrightarrow|f(xe^{i\sigma'})|^2-1 \tag{2·27}$$

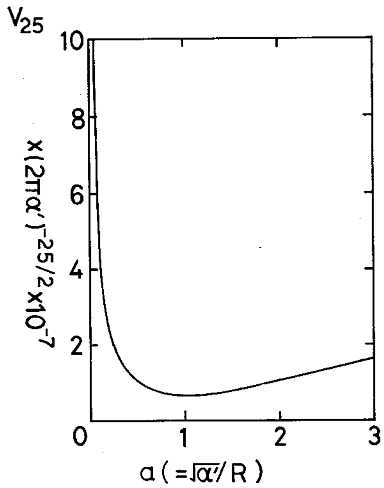

Fig. 3. The one-loop vacuum energy as a function of the parameter $a=\sqrt{\alpha'}/R$ when one dimension is compactified on torus. It is invariant under the exchange $a \longleftrightarrow a^{-1}$. The asymptotic behavior for $a \longrightarrow 0$ is given by Eq. (2·31).

which amounts to subtracting tachyon contributions ($N=\tilde{N}=0$) in the fundamental region.*) The corresponding subtraction procedure in integration regions other than the fundamental region can be found by applying the modular transformation. It turns out that the prescription becomes a subtraction of a singularity at $x=1$ (ultraviolet divergence) in other regions. We finally arrive at the finite expression for the vacuum energy density

$$V_{25}=\frac{-1}{4\pi(2\pi\alpha')^{25/2}} \times \int_F dx d\sigma' \left(\ln\frac{1}{x}\right)^{-27/2} \times x^{-3}(|f(xe^{i\sigma'})|^2-1) \times \sum_{m,l=-\infty}^{\infty} e^{-i\sigma' ml} x^{\alpha^2 m^2/2+l^2/2\alpha^2}, \tag{2·28}$$

where F denotes the fundamental region (2·26).

The vacuum energy is clearly invariant[11] under $a\to a^{-1}$, namely

$$V_{25}(R)=V_{25}(\alpha'/R). \tag{2·29}$$

Therefore $R=\sqrt{\alpha'}$ is a stationary point

$$dV_{25}/dR=0 \quad \text{at} \quad R=\sqrt{\alpha'}, \tag{2·30}$$

where the affine Kac-Moody algebra for $SU(2)\times SU(2)$ is realized. In the limit $R\to\infty$, the vacuum energy is found to be proportional to $2\pi R$ (the volume of the compactified space) and the proportionality constant Λ (vacuum energy density in the 26-dimensional flat space) is found to be positive

$$V_{25}/2\pi R \longrightarrow \text{constant}=\Lambda \qquad \text{as } R\to\infty, \tag{2·31}$$

$$\Lambda \equiv \frac{-1}{4\pi(2\pi\alpha')^{13}} \int_F dx d\sigma' \left(\ln\frac{1}{x}\right)^{-14} x^{-3}(|f(xe^{i\sigma'})|^2-1).$$

$$\approx 2.10\times10^{-8}\cdot(2\pi\alpha')^{-13}. \tag{2·32}$$

Numerical evaluation of the second derivative d^2V_{25}/dR^2 at $R=\sqrt{\alpha'}$ shows that the extrema is a local minimum. We have numerically evaluated $V_{25}(R)$ as a function of R and found that $R=\sqrt{\alpha'}$ is the absolute minimum as shown in Fig. 3. Therefore we conclude that the affine Kac-Moody algebra for $SU(2)\times SU(2)$ is realized as the absolute minimum of the vacuum energy when one dimension is compactified on torus.

*) Unfortunately this subtraction is not the same as the subtraction of the vacuum energy of the uncompactified string, since tachyons with nonzero discrete momenta or winding numbers give R-dependent divergent contributions.

§ 3. Compactification of r dimensions

Similarly to the one-dimensional torus, compactification on r-dimensional torus introduces r-dimensional discrete momenta $\boldsymbol{p}$ and winding number vector $\boldsymbol{L}$ into the normal mode expansion

$$\begin{aligned}\boldsymbol{X}(\tau,\sigma)&=\boldsymbol{x}+2\alpha'\boldsymbol{p}\tau+2\boldsymbol{L}\sigma\\&+\sqrt{2\alpha'}\frac{i}{2}\sum_{n\neq 0}\frac{1}{n}(\boldsymbol{\alpha}_n e^{-2in(\tau-\sigma)}+\tilde{\boldsymbol{\alpha}}_n e^{-2in(\tau+\sigma)})\,.\end{aligned}\qquad(3\cdot1)$$

It is sometimes more convenient to use right- and left-moving decomposition of momentum and winding number

$$\begin{aligned}\boldsymbol{W}_R&=\sqrt{2\alpha'}\boldsymbol{p}/2-\boldsymbol{L}/\sqrt{2\alpha'}\,,\\\boldsymbol{W}_L&=\sqrt{2\alpha'}\boldsymbol{p}/2+\boldsymbol{L}/\sqrt{2\alpha'}\,,\\\boldsymbol{X}(\tau,\sigma)&=\boldsymbol{x}+\sqrt{2\alpha'}(\tau-\sigma)\boldsymbol{W}_R+\sqrt{2\alpha'}(\tau+\sigma)\boldsymbol{W}_L+\cdots\,.\end{aligned}\qquad(3\cdot2)$$

Since the central charge of the Virasoro algebra is unchanged, the torus compactification satisfies invariance under infinitesimal reparametrizations. Constraint (2·8), which assures the independence on the choice of σ-coordinate origin, becomes

$$N-\tilde{N}=\boldsymbol{p}\cdot\boldsymbol{L}=\frac{1}{2}(\boldsymbol{W}_L{}^2-\boldsymbol{W}_R{}^2)\,.\qquad(3\cdot3)$$

Consistent compactification requires the right-hand side of (3·3) to be integers. One possibility is to consider an arbitrary nonsingular lattice specified by basis vectors $\boldsymbol{E}_i$ ($i=1,\cdots,r$) and identify points in real space $\boldsymbol{x}$ which differ by integer multiples of $2\pi\sqrt{\alpha'}\boldsymbol{E}_i$. Winding number vectors $\boldsymbol{L}$ are then given by

$$\boldsymbol{L}=\sqrt{\alpha'}\sum_i l_i\boldsymbol{E}_i\,,\qquad l_i=\text{integer}\,.\qquad(3\cdot4)$$

Discrete momenta are given by introducing inverse lattice basis vectors $\boldsymbol{e}_i$ ($i=1,\cdots,r$)

$$\boldsymbol{e}_i\cdot\boldsymbol{E}_j=\delta_{ij}\,,\qquad(3\cdot5)$$

$$\boldsymbol{p}=\sum_i m_i\boldsymbol{e}_i/\sqrt{\alpha'}\,,\qquad m_i=\text{integer}\,.\qquad(3\cdot6)$$

Then constraint (3·3) becomes

$$N-\tilde{N}=\sum_i m_i l_i\,.\qquad(3\cdot7)$$

The mass operator in Eq. (2·7) becomes

$$\begin{aligned}\mathcal{M}=\alpha'(\text{mass})^2/2&=N+\tilde{N}-2+2\alpha'\left(\frac{\boldsymbol{p}}{2}\right)^2+\frac{1}{2\alpha'}\boldsymbol{L}^2\\&=N+\tilde{N}-2+\frac{1}{2}\sum_{i,j}(m_i g_{ij}m_j+l_i g_{ij}^{-1}l_j)\,,\end{aligned}\qquad(3\cdot8)$$

where g_{ij} is the metric of the lattice defined by the basis vectors $\boldsymbol{e}_i$

$$g_{ij}=\boldsymbol{e}_i\cdot\boldsymbol{e}_j\,,$$
$$g_{ij}^{-1}=\boldsymbol{E}_i\cdot\boldsymbol{E}_j\,. \tag{3·9}$$

One should note that sizes and angles of the lattice are arbitrary and can be changed continuously for this class of compactifications. It has been noted previously that this class of compactification cannot produce affine Kac-Moody algebras of *simple* groups except the $SU(2)$ case in the previous section.[10),18)] Let us examine in detail possible additional massless vector bosons besides the usual Kaluza-Klein $U(1)$ gauge bosons in general r-dimensional case. The mass operator (3·8) shows that $N+\tilde{N}\le 1$ is required for massless particles with nontrivial winding numbers and/or discrete momenta. Since $N=\tilde{N}=0$ gives tachyons, we only need to consider $N=1$, $\tilde{N}=0$ or $N=0$, $\tilde{N}=1$ cases. Defining

$$\boldsymbol{v}=\sqrt{\alpha'}\boldsymbol{p}=\sum_i m_i\boldsymbol{e}_i\,,$$
$$\boldsymbol{V}=\boldsymbol{L}/\sqrt{\alpha'}=\sum_i l_i\boldsymbol{E}_i\,, \tag{3·10}$$

massless condition and constraint (3·7) becomes

$$\boldsymbol{v}^2+\boldsymbol{V}^2=2\,,\qquad \boldsymbol{v}\cdot\boldsymbol{V}=\pm 1\,, \tag{3·11}$$

which are equivalent to

$$\boldsymbol{v}=\pm\boldsymbol{V}\,,\qquad \boldsymbol{v}^2=1\,. \tag{3·12}$$

Since the vectors $\boldsymbol{v}$ and $\boldsymbol{V}$ are on two lattices defined by $\boldsymbol{e}_i$ and $\boldsymbol{E}_i$ which are inverse of each other, their magnitude can be equal to unity, if and only if two lattices are the same and (l_i) and (m_i) are unit vectors

$$\boldsymbol{e}_i=\boldsymbol{E}_i\,,$$
$$(l_i),(m_i)=\text{one of }(1,0,\cdots,0),\cdots,(0,\cdots,0,1)\,. \tag{3·13}$$

The resulting configuration is nothing but a direct product of r-copies of the one-dimensional torus compactification at the symmectric point $R=\sqrt{\alpha'}$. Hence we find that the first class of r-dimensional torus compactification can give only the affine Kac-Moody algebra*) of $[SU(2)\times SU(2)]^r$ at the symmetric point $g_{ij}=g_{ij}^{-1}=\delta_{ij}$.

As the second class of compactifications which make the right-hand side of (3·3) integers, we can take the string compactifications for the affine Kac-Moody algebras.[8),9),13)] It was explicitly constructed in Ref. 13) as

$$\boldsymbol{w}_R=\boldsymbol{w}_0+\boldsymbol{p}_R\,,\qquad \boldsymbol{w}_L=\boldsymbol{w}_0+\boldsymbol{p}_L\,, \tag{3·14}$$

where $\boldsymbol{p}_L$ and $\boldsymbol{p}_R$ run over root lattice Λ_R of a simply-laced group G[9)] and $\boldsymbol{w}_0$ are minimal weights of distinct irreducible representations of the center of G. This class of compactification admits only particular lattices which cannot be changed continuously.**)

*) It is of course possible to have an incomplete realization, $[SU(2)\times SU(2)]^l$ $0\le l\le r$, if only l out of r basis vectors $\boldsymbol{e}_i$ are of unit length and orthogonal to each other and to all other basis vectors.

**) A direct product of two classes of compactifications is of course possible.

Self-dual lattices, which exist in $8\cdot n$ dimensions, are most special and even allow different groups for left- and right-movers ($p_R \in \Lambda_R$, $p_L \in \tilde{\Lambda}_R$, $\Lambda_R \neq \tilde{\Lambda}_R$).[13] This fact enabled the construction of the heterotic string.[10]

The one-loop vacuum energy density in $26-r$ dimensions can easily be obtained for the first class of r-dimensional torus, by generalizing Eq. (2·13)

$$V_{26-r}=\frac{-1}{4\pi(2\pi\alpha')^{(26-r)/2}}\int_0^1 dx\int_{-\pi}^{\pi} d\sigma'\left(\ln\frac{1}{x}\right)^{-14} x^{-3}|f(xe^{i\sigma'})|^{-48}\cdot F_r\,, \tag{3·15}$$

$$F_r=\left(\ln\frac{1}{x}\right)^{r/2}\sum_{m_i,l_i=-\infty}^{\infty} e^{-i\sigma' m_i l_i} x^{1/2(m_i g_{ij} m_j + l_i g_{ij}^{-1} l_j)}\,. \tag{3·16}$$

Changing x,σ' to τ in Eq. (2·16), we obtain

$$V_{26-r}=\frac{-\pi}{(2\pi\alpha')^{(26-r)/2}}\int d^2\tau(2\pi\tau_2)^{-14}e^{4\pi\tau_2}|f(e^{2\pi i\tau})|^{-48}\cdot F_r\,, \tag{3·17}$$

$$F_r(\tau)=(2\pi\tau_2)^{r/2}\sum_{m_i,l_i}\exp(-\pi M^T A M)\,, \tag{3·18}$$

where M is a $2r$ component column vector and A is a $2r$ by $2r$ matrix

$$M^T=(m_1,\cdots,m_r,l_1,\cdots,l_r)\,, \tag{3·19}$$

$$A=\begin{bmatrix}\tau_2 g_{ij} & i\tau_1\delta_{ij}\\ i\tau_1\delta_{ij} & \tau_2 g_{ij}^{-1}\end{bmatrix}. \tag{3·20}$$

The first modular transformation (2·20) is trivially satisfied. To show invariance under the second modular transformation (2·21), we use a generalization of the Fourier transform method in Ref. 7). Let us consider an auxiliary expression

$$F_r(\tau\,;x)=(2\pi\tau_2)^{r/2}\sum_M\exp\{-\pi(M+x)^TA(M+x)\}\,,$$

$$x^T=(x_1,\cdots,x_{2r})\,, \tag{3·21}$$

which is periodic in x_i, $i=1,\cdots,2r$ with period 1. Hence it can be expanded in a Fourier series

$$F_r(\tau\,;x)=\left[\frac{(2\pi\tau_2)^r}{\det A}\right]^{1/2}\sum_M\exp(-\pi M^TA^{-1}M+2\pi i M^Tx)\,,$$

$$A^{-1}=\frac{1}{|\tau|^2}\begin{bmatrix}\tau_2 g_{ij}^{-1} & -i\tau_1\delta_{ij}\\ -i\tau_1\delta_{ij} & \tau_2 g_{ij}\end{bmatrix},$$

$$\det A=|\tau|^{2r}\,. \tag{3·22}$$

Putting $x=0$, we find the modular invariance of the correction factor

$$F_r(\tau)=F_r(-1/\tau)\,. \tag{3·23}$$

Since other factors together with the integration measure in Eq. (3·17) is modular invariant,[2] V_{26-r} is invariant under (2·21). Even after restricted to the fundamental region, the vacuum energy density V_{26-r} is still divergent because of tachyon contributions, similarly to the one-dimensional case. Applying the same subtraction prescription (2·27) to this case, we arrive at the finite expression

$$V_{26-r}=\frac{-1}{4\pi(2\pi\alpha')^{(26-r)/2}}\int_F dx d\sigma'\left(\ln\frac{1}{x}\right)^{-14}x^{-3}(|f(xe^{i\sigma'})|^{-48}-1)\cdot F_r. \tag{3·24}$$

For the second class corresponding to the root lattice Λ_R of a simply-laced group G, the vacuum energy density is given by the same expression as (3·15) or (3·17) except that the correction factor F_r is replaced by

$$F_r'=(2\pi\tau_2)^{r/2}\sum_{w_0}{}'\sum_{p_R,p_L\in\Lambda_R}\exp\{i\pi\tau(\boldsymbol{w}_0+\boldsymbol{p}_R)^2-i\pi\tau^*(\boldsymbol{w}_0+\boldsymbol{p}_L)^2\}, \tag{3·25}$$

where Σ' is the finite sum over minimal weights $\boldsymbol{w}_0$ of all distinct irreducible representations of the center of G.[13] The modular invariance of the correction factor F_r' was shown previously.[13),18] Also for this case, we choose the fundamental region as the integration region and apply the same subtraction (2·27) to remove divergent tachyon contributions.

The vacuum energy density for the first class tori given in Eqs. (3·24) and (3·16) is clearly invariant under the interchange

$$g_{ij}\longleftrightarrow g_{ij}^{-1}. \tag{3·26}$$

Hence the symmetric point $g_{ij}=\delta_{ij}$ is an extremum of the vacuum energy

$$\partial V_{26-r}/\partial g_{ij}=0 \qquad \text{at} \quad g_{ij}=\delta_{ij}. \tag{3·27}$$

At this extremum the affine Kac-Moody algebra for $[SU(2)\times SU(2)]^r$ is realized. From the symmetry of integrand (3·16) at the symmetric point $g_{ij}=\delta_{ij}$, we find that only diagonal elements of second derivative matrix are nonvanishing

$$\partial^2 V_{26-r}/\partial g_{11}^2=\cdots=\partial^2 V_{26-r}/\partial g_{rr}^2\neq 0,$$
$$\partial^2 V_{26-r}/\partial g_{12}^2=\cdots=\partial^2 V_{26-r}/\partial g_{r-1r}^2\neq 0,$$
$$\partial^2 V_{26-r}/\partial g_{ij}\partial g_{kl}=0 \qquad \text{otherwise}. \tag{3·28}$$

Table I. The nonvanishing second derivatives (3·28) at the symmetric point $g_{ij}=\delta_{ij}$ of the first class lattice when r dimensions are compactified. All quantities are in units of $(2\pi\alpha')^{-(26-r)/2}$.

r	$\partial^2 V_{26-r}/\partial g_{11}^2$	$\partial^2 V_{26-r}/\partial g_{12}^2$
2	1.97×10^{-7}	-2.05×10^{-8}
3	5.55×10^{-7}	-5.93×10^{-8}
4	1.61×10^{-6}	-1.71×10^{-7}
5	4.66×10^{-6}	-4.93×10^{-7}
6	1.35×10^{-5}	-1.42×10^{-6}
7	3.90×10^{-5}	-4.10×10^{-6}
8	1.13×10^{-4}	-1.18×10^{-5}
9	3.26×10^{-4}	-3.40×10^{-5}
10	9.43×10^{-4}	-9.79×10^{-5}
11	2.73×10^{-3}	-2.82×10^{-4}
12	7.88×10^{-3}	-8.09×10^{-4}
13	2.28×10^{-2}	-2.32×10^{-3}
14	6.57×10^{-2}	-6.66×10^{-3}
15	1.90×10^{-1}	-1.91×10^{-2}
16	5.47×10^{-1}	-5.47×10^{-2}

We have numerically computed the second derivatives as shown in Table I and found that the extremum is unstable. On the other hand we find that V_{26-r} is proportional to the volume of the compactified space in the limit of large size lattices: $g_{ij}^{-1}=\lambda\hat{g}_{ij}^{-1}$, $\lambda\to\infty$ with $\hat{g}_{ij}$ fixed

$$V_{26-r}\longrightarrow(2\pi\sqrt{\alpha'})^r\det g_{ij}^{-1}\cdot\Lambda, \tag{3·29}$$

where the proportionality constant Λ is positive and is given by Eq. (2·32). Combined with symmetry (3·26), we see that the effective potential increases inde-

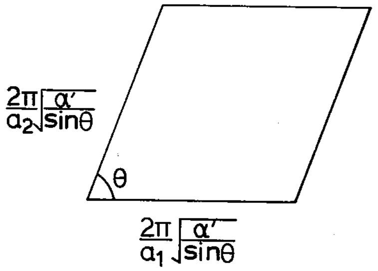

Fig. 4. The parametrization of Eq. (3·30).

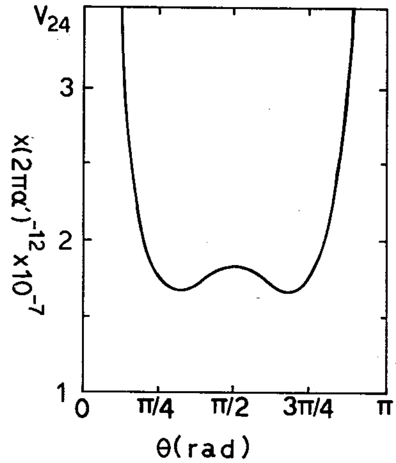

Fig. 5. The one-loop vacuum energy as a function of the angle θ at $a_1=a_2=1$ when two dimensions are compactified.

finitely for very large and very small size lattices. Hence we expect that the most favorable torus compactification in the first class should occur at a certain lattice whose size is of order $\sqrt{\alpha'}$.

To study the stability and the global behavior in detail, we take the $r=2$ case and evaluated the vacuum energy density V_{24} numerically. A convenient parametrization for $r=2$ is given by a_1, a_2 and θ as

$$g_{11}=a_1^2/\sin\theta\,,$$

$$g_{22}=a_2^2/\sin\theta\,,$$

$$g_{12}=-a_1a_2\cot\theta\,. \tag{3·30}$$

This lattice corresponds, in physical space $\boldsymbol{X}(\tau,\sigma)$, to a periodicity unit of the parallelogram whose sides have length $2\pi\sqrt{\alpha'}/(a_i\sqrt{\sin\theta})$ and make an angle θ as shown in Fig. 4. Symmetry (3·26) of the vacuum energy density becomes in this parametrization

$$a_i\longleftrightarrow a_i^{-1} \quad \text{and} \quad \theta\longleftrightarrow\pi-\theta\,. \tag{3·31}$$

From the symmetry property of the integrand we find that the vacuum energy is stationary for two directions a_1 and a_2 at $a_1=a_2=1$ for any θ

$$\partial V_{24}/\partial a_1=\partial V_{24}/\partial a_2=0\,, \tag{3·32}$$

and moreover

$$\partial^2 V_{24}/\partial\theta\partial a_1=\partial^2 V_{24}/\partial\theta\partial a_2=0\,. \tag{3·33}$$

By evaluating the vacuum energy density numerically along this symmetric line, it is found that V_{24} $(a_1=a_2=1,\theta)$ has a local maximum at $\theta=\pi/2$ and a minimum at θ_0 near $\pi/3$ (and also at $\pi-\theta_0$) as shown in Fig. 5. Useful formulas for the numerical evaluation is summarized in the Appendix. We also find that the 2×2 second derivative matrix $\partial^2 V_{24}/\partial a_i\partial a_j$ is positive at $a_1=a_2=1$ for not too small θ. Thus we obtain that the extremum corresponding to the affine Lie algebra for $[SU(2)\times SU(2)]^2$ is an unstable saddle point, whereas the point $\theta=\theta_0$, $a_1=a_2=1$ is at least a local minimum

$$V_{24}(a_1=a_2=1,\,\theta=\theta_0)\simeq1.65\times10^{-7}\cdot(2\pi\alpha')^{-12}\,. \tag{3·34}$$

By evaluating the vacuum energy density for intermediate values of a_1, a_2 and θ numerically, we find that the local minimum is in fact a global mimimum. One should note that this minimum configuration does not realize any affine Kac-Moody algebras, even though it resembles the lattice for the $SU(3)\times SU(3)$ affine Kac-Moody algebra. In fact we have seen already that the first class compactification can give massless solitons only for $[SU(2)\times SU(2)]^2$ or $SU(2)\times SU(2)$.

The only remaining consistent compactification for $r=2$ is the case of the affine Kac-Moody algebra for $SU(3)\times SU(3)$. The vacuum energy density in this case is given by

$$V_{24}=\frac{-1}{4\pi(2\pi\alpha')^{12}}\int_F dx d\sigma'\left(\ln\frac{1}{x}\right)^{-14} x^{-3}\cdot(|f(xe^{i\sigma'})|^{-48}-1)\cdot F'_2\,, \tag{3·35}$$

$$F'_2=2\pi\tau_2 \sum_{n_{Ri},n_{Li}=-\infty}^{\infty} e^{2\pi i\Phi}\cdot\Psi\,,$$

$$\Phi=\tau(n_{R1}^2+n_{R2}^2-n_{R1}n_{R2})-\tau^*(n_{L1}^2+n_{L2}^2-n_{L1}n_{L2})\,,$$

$$\Psi=1+e^{i\pi\{\tau(2/3+2n_{R1})-\tau^*(2/3+2n_{L1})\}}+e^{i\pi\{\tau(2/3+2n_{R2})-\tau^*(2/3+n_{L2})\}}\,. \tag{3·36}$$

We have evaluated the vacuum energy density numerically

$$V_{24}(SU(3)\times SU(3))\simeq 1.28\times 10^{-7}(2\pi\alpha')^{-12} \tag{3·37}$$

and found that it is lower than the minimum of the first class (3·34). It is intriguing that the higher symmetry configuration realizing the affine Kac-Moody algebra has the lower vacuum energy.

In conclusion, our result so far seems to suggest that the vacuum energy for torus compacifications favors higher symmetry configurations and hence the affine Kac-Moody algebras tend to be realized. On the other hand, we should have in mind that the more satisfactory resolution of the tachyon problem in the bosonic string theory has to be found before the choice and stability of the string vacuum configuration is really settled. This question may be more properly answered by finding the effective action or equations of motion of string theories.

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Appendix

Here we summarize useful formulas for numerical evaluation of the one-loop vacuum energy integral (3·24). As we mentioned in the text, we must perform this integral in the fundamental region. Let us use x and σ' as integration variables. Then the integral becomes

$$\int_F dx d\sigma'=\int_0^{e^{-2\pi}}dx\int_{-\pi}^{\pi}d\sigma'+\int_{e^{-2\pi}}^{e^{-\sqrt{3}\pi}}dx\left(\int_{-\pi}^{-\theta(x)}d\sigma'+\int_{\theta(x)}^{\pi}d\sigma'\right),$$

$$\theta(x)=\sqrt{4\pi^2-(\ln x)^2}\ . \tag{A·1}$$

In evaluating this integral, we integrate over σ' first. To do this we must know the σ'-dependence of the mode sum part $(|f(xe^{i\sigma'})|^{-48}-1)$. We can see easily that this mode sum part depends on σ' only in the form of cosine, and we can write

$$(|f|^{-48}-1)=\sum_{n=0}^{\infty}O_n(x)\cos n\sigma'\ , \tag{A·2}$$

where $O_n(x)$ is a polynomial of x and its lowest power is n except $O_0(x)$. Since we have subtracted the tachyon contribution as in Eq. (2·27), $O_0(x)$ contains only x^2 or higher powers of x.

After integration over σ', the integral becomes

$$V_{26-r}=-\frac{1}{4\pi}(2\pi\alpha')^{-(26-r)/2}\Bigg\{2\sum_{(m_il_i)}^{\Sigma_i m_i l_i=0}\Big[I^{(1)}\Big(14-\frac{r}{2},M\Big)+I^{(2)}\Big(14-\frac{r}{2},M\Big)\Big]$$
$$+\sum_{(m_i,l_i)}^{\Sigma_i m_i l_i\neq 0}\Big[I^{(3)}\Big(|\sum_i m_i l_i|,14-\frac{r}{2},M\Big)+I^{(4)}\Big(|\sum_i m_i l_i|,14-\frac{r}{2},M\Big)\Big]\Bigg\}\ ,$$

$$M=\sum_{ij}(m_i g_{ij} m_j+l_i g_{ij}^{-1} l_i)\ , \tag{A·3}$$

$$I^{(1)}(a,b)=\int_0^{e^{-2\pi}}dx\,x^{b/2-3}\Big(\ln\frac{1}{x}\Big)^{-a}\pi O_0(x)\ ,$$

$$I^{(2)}(a,b)=\int_{e^{-2\pi}}^{e^{-\sqrt{3}\pi}}dx\,x^{b/2-3}\Big(\ln\frac{1}{x}\Big)^{-a}\Big[(\pi-\theta(x))\,O_0(x)-\sum_{n>0}\frac{\sin(n\theta(x))}{n}O_n(x)\Big]\ ,$$

$$I^{(3)}(|\sum_i m_i l_i|,a,b)=\int_0^{e^{-2\pi}}dx\,x^{b/2-3}\Big(\ln\frac{1}{x}\Big)^{-a}\pi O_{|\Sigma m_i l_i|}(x)\ ,$$

$$I^{(4)}(|\sum_i m_i l_i|,a,b)=\int_{e^{-2\pi}}^{e^{-\sqrt{3}\pi}}dx\,x^{b/2-3}\Big(\ln\frac{1}{x}\Big)^{-a}$$
$$\times\Big[-\frac{2\sin(|\sum_i m_i l_i|\theta(x))}{|\sum_i m_i l_i|}O_0(x)+\Big(\pi-\theta(x)-\frac{\sin(2|\sum_i m_i l_i|\theta(x))}{2|\sum_i m_i l_i|}\Big)O_{|\Sigma m_i l_i|}(x)$$
$$+\sum_{n>0}^{n\neq|\Sigma_i m_i l_i|}\Big(-\frac{\sin(n+\sum_i m_i l_i)\theta(x)}{n+\sum_i m_i l_i}-\frac{\sin(n-\sum_i m_i l_i)\theta(x)}{n-\sum_i m_i l_i}\Big)O_n(x)\Big]. \tag{A·4}$$

Explicit forms of $O_n(x)$ are as follows:

$$O_0(x)=576x^2+104976x^4+O(x^6)\ ,$$
$$O_1(x)=48x+15552x^3+O(x^5)\ ,$$
$$O_2(x)=648x^2+153600x^4+O(x^6)\ ,$$
$$O_3(x)=6400x^3+O(x^5)\ ,$$
$$O_4(x)=51300x^4+O(x^6)\ . \tag{A·5}$$

We were not able to perform the integral analytically, but evaluated it numerically. Because we restricted the integration region to the fundamental region, x is very small $(\sim 10^{-3})$ and a power series expansion in x converges rapidly. Hence it is sufficient to use first few terms in $O_n(x)$.

The vacuum energy for the $SU(3)\times SU(3)$ case (3·35) and (3·36) involves an integral which can be treated similarly.

References

1) For a reiew, J. Scherk, Rev. Mod. Phys. **47** (1975), 123.
2) J. Schwarz, Phys. Rep. **89** (1982), 223.
3) See for example, A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).
4) J. A. Shapiro, Phys. Rev. **D5** (1972), 1945.
5) D. Gross, J. Harvey, E. Martinec and R. Rohm, Princeton Preprint "Heterotic String Theory II".
6) E. Cremmer and J. Scherk, Nucl. Phys. **B103** (1976), 399.
7) M. Green, J. Schwarz and L. Brink, Nucl. Phys. **B198** (1982), 474.
8) I. B. Frenkel and V. G. Kac, Inv. Math. **62** (1980), 23.
G. Segal., Comm. Math. Phys. **80** (1981), 301.
9) P. Goddard and D. Olive, ed. J. Lepowsky et al. (MSRI publication No. 3, Springer Verlag, 1984), p. 51.
T. Eguchi and K. Higashijima, talk presented at Niels Bohr Centennial, Copenhagen, 1985.
10) D. Gross, J. Harvey, E. Martinec and R. Rohm, Nucl. Phys. **B256** (1985), 253.
11) K. Kikkawa and M. Yamasaki, Phys. Lett. **149B** (1984), 357.
12) R. Rohm, Nucl. Phys. **B237** (1984), 553.
13) A. Casher, F. Englert, H. Nicolai and A. Taormina, CERN Preprint TH. 4220/85.
14) M. Ademollo et al. Nucl. Phys. **B94** (1975), 221.
15) K. Bardakci, Nucl. Phys. **B68** (1974), 331; **B70** (1974), 397; **B133** (1978),297.
16) A. Erdelyi et al., *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), vol. 2, chap. 13.
17) J. Polchinski, Texas Preprint UTTG-13-85.
18) F. Englert and A. Neveu, CERN Preprint TH. 4168/85.