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Vacuum energy in smooth background fields

by

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Abstract

We consider the ground state energy of a scalar field in the background of a general potential which dependes on one coordinate. We consider a general expression following from the analytical properties of the one dimensional scattering matrix. We show, that reflections give a positive and bound states a negative contribution to the effective potential and we calculate explicitely two simple examples, the square well potential and a picewise oscillatory potential. We demonstrate our formulas by an easy rederivation of the mass of the kink.

1 Introduction

The ground state energy is the response of the vacuum to some external conditions or background fields. The most popular example is the Casimir effect. Important applications arise for spaces with nontrivial topologies, especially in connection with spontaneous compactification, within external gravitational fields, in the bag model and other areas. Sharp boundary conditions are often a good approximation of a physical situation. However, there is a interest in calculating the ground state energy in cases with smooth background fields too. Possible applications include nonideal boundaries, general background fields and metrics and others. Special interest comes from classical solutions of field equations like solitons, which can be considered as smooth background fields when calculating quantum fluctuations around them.

The calculation of the ground state energy is quite easy and powerful methods – the zeta function method [1] and the heat kernel expansion for instance – are known for a variety of different boundary conditions and topologies and also in constant background fields. There is a full understanding of the renormalization. However, the calculation of the bound state energy in general background fields is still a difficult task. In the general case one is left with perturbative expansions with respect to the background field or with respect to its derivatives.

The aim of the present paper is to make a step forward in the investigation of the ground state energy a general background field. For this reason we consider the effective potential[1]

$$V_{\rm eff} = \frac{1}{2}\,{\rm Tr}\log(\Box - m^2 - V(x)) \tag{1}$$

in the external potential $V(x)$. Then the problem is to find the spectrum of this operator. For a time independent potential $V(x)$ this is equivalent to a nonrelativistic Schrödinger equation which is a well investigated object. Therefor it is natural to adopt these results. There are several attempts to do so in literature. For instance this idea has been proposed in [2] and recently in [3]. Also, the expression of the effective potential as a frequency sum over the mode density or the scattering phase shift is commonly used, see [4] or recently [6]. Actual interest is in connection with chiral fields [5] and the evolution of bubble-walls [7].

In the present paper we restrict ourself to the case when $V(x)$ depends on one coordinate only and decreases as $x \to \infty$ so that $\int_{-\infty}^{\infty} |V(x)|(1+|x|){\rm d}x$ is finite. This case is the simplest one. It is generic in (1+1) dimensions and it serves as a model in (3+1) dimensions. Using well known properties of the S-matrix of the corresponding nonrelativistic Schrödinger equation we give a closed representation for the effective potential. By means of an analytic continuation we obtain a representation which incoporates the bound states in a very natural manner and which has a improved convergence. Using this representation we consider two simple examples - the square well potential and a picewise quadratic potential - and calculate the ground state energy numerically. A different representation is obtained in using the analytical properties which allow for a general conclusion concerning the sign of the effective

[1]Often this termin is used in the case of a constant background field and the corresponding quantity in a nonconstant background field is called effective action.

potential. A explicit calculation is possible for all reflectionless potentials. We give the corresponding formulas. We apply them to the kink model in (1+1) dimensions and we rederive the corresponding result [11].

The case of a potential depending on one coordinate which is considered here, is of restricted interest. The next step will be the extension of this method to a spherical symmetric potential, for instance.

2 Modesummation and analytic continuation

The effective potential requires for its renormalization the imbedding into a external system. In order to have a concrete model we consider the Lagrange density

$$L = \frac{1}{2}\Phi\left(\Box - M^2 - \lambda\Phi^2\right)\Phi + \frac{1}{2}\varphi\left(\Box - m^2 - \lambda'\Phi^2\right)\varphi\,. \tag{2}$$

The field Φ should be considered as a classical background field depending on one coordinate. By means of

$$V(x_1) = \lambda'\Phi^2 \tag{3}$$

it defines the potential in (1) for the field $\varphi(x)$, which should be quantized in the usual way. The energy density per unit area of the plane perpendicular to x_1 is given by

$$E = \frac{1}{2}V_g + \frac{M^2}{2}V_1 + \lambda V_2 + V_{\rm eff}\,, \tag{4}$$

whith the following notations: $V_g \equiv \int_{-\infty}^{\infty} \mathrm{d}x_1 \left(\frac{\partial}{\partial x_1}\Phi(x_1)\right)^2$, $V_1 \equiv \int_{-\infty}^{\infty} \mathrm{d}x_1 \left(\Phi(x_1)\right)^2$ and $V_2 \equiv \int_{-\infty}^{\infty} \mathrm{d}x_1 \left(\Phi(x_1)\right)^4$. The effective potential $V_{\rm eff}$ is given by (1), a regularization is provided. We use the regularization known from the zeta function method

$$V_{\rm eff} = \frac{1}{2}\frac{\partial}{\partial s}\mathrm{Tr}(\Box - m^2 - V(x_1))^{-s}\mu^{2s}\Big|_{s=0}\,. \tag{5}$$

The parameter μ has the dimension of a mass and appears in order to adjust the dimensions.

Instead of the effective potential one can use the ground state energy (for a detailled discussion see [12])

$$E_0 = \frac{1}{2}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\sum_n \sqrt{k_1^2 + k_2^2 + \omega_n^2 + m^2}^{(1-2s)}\mu^{2s} \tag{6}$$

with s to zero in the end.

The system has to be considered within a large box $-L \le x_1 \le L$ because E_0 as well as $V_{\rm eff}$ contain a contribution which is proportional to L. It appears as a consequence of the translational invariance which is unbroken in the case when the potential $V(x)$ is absent. Therefor, that contribution is independent on $V(x)$ and can be dropped. The corresponding Schrödinger equation reads

$$\left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = k^2\,\psi(x)\,. \tag{7}$$

For $-L \leq x_1 \leq L$, the eigenvalues $k \to \omega_n$ are discrete.

Because the potential depends on one coordinate only, after integration over the momenta corresponding to the directions with unbroken translational invariance the effective potential can be written in the form

$$V_{\rm eff} = -\frac{1}{2}\frac{\partial}{\partial s}\frac{\Gamma(s-3/2)}{8\pi^{3/2}\Gamma(s)}\sum_n \left(\omega_n^2+m^2\right)^{3/2-s}\Big|_{s=0}. \tag{8}$$

The frequency sum in this expression is the essential quantity to be calculated. It is known to have a single pol at $s=0$:

$$\sum_n \left(\omega_n^2+m^2\right)^{3/2-s} = \frac{1}{s}\Sigma_{-1}+\Sigma_0+O(s), \tag{9}$$

where Σ_{-1} is the residuum and Σ_0 is the regular part.

Using this, the effective potential takes the form

$$V_{\rm eff} = -\frac{\Sigma_{-1}}{6\pi}\left(\frac{4}{3}+\ln\frac{\mu}{2}\right)-\frac{\Sigma_0}{12\pi}. \tag{10}$$

A similar calculation yields for the ground state energy

$$E_0 = -\frac{1}{s}\frac{\Sigma_{-1}}{12\pi}-\frac{\Sigma_{-1}}{6\pi}\left(\frac{1}{3}+\ln\mu\right)-\frac{\Sigma_0}{12\pi}+O(s). \tag{11}$$

The renormalization will be discussed later on. In order to calculate the frequency sum (9) we use the following well known properties of the 1-dimensional Schrödinger equation. Under the asymption of a sufficiently fast decreasing potential $V(x)$ the two independent solutions can be choosen to have the asymptotics

$$\begin{aligned} \psi_1 &\underset{x\to-\infty}{\sim} e^{ikx}+s_{12}\,e^{-ikx}, & \psi_1 &\underset{x\to\infty}{\sim} s_{11}\,e^{ikx} \\ \psi_2 &\underset{x\to-\infty}{\sim} s_{22}\,e^{-ikx}, & \psi_2 &\underset{x\to\infty}{\sim} s_{21}\,e^{ikx}+e^{-ikx}. \end{aligned} \tag{12}$$

The S-matrix $\mathbf{S} = (s_{ij})$ is unitary. The coefficient $s_{11}(k)$ is a meromorphic function of k with possibly a finite number of simple poles on the upper half of the imaginary axis at $k = i\kappa$, where κ_n are the corresponding bound state energies. The reflection and the transmission coefficients are given by $T(k) = |s_{11}(k)|^2$ and $R(k) = |s_{12}(k)|^2$, respectively. Obviously, $1 = R + T$ holds.

At $x = \pm L$ we impose Dirichlet boundary conditions to $\psi(x)$. Other conditions could be choose well, the difference between distinct choices can be shown to be independent on the potential.

Under these conditions, the eigenvalues ω_n are solutions of the equations

$$(s_{11}\pm s_{21})e^{ikL}\pm e^{-ikL}=0.$$

These conditions correspond to the linear combinations $\psi_1 \pm \psi_2$, which form a independent set of solutions too. Now, we express the sum (9) in the form

$$\sum_{\omega_n} \left(\omega_n^2 + m^2\right)^{\frac{3}{2}-s} = \sum_{\kappa_n} \left(-\kappa_n^2 + m^2\right)^{\frac{3}{2}-s} + \int_\gamma \frac{\mathrm{d}k}{2\pi \mathrm{i}} \left(k^2 + m^2\right)^{\frac{3}{2}-s} \frac{\partial}{\partial k} \ln \left[\left((s_{11} - s_{21})\,\mathrm{e}^{\mathrm{i}kL} - \mathrm{e}^{-\mathrm{i}kL}\right) \left((s_{11} + s_{21})\,\mathrm{e}^{\mathrm{i}kL} + \mathrm{e}^{-\mathrm{i}kL}\right)\right] \tag{13}$$

where the integration path γ goes from ∞ above the real axis to zero and further below the axis from zero to ∞ so that the points $k = \omega_n > 0$, which are poles of the integrand, are inside. In the r.h.s. of this formula, the sum corresponds to the bound states $k = \mathrm{i}\kappa_n$ and the integral corresponds to the contribution of the continuous part of the spectrum in the limit $L \to \infty$.

To perform the limit $L \to \infty$ we consider the upper half part of γ. By means of $k \to k + \mathrm{i}\epsilon$ $(\epsilon > 0)$ we have

$$\ln \left[\left((s_{11} - s_{21})\,\mathrm{e}^{\mathrm{i}kL} - \mathrm{e}^{-\mathrm{i}kL}\right) \left((s_{11} + s_{21})\,\mathrm{e}^{\mathrm{i}kL} + \mathrm{e}^{-\mathrm{i}kL}\right)\right] = -2\mathrm{i}kL + 2\epsilon L - \mathrm{i}\pi + O(\mathrm{e}^{-\epsilon L})\,.$$

For the lower half part we note $k \to k - \mathrm{i}\epsilon$ and have

$$\ln \left[\left((s_{11} - s_{21})\,\mathrm{e}^{\mathrm{i}kL} - \mathrm{e}^{-\mathrm{i}kL}\right) \left((s_{11} + s_{21})\,\mathrm{e}^{\mathrm{i}kL} + \mathrm{e}^{-\mathrm{i}kL}\right)\right] = 2\mathrm{i}kL + 2\epsilon L + \ln\left(s_{11}^2 - s_{21}^2\right) + O(\mathrm{e}^{-\epsilon L})\,.$$

In the limit $L \to \infty$ we obtain up to exponentially small contributions

$$\sum_{\omega_n} \left(\omega_n^2 + m^2\right)^{\frac{3}{2}-s} = \sum_{\kappa_n} \left(-\kappa_n^2 + m^2\right)^{\frac{3}{2}-s} + 2L \int_0^\infty \frac{\mathrm{d}k}{\pi} \left(k^2 + m^2\right)^{\frac{3}{2}-s} + \int_0^\infty \frac{\mathrm{d}k}{2\pi \mathrm{i}} \left(k^2 + m^2\right)^{\frac{3}{2}-s} \frac{\partial}{\partial k} \ln\left(s_{11}^2 - s_{21}^2\right) + O(\mathrm{e}^{-\epsilon L})\,. \tag{14}$$

The second term in the r.h.s. is the contribution, which is independent on $V(x)$ and which is proportional to L. It will be dropped. Now, the integration path can be turned to the imaginary axis: $k \to \mathrm{i}k$. Using the relation

$$s_{11}^2 - s_{21}^2 = \frac{s_{11}(k)}{s_{11}(-k)} = \mathrm{e}^{2\mathrm{i}\delta(k)}\,,$$

which is a consequence of the unitarity of $\mathbf{S}$ and where $\delta(k)$ is the scattering phase, we obtain

$$\sum_{\omega_n} \left(\omega_n^2 + m^2\right)^{\frac{3}{2}-s} = -\frac{\cos \pi s}{\pi} \int_m^\infty \mathrm{d}k \left(k^2 - m^2\right)^{\frac{3}{2}-s} \frac{\partial}{\partial k} \ln s_{11}(\mathrm{i}k) \tag{15}$$

up tp terms independent on $V(x)$. The contribution from the boundstates has been canceled by the extra contributions appearing from the poles at $k = \mathrm{i}\kappa_n$ when turning

the integration contour to the imaginary axis. This formula connects the ground state energy with the **S**-matrix taken at imaginary momentum k. It it remarkable that the contributions of the bound states are not explicit here. They are implicit present, of course.

Now, let us discuss the renormalization. As it was shown in [8], the function $\log s_{11}$ has the following asymptotics as $k \to \infty$:

$$\log s_{11}(ik) \approx -\frac{\lambda' V_1}{2k} + \frac{\lambda'^2 V_2}{(2k)^3} + O(k^{-5})$$

with V_1 and V_2 defined below eq.(4). Inserting this two terms into the representation (15) of the frequency sum, we can calculate the resulting residuum at $s=0$ and the regular part according to formula (9). We denote them by Σ_{-1}^{as} and Σ_0^{as} correspondingly. Now, the effective potential V_{eff} splits into two parts $V_{\text{eff}} = V_{\text{eff}}^{\text{as}} + V_{\text{eff}}^{\text{sub}}$ with

$$V_{\text{eff}}^{\text{as}} = \frac{\lambda'}{32\pi^2}\left(\log\frac{m^2}{\mu^2} - 1\right) m^2 V_1 + \frac{\lambda'^2}{64\pi^2}\log\frac{m^2}{\mu^2} V_2 \,. \tag{16}$$

Inserting the last line into formula (4) for the energy of the whole system and using (3) it is clear, that these terms perform a renormalization of the mass M and the coupling λ of the field Φ:

$$M^2 \to M^2 + \frac{m^2\lambda'}{16\pi^2}\left(\log\frac{m^2}{\mu^2} - 1\right) \tag{17}$$

$$\lambda \to \lambda + \frac{\lambda'^2}{64\pi^2}\log\frac{m^2}{\mu^2}\,.$$

This renormalization is a finite one. The reason is the use of the zeta function which yields finite results. When using the formula (6) for the ground state energy instead, the same calculation can be performed. In that case, the renormalization reads

$$M^2 \to M^2 + \frac{m^2\lambda'}{16\pi^2}\left(\log\frac{m^2}{4\mu^2} + 1 - \frac{1}{s}\right) \tag{18}$$

$$\lambda \to \lambda + \frac{\lambda'^2}{64\pi^2}\left(\log\frac{m^2}{4\mu^2} + \frac{1}{2} - \frac{1}{s}\right)\,.$$

It is infinite, as usual in quantum field theory. Also, from general theory of renormalization, these coefficients can be connected with the renormalization group functions, especially with the beta function, of course. Remark, that in this case there is no renormalization of the kinetic term, i.e., of that term which contains derivatives of the field Φ.

Substracting the asymptotic terms (16) out from $\log s_{11}(ik)$ in (15), the integral becomes finite for $s=0$. The corresponding contribution to the effective potential reads:

$$V_{\text{eff}}^{\text{sub}} = \frac{1}{12\pi^2}\int_m^\infty \mathrm{d}k\ (k^2-m^2)^{3/2}\frac{\partial}{\partial k}\left[\log s_{11}(ik) + \frac{\lambda' V_1}{2k} - \frac{\lambda'^2 V_2}{8k^3}\right]\,. \tag{19}$$

3 Two simple examples

In this section we apply the above formulas to two simple examples for the potential $V(x)$, namely the square well potential

$$V_{\rm sq}(x) = \begin{cases} 0 & |x| > L \\ V_0 & |x| < L \end{cases} \tag{20}$$

and the picewise oscillatory potential

$$V_{\rm os}(x) = \begin{cases} 0 & |x| > L \\ V_0(1 - |x|/L)^2 & |x| \le L \end{cases} . \tag{21}$$

Let us first consider the square well potential $V_{\rm sq}$ (20). The contribution V_g to the energy, which results from the gradient of the field Φ, is infinite, the other two parts are simply $V_1 = LV_0$ and $V_2 = LV_0^2$. Therefor, the total energy is not a meaningful quantity, nevertheless the effective potential can be calculated and the renormalization can be carried out.

The **S**-matrix for this potential is well known and it reads (already after the rotation to the imaginary axis $k \to {\rm i}k$):

$$s_{11}({\rm i}k) = \frac{4kq\ {\rm e}^{2kL}}{(k+q)^2\ {\rm e}^{2qL} - (k-q)^2\ {\rm e}^{-2qL}} \tag{22}$$

with $q = \sqrt{k^2 + V_0}$. This expression can be inserted into $V_{\rm eff}^{\rm sub}$ (19). After some simple calculations we obtain:

$$\begin{aligned} V_{\rm eff}^{\rm sub} \;=\; & \frac{L}{32\pi^2}\left(\left(m^2+V_0\right)^2 \log\left(1+\frac{V_0}{m^2}\right) - m^2V_0 - \frac{3}{2}V_0^2\right) \\ & + \frac{1}{12\pi^2}\int_m^\infty {\rm d}k\ (k^2-m^2)^{3/2}\ \frac{(k-q)^2}{kq^2} \\ & - \frac{1}{3\pi^2}\int_m^\infty {\rm d}k\ (k^2-m^2)^{3/2}\ \frac{1}{q}\frac{1+kL}{\left(\frac{k+q}{k-q}\right)^2 {\rm e}^{4qL} - 1} . \end{aligned} \tag{23}$$

It is represented in Fig. 1 for several values of the parameters L and V_0. The first term within this expression is proportional to L, the 'length of the potential well'. Its sign depends on V_0/m^2 and takes both values, positive and negative. The second part depends on V_0/m^2 only. The third part depends on all parameters. For large L it is proportional to $\exp(-4\sqrt{m^2+V_0}L)$. The convergence of the k-integral is a consequence of the subtractions performed. In the second term the convergence of the integral is by powers of k. In the third term, which is the most complicated one, the integrand falls off exponentially as $k \to \infty$ and the integral converges very easy. This is a result of rotating the integration contur in $V_{\rm eff}$ (15).

The representation (23) is valid for both signs of V_0, i.e., for repulsive as well as for attractive potentials $V(x_1)$. In the last case, there is always at last one bound state. Its contribution to the frequency sum is taken into account by formula (19)

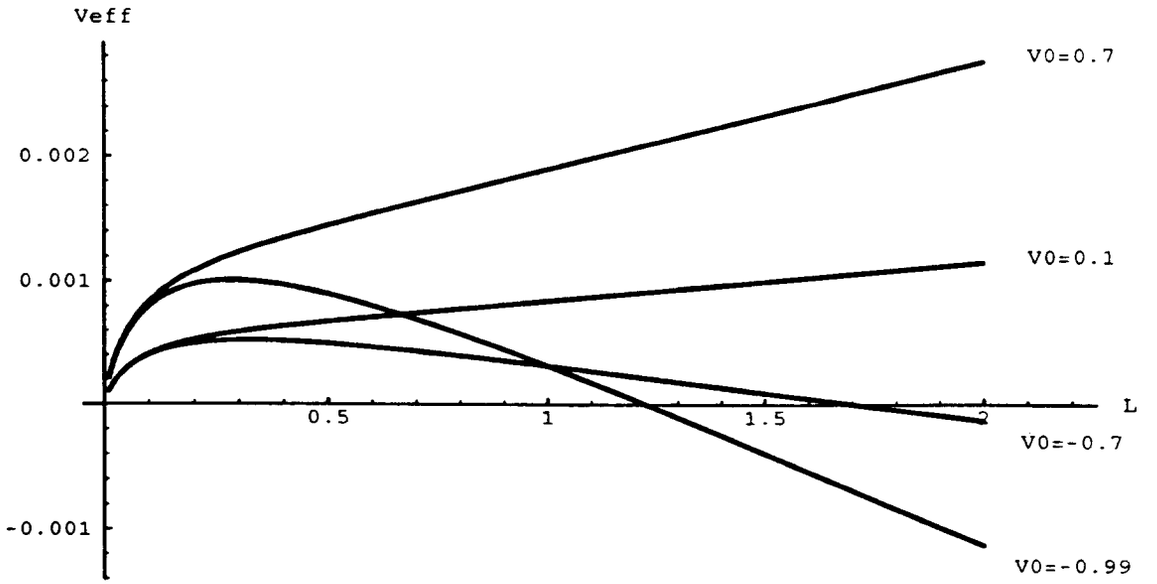

Fig. 1 The effective potential for the square well potential as function of the length L of the well for some values of its heigth V_0

automatically. In the case $V_0 < -m < 0$, the potential is overcritical with respect to particle creation. Correspondingly, $V_{\text{eff}}^{\text{sub}}$ acquires a imaginary part.

Now we turn to the picewise oscillatory potential V_{os} (21). The scattering matrix can be obtained in the following way. We consider the solution of the Schrödinger equation (7)

$$\psi(x) = \begin{cases} e^{ikx} + s_{12}e^{-ikx} & x < -L \\ \alpha_1 u(-x) + \beta_1 v(-x) & -L \le x \le 0 \\ \alpha_2 u(x) + \beta_2 v(x) & 0 \le x \le L \\ s_{11}e^{ikx} & L < x \end{cases},$$

where $u(x)$ and $v(x)$ are two independent solutions in the interval $0 \le x \le L$, α and β being constants. This function and its first derivative have to be continuous at $x = 0$ and $x = \pm L$. From these conditions, one obtains

$$s_{11}(k) = \frac{ik\, W^2\, e^{-2ik}}{(Q\, u(0) - P\, v(0))\,(Q\, u'(0) - P\, v'(0))} \tag{24}$$

with $W = uv' - u'v$ and where the abbreviations $P = u'(L) - ik\, u(L)$ and $Q = v'(L) - ik\, v(L)$ are used. These solutions can be expressed by hypergeometric functions. We choose the following combinations [9]:

$$u(x) = U(a, y) \tag{25}$$

$$v(x) = U(a, -y) \tag{26}$$

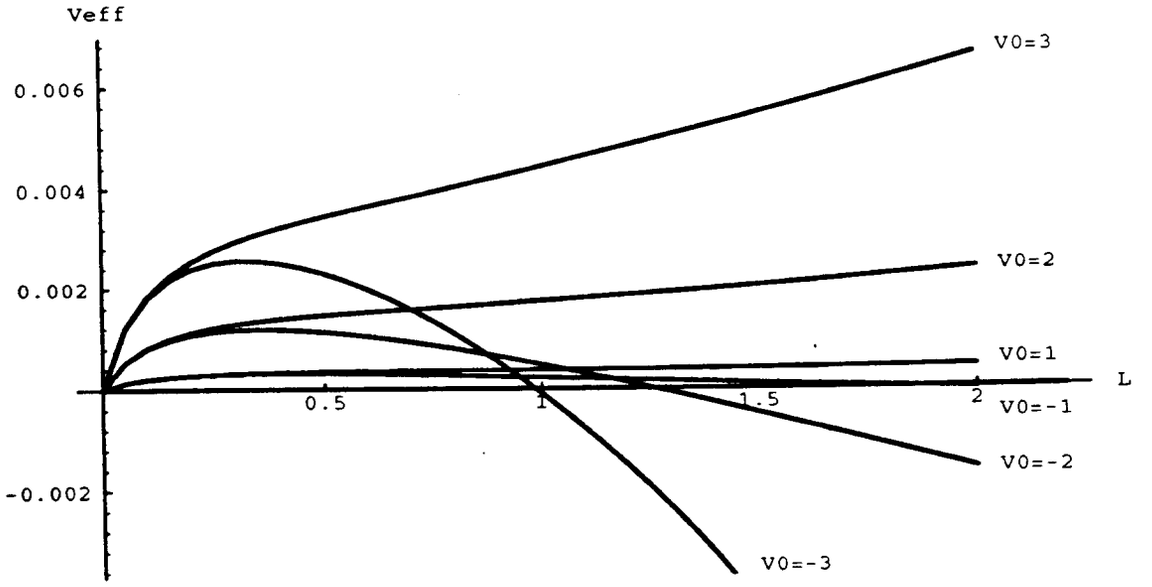

Fig. 2 he effective potential for the picewise oscillatory potential V_{os} as function of the length L for some values of the heigth V_0 for $m = 1$

with the notations

$$a \equiv \frac{-k^2 L}{2\sqrt{V_0}}, \qquad y \equiv \sqrt{\frac{2\sqrt{V_0}}{L}\,(L-x)}$$

and

$$U(a,y) = \frac{\sqrt{\pi}}{2^\alpha}\left[\frac{{}_1F_1(\alpha,1/2;y^2/2)}{\Gamma(\alpha+1/2)} - \frac{\sqrt{2}y\ {}_1F_1(\alpha+1/2,3/2;y^2/2)}{\Gamma(\alpha)}\right] e^{-y^2/4}$$

with $\alpha \equiv a/2 + 1/4$. From [9] we find $W = -\sqrt{4\pi\sqrt{V_0}/L}/\Gamma(a+1/2)$ in this case. Using these formulas, we obtain

$$Q\ u(0) - P\ v(0) = -\sqrt{\frac{4\pi\sqrt{V_0}}{L}}\left[{}_1F_1\left(\alpha,\frac{1}{2};\frac{\bar{L}^2}{2}\right) + \sqrt{a}\bar{L}_1F_1\left(\alpha+\frac{1}{2},\frac{3}{2};\frac{\bar{L}^2}{2}\right)\right] e^{-\bar{L}^2/4}$$

and

$$Q\ u'(0) - P\ v'(0) \equiv \left\{2\sqrt{\frac{2\pi}{L}}V_0^{3/4}\left(-\frac{1}{2}{}_1F_1\left(\alpha,\frac{1}{2};\frac{\bar{L}^2}{2}\right) + {}_1F_1'\left(\alpha,\frac{1}{2};\frac{\bar{L}^2}{4}\right)\right)\right.$$

$$\left. +\sqrt{2\pi a}\frac{2\sqrt{V_0}}{L}\left(\left(1-\frac{\bar{L}^2}{2}\right){}_1F_1\left(\alpha+\frac{1}{2},\frac{3}{2};\frac{\bar{L}^2}{2}\right) + \bar{L}_1F_1'\left(\alpha+\frac{1}{2},\frac{3}{2};\frac{\bar{L}^2}{2}\right)\right)\right\} e^{-\bar{L}^2/4}$$

with $\bar{L} \equiv \sqrt{2L\sqrt{V_0}}$ and ${}_1F'_1(a,b;z)$ is the derivative of the hypergeometric function with respect to z. These formulas are sufficient to calculate $s_{11}(k)$, also the rotation

$k \to ik$ can be performed. Before carrying out the calculation of the effective potential the subtractions have to be performed. From (21) we calculate $V_1 = \frac{2}{3}V_0 L$ and $V_2 = \frac{2}{5}V_0^2 L$. Inserting all these into the effectice potential $V_{\rm eff}^{\rm sub}$ (19), the integral in finite and can be calculated numerically for any values of the parameters. The result is represented in Fig. 2 as a function of L for several values of the depth V_0. For negative V_0, the potential is attractive. In that case one has to perform the formal substitution $a \to -ia$ and $\bar{L} \to \sqrt{i}\bar{L}$ in the above formulas. For sufficiently large values of V_0, the potential becomes overcritical and the effective potential takes complex values.

4 Analytical properties and reflectionless potentials

Consider the Schrödinger equation (7). The set $\{R(k), \beta_n, \kappa_n (n = 0, 1, ..., N)\}$, where $R(k)$ is the reflection coefficient, the κ_n are the bound state energies, the β_n are some numbers and N is the number of bound states, is called scattering data. The potential $V(x)$ can be restored from the scattering data uniquely. In general, by means of its analytic properties, the coefficient $s_{11}(k)$ can be represented in the form

$$\log s_{11}(k) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1-|R(q)|^2)}{k-q+i\epsilon}\,dq + \sum_{n=1}^{N} \log\frac{k+i\kappa_n}{k-i\kappa_n}\,, \qquad (27)$$

where $\epsilon \to +0$ and $R(q) = 1 - |s_{11}(q)|^2$ is the reflection coefficient and the sum goes over the bound states. The continuation to the positive imaginary axis, i.e., $k \to ik$ yields

$$\log s_{11}(k) = \frac{k}{\pi}\int_0^{\infty} \frac{\log|s_{11}(q)|)}{q^2+k^2}\,dq + \sum_{n=1}^{N} \log\frac{k+\kappa_n}{k-\kappa_n}$$

where $s_{11}(-k) = \overline{s_{11}(k)}$ has been used. In order to perform the renormalization we use the formulas [8]

$$\lambda' V_1 \equiv \int_{-\infty}^{\infty} V(x)dx = \frac{-2}{\pi}\int_{-\infty}^{\infty} \log|s_{11}(q)|\,dq - 4\sum_{n=1}^{N}\kappa_n$$

$$\lambda'^2 V_2 \equiv \int_{-\infty}^{\infty} V(x)^2 dx = \frac{-8}{\pi}\int_{-\infty}^{\infty} q^2 \log|s_{11}(q)|\,dq + \frac{16}{3}\sum_{n=1}^{N}\kappa_n^3$$

and obtain from (19)

$$V_{\rm eff}^{sub} = \frac{1}{12\pi^2}\int_m^{\infty} dk(k^2-m^2)^{3/2}\frac{\partial}{\partial k}\left\{\frac{2}{\pi}\int_0^{\infty} dq \frac{q^4}{k^2(k^2+q^2)}\log|s_{11}(q)| + \sum_{n=1}^{N}\left(\log\frac{k+\kappa_n}{k-\kappa_n} - 2\frac{\kappa_n}{k} + \frac{2}{3}\left(\frac{\kappa_n}{k}\right)^3\right)\right\}. \qquad (28)$$

This formula can be simplified

$$V_{\rm eff}^{sub} = \frac{1}{8\pi^3}\int_0^\infty {\rm d}q\; q\; \left\{-2q+\sqrt{q^2+m^2}\log\frac{\sqrt{q^2+m^2}+q}{\sqrt{q^2+m^2}-q}\right\}\; \log\frac{1}{|s_{11}(q)|}$$
$$-\frac{1}{6\pi^2}\sum_{n=1}^{N}\left[(m^2-\kappa_n^2)^{3/2}\arcsin\frac{\kappa_n}{m}+\frac{4}{3}\kappa_n^3-\kappa_n m^2\right]. \qquad (29)$$

From this formula it follows that the bound states give a negative contribution to the effective potential whereas the integral in r.h.s. of this formula, which results from the reflections (it vanishes for $|s_{11}|=1$, i.e., for a vanishing reflection coefficient), gives a positive contribution (note $|s_{11}|\leq 1$ and the expressions in both figure brackets in (29) are nonnegative). This conclusion is different from what one would expect from the known fact, that a repulsive resp. attractive potential yields a negativerep. attractive phase shift.

An important special case are the reflectionless potentials. In that case we have $|s_{11}(k)|=1$ and $s_{11}(k)$ is a rational function

$$s_{11}(k)=\prod_{n=1}^{N}\frac{k+{\rm i}\kappa_n}{k-{\rm i}\kappa_n}. \qquad (30)$$

The potential $V(x)$ can be restored from the scattering data explicitly

$$V(x)=-2\frac{d^2}{dx^2}\log\det\mathbf{A}\;, \qquad (31)$$

where the matrix $\mathbf{A}$ is

$$\mathbf{A}_{nm}=\delta_{nm}+\frac{\beta_n}{\kappa_n+\kappa_m}{\rm e}^{-2\kappa_n x}.$$

The effective potential is in this case given by the second term in r.h.s. of formula (29)

$$V_{\rm eff}^{sub}=-\frac{1}{6\pi^2}\sum_{n=1}^{N}\left[(m^2-\kappa_n^2)^{3/2}\arcsin\frac{\kappa_n}{m}+\frac{4}{3}\kappa_n^3-\kappa_n m^2\right].$$

It is completely negative. The contribution of one bound state to this formula is shown in Fig. 3a.

5 The Kink

In this section we demonstrate how our method reproduces the well known correction to the mass of the kink. For this purpose we need to rewrite the basic formulas in (1+1) dimensions. The formulas (2)-(5) remain unchanged, instead of (8) we obtain

$$V_{\rm eff}=-\frac{1}{2}\frac{\partial}{\partial s}\frac{\Gamma(s-1/2)}{2\pi^{1/2}\Gamma(s)}\sum_n\left(\omega_n^2+m^2\right)^{1/2-s}\Big|_{s=0}.$$

Expression (9) is independent on the dimension and (10) becomes

$$V_{\rm eff}=\Sigma_{-1}\left(1+\ln\frac{\mu}{2}\right)-\frac{1}{2}\Sigma_0\,.$$

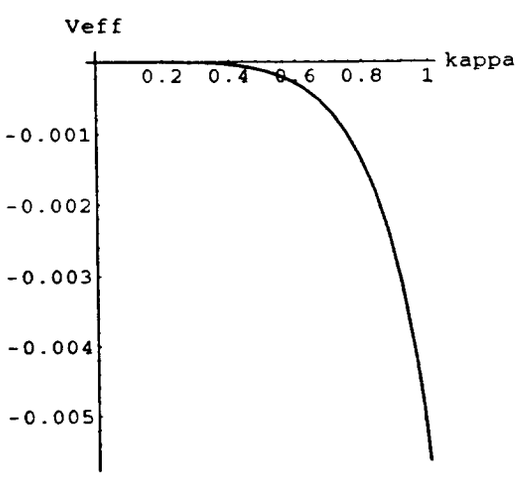

Fig. 3a The contribution of one bound state to the effective potential for a reflectionless potential in the (3+1) dimensional case

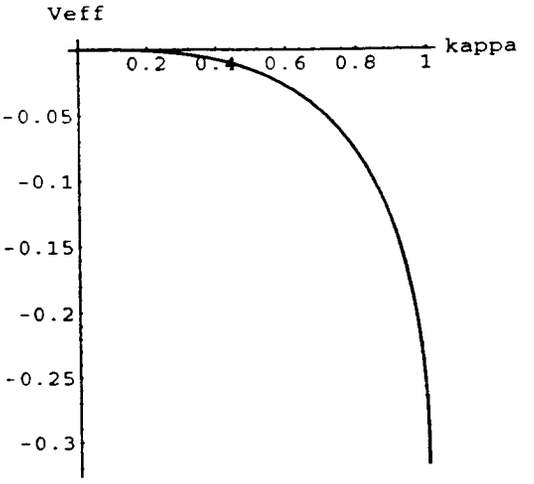

Fig. 3b The contribution of one bound state to the effective potential for a reflectionless potential in the (1+1) dimensional case

The renormalization has been changed in that sense, that only one subtraction is necessary and only the mass term is renormalized. For the effective potential we obtain

$$V_{\rm eff}^{\rm sub} = \frac{1}{2\pi}\int_m^\infty {\rm d}k\ (k^2-m^2)^{1/2}\frac{\partial}{\partial k}\left[\log s_{11}({\rm i}k) + \frac{\lambda' V_1}{2k}\right] .$$

In the case of a reflectionless potential we note (30) and this integral can be calculated. We obtain for the effective potential

$$\begin{aligned} V_{\rm eff}^{(i)} &= \sum_{n=1}^{N}\frac{1}{2\pi}\int_m^\infty {\rm d}k\ (k^2-m^2)^{1/2}\frac{\partial}{\partial k}\left[\log\frac{k+\kappa_n}{k-\kappa_i} - 2\frac{\kappa_n}{k}\right] \\ &= \sum_{n=1}^{N}\frac{1}{\pi}\left[(m^2-\kappa_n^2)^{1/2}\arcsin\frac{\kappa_n}{m} - \kappa_n\right] . \end{aligned} \tag{32}$$

This function is very closed to that in the (3+1) dimensional case. It is shown in Fig. 3b. In this form it was first derived in [10] in connection with static solitons.

These formulas can easily applied to the kink. It is a static solution $\Phi_{\rm cl}$ of the scalar Φ^4 theory in the case of a mass term with 'wrong sign'

$$S(\Phi) = \frac{1}{2}\int {\rm d}x\ \left(\left(\frac{\partial\Phi}{\partial x^\mu}\right)^2 + M^2\Phi^2 - \frac{\lambda}{2}\Phi^4\right) .$$

It reads $\Phi_{\rm cl}(x) = \frac{M}{\sqrt{\lambda}}\tanh\frac{mx}{\sqrt{2}}$,. By means of a expansion of the action around the classical solution $\Phi = \Phi_{\rm cl} + \varphi$, i.e.,

$$S(\Phi_{\rm cl}+\varphi) = S(\Phi_{\rm cl}) + \frac{1}{2}\int {\rm d}x\ \varphi(x)\frac{\delta^2 S(\Phi_{\rm cl})}{\delta\varphi^2}\varphi(x) + \ldots$$

we obtain the Lagrangean (2) with $m = \sqrt{2}M$ and $V(x_1) = 3M^2/\cosh^2(mx_1/\sqrt{2})$. This is the well known Eckhart potential. It has 2 bound states $\kappa_1 = M/\sqrt{2}$ and $\kappa_2 = \sqrt{2}M$. Inserting these values into formula (32) we obtain immediately

$$V_{\rm eff}^{\rm sub} = \left(\frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2}\pi}\right) M$$

in aggreement with [11].

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