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Vacuum Structure and Chiral Charge Quantization in the Large N Limit

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The chiral structure of the vacuum is analysed with the help of the effective $U(1)$ Lagrangian. The origin of the vacuum angle θ is discussed in this large N approach. The angle θ is shown to be an angular variable with period 2π even if we include $1/N$ nonleading terms, although this periodicity disappears in the large N limit. It is argued that "topological charge" and chiral charges are quantized in integer and even integer units, respectively. The total space of states is also explicitly constructed with these charge eigenstates and the conventional Fock space.

§ 1. Introduction

One of the old problems in hadron physics is the $U(1)$ problem or the problem of η meson.[1] Gauge theories of quarks such as Quantum Chromodynamics (QCD) which have chiral $SU(L)\otimes SU(L)$ symmetries necessarily have additional $U(1)$ symmetry. The question is why this symmetry leaves no trace empirically; that is, why is there no light flavor-singlet pseudoscalar meson? A closely related question is how one can explain the width of the $\eta\to 3\pi$ decay.

There have been attempts to resolve this problem at a formal level[2] using the instanton solutions of pure Yang-Mills theories.[3] Unfortunately, this approach was severely criticized by Crewther,[4] who showed that the consistency of the anomalous Ward-Takahashi (WT) identities of QCD is not respected in such formulation. Also, it seems that no simple picture based on the quark model can be obtained in this approach.[5]

Recently it has been suggested that this problem might be resolved in the large N limit of QCD with N colors.[6] This approach in fact does not suffer from the difficulty pointed out by Crewther with respect to the WT identities and also gives a simple quark annihilation picture as a resolution of the problem. Moreover, using an effective $U(1)$ Lagrangian,[7] the massless mode responsible for the spontaneous breakdown of chiral $U(1)$ symmetry is shown to decouple from the physical sector[8),9] and the $\eta\to 3\pi$ decay is computed with semi-quantitative success.[8),10] This massless mode which plays an important role here is coupled to the "topological current" K_μ.

On the other hand, Crewther argued on the basis of the anomalous WT identities that fractional "topological charges" $\nu[\equiv\int\partial^\mu K_\mu(x)d^4x]$ are necessary.[4] He further argued that how ν is quantized depends on the relations among light quark masses but that the vacuum angle θ is an angular variable with period 2π.[11)]

It is probable that the gluon sector ν has some correlations with the quark sector because they may couple strongly through $q\bar{q}$ annihilation diagrams. However, it seems that there are some confusions concerning chirality angles and θ parameter[12)] and that the vacuum structure is not fully analysed. Also the reason why θ is introduced is somewhat blurred. (If we do not consider instantons, it is necessary to explain why θ is present.) It is thus worth while investigating the vacuum structure and charge quantization problems within the picture of $1/N$ expanded QCD. This paper presents a self-contained analysis of these problems using the above-mentioned effective $U(1)$ Lagrangian.

Our point of view in this paper is to regard the $U(1)$ Lagrangian as the effective theory of $1/N$ expanded QCD; in fact, this Lagrangian is almost uniquely derived by considering anomaly structure of QCD and $1/N$ expansion.[7b)] It may be then suspected to be invalid to pose some physical-state conditions at the level of effective theory, once appropriate subsidiary conditions are posed in the underlying QCD. We assume that it is possible to investigate problems by using only effective degrees of freedom. In fact, the decoupling of the massless mode K_μ can be achieved either at the level of the effective theory[8)] or at that of the underlying theory.[9)]

The present paper is then organized as follows. In § 2, we explain the origin of θ parameter in our $U(1)$ Lagrangian. The physical state condition is specified by this vacuum angle which is not originally contained in the Lagrangian.[13)] Our discussion then proceeds to the problem of θ-dependence of this theory (§ 3). Although part of such analysis has already been performed in the literature,[7b)] we briefly discuss it here for completeness and convenience for later discussion. In particular it is shown that physics is periodic in θ with period 2π even if we include $1/N$ nonleading terms, although *this is not the case in the large N limit.*

In § 4, we analyse the vacuum structure and charge quantization problems in our large N approach. It turns out that the vacuum has a structure similar to the two-dimensional Schwinger model,[14),15)] having degeneracy for finite chiral rotations. It is clear that "topological charge" $K[\equiv \int K_0(x) d^3x]$*) and the chiral charge $\tilde{Q}_5^{(0)}[\equiv Q_5^{(0)} - 2L\cdot\sqrt{(2/L)}K]$ are ill-defined due to spontaneous breakdown of the chiral $U(1)$ symmetry (that is, contributions from the massless mode K_μ) and explicit violations by mass terms. We propose to define these charges as the generators of the transformations by taking "reducible representations" in the chiral sector[14)] and then argue that $\sqrt{(L/2)}\,\tilde{Q}_5^{(0)}$ is quantized in even integer units. If we could neglect quark sector $Q_5^{(0)}$, then $\sqrt{(L/2)}\,\tilde{Q}_5^{(0)} \simeq 2LK$, which would imply

*) We use the same terminology "topological charge" for ν and K, the difference between which is that ν is given by the difference of K at positive and negative infinite times.

that K (and therefore ν) is quantized in $1/L$ units. However, this cannot be justified. Instead, we shall argue that ν is quantized in integer units. The total space of states is explicitly constructed in terms of the eigenstates of these charges and the conventional Fock space.

Finally, in § 5, we briefly discuss the main results obtained here. Especially, a loophole to Crewther's argument[11] for fractional ν is discussed in our model.

§ 2. The origin of θ

We start with the following $U(1)$ Lagrangian:[7),8)]

$$\mathcal{L}=\frac{F_\pi^2}{16}\mathrm{Tr}\{\partial_\mu M\cdot\partial^\mu M^\dagger\}+\frac{F_\pi^2}{16}\mathrm{Tr}\{\mathcal{M}(M+M^\dagger)\}$$
$$+\frac{2L^2}{F_s^2(m_s^2-m^2)}(\partial_\mu K^\mu)^2+\frac{i}{2}(\partial_\mu K^\mu)\cdot\ln\det\frac{M}{M^\dagger}\,, \tag{2·1}$$

where M is an $L\times L$ matrix field which can be expressed nonlinearly by L^2 pseudoscalar mesons

$$M=\exp\left\{i\frac{2}{F_\pi}\varPi\right\},\qquad \varPi=\sum_{i=1}^{L^2-1}\Pi_i\lambda^i+S\lambda^0 \tag{2·2}$$

with λ^a ($a=0, 1, \cdots, L^2-1$) being $U(L)$ matrices normalized as

$$\mathrm{Tr}\{\lambda^a\lambda^b\}=2\delta^{ab}\,. \tag{2·3}$$

The vector field K^μ corresponds in QCD to the well-known "topological" current

$$K^\mu=\frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\lambda\sigma}A_\nu^a\left(\partial_\lambda A_\sigma^a+\frac{1}{3}gf^{abc}A_\lambda^b A_\sigma^c\right), \tag{2·4}$$

which is gauge variant but its divergence

$$\partial_\mu K^\mu=\frac{g^2}{32\pi^2}F_{\mu\nu}^a\tilde{F}^{a\mu\nu}\,, \tag{2·5}$$

is gauge invariant. $\mathcal{M}=\mathrm{diag}(\mu_1^2, \mu_2^2, \cdots, \mu_L^2)$ [$\simeq\mathrm{diag}(m_\pi^2, m_\pi^2, 2m_K^2-m_\pi^2)$ for $L=3$] is the mass matrix for flavor-nonsinglet pseudoscalar mesons and is proportional to the light quark mass matrix. m_s^2 is the flavor-singlet meson mass, $m^2=(\mu_1^2+\cdots+\mu_L^2)/L$ and finally F_π is the decay constant of flavor-nonsinglet pseudoscalar meson with $F_s=\sqrt{(L/2)}\,F_\pi$.

The Lagrangian (2·1) is invariant under chiral $SU(L)\otimes SU(L)$ transformation in the chiral limit $\mathcal{M}\to 0$. However, it no longer has invariance under chiral $U(1)$ transformation owing to the last "anomaly" term.*) This statement may be somewhat misleading. We usually say that such a system is chiral $U(1)$ in-

*) If we modify this term to $-(i/2)K_\mu\cdot\partial^\mu\ln\det(M/M^\dagger)$ by partial integrations, then Eq. (2·1) is manifestly invariant under chiral $U(1)$ transformation but is not gauge invariant.

variant since the change in $\mathcal{L}$ is merely total divergence $\partial_\mu K^\mu$. However, it is known that this "invariance" is (spontaneously) broken because K_μ is a massless mode.

Another transformation under which (2·1) is invariant is the "gauge" one

$$K_\mu \to K_\mu + \varepsilon_{\mu\nu\lambda\sigma}\partial^\nu \Lambda^{\lambda\sigma}\,, \tag{2·6}$$

which reflects color gauge invariance of the underlying QCD. To quantize this system, we introduce gauge fixing term[8)]

$$\frac{L^2}{F_s{}^2(m_s{}^2 - m^2)\alpha}(\partial^\mu K^\nu - \partial^\nu K^\mu)^2\,. \tag{2·7}$$

Then equation of motion for K_μ is given by

$$-\partial_\mu\left\{\frac{4L^2}{F_s{}^2(m_s{}^2 - m^2)}\partial_\nu K^\nu + \frac{i}{2}\ln\det\frac{M}{M^\dagger}\right\}$$

$$= \frac{4L^2}{F_s{}^2(m_s{}^2 - m^2)\alpha}\partial^\nu(\partial_\nu K_\mu - \partial_\mu K_\nu), \tag{2·8}$$

from which one obtains

$$\Box(\partial_\mu K_\nu - \partial_\nu K_\mu) = 0\,. \tag{2·9}$$

Thus one can impose the following subsidiary condition for K_μ:

$$(\partial_\mu K_\nu - \partial_\nu K_\mu)^{(+)}|\text{phys}\rangle = 0\,. \tag{2·10}$$

Also, Eq. (2·8) yields

$$\Box\left\{\frac{4L^2}{F_s{}^2(m_s{}^2 - m^2)}\partial_\mu K^\mu + \frac{i}{2}\ln\det\frac{M}{M^\dagger}\right\} = 0\,. \tag{2·11}$$

The physical state condition consistent with Eqs. (2·8) and (2·10) is

$$\left\langle\frac{4L^2}{F_s{}^2(m_s{}^2 - m^2)}\partial_\mu K^\mu + \frac{i}{2}\ln\det\frac{M}{M^\dagger}\right\rangle_{\text{phys}} = C\,. \tag{2·12}$$

The "integration" *constant* C is usually set to zero. However, there is no reason for C to be zero; it should be fixed by the boundary condition at infinity. To fix C, we may add to the Lagrangian (2·1) the well-known term

$$\theta \cdot \partial^\mu K_\mu\,. \tag{2·13}$$

From the Lagrangian (2·1)+(2·7)+(2·13), one computes the effective potential with the help of Eqs. (2·10) and (2·12), whose stationary point gives

$$C = -\theta\,. \tag{2·14}$$

This origin of θ parameter as an integration constant was first noted by Coleman[15)] in the two-dimensional Schwinger model and has recently been emphasized by Aurilia, Nicolai and Townsend[16)] in a four-dimensional model. It should be noticed here that θ has nothing to do with the "topological charge" at this stage.

§ 3. θ-dependence

Now that we have obtained our complete Lagrangian

$$\mathcal{L}_\theta=\frac{F_\pi^2}{16}\mathrm{Tr}\{\partial_\mu M\cdot\partial^\mu M^\dagger\}+\frac{F_\pi^2}{16}\mathrm{Tr}\{\mathcal{M}(M+M^\dagger)\}$$

$$+\frac{1}{2F_s^2(m_s^2-m^2)}\left\{2L(\partial_\mu K^\mu)+\frac{F_s^2(m_s^2-m^2)}{2L}\left(\theta+\frac{i}{2}\ln\det\frac{M}{M^\dagger}\right)\right\}^2$$

$$-\frac{F_s^2(m_s^2-m^2)}{8L^2}\left(\theta+\frac{i}{2}\ln\det\frac{M}{M^\dagger}\right)^2+(\text{gauge fixing term}),\qquad(3\cdot1)$$

we can investigate θ-dependence of this theory.[7b)] This is our main task in this section.

3.1. *Minimizing the vacuum energy*

From Eq. (3·1) together with the subsidiary conditions (2·10) and (2·12), we see that the effective potential is given by

$$V(M)=\frac{F_\pi^2}{16}\left[-\mathrm{Tr}\{\mathcal{M}(M+M^\dagger)\}+\frac{m_s^2-m^2}{L}\left(\theta+\frac{i}{2}\ln\det\frac{M}{M^\dagger}\right)^2\right].\qquad(3\cdot2)$$

Since $\mathcal{M}$ is diagonal, we look for a minimum of $V(M)$ with a diagonal matrix

$$\langle M\rangle=\mathrm{diag}(e^{i\varphi_1},e^{i\varphi_2},\cdots,e^{i\varphi_L}),\qquad(3\cdot3)$$

which yields the vacuum energy

$$E(\theta,\varphi_i)=\frac{F_\pi^2}{8}\left\{-\sum_{i=1}^{L}\mu_i^2\cos\varphi_i+\frac{m_s^2-m^2}{2L}(\theta-\sum_{i=1}^{L}\varphi_i)^2\right\}.\qquad(3\cdot4)$$

From the minimization condition, we get

$$\mu_i^2\sin\varphi_i=\frac{m_s^2-m^2}{L}(\theta-\sum_{j=1}^{L}\varphi_j).\quad(i:\text{ not summed})\qquad(3\cdot5)$$

When any one of μ_i^2, say μ_1^2, vanishes, this set of equations has the solution $\varphi_1=\theta$, $\varphi_2=\cdots=\varphi_L=0$. In this case, θ completely disappears if one rotates M by $\hat{M}=M\langle M\rangle^{-1}$ and has no physical effects.

We note that Eq. (3·3) is equivalent to regarding the fields S, Π_3 and Π_8 (for $L=3$) as having vacuum expectation values (VEV's)

$$\sqrt{\frac{2}{3}}\left\langle\frac{2}{F_\pi}S\right\rangle=\frac{\varphi_1+\varphi_2+\varphi_3}{3},\qquad\left\langle\frac{2}{F_\pi}\Pi_3\right\rangle=\frac{\varphi_1-\varphi_2}{2},$$

$$\left\langle\frac{2}{F_\pi}\Pi_8\right\rangle=\frac{\varphi_1+\varphi_2-2\varphi_3}{2\sqrt{3}}.\qquad(3\cdot6a)$$

In fact, solving Eq. (3·5) for $L=3$ and small θ recovers our previous results.[13]

$$\langle S\rangle=\frac{F_s}{6}\frac{(m_s^2-m^2)(4m_K^2-m_\pi^2)}{m_s^2(4m_K^2-m_\pi^2)-\frac{8}{3}(m_K^2-m_\pi^2)^2}\theta\,,$$

$$\langle \Pi_8\rangle=\frac{F_s}{6}\frac{2\sqrt{2}(m_s^2-m^2)(m_K^2-m_\pi^2)}{m_s^2(4m_K^2-m_\pi^2)-\frac{8}{3}(m_K^2-m_\pi^2)^2}\theta\,. \tag{3·6b}$$

3.2. *Periodicity*

It is immediately clear from Eq. (3·5) that θ is an angular variable with period 2π since 2π change in θ is compensated by the change in φ_i which is periodic by 2π. It should be stressed that this periodicity has nothing to do with the "topological" charge quantization but is a consequence of periodicity in chiral phases φ_i.

It is especially interesting to note here that *θ ceases to be an angular variable in the large N limit or in the absence of the coupling of "gluon" and matter sectors.* In fact, in this limit the last term in Eq. (2·1) is absent because $F_\pi\sim O(\sqrt{N})$ and $m_s^2-m^2\sim O(1/N)$, and the θ-dependence of the vacuum energy is $\sim\theta^2$, in sharp contrast with dilute instanton gas calculation [$\sim\cos\theta$].[17]

Let us see in more detail how this periodicity is realized for the special case $\mu_1^2=\mu_2^2=\mu_3^2\equiv\mu^2\ll m_s^2-m^2$ ($L=3$).[7b),11),12)] Then Eq. (3·5) reduces to

$$\varphi_1+\varphi_2+\varphi_3=\theta\,, \tag{3·7a}$$

$$\sin\varphi_1=\sin\varphi_2=\sin\varphi_3\,, \tag{3·7b}$$

and the vacuum energy (3·4) is now given by

$$E=-\frac{F_\pi^2}{8}\mu^2(\cos\varphi_1+\cos\varphi_2+\cos\varphi_3). \tag{3·8}$$

The minimum energy solutions of Eq. (3·7) are

$$\varphi_1-2\pi=\varphi_2=\varphi_3=\frac{\theta-2\pi}{3}\qquad\text{for}\quad -3\pi<\theta<-\pi\,,$$

$$\varphi_1=\varphi_2=\varphi_3=\frac{\theta}{3}\qquad\text{for}\quad -\pi<\theta<\pi\,,$$

$$\varphi_1+2\pi=\varphi_2=\varphi_3=\frac{\theta+2\pi}{3}\qquad\text{for}\quad \pi<\theta<3\pi\,, \tag{3·9}$$

and this behavior repeats itself for the whole range of θ.*) $E(\theta)$ is depicted in Fig. 1.

*) Note that φ_i's can be determined modulo 2π.

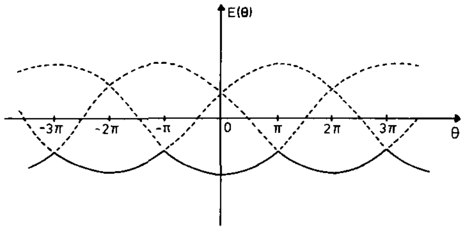

Fig. 1. θ-dependence of the vacuum energy for $\mu_1^2=\mu_2^2=\mu_3^2 \ll m_s^2-m^2$ $(L=3)$. The solid line represents the physical vacuum.

From this figure, we see that the vacuum achieves 2π periodicity by changing three different branches of states. Each branch has 6π period but becomes degenerate at $\theta=\pm\pi$, $\pm 3\pi$, $\cdots$, thus enabling the physical states to become 2π periodic in θ. Once again, we emphasize that the fact that each branch of states has 6π (or $2L\pi$ in general) period, which is often considered to imply fractional ν, has nothing to do with "topological" charge quantization (either fractional or integral). It is a consequence of the nature of the solution of Eq. (3·7).*)

Finally, we point out that this periodicity in θ holds even if we take $1/N$ nonleading terms into account. Indeed, if we neglect higher derivatives of $\partial_\mu K^\mu$ (allowing this possibility leaves the following discussion essentially unchanged), the most general Lagrangian is[7b)]

$$\mathcal{L}=\mathcal{L}_0(M, M^\dagger)+(\partial_\mu K^\mu)\left(\theta+\frac{i}{2}\ln\det\frac{M}{M^\dagger}\right)+\frac{2L^2}{F_s^2(m_s^2-m^2)}(\partial_\mu K^\mu)^2$$
$$+\sum_{k=1}^{\infty}\mathcal{L}_k(M, M^\dagger)(\partial_\mu K^\mu)^k\,, \tag{3·10}$$

where the first line is the same as (3·1), and the N-dependence of each term is given by

$$F_\pi^2\sim O(N), \qquad \mathcal{L}_0\sim O(N),$$
$$\mathcal{L}_{2k}\sim O(N^{2-2k}), \qquad \mathcal{L}_{2k-1}=0\,. \quad (k=1, 2, \cdots) \tag{3·11}$$

The important point is that the form of the second term in (3·10) is left unchanged because there is no higher order correction to the anomaly and that other terms are functions of M, $M^\dagger$ and $\partial^\mu K_\mu$. The minimization condition with respect to the VEV of $\partial_\mu K^\mu$ now reads

$$\sum_{j=1}^{L}\varphi_j-\theta+f(\varphi_i, \langle\partial_\mu K^\mu\rangle)=0 \qquad (i=1, 2, \cdots, L) \tag{3·12a}$$

with

*) It was in fact pointed out[18)] that $2L\pi$ periodicity for some amplitudes could be obtained also by using instantons, contrary to the usual belief.

$$f(\varphi_i+2\pi, \langle\partial_\mu K^\mu\rangle)=f(\varphi_i, \langle\partial_\mu K^\mu\rangle). \qquad (3\cdot12\text{b})$$

Equation (3·12b) is a consequence of the above-mentioned property. It is then clear that the change $\theta\to\theta+2\pi$ has no physical effects. Thus physics is always periodic in θ by 2π *if we correctly take into account the "gluon" and matter coupling.*

3.3. *Modifying the Lagrangian*

Let us now rewrite the Lagrangian in terms of fields with no VEV's, $\hat{M}=M\langle M\rangle^{-1}$. Eliminating $\partial_\mu K^\mu$ and substituting $\hat{M}$ into the $U(1)$ Lagrangian yield[7b)]

$$\begin{aligned}\mathcal{L}=&\frac{F_\pi^{\,2}}{16}\operatorname{Tr}\{\partial_\mu\hat{M}\cdot\partial^\mu\hat{M}\}\\&+\frac{F_\pi^{\,2}}{16L}\left[\frac{m_S^{\,2}-m^2}{4}\left(\ln\det\frac{M}{M^\dagger}\right)^2+L\operatorname{Tr}\{\mathcal{M}(\theta)(\hat{M}+\hat{M}^\dagger-2)\}\right.\\&\left.-i(m_S^{\,2}-m^2)\Big(\theta-\sum_{i=1}^{L}\varphi_i\Big)\operatorname{Tr}\left\{\ln\frac{\hat{M}}{\hat{M}^\dagger}-(\hat{M}-\hat{M}^\dagger)\right\}\right]-E(\theta,\varphi_i),\end{aligned} \qquad (3\cdot13)$$

where $E(\theta, \varphi_i)$ is given by Eq. (3·4) and

$$\mathcal{M}(\theta)=\operatorname{diag}(\mu_1^{\,2}\cos\varphi_1, \cdots, \mu_L^{\,2}\cos\varphi_L), \qquad (3\cdot14)$$

where φ_i's are determined by Eqs. (3·5).

This form of Lagrangian is most convenient for practical applications. The last term in the square bracket in Eq. (3·13) gives the famous CP-violating effect.[7b),13),19)] We note that this term is $SU(L)$ singlet, consistent with the well-known theorem due to Nuyts.[20)]

§ 4. Charge quantization and physical states

Although the analysis presented in the previous section is enough for practical applications, the vacuum structure is not fully clarified. We shall investigate this and charge quantization problems in this section, putting $L=3$ explicitly.

4.1. *Chiral charge quantization*

Our main concern is now the chiral structure of the theory. In order to investigate this, consider the gauge-invariant chiral $U(1)$ current

$$j_{5\mu}^{(0)}=\frac{i}{8}F_\pi^{\,2}\operatorname{Tr}\{\lambda^0(\partial_\mu M\cdot M^\dagger-M\cdot\partial_\mu M^\dagger)\}, \qquad (4\cdot1)$$

which consists of matter fields alone and satisfies

$$\partial^\mu j_{5\mu}^{(0)} = 6\sqrt{\frac{2}{3}}\partial^\mu K_\mu - \frac{iF_\pi{}^2}{8}\sqrt{\frac{2}{3}}\,\mathrm{Tr}\{\mathcal{M}(M-M^\dagger)\}. \tag{4·2}$$

Therefore, this current is not conserved in the chiral limit $\mathcal{M}\to 0$:

$$\partial^\mu j_{5\mu}^{(0)} = 6\sqrt{\frac{2}{3}}\partial^\mu K_\mu \neq 0\,. \tag{4·3}$$

Still we can construct a conserved but gauge-variant current

$$\tilde{j}_{5\mu}^{(0)} = j_{5\mu}^{(0)} - 6\sqrt{\frac{2}{3}}K_\mu\,. \tag{4·4}$$

It is easy to see that charges $Q_5^{(0)}$ and $\tilde{Q}_5^{(0)}$ defined by $j_{5\mu}^{(0)}$ and $\tilde{j}_{5\mu}^{(0)}$, respectively, generate the same transformations of M:

$$\overset{(\sim)}{Q}{}_5^{(0)} \equiv \int \overset{(\sim)}{j}{}_{5,0}^{(0)}(x)\,d^3x\,, \tag{4·5}$$

$$\exp(i\overset{(\sim)}{Q}{}_5^{(0)}\alpha)M\exp(-i\overset{(\sim)}{Q}{}_5^{(0)}\alpha) = M\exp(-2i\alpha\lambda^0). \tag{4·6}$$

This is because the difference $Q_5^{(0)} - \tilde{Q}_5^{(0)}$ commutes with the pseudoscalar meson fields at equal times:

$$[Q_5^{(0)} - \tilde{Q}_5^{(0)}, M] = 0\,. \tag{4·7}$$

However, $Q_5^{(0)}$ does not commute with $\partial^\mu K_\mu$ but $\tilde{Q}_5^{(0)}$ does at equal times:

$$[\tilde{Q}_5^{(0)}, \partial^\mu K_\mu] = 0\,,\quad [Q_5^{(0)}, \partial^\mu K_\mu] = \frac{1}{6}\sqrt{\frac{3}{2}}F_\pi{}^2(m_s{}^2 - m^2). \tag{4·8}$$

Hence, *the correct charge that generates the chiral $U(1)$ transformation is not the gauge-invariant $Q_5^{(0)}$ but the (gauge-variant) $\tilde{Q}_5^{(0)}$.**) However, the charge $\tilde{Q}_5^{(0)}$ is not well-defined because of the contributions from the massless mode K_μ. This is to be expected because in the chiral limit this chiral $U(1)$ invariance is (spontaneously) broken.

On the other hand, since chiral $U(1)$ transformation shifts the value of the field S at infinity, the transformation generated by the charge cannot be compatible with the physical subspace specified by Eqs. (2·10) and (2·12). Moreover, in the presence of mass terms, the chiral charge is not conserved. This does not prevent us, however, from defining the chiral charge by a generator of the chiral $U(1)$ transformation. The same fact also applies to other chiral charges. Here we concentrate on the "diagonal" charges

$$Q_5^{(i)} = \int j_{5,0}^{(i)}(x)\,d^3x\,,\qquad (i=3, 8) \tag{4·9}$$

*) A similar phenomenon is known to occur in the Schwinger model. See T. Kugo.[14]

where

$$j_{5\mu}^{(i)}=\frac{i}{8}F_\pi{}^2\ \mathrm{Tr}\{\lambda^i(\partial_\mu M\cdot M^\dagger-M\cdot\partial_\mu M^\dagger)\},\tag{4·10}$$

are the flavor-nonsinglet chiral currents.

In order to implement these charges, we proceed as follows. First, we write M in the factorized form $\hat{M}\cdot\exp(i\sum_{i=0,3,8}\lambda^i\hat{\chi}_i)$ in terms of massive fields $\hat{\Pi}$'s and constant fields ($\hat{\chi}_0$, $\hat{\chi}_3$, $\hat{\chi}_8$), and propose that the chiral transformations act only on the constant fields:*)

$$\exp(i\alpha\tilde{Q}_5^{(0)})\exp\left(i\frac{2}{F_\pi}\hat{S}\lambda^0\right)\exp(-i\alpha\tilde{Q}_5^{(0)})=\exp\left(i\frac{2}{F_\pi}\hat{S}\lambda^0\right),\ \text{etc.},\tag{4·11}$$

$$\exp(i\alpha\tilde{Q}_5^{(0)})\exp(i\hat{\chi}_0\lambda^0)\exp(-i\alpha\tilde{Q}_5^{(0)})=\exp\{i(\hat{\chi}_0-2\alpha)\lambda^0\},\ \text{etc.}\tag{4·12}$$

We can express (S, Π_3, Π_8) as functions of the massive and constant fields although these functions except for S are complicated due to nonabelian nature of the chiral group. It is convenient to introduce linear combinations of these charges

$$Q_1=\frac{1}{6}\left(2\sqrt{\frac{3}{2}}\tilde{Q}_5^{(0)}+3Q_5^{(3)}+\sqrt{3}Q_5^{(8)}\right),\tag{4·13a}$$

$$Q_2=\frac{1}{6}\left(2\sqrt{\frac{3}{2}}\tilde{Q}_5^{(0)}-3Q_5^{(3)}+\sqrt{3}Q_5^{(8)}\right),\tag{4·13b}$$

$$Q_3=\frac{1}{6}\left(2\sqrt{\frac{3}{2}}\tilde{Q}_5^{(0)}-2\sqrt{3}Q_5^{(8)}\right),\tag{4·13c}$$

which generate the following chiral transformation:

$$\exp(i\sum_{i=1}^{3}\alpha_iQ_i)\exp\left\{i\begin{pmatrix}\hat{\varphi}_1 & & 0\\ & \hat{\varphi}_2 & \\ 0 & & \hat{\varphi}_3\end{pmatrix}\right\}\exp(-i\sum_{i=1}^{3}\alpha_iQ_i)$$

$$=\exp\left\{i\begin{pmatrix}\hat{\varphi}_1-2\alpha_1 & & 0\\ & \hat{\varphi}_2-2\alpha_2 & \\ 0 & & \hat{\varphi}_3-2\alpha_3\end{pmatrix}\right\},\tag{4·14}$$

where $\hat{\varphi}_i$ ($i=1, 2, 3$) are appropriate linear combinations of the constant fields $\hat{\chi}_i$ ($i=0, 3, 8$)

$$\hat{\varphi}_1=\sqrt{\frac{2}{3}}\hat{\chi}_0+\hat{\chi}_3+\frac{\hat{\chi}_8}{\sqrt{3}},\quad \hat{\varphi}_2=\sqrt{\frac{2}{3}}\hat{\chi}_0-\hat{\chi}_3+\frac{\hat{\chi}_8}{\sqrt{3}},\quad \hat{\varphi}_3=\sqrt{\frac{2}{3}}\hat{\chi}_0-\frac{2}{\sqrt{3}}\hat{\chi}_8.\tag{4·15}$$

*) Such constant fields were first introduced by Lowenstein and Swieca and subsequently formulated in the same form as the one presented here by Kogut and Susskind in the *massless* Schwinger model.[14] In this case, the meson field is simply the *sum* of massive and constant fields.

Since Q_i's do not commute only with the constant fields $\hat{\varphi}_i$, we can give definite meanings to these charges, which are originally ill-defined, *by defining them as the canonical conjugates of* $\hat{\varphi}_i$*'s*[14)]

$$Q_i = -2\Pi_{\hat{\varphi}_i}, \qquad (i=1, 2, 3) \tag{4·16}$$

so as to satisfy Eq. (4·14). The degrees of freedom $\hat{\varphi}_i$ and $\Pi_{\hat{\varphi}_i}$ are independent of the fields $\hat{S}$, $\hat{\Pi}_3$ and $\hat{\Pi}_8$ and therefore once they are fixed, their eigenvalues are not changed. Only chiral transformations generated by the charges (4·16) relate those states with different eigenvalues for $\hat{\varphi}_i$.

We note that $\hat{\varphi}_i$'s are angular variables. This follows from the fact that the chiral group is compact. It is then clear from Eq. (4·14) that the generators of transformations

$$e^{in\pi Q_i}, \qquad (i=1, 2, 3) \tag{4·17}$$

are unit operators for any integer n, because the transformations generated by them leave all the fields unchanged. This implies that only the states with even integer Q_i are allowed.

The total space of states is thus a direct product of the conventional Fock space specified by the conditions similar to Eqs. (2·10) and (2·12) and a space spanned by vectors with integer $\frac{1}{2}Q_i$ $(i=1, 2, 3)$

$$\begin{aligned}\Pi_{\hat{\varphi}_i}|n_1, n_2, n_3\} &= -\frac{1}{2}Q_i|n_1, n_2, n_3\} \\ &= n_i|n_1, n_2, n_3\}. \end{aligned} \tag{4·18}$$

The operators $\sigma_\pm^{(i)} = \exp(\pm i\hat{\varphi}_i)$ are raising and lowering operators:

$$\sigma_\pm^{(1)}|n_1, n_2, n_3\} = |n_1 \pm 1, n_2, n_3\}, \quad \text{etc}., \tag{4·19}$$

which can be easily seen if we notice the commutation relations

$$[\sigma_\pm^{(i)}, \Pi_{\hat{\varphi}_i}] = \mp\sigma_\pm^{(i)}, \tag{4·20}$$

The use of these operators allows us to construct the states $|n_1, n_2, n_3\}$ by the standard procedure. The eigenvectors of $\hat{\varphi}_i$'s are also easily constructed*)

$$|\varphi_1, \varphi_2, \varphi_3\} = \sum_{n_1,n_2,n_3} \exp(-i\sum_{i=1}^{3} n_i\varphi_i)|n_1, n_2, n_3\} \tag{4·21}$$

with normalization condition

$$\begin{aligned}\{\varphi_1', \varphi_2', \varphi_3'|\varphi_1, \varphi_2, \varphi_3\} &= \sum_{n_1,n_2,n_3} \exp(-i\sum_{i=1}^{3} n_i(\varphi_i - \varphi_i')) \\ &= \prod_{i=1}^{3} 2\pi\delta(\varphi_i - \varphi_i'). \end{aligned} \tag{4·22}$$

*) It is well known that the cluster property of the theory is violated in terms of the charge eigenstates but is repaired if we use the eigenstates of $\hat{\varphi}_i$'s.

The eigenvalues φ_i's for $\hat{\varphi}_i$'s should be identified with those already introduced in Eq. (3·3). The vacua with different φ_i's are related by chiral transformations but consist of different vector spaces in the sense of Eq. (4·22).

4.2. *"Topological charge" quantization*

Let us now turn our attention to the "topological charge" defined formally by

$$K=\int K_0(x)d^3x\ . \tag{4·23}$$

Although this is neither conserved nor well-defined because K_μ is a massless mode, it is easy to see from the canonical commutation relations that K generates the following transformations at equal times:

$$\begin{aligned} &e^{i\alpha K}Me^{-i\alpha K}=M\ ,\\ &e^{i\alpha K}K_\mu e^{-i\alpha K}=K_\mu\ ,\\ &e^{i\alpha K}\partial^\mu K_\mu e^{-i\alpha K}=\partial^\mu K_\mu+\alpha F_s^2(m_s^2-m^2)/4L^2\ . \end{aligned} \tag{4·24}$$

So just as in the case of the chiral charges, we can make K well-defined by writing as

$$\begin{aligned} &\partial^\mu K_\mu=\widehat{\partial^\mu K_\mu}+F_s^2(m_s^2-m^2)\hat{\beta}/4L^2\ ,\\ &K=\Pi_{\hat{\beta}}\ , \end{aligned} \tag{4·25}$$

where it is understood that only $\hat{\beta}$ undergoes the transformation

$$e^{i\alpha K}\hat{\beta}e^{-i\alpha K}=\hat{\beta}+\alpha\ . \tag{4·26}$$

Then from Eq. (3·1), we note that $e^{i2n\pi K}$ also can be regarded as a unit operator. This implies that topological charge K is quantized in integer units, but this fact has no apparent relation with instantons.

It is an easy task to construct eigenvectors of $\hat{\beta}$. We do not repeat it here since it is essentially the same as for the case of the chiral charges. The total space of states is now a direct product of the conventional Fock space specified by Eq. (2·10) and

$$\left\langle \frac{4L^2}{F_s^2(m_s^2-m^2)}\partial^\mu K_\mu-\frac{2L}{F_s}\hat{S}\right\rangle_{\rm phys}=0\ , \tag{4·27}$$

and a space spanned by eigenvectors of $\hat{\varphi}_i$'s and $\hat{\beta}$ with the eigenvalue β of $\hat{\beta}$ identified as

$$\beta=L\sqrt{\frac{2}{L}}\chi_0-\theta\ . \tag{4·28}$$

Note that Eq. (4·28) is in agreement with the modified form of the partial conservation of the $U(1)$ current[13] [Eqs. (2·12) and (2·14)]

$$2L\langle\partial^\mu K_\mu\rangle_{\rm phys}=F_s(m_s{}^2-m^2)\Big(\langle S\rangle_{\rm phys}-\frac{F_s}{2L}\theta\Big),$$

which could be obtained also if we could get the "equation of motion" for $\partial^\mu K_\mu$ directly by taking the variations of the $U(1)$ Lagrangian with respect to $\partial^\mu K_\mu$ itself.

§ 5. Discussion

We have examined the vacuum structure of our effective $U(1)$ Lagrangian, which is very similar to that of Schwinger model.[14] In order to make chiral charges well-defined, we have proposed to define them as the generators of the corresponding transformations. The chiral charges Q_1, Q_2 and Q_3 are then quantized in even integer units. This fact implies that

$$\sqrt{\frac{3}{2}}\tilde{Q}_5^{(0)}=Q_1+Q_2+Q_3\,, \tag{5·1}$$

is also quantized in even integer units. *If we could neglect the matter contribution* $Q_5^{(0)}$, this would imply that ν is quantized in $1/3$ (or $1/L$ in general) units. However, this is not allowed. In fact, ν is quantized in integer units although this is not related to instantons in view of the θ-dependence of "pure Yang-Mills theory" (large N limit). The total physical space is constructed by using these charge eigenstates and the conventional Fock space.

Let us finally examine Crewther's argument[11] for fractional ν by the example in § 3.2. He writes the two degenerate vacuum states near $\theta=\pi$ as

$$|{\rm vac}\rangle_1=\sum_m e^{im\pi}|m\rangle\,, \tag{5·2}$$

$$|{\rm vac}\rangle_2=\sum_m e^{-im\pi}|m\rangle\,, \tag{5·3}$$

in terms of "eigenstates" $|m\rangle$ of K with eigenvalue m. By requiring $|{\rm vac}\rangle_1 \neq e^{i\xi}|{\rm vac}\rangle_2$, i.e.,

$$\sum_m\{e^{im\pi}-e^{i(\xi-m\pi)}\}|m\rangle\neq 0\,, \tag{5·4}$$

he concludes that states of $m'-m\neq$integer are necessary. This reasoning depends on the equivalence of those states $|m\rangle$ appearing in Eqs. (5·2) and (5·3). However, they are also characterized by different chirality angles φ_i [see Eq. (3·9) and Fig. 1]*) and consist of different vector spaces in our model, as we have shown in § 4. This means that Eq. (5·4) is not valid, as several authors have recently suggested.[12),18)] These states are simply related by chiral trans-

*) To be precise, the vacuum states but not $|m\rangle$ are characterized by φ_i's.

formations and no fractional charges emerge. Our analysis indicates that the argument for fractional ν depends on neglecting matter sector.

It is particularly interesting to note that the periodicity in θ and "topological charge" quantization in integer units are essentially due to the coupling of the matter and "gluon" sector, which may be expected in naive quark annihilation picture.

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